1 What Do Engineers Do?

This textbook is concerned with the mathematical skills that are essential to students and practitioners of all branches of engineering. The primary purpose of this brief book is to enumerate and discuss *only those mathematical skills that engineers use most often*. These represent the math skills that both engineering students and engineering practitioners must be able to recall immediately and understand thoroughly to be successful engineers. We will not stray into esoteric topics in mathematics. If a topic is not encountered frequently in the daily study or practice of engineering, it is not discussed here. The goal of this book is to give the reader lasting and functional use of these essential mathematical skills, and to categorize the skills into functional groups to promote their retention.

Students studying for a batchelor of science degree in one of the several engineering specialties (i.e., electrical, mechanical, civil, biomedical, environmental, aeronautical, etc.) are required to complete several courses in general mathematics. Prior to enrolling in a university, students are generally required to have completed high school courses covering algebra and trigonometry. At the author's institution, which is fairly typical, students are required to complete a semester of differential calculus and a semester of integral calculus in the freshman year. In addition, a course in differential equations is required to be completed in the sophomore year. Many engineering programs also require a semester in the junior year on multivariable mathematics. All of this mathematical material represents a rather formidable amount of detail and can leave a student overwhelmed by its sheer volume. However, in a student's everyday studies in engineering, as well as in professional engineering practice, the vast majority of those skills and concepts are never used or are used so infrequently that they are quickly forgotten. The majority of the math skills that students will use on a frequent basis represents only a small portion of the math topics studied in their math courses. It therefore makes sense for students to concentrate on and commit to memory and immediate usability only those skills that are used frequently, deferring those math topics that will be used infrequently to being "looked-up" when they arise. The healthy view of the role of mathematics in engineering is as a tool to understand the behavior of the particular engineering system being studied—in the same way that language is a means of communicating. If the student is distracted in a particular

Essential Math Skills for Engineers, By Clayton R. Paul

engineering course by struggling to remember frequently encountered math skills, learning the particular engineering topic being studied will not take place. This book is intended to cover only those math skills used most frequently by engineering undergraduate students and the majority of engineering practitioners.

Even though students do not use all these math skills on a frequent basis, they nevertheless benefit from being exposed to the majority of the mathematical concepts and mathematical sophistication they will study in their math courses; engineering study is also intended to be an "education." Students who go on to graduate school and pursue advanced degrees in engineering will need more math skills and topics the farther they go. However, the success of students in a graduate engineering program will also rely primarily on their having a solid understanding of the undergraduate program, which requires a solid understanding of the basic mathematics covered in this book. But it is important to remember that the majority of undergraduate engineering students will not pursue a graduate engineering degree. Hence, both students and practitioners must firmly understand and be able easily to recall these few but critically important math skills if they are to be successful in their engineering studies and in their future engineering practice. Students tend to be "overwhelmed" by the vast body of mathematical knowledge they have encountered in their math courses and are not readily able to compartmentalize which skills are most important to commit to immediate memory. They have not completed their studies and are therefore not able to discern this for themselves. We teachers of engineering easily forget this important fact. We have been studying engineering for many years and it is abundantly clear which math skills are frequently encountered. Students do not possess our broad overall view of the situation.

It is for this reason that I have undertaken to write this brief textbook on the essential math skills required of engineers. I have been studying engineering for over 50 years and have been teaching the subject for over 40 years. Although the majority of that time was spent in academia, I have also spent a substantial portion of it involved in the concurrent practice of engineering in industry. It has become very clear to me after this lengthy experience that the primary attribute which determines whether a student will be successful in his or her engineering studies is the person's ability for immediate recall and successful use of this small but important body of math skills. This observation also applies to engineering professionals.

I am frequently asked the question: "What do engineers do?" I have reduced my answer to the following succinct summary:

> Engineers develop and analyze mathematical models of physical systems for the purpose of designing those physical systems to perform a specific task.

So the unique function of engineers is *design*. The next question that is asked is: "Why do they do that?" My answer to this question is:

> Engineers do that so that they can develop *insight* into the behavior of the physical system in order to construct a mathematical model which when implemented in a physical system will accomplish certain design goals.

Hence, *mathematics is at the heart of engineering design*. This makes it abundantly clear that any student or practitioner of engineering must be fluent in the mathematical skills that they encounter frequently. Being able to use mathematical skills alone will not make you a competent engineer, but not being able to use those skills will handicap your ability to become a competent engineer.

Engineering systems and the design problems associated with them are much too complicated today (and will no doubt become increasingly so in the future) to be understood by one's ordinary "life experiences." In fact, if this were so, engineering companies would not be at the top of industry jobs paying the highest salaries. Many students today seem to believe that the digital computer will make engineering easy and obviate the need for them to learn mathematics. Numerous computer programs exist that can solve (give a numerical answer to) the complicated mathematical equations that describe engineering systems. If the design of engineering systems were that simple, industries could, perhaps, simply pay minimum wage to someone and train the person (in a very short time) to use that computer program. But that would be missing the point of what engineers do. These computer programs give a numerical "answer," but they give little, if any, insight into how a particular engineering system behaves. To obtain that insight (which is critical to design), we must understand what the mathematics governing that system is telling us about its behavior. Without that understanding we would have to construct a physical prototype of the anticipated system and change the structure and parameters of that prototype endlessly and randomly, with no clear indication of how to arrive at an optimal solution to the design problem desired.

It is vitally important to keep in mind the important fact that the physical laws that govern the behavior of all physical systems are given in mathematical terms. Therefore, if we are to understand how to design that physical system to perform a particular task, we must understand how to solve these governing mathematical equations.

In the remaining part of this chapter I give examples of engineering systems and the equations that govern them. Examples such as these pervade all branches of engineering and are too numerous to give here, so I concentrate on a few disciplines: electrical engineering, mechanical engineering, and civil engineering. I do not go into how the equations governing a particular system are derived from the specific laws of the discipline, as that is the goal of your particular engineering courses. The goal in this book is to give you the mathematical tools to solve those equations in order to be able to understand what they are telling you about the behavior of that system.

Example 1 (Electrical Engineering) Electrical engineers are concerned with electrical systems whose important variables—voltage, current, and power—are to be determined for a particular interconnection of electric elements (an electric circuit). Electrical systems are governed by Maxwell's equations. These are complicated partial differential equations (Chapter 6). However, they can be approximated for a large majority of electrical systems by two simple governing equations: Kirchhoff's voltage law (KVL), which governs the voltages of a particular interconnection of electrical elements that comprise an electric circuit, and Kirchhoff's current law (KCL), which governs the currents of the circuit. The individual electrical elements of the circuit (resistors, capacitors, inductors, diodes, transistors, etc.) relate the element's voltage to its current.

The electric circuit shown in Figure 1.1 consists of a voltage source and resistors. It is desired to determine the value of resistor R such that the output voltage is $V = 3$V. The circuit can be "solved" by writing the mesh-current equations as

$$7I_1 - 3I_2 = 10$$
$$-3I_1 + (5 + R)I_2 = 0$$

These are two simultaneous, linear, algebraic equations that can be solved using Cramer's rule (Chapter 3) as

$$I_2 = \frac{\begin{vmatrix} 7 & 10 \\ -3 & 0 \end{vmatrix}}{\begin{vmatrix} 7 & -3 \\ -3 & (5+R) \end{vmatrix}} = \frac{30}{26 + 7R}$$

The desired voltage is $V = 3$V and $V = RI_2$. Substituting $I_2 = 3/R$ gives $R = 78/9\,\Omega$.

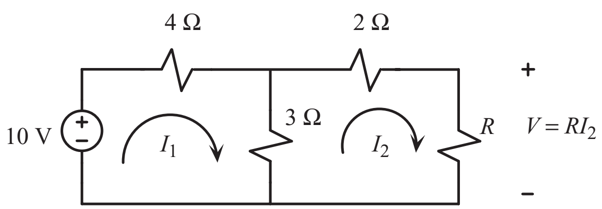

Fig. 1.1. Example 1: electrical engineering.

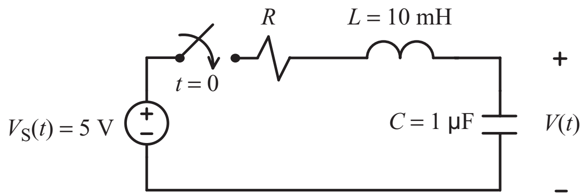

Fig. 1.2. Example 2: electrical engineering.

Example 2 (Electrical Engineering) The electric circuit shown in Figure 1.2 is composed of a 5-V voltage source, a resistor whose value is to be determined, a 10-mH inductor, a 1-μF capacitor, and a switch that closes at $t = 0$. The design task here is to determine the value of R and perhaps add other components so that the output voltage $V(t)$ rises smoothly and rapidly to 5V at $t = 1$ ms. The differential equation relating the output voltage to the source voltage is

$$\frac{d^2V(t)}{dt^2} + \frac{R}{L}\frac{dV(t)}{dt} + \frac{1}{LC}V(t) = \frac{1}{LC}V_S(t)$$

The complete solution requires two initial conditions. The initial conditions at $t = 0^+$ (just after the switch closes) are $V(0^+) = 0$ and $dV(t)/dt|_{t=0^+} = 0$. This is a second-order, linear, constant-coefficient, ordinary differential equation whose solution we determine in Chapter 4. The general form of the solution is

$$V(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} + 5 \qquad t \geq 0$$

where the (as yet) undetermined constants C_1 and C_2 are determined by applying the two initial conditions. The exponents of the exponential terms, p_1 and p_2, are roots of the quadratic equation

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = (p - p_1)(p - p_2) = 0$$

We solve this quadratic equation in Chapter 2, giving the two roots as

$$p_1,\ p_2 = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

In Chapter 2 we also investigate the exponential function e^{pt}, which arises quite frequently throughout engineering. There are three possibilities for these two roots:

Case I: roots real and distinct:

$$p_1 \neq p_2 \quad \text{if} \left(\frac{R}{L}\right)^2 > \frac{4}{LC}$$

Case II: roots real and equal:

$$p_1 = p_2 = p \quad \text{if} \left(\frac{R}{L}\right)^2 = \frac{4}{LC}$$

Case III: roots complex:

$$p_1,\, p_2 = \alpha \pm j\beta \quad \text{if} \left(\frac{R}{L}\right)^2 < \frac{4}{LC}$$

Cases I and II rise smoothly to the final value of 5V, but case II does so very rapidly. Its general solution is

$$V(t) = C_1 e^{pt} + tC_2 e^{pt} + 5$$

However, case III has an oscillatory behavior, with the solution oscillating about the final value of 5V but eventually settling down to the desired value of 5V:

$$V(t) = K_1 e^{\alpha t} \sin \beta t + K_2 e^{\alpha t} \cos \beta t + 5$$

This is a very undesirable result, so we choose case II. For the two roots to be identical, we must have

$$\left(\frac{R}{L}\right)^2 = \frac{4}{LC}$$

or

$$R = 2\sqrt{\frac{L}{C}}$$

The two roots will be $p_1 = p_2 = p = -R/2L$ and hence are negative, so that the two exponential pieces of the solution will decay to zero, leaving the desired solution of 5V. These two exponential parts of the solution will have decayed sufficiently to be considered close enough to zero in a time of approximately

$$\frac{5}{|p|} = 10\frac{L}{R} = 1 \times 10^{-3}\ \text{s}$$

Substituting the value of $L = 10 \times 10^{-3}$ gives the desired value of $R = 100\,\Omega$. Solving $R = 2\sqrt{L/C}$ with the now known values of R and L requires that $C = 4\,\mu\text{F}$. Therefore, we must add another 3-μF capacitor in parallel with the existing 1-μF capacitor. (Capacitors in parallel add like resistors in series.) A plot of the response with these values from $t = 0$ to $t = 2\,\text{ms}$ is shown in Figure 1.3, indicating a successful design.

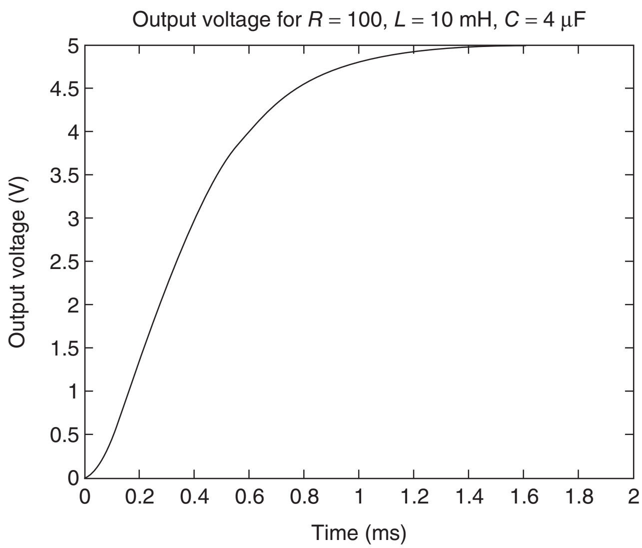

Fig. 1.3. The solution for repeated roots: $R = 100\,\Omega$, $L = 10\,\text{mH}$, $C = 4\,\mu\text{F}$.

The plots in Figure 1.4 compare how the solution behaves for

$R = 100\,\Omega,\quad L = 10\,\text{mH},\quad C = 4\,\mu\text{F}$ (case II: the desired solution)

$R = 100\,\Omega,\quad L = 10\,\text{mH},\quad C = 10\,\mu\text{F}$ (case I: distinct roots)

$R = 100\,\Omega,\quad L = 10\,\text{mH},\quad C = 0.05\,\mu\text{F}$ (case III: the oscillatory solution)

This example has illustrated clearly that being able to solve the differential equation where the coefficients are symbols (one of the objectives of this book) as opposed to a numerical solution (solving the differential equation where the coefficients are numbers) has allowed us to go immediately to the optimum solution for the design rather than guessing a set of element values and solving for a numerical result, changing our guess, and re-solving and continuing in a never-ending fashion with no hope of arriving quickly at the optimum solution. This clearly illustrates that simply being able to solve a specific differential equation whose coefficients are numbers using a computer is not the answer to an effective and efficient method of engineering design. A numerical computer solution is simply an analysis of a given situation; design is the synthesis of a solution.

Example 3 (Mechanical Engineering) In dynamic mechanical engineering systems (systems that vary with time) the important variables are position, velocity, acceleration, and force. The equations describing such a system are obtained from free-body diagrams and D'Alembert's principle or Newton's

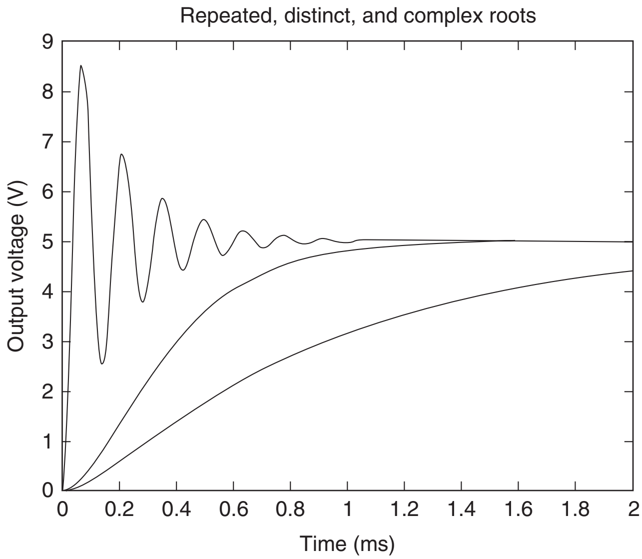

Fig. 1.4. Comparison of the solution for the three cases.

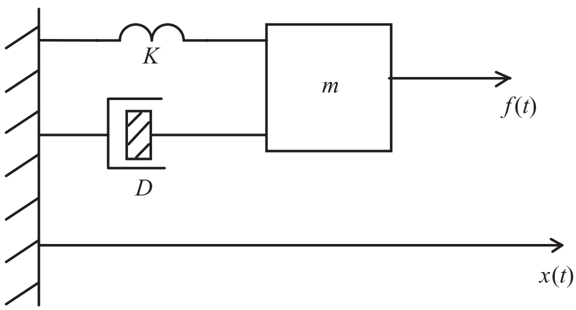

Fig. 1.5. Dynamic model of a shock absorber.

laws. An example of a possible automobile shock absorber system is shown in Figure 1.5. The mass of the upper part of the automobile is represented by the block containing m, the spring connected between the axle and the body is represented by K, and the shock absorber is represented by D. The differential equation relating the position of the upper part of the automobile, $x(t)$, to the force applied is

$$\frac{d^2x(t)}{dt^2}+\frac{D}{m}\frac{dx(t)}{dt}+\frac{K}{m}=\frac{1}{m}f(t)$$

When the automobile strikes a bump in the road, it is given an initial displacement and initial velocity upward. This is again a second-order, linear, constant-coefficient ordinary differential equation that we study in Chapter 4. It is also identical in form to that of the electric circuit investigated previously. To provide a smooth ride for passengers, the vehicle position should return rapidly and smoothly to the original position with minimal oscillatory movement about the steady-state position. Hence, the general forms of the solution for the position $x(t)$ are very similar to the electrical circuit problem. The exponential parts of the solution will again have three forms, depending on whether the roots of the characteristic equation

$$p^2 + \frac{D}{m}p + \frac{K}{m} = (p - p_1)(p - p_2) = 0$$

which are

$$p_1,\ p_2 = -\frac{D}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{D}{m}\right)^2 - 4\frac{K}{m}}$$

are real and distinct, real repeated, or complex:

Case I: roots real and distinct:

$$p_1 \neq p_2 \quad \text{if } \left(\frac{D}{m}\right)^2 > 4\frac{K}{m}$$

Case II: roots real and equal:

$$p_1 = p_2 = p \quad \text{if } \left(\frac{D}{m}\right)^2 = 4\frac{K}{m}$$

Case III: roots complex:

$$p_1,\ p_2 = \alpha \pm j\beta \quad \text{if } \left(\frac{D}{m}\right)^2 < 4\frac{K}{m}$$

Once again, for the vehicle body to return smoothly and rapidly to its original position, with no oscillation, the desired case is case II for repeated roots, from which we can determine the desired relation between K, D, and m.

Example 4 (Mechanical Engineering) Mechanical engineers are frequently concerned with vibrations of beams, rods, and tension cables. Strong oscillations can destroy a system. To describe the general problem, we focus on the vibrating string problem. A string or cable has its two endpoints held fixed, and it is given an initial upward displacement and velocity of displacement as shown in Figure 1.6. The variable $u(x, t)$ gives the vertical position of the string

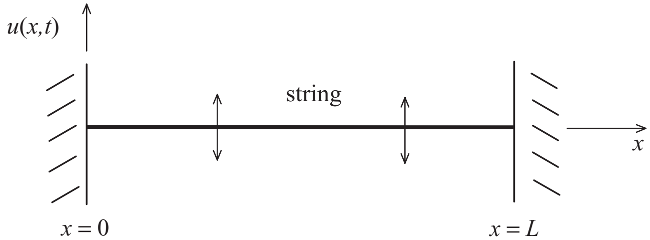

Fig. 1.6. The vibrating string problem.

as a function of time t and position along the string x. This problem is governed by the "wave equation":

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

This is a linear, constant-coefficient, partial differential equation (PDE) whose solution process we study in Chapter 6. Partial derivatives are required here because the displacement of the string u is a function of two independent variables, t (time) and x (position along the string). The general solution to this PDE is of the form

$$u(x,t) = f^+(x - vt) + f^-(x + vt)$$

The functions $f^+(x - vt)$ and $f^-(x + vt)$ are to be determined by the boundary conditions and are said to be "traveling waves" in the same fashion as when two people hold the ends of a jump rope and one gives a rapid snap to that end to the rope. The function $f^+(x - vt)$ is a wave traveling in the $+x$ direction, and the function $f^-(x + vt)$ is a wave traveling in the $-x$ direction. Observe that the solutions can only be a function of x, v, and t as $x - vt$ and $x + vt$. The combination of these two oppositely traveling waves results in a "standing wave" such as that seen in a vibrating violin string.

Example 5 (Civil Engineering) One of the many design problems that civil engineers encounter is the design of bridge trusses. A simple such structure is shown in Figure 1.7. The stresses in each member shown as vectors T_1, T_2, and T_3 are to be determined. The structure is symmetric about its middle so that the stress vectors in the members of the right half of the structure are not shown but are the same as those in the corresponding members of the left half of the structure. A car is stopped at the middle of the bridge and exerts a force F downward at that point. The structure is assumed to be in equilibrium. Summing the vertical force components and summing the horizontal force components at points A, B, and C and using trigonometry (Chapter 2) gives four equations:

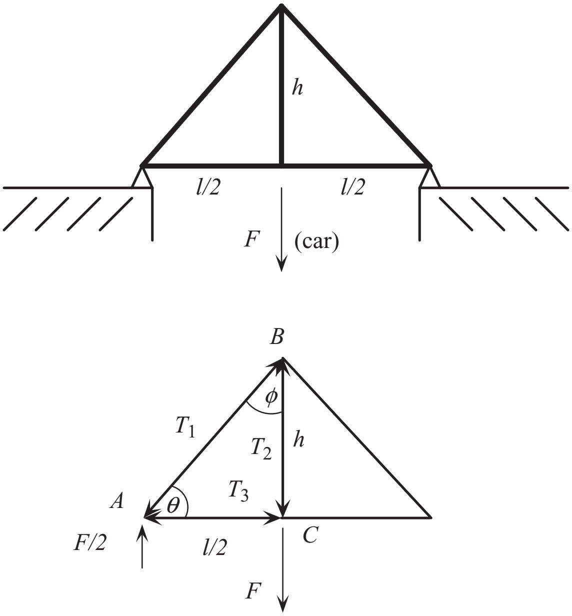

Fig. 1.7. Truss design.

$$-T_1 \sin\theta + \frac{F}{2} = 0$$
$$-T_1 \cos\theta - T_3 = 0$$
$$2T_1 \cos\phi + T_2 = 0$$
$$-T_2 - F = 0$$

The dimensions of the truss members, h and l, are known, as is the force exerted by the car's weight, F, so there are five unknowns: T_1, T_2, T_3, θ, and ϕ. From trigonometry (Chapter 2) we see that $\theta = \tan^{-1}(h/(l/2))$, $\phi = \tan^{-1}((l/2)/h)$, and $\theta + \phi = 90°$. We develop the following trigonometric identity in Chapter 2:

$$\cos\phi = \cos(90° - \theta)$$
$$= \sin\theta$$

Substituting this to eliminate ϕ and eliminating redundant equations gives two equations:

$$2T_1 \sin\theta = F$$
$$T_1 \cos\theta + T_3 = 0$$

Therefore, there are only two unknowns, T_1 and T_3. Arranging the equations above in matrix form gives

$$\begin{bmatrix} 2\sin\theta & 0 \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_3 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

which are simultaneous, linear, algebraic equations that can be solved by the methods of Chapter 3. These simple equations are rather easy to solve by substitution, giving $T_1 = F/2\sin\theta$ (a tensile stress) and $T_3 = -T_1\cos\theta$ (a compressive stress). But more complicated structures having many beams result in large systems of simultaneous equations that require systematic solution methods which we study in Chapter 3.

Example 6 (Discrete-Time Systems and the Approximate Solution of Differential Equations) The digital technology of today allows powerful computing and data processing. Analog-to-digital converters (A-D converters) allow analog signals to be converted to digital data, which can be processed by computers much faster than by the analog processing used generally a few years ago. Similarly, digital computers can provide rapid but approximate solutions to differential equations (even nonlinear ones, most of which are virtually impossible to solve by hand) by discretizing the variables into discrete increments. For example, consider the first-order ordinary differential equation

$$\frac{dy(t)}{dt} + ay(t) = K$$

where a and K are known. The general solution to this is determined in Chapter 4 subject to the known initial condition $y(0)$ as

$$y(t) = \left[y(0) - \frac{K}{a} \right] e^{-at} + \frac{K}{a}$$

Instead, let us solve this by breaking the t (time) axis into equal increments Δt and approximate the derivative as

$$\frac{dy(t)}{dt} \cong \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

Substituting this into the original differential equation gives the "recursion relation"

$$y_{n+1} = (1 - a\Delta t)\, y_n + \Delta t\, K$$

where we have denoted the value of y at each of the discrete times $t_n = n\Delta t$ as

$$y_n \equiv y(n\Delta t)$$

We can solve these *recursively* as

$$y_1 = (1 - a\,\Delta t)\,y_0 + \Delta t\,K$$
$$y_2 = (1 - a\,\Delta t)\,y_1 + \Delta t\,K$$
$$y_3 = (1 - a\,\Delta t)\,y_2 + \Delta t\,K$$
$$\vdots$$

Solving the first equation [a, $y_0 = y(0)$, Δt, and K are known] gives y_1. Substituting this into the second equation, we can solve for y_2, and so on. Thus, we can arrive at an approximation in a "marching in time" fashion. But to achieve sufficient accuracy, the time increment Δt must be chosen small enough. Hence, we may need very many calculations to reach the solution at a desired time. This recursion relation can be written as a difference equation (Chapter 5) by placing the unknowns on the left and the known quantities on the right:

$$y_{n+1} - (1 - a\,\Delta t)\,y_n = \Delta t\,K$$

In Chapter 5 we determine how to obtain a closed-form equation for the answer *at any time increment*:

$$y_n = \left(y_0 - \frac{K}{a}\right)(1 - a\,\Delta t)^n + \frac{K}{a}$$

without having to go through the iterative process (i.e., by going directly to the answer)!

The examples above have shown two important things. First, the equations governing engineering systems fall into a few general categories (simultaneous algebraic equations, ordinary differential equations, difference equations, and partial differential equations). To design engineering systems to accomplish a specific task, it is imperative that you know how to solve these types of equations, and because of the frequency with which you will encounter them, you must be able to recognize them immediately and must have immediate recall of that solution ability. Otherwise, you will have little hope of becoming competent engineers. Second, there are a number of miscellaneous math skills, such as trigonometry and logarithms, that are involved routinely in their solution, and the interpretation of that solution (which is critical to your obtaining "engineering insight" into how engineering systems behave). This book is dedicated to giving you those math skills. If you wish to become a competent engineer, you must learn the essential math skills of this book and have immediate recall of them. If you do, you will proceed successfully toward your goal of becoming a competent engineer. With a mastery of the math skills in this book, anyone can become a competent engineer. Without this mastery, it is doubtful that you will achieve your goal.

When you study this book it is important that you adopt the following attitude about learning the material.

> *DO NOT take the attitude that you only need to memorize the equations and their solution. If you do, you will be no better off than when you started since that mode of thinking will not provide you with the long-term ability to use these important math skills and they will be quickly forgotten. Try to understand, in simple terms, what the equations and their solution are "trying to tell you" and why that makes sense. This is how you will develop engineering "insight" and become successful engineers. I will not "prove" the results in the traditional mathematical meaning of a "rigorous" proof. Instead, I will provide you with simple, "commonsense" explanations of the logical meaning of the equations and their solution, as well as simple and "commonsense" methods for obtaining their solution. If you try to think through my commonsense explanations of the solution process, you should be able to extend these results to obtain other useful results on your own. If you adopt this attitude of study, you will achieve the desired long-term retention of these math skills and will be able to use them effectively in your daily study or work.*

I will place a box around certain important results. This is not intended to mean that the result is something you should memorize but is something you should focus on.