

1

Accelerated Life Testing

Vilijandas Bagdonavičius and Mikhail Nikulin

1.1 Introduction

Failures of highly reliable units are rare and it may be not possible to gather the failure time data needed for reliability estimation. One way of obtaining failures during the time given for experiments is to apply methods of *accelerated life testing (ALT)*. In ALT units are tested at higher than usual (design) stress conditions.

The purpose of accelerated life testing (ALT) is to give estimators of the main reliability characteristics of units under usual (design) stress using data of accelerated experiments when units are tested at higher than usual stress conditions. Examples of stresses or accelerating factors are numerous: voltage, temperature, humidity, load, and so on.

Statistical inference from ALT can be done using models that describe dependence of the lifetime distribution on stress. A number of such models were proposed by engineers who considered physics of failure formation process of certain products or by statisticians (see Refs. [1-3,5-8,10-15,20,21,24-26,29-33,36,40-42]).

The lifetime distribution under any stress can be defined by many functions such as the cumulative distribution function, the survival function, and the probability density function. But the sense of

accelerated life models is best seen if they are formulated in terms of the hazard rate function.

Suppose at first that a stress x is a deterministic possibly multidimensional time function:

$$x(t) = (x_1(t), \dots, x_m(t))^T, \quad t \geq 0,$$

where x_i is a univariate stress component.

Denote by T the failure time and

$$S(t|x) = \mathbf{P}\{T > t|x\}, \quad t \geq 0,$$

the *survival function* of T given x . Let $F(t|x) = 1 - S(t|x)$ be the *cumulative distribution function* of T given x . The *conditional hazard function* of T given x is

$$\begin{aligned} \lambda(t|x) &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbf{P}\{T \in (t, t+h] \mid T > t, x\} \\ &= -\frac{S'(t|x)}{S(t|x)}. \end{aligned}$$

Denote by

$$\Lambda(t|x) = \int_0^t \lambda(u|x) du = -\ln\{S(t|x)\}$$

the *conditional cumulative hazard function* of T given x .

Each specified accelerated life model relates the hazard function (or other function) to the stress in some particular way.

2 Accelerated Life Testing

We say that a stress x_2 is higher than a stress x_1 and we write $x_2 > x_1$, if for any $t \geq 0$ the inequality $S(t|x_1) \geq S(t|x_2)$ holds and exists $t_0 > 0$ such that $S(t_0|x_1) > S(t_0|x_2)$. We say also that the stress x_2 is *accelerated* with respect to the stress x_1 .

1.2 Types of Stresses Used in ALT

The types of stresses most often used in ALT are: constant in time stresses, step-stresses, progressive (monotone) stresses, cyclic stresses, and random stresses.

The most common case is when the stress is constant and one-dimensional but more than one accelerated stress may be used.

Sometimes step-stresses are used: units are placed on test at an initial low stress and if they do not fail in a predetermined time t_1 , the stress is increased. If they do not fail in a predetermined time $t_2 > t_1$, the stress is increased once more, and so on. Thus step-stresses have the form

$$x(u) = \begin{cases} x_1, & 0 \leq u < t_1, \\ x_2, & t_1 \leq u < t_2, \\ \dots & \dots \\ x_m, & t_{m-1} \leq u < t_m, \end{cases} \quad (1)$$

where x_1, \dots, x_m are constant stresses. In particular, step-stresses of the form

$$x(u) = \begin{cases} x_1, & 0 \leq u < t_1, \\ x_2, & u \geq t_1, \end{cases} \quad (2)$$

are used.

1.3 Sedyakin's Principle and Model

Accelerated life models could be at first formulated for constant explanatory variables. Nevertheless, before formulating them, let us consider a method for generalizing such models to the case of time-varying stresses.

In 1966, Sedyakin formulated his famous *physical principle in reliability*, which states that for two identical populations of units functioning under different stresses x_1 and x_2 , two moments t_1 and t_2 are equivalent if the probabilities of survival until these moments are equal:

$$\begin{aligned} \mathbf{P}\{T > t_1|x_1\} \\ &= S(t_1|x_1) = S(t_2|x_2) \\ &= \mathbf{P}\{T > t_2|x_2\}, \quad x_1, x_2 \in E_1. \end{aligned}$$

If after these equivalent moments the units of both groups are observed under the same stress x_2 , i.e., the first population is observed under the step-stress x of the form (2) and the second all time under the constant stress x_2 , then for all $s > 0$

$$\lambda(t_1 + s|x) = \lambda(t_2 + s|x_2). \quad (3)$$

Bagdonavičius [1] generalized the model of Sedyakin to the case of any time-varying stresses by supposing that the hazard rate $\lambda(t|x)$ at any moment t is a function of the value of the stress at this moment and of the probability of survival until this moment. It is formalized by the following definition.

Definition 1.3.1 We say that *Sedyakin's model* (SM) holds on a set of stresses E if there exists on $E \times \mathbf{R}^+$ a positive function g such that for all $x \in E$

$$\lambda(t|x) = g(x(t), S(t|x)). \quad (4)$$

The fact that the SM does not give relations between the survival under different constant stresses is a cause of non-applicability of this model for estimation of reliability under the design (usual) stress from accelerated experiments. On the other hand, restrictions of this model when not only rule (4) but also some relations between survival under different constant stresses are assumed can be considered. These narrower models can be formulated

by using models for constant stresses and rule (4). For example, the well-known and mostly used accelerated failure time model for time-varying stresses is a restriction of the SM when the survival functions under constant stresses differ only in scale.

1.4 Model of Sedyakin for Step-Stresses

The mostly used time-varying stresses in accelerated life testing are step-stresses (1) or (2). Let us consider the meaning of rule (4) for these step-stresses.

Proposition 1.4.1 *If the SM holds for stresses of the form (2), then the survival function and the hazard rate under such stress satisfy the equalities*

$$S(t|x) = \begin{cases} S(t|x_1), & 0 \leq t < t_1, \\ S(t - t_1 + t_1^*|x_2), & t \geq t_1, \end{cases} \quad (5)$$

and

$$\lambda(t|x) = \begin{cases} \lambda(t|x_1), & 0 \leq t < t_1, \\ \lambda(t - t_1 + t_1^*|x_2), & t \geq t_1, \end{cases} \quad (6)$$

respectively; the moment t_1^* is determined by the equality $S(t_1|x_1) = S(t_1^*|x_2)$.

We see that in the SM the survival function under the step-stress is obtained from the survival functions under constant stresses by the rule of time-shift.

From the proposition it follows that for all $s \geq 0$

$$\lambda(t_1 + s|x) = \lambda(t_1^* + s|x_2). \quad (7)$$

It is the model proposed by Sedyakin [36].

Set $t_0 = 0$. The rule of time-shift holds for stepwise stresses of the form (1), too.

Proposition 1.4.2 *If the SM holds for stresses of the form (1), then the survival function $S(t|x)$ under such stress satisfies*

the equalities:

$$S(t|x) = S(t - t_{i-1} + t_{i-1}^*|x_i), \quad \text{if} \\ t \in [t_{i-1}, t_i], \quad i = 1, 2, \dots, m, \quad (8)$$

where t_i^* satisfy the equations

$$S(t_1|x_1) \\ = S(t_1^*|x_2), \dots, S(t_i - t_{i-1} + t_{i-1}^*|x_i) \\ = S(t_i^*|x_{i+1}), \quad i = 1, \dots, m - 1. \quad (9)$$

In the literature on ALT (see Refs. [11,31,29]) the last model is also called the *basic cumulative exposure model*.

We note here that the SM can be inappropriate in situations of periodic and quick change of the stress level or when switch-up of the stress from one level to another can imply failures or shorten the life; see Bagdonavičius and Nikulin [5] for models with cyclic stresses.

1.5 Accelerated Failure Time Model

The most used model in ALT is the *accelerated failure time (AFT)* model. We start from the definition of this model for constant stresses.

Suppose that under different constant stresses x from a set E_1 the survival functions differ only in scale:

$$S(t|x) = S_0\{r(x)t\}, \quad (10)$$

where the survival function S_0 , called the *baseline survival function*, does not depend on x . For any fixed x the value $r(x)$ can be interpreted as the *time-scale change constant* or the acceleration (deceleration) constant.

Applicability of this model in accelerated life testing was first noted by Pieruschka [45]. It is the most simple and the most used model in failure time regression data analysis and ALT.

Under the AFT model on E_1 the conditional distribution of the random variable

$$R = r(x)T$$

4 Accelerated Life Testing

given x does not depend on $x \in E_1$ and its survival function is S_0 . Denote by m and σ^2 the mean and the variance of R , respectively. In this notation we have

$$\mathbf{E}(T|x) = m/r(x), \quad \mathbf{Var}(T|x) = \sigma^2/r^2(x),$$

and hence the coefficient of variation

$$\frac{\mathbf{E}(T|x)}{\sqrt{\mathbf{Var}(T|x)}} = \frac{m}{\sigma}$$

does not depend on x .

The survival functions under any $x_1, x_2 \in E_1$ are related in the following way:

$$S(t|x_2) = S(\rho(x_1, x_2)t|x_1),$$

where the function $\rho(x_1, x_2) = r(x_2)/r(x_1)$ shows the degree of scale variation. It is evident that $\rho(x, x) = 1$.

Bagdonavičius [1] showed that if Sedyakin's model holds on a set E of time-varying stresses and the AFT model holds on the subset $E_1 \subset E$ of constant over time stresses, then for any $x \in E$

$$S(t|x) = S_0 \left(\int_0^t r\{x(u)\} du \right). \quad (11)$$

Definition 1.5.1 The AFT model holds on E if there exists on E a positive function r and a survival function S_0 such that for any $x \in E$ the equality (11) holds.

The next two proposals give the forms of the survival functions under step-stresses in the AFT model.

Proposition 1.5.1 If the AFT model holds for stresses of the form (2), then the survival function under any such stress verifies the equality

$$S(t|x) = \begin{cases} S(t|x_1), & 0 \leq t < t_1, \\ S(t - t_1 + t_1^*|x_2), & t \geq t_1, \end{cases} \quad (12)$$

where

$$t_1^* = \frac{r(x_1)}{r(x_2)} t_1. \quad (13)$$

Proposition 1.5.2 If the AFT model holds for stresses of the form (1), then the survival function under any such stress verifies the equalities:

$$S(t|x) = S_0 \left\{ \sum_{j=1}^{i-1} r(x_j)(t_j - t_{j-1}) + r(x_i)(t - t_{i-1}) \right\}, \\ t \in [t_{i-1}, t_i], \quad i = 1, 2, \dots, m.$$

1.6 Parametric, Semiparametric, and Nonparametric AFT Models

The AFT models (10) or (11) are nonparametric if the functions $S_0(t)$ and $r(x)$ are unknown.

Often the function r is written in the following parametric form:

$$r(x) = e^{-\beta^T z}, \quad (14)$$

where $\beta = (\beta_0, \dots, \beta_m)^T$ is a vector of unknown parameters, and

$$z = (1, z_1, \dots, z_m)^T \\ = (1, \varphi_1(x), \dots, \varphi_m(x))^T$$

is a vector of specified functions φ_i of stresses. In this case the AFT model is given by the next formula:

$$S(t|x) = S_0 \left(\int_0^t e^{-\beta^T z(u)} du \right), \quad x \in E. \quad (15)$$

If stresses are constant over time, then

$$S(t|x) = S_0 \left(e^{-\beta^T z} t \right), \quad x, x_j \in E_1, \quad (16)$$

and the logarithm of the failure time T under x may be written as

$$\ln\{T\} = \beta^T z + \varepsilon, \quad (17)$$

where the survival function of the random variable ε is $S(t) = S_0(\ln t)$. It does not depend on x .

The choice of the functions $\varphi_j(x)$ is very important in accelerated life testing because the *usual stress is not in the region of the stresses used in the experiment*, and bad choice of the model may give bad estimators of the reliability characteristics under the usual stress.

Let us discuss the choice of the functions φ_i for univariate x . Suppose that $m = 1$. Then $z = (1, z_1)$, $z_1 = \varphi(x)$. The most applied examples are:

1. *Log-linear model:* $z_1 = x$. Then $e^{-\beta^T z} = e^{-\beta_0 - \beta_1 x}$.
2. *Power rule model:* $z_1 = \ln x$. Then $e^{-\beta^T z} = \alpha x^{-\beta_1}$.
3. *Arrhenius model:* $z_1 = 1/x$. Then $e^{-\beta^T z} = e^{\beta_0 - \beta_1/x}$.
4. *Meeker-Luvalle model:* $z_1 = \ln \frac{x}{1-x}$. Then $e^{-\beta^T z} = \alpha \left(\frac{x}{1-x} \right)^{-\beta_1}$.

The Arrhenius model is used to model product life when the explanatory variable is the temperature; the power rule model when the explanatory variable is voltage, mechanical loading; and the log-linear model is applied in endurance and fatigue data analysis, testing various electronic components. The model of Meeker-Luvalle is used when x is the proportion of humidity.

Sometimes a better choice is taking $m = 2$. Then $z = (1, z_1, z_2)^T$.

5. *Eyring model:* $z_1 = \ln x$, $z_2 = 1/x$, $\beta_1 = 1$. Then $e^{-\beta^T z} = e^{-\beta_0 - \beta_1 z_1 - \beta_2 z_2} = \alpha x e^{-\beta_2/x}$. The Eyring model is often applied when the explanatory variable x is the temperature.

In dependence on the choice of the function φ , the power rule AFT model, the Ar-

rhenius AFT model, and other models may be considered.

The AFT models (15) or (16) may be considered as semiparametric (if the function S_0 is unknown) or parametric (the function S_0 belongs to some parametric class of survival functions).

The parametric AFT model is usually written in the following way. Suppose that S_0 belongs to a specified class of survival functions:

$$S_0(t, \gamma), \quad \gamma \in G \subset \mathbf{R}^q.$$

The parametric AFT model has the form

$$S(t|x) = S_0 \left(\int_0^t e^{-\beta^T z(u)} du, \gamma \right). \quad (18)$$

If the covariates are constant over time, then

$$S(t|z) = S_0 \left(e^{-\beta^T z} t, \gamma \right). \quad (19)$$

Exponential, Weibull, loglogistic, and log-normal families of the survival functions S_0 are mostly used. So the Weibull-power rule AFT model, loglogistic-Arrhenius AFT model, and other models may be considered.

1.7 Time-Dependent Regression Coefficients

More flexible models can be obtained by supposing that the coefficients β are time-dependent, considering the model

$$S(t|x) = S_0 \left\{ \int_0^t e^{-\beta^T(u)x(u)} du \right\}. \quad (20)$$

It is the AFT model with time-dependent regression coefficients. Usually the coefficients $\beta_i(t)$ are used in the form

$$\beta_i(t) = \beta_i + \gamma_i g_i(t), \quad (i = 1, 2, \dots, m),$$

where $g_i(t)$ are some specified deterministic functions or realizations of predictable processes. In such a case the AFT model with

6 Accelerated Life Testing

time-dependent coefficients and constant or time-dependent stresses can be written in the usual form (15) with different interpretation of the stresses. Indeed, set

$$\begin{aligned}\theta &= (\theta_0, \theta_1, \dots, \theta_{2m})^T \\ &= (\beta_0, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_m)^T, \\ z &= (1, z_1, \dots, z_{2m})^T \\ &= (1, x_1, \dots, x_m, x_1 g_1, \dots, \\ &\quad x_m g_m)^T.\end{aligned}\quad (21)$$

Then

$$\begin{aligned}\beta^T(u)x(u) &= \beta_0 + \sum_{i=1}^m (\beta_i + \gamma_i g_i(t))x_i(t) \\ &= \theta^T z(u).\end{aligned}$$

So the AFT model with the time-dependent regression coefficients can be written in the standard form

$$S(t|x) = S_0 \left\{ \int_0^t e^{-\theta^T z(u)} du \right\}, \quad (22)$$

where the unknown parameters and the explanatory variables are defined by (21).

1.8 Plans of Experiments and Estimation in ALT under the AFT Model

As was said before, the purpose of ALT is to give estimators of the main reliability characteristics: *the reliability function* $S(t|x^{(0)})$, *the p -quantile* $t_p(x^{(0)})$ and *the mean value* $m(x_0)$ under usual (design) stress $x^{(0)}$, using data of accelerated experiments when units are tested at higher than usual stress conditions.

Different plans of experiments are used in ALT. We shall give the most used plans of experiments when the AFT model is used.

1.8.1 The First Plan of Experiments

Let $x_0 = (1, x_{01}, \dots, x_{0m})^T$ be the usual stress. Generally accelerated life testing experiments are done under a one-dimensional stress ($m = 1$), sometimes under two-dimensional ($m = 2$).

Let x_1, \dots, x_k be constant over time *accelerated stresses*:

$$x_0 < x_1 < \dots < x_k;$$

here $x_i = (1, x_{i1}, \dots, x_{im})^T$. The usual stress x_0 is *not used* during experiments.

According to the first plan of experiments k groups of units are tested. The i th group of n_i units, $\sum_{i=1}^k n_i = n$, is tested under the stress x_i . The data can be complete or right censored.

If the AFT model is considered as fully parametric, then it is written in the form (19) with the baseline survival function $S_0(t, \gamma)$ taken from some class of parametric distributions, such as Weibull, log-normal, loglogistic, etc., and z defined by the power rule, the Arrhenius, or another model. In this case the parameters β and γ are estimated using the parametric method of maximum likelihood. If $\hat{\beta}$ and $\hat{\gamma}$ are the ML estimators, then

$$\begin{aligned}\hat{S}(t|x_0) &= S_0 \left(e^{-\hat{\beta}^T z_0} t, \hat{\gamma} \right), \\ \hat{t}_p(x_0) &= e^{\hat{\beta}^T z_0} S_0^{-1}(1-p, \hat{\gamma}), \\ \hat{m}(x_0) &= e^{\hat{\beta}^T z_0} \int_0^\infty S_0(u, \hat{\gamma}) du,\end{aligned}\quad (23)$$

where $z_0 = (1, \varphi_1(x_0), \dots, \varphi_m(x_0))^T$. Asymptotic confidence intervals for the estimated reliability characteristics $S(t|x_0)$, $\hat{t}_p(x_0)$, and $\hat{m}(x_0)$ can also be considered.

Parametric AFT models with constant stresses were considered in Singpurwalla [39], Singpurwalla et al. [40], Klein and Basu [22,23], Glaser [18], Meeker [27], Nelson [31], Hirose [19], Rodrigues et al. [33], Meeker and Escobar [29], and others.

If the AFT model is considered as semi-parametric, then it is written in the form (16) with unknown baseline survival function $S_0(t)$. Methods of estimation of β , S_0 , and the reliability characteristics $S(t|x_0)$, $t_p(x_0)$, and $\hat{m}(x_0)$ are given in Schmoyer [34], Bagdonavičius and Nikulin [6].

If the form of the function r is completely unknown and this plan of experiments is used, the function S_0 cannot be estimated even if it is supposed to know a parametric family to which it belongs.

For example, if $S_0(t) = e^{-(t/\theta)^\lambda}$, then for constant stresses

$$S(t|x) = \exp \left\{ - \left(\frac{r(x)}{\theta} t \right)^\lambda \right\}, \quad x \in E_1.$$

Under the given plan of experiments the parameters

$$\lambda, \frac{r(x_1)}{\theta}, \dots, \frac{r(x_k)}{\theta}, \quad x_i \in E_1$$

and the functions $S(t|x_1), \dots, S(t|x_k)$ may be estimated. Nevertheless, the function $r(x)$ being completely unknown, the parameter $r(x_0)$ cannot be written as a known function of these estimated parameters. So $r(x_0)$ and, consequently, $S(t|x_0)$ cannot be estimated.

1.8.2 The Second Plan of Experiments

In step-stress accelerated life testing the second plan of experiments is as follows: n units are placed on test at an initial low stress and if it does not fail in a predetermined time t_1 , the stress is increased and so on. Thus, all units are tested under the step-stress x of the form:

$$x(u) = \begin{cases} x_1, & 0 \leq u < t_1, \\ x_2, & t_1 \leq u < t_2, \\ \dots & \dots \\ x_k, & t_{k-1} \leq u < t_k, \end{cases} \quad (24)$$

where $x_j = (1, x_{j1}, \dots, x_{jm})^T$, $t_0 = 0, t_k = \infty$.

In this case the function $r(x)$ should be also parameterized because, even when the usual stress is used until the moment t_1 , the data of failures occurring after this moment do not give any information about the reliability under the usual stress when the function $r(x)$ is unknown.

If the AFT model in considered as fully parametric, then it is written in the form (18) with the baseline survival function $S_0(t, \gamma)$ taken from some class of parametric distributions, such as Weibull, log-normal, loglogistic, etc., and z defined by the power rule, the Arrhenius, or another model.

From Proposition 2 it follows that the AFT model can be written in the form: if $t \in [t_{i-1}, t_i), i = 1, \dots, k$, then

$$S(t|x) = S_0 \left\{ \mathbf{1}_{\{i>1\}} \sum_{j=1}^{i-1} e^{-\beta^T z_j} (t_j - t_{j-1}) + e^{-\beta^T z_i} (t - t_{i-1}), \gamma \right\}. \quad (25)$$

$z_j = (1, \varphi_1(x_{j1}), \dots, \varphi_m(x_{jm}))^T$. In this case the parameters β and γ are estimated using the parametric method of maximum likelihood. If $\hat{\beta}, \hat{\gamma}$ are the ML estimators, then $S(t|x_0)$, $t_p(x_0)$, and $m(x_0)$ are estimated by (22). Asymptotic confidence intervals for these reliability characteristics can also be considered.

Parametric analysis of AFT models with step-stresses can be found in Shaked and Singpurwalla [38], Nelson [31], Meeker and Escobar [29], Balakrishnan et al. [10], and others.

If the AFT model is considered as semi-parametric, then it is written in the following form: if $t \in [t_{i-1}, t_i), i = 1, \dots, k$, then

$$S(t|x) = S_0 \left\{ \mathbf{1}_{\{i>1\}} \sum_{j=1}^{i-1} e^{-\beta^T z_j} (t_j - t_{j-1}) + e^{-\beta^T z_i} (t - t_{i-1}) \right\}, \quad (26)$$

with unknown baseline survival function $S_0(t)$. Methods of estimation of β , S_0 ,

and the reliability characteristics $S(t|x_0)$, $t_p(x_0)$, and $\hat{m}(x_0)$ are given in Bagdonavičius and Nikulin [6].

Application of the first two plans may not give satisfactory results because assumptions on the form of the function $r(x)$ are done. These assumptions cannot be statistically verified because of lack of experiments under the usual stress.

If the function $r(x)$ is completely unknown, and the coefficient of variation (defined as the ratio of the standard deviation and the mean) of failure times is not too large, the following plan of experiments may be used.

1.8.3 The Third Plan of Experiments

Suppose that the failure time under the usual stress x_0 takes large values and most of the failures occur after the moment t_2 given for the experiment. According to this plan two groups of units are tested:

- (a) the first group of n_1 units under a constant accelerated stress x_1 ;
- (b) the second group of n_2 units under a step-stress: time t_1 under x_1 , and after this moment under the usual stress x_0 until the moment t_2 , i.e., under the stress

$$x_2(t) = \begin{cases} x_1, & 0 \leq t \leq t_1, \\ x_0, & t_1 < t \leq t_2. \end{cases} \quad (27)$$

Units use much of their resources until the moment t_1 under the accelerated stress x_1 , so after the switch-up failures occur in the interval $[t_1, t_2]$ even under usual stress. Under this plan of experiments the AFT model is written in the form

$$\begin{aligned} S(u|x_1) &= S_0(ru|x_0), \\ S(u|x_2) &= S_0(r(u \wedge t_1) \\ &\quad + (u - t_1) \vee 0|x_0), \end{aligned} \quad (28)$$

where $r = r(x_1)/r(x_0)$, $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$.

Estimation of $S(t|x_0)$ when both functions $S_0(t)$ and $r(x)$ are completely unknown is considered in Bagdonavičius and Nikulin [6].

The third plan may be modified. The moment t_1 may be chosen as random. The most natural is to choose t_1 as the moment when the failures begin to occur.

1.9 Other Models

The AFT model is rather restrictive because under this model the failure time distributions under different constant stresses differ only in scale.

Parametric and semiparametric estimation procedures for the models different from the AFT model or for time-varying non-step stresses can be found in Viertl [42], Bhattacharyya and Stoejoeti [11], Bagdonavičius [2], Nelson [31], Doksum and Hoyland [15], Bagdonavičius and Nikulin [3,5,6], Meeker and Escobar [29], Bagdonavičius et al. [7], Chen [14], and Balakrishnan et al. [10].

A natural extension of the AFT model is the *changing shape and scale* (CHSS) model. For constant stresses [44,31,29] this extension is obtained by supposing that under different stresses not only scale but also shape parameters are different: there exist positive functions of stress $\theta(x)$ and $\alpha(x)$ such that for all stresses x from a set E_0

$$S(t|x) = S_0 \left((t/\theta(x))^{\alpha(x)} \right). \quad (29)$$

Meeker and Escobar [29] note that having $\alpha(x)$ increase is typical in fatigue data. As example, see low-cycle fatigue data on a nickel-base superalloy given in Nelson [31]. These data were also analyzed by Meeker and Escobar [29].

The CHSS model holds in a set E of time-varying stresses if for any $x \in E$ (see

Ref. [6])

$$S(t|x) = S_0 \left(\int_0^t r[x(u)] u^{\alpha(x(u))-1} du \right),$$

where $r, \alpha : E \rightarrow \mathbf{R}_+$. If the stress is constant, then the last model implies the model (28) with $\theta(x) = (\alpha(x)/r(x))^{1/\alpha(x)}$.

For parametric estimation methods using the CHSS model with constant stresses see Nelson [31] and Meeker and Escobar [29]; for parametric and nonparametric estimation methods with constant and step-stresses see Bagdonavičius and Nikulin [43].

A very famous survival regression model is the *proportional hazards (PH)* or Cox model, which means that the hazard function under any stress has the following form:

$$\lambda(t|x) = r\{x(t)\}\lambda_0(t),$$

where $\lambda_0(t)$ is baseline stress that does not depend on stress.

In the particular case of step-stresses the PH model is called the *tampered failure rate (TFR)* model (see Ref. [11]). A particular case of TFR model when the distribution of failure time under usual constant stress is Weibull was considered by Khamis and Higgins [21].

Note that application of the PH model when stresses are time varying is not natural because the PH model has “absence of memory” property: the hazard rate at any moment t depends only on stress applied at this moment and does not depend on the stress used before the moment t . If units are aging, then they use more of their resource (age quicker) under high stress than under low stress, which contradicts the “absence of memory” property.

Bagdonavičius and Nikulin [6] and Bagdonavičius et al. [7] considered *generalized proportional hazards (GPH)* models that “have memory”: the models have the form

$$\lambda(t|x) = r\{x(t)\}q\{S(t|x)\}\lambda_0(t),$$

where q (and possibly r) have specified forms. The factor $q\{S(t|x)\}$ shows dependence of the hazard rate on the “past”—the greater the stress applied in the time interval $[0, t]$ the smaller the value $S(t|x)$ of the survival function. Parametric estimation in ALT with step-stresses under the GPH models is given in Bagdonavičius et al. [7]. Semiparametric methods (r parameterized, λ_0 unknown) are given in Bagdonavičius and Nikulin [6].

Statistical methods for analysis of ALT when a process of production is unstable are given in Bagdonavičius and Nikulin [3], using modification of the principle of “heredity” in reliability (see Ref. [20]).

References

1. Bagdonavičius, V. (1978). Testing the hypothesis of the additive accumulation of damages. *Probab. Theory Appl.*, **23**, 403–408.
2. Bagdonavičius, V. (1990). Accelerated life models when the stress is not constant. *Kybernetika*, **26**, 289–295.
3. Bagdonavičius, V. and Nikulin, M. (1997). Accelerated life testing when a process of production is unstable, *Stat. Prob. Letters*, **35**, 269–275.
4. Bagdonavičius, V., Gerville-Réache, L., Nikoulina, V., and Nikulin, M. (2000). Expériences accélérées: analyse statistique du modèle standard de vie accéléré. *Revue de Statistique Appliquée*, XLVIII, 3, 5–38.
5. Bagdonavičius, V. and Nikulin, M. (2001). Estimation of cycling effect on reliability. In *Probability and Statistical Models with Applications*, Ch. Charalambides, M. Koutras, and N. Balakrishnan, eds., pp. 537–545. Chapman and Hall/CRC, London.
6. Bagdonavičius, V. and Nikulin, M. (2002). *Accelerated Life Models*. Chapman & Hall/CRC, Boca Raton.
7. Bagdonavičius, V., Gerville-Rache, L., and Nikulin, M. (2002). Parametric infer-

- ence for step-stress models. *IEEE Transactions on Reliability*, **51**, 27–31.
8. Bagdonavičius, V., Cheminade, O., and Nikulin, M. (2004). Statistical planning and inference in accelerated life testing using the CHSS model. *Journal of Statistical Planning and Inference*, **126**, 535–551.
 9. Bagdonavičius, V., Kruopis, J., and Nikulin, M. (2010). *Testing Hypotheses in Nonparametric Models. Complete and Censored Data*. ISTE & Wiley, London. (In production).
 10. Balakrishnan, N., Beutner, E., and Kateri, M. (2009). Order restricted inference for exponential step-stress models. *IEEE Transaction on Reliability*, **56**, 132–142.
 11. Bhattacharyya, G. K. and Stoejoeti, Z. (1989). A tampered failure rate model for step-stress accelerated life test. *Commun. Statist., Theory and Methods*, **18**, 1627–1643.
 12. Cox, D. R. (1975). Partial likelihood. *Biometrika*, **62**, 269–276.
 13. Cox, D. R. and Oakes, D. (1984). *Analysis of Survival Data*. Chapman and Hall, New York.
 14. Chen, Y. Q. (2001). Accelerated hazards regression model and its adequacy for censored survival data. *Biometrics*, **57**, 853–860.
 15. Doksum, K. A. and Hoyland, A. (1992). Models for variable-stress accelerated life testing experiment based on Wiener processes and the inverse Gaussian distribution. *Technometrics*, **34**, 74–82.
 16. Gertsbakh, L. B. and Kordonskiy, K. B. (1969). *Models of Failure*. Springer-Verlag, New York.
 17. Gerville-Réache, L. and Nikoulina, V. (1999). Analysis of reliability characteristics estimators in accelerated life testing. In *Statistical and Probabilistic Models in Reliability*, D. Ionescu and N. Limnios, eds., pp. 91–100. Birkhauser, Boston, MA.
 18. Glaser, R. E. (1984). Estimation for a Weibull accelerated life testing model. *Naval Research Logistics Quarterly*, **31**, 559–570.
 19. Hirose, H. (1993). Estimation of threshold stress in accelerated life-testing. *IEEE Transaction on Reliability*, **42**, 650–657.
 20. Kartashov, G. D. and Perrote, A. I. (1968). On the principle of “heredity” in reliability theory. *Engin. Cybernetics*, **9**, 231–245.
 21. Khamis, I. H. and Higgins, J. J. (1998). A new model for step-stress testing. *IEEE Transactions on Reliability*, **47**, 131–134.
 22. Klein, J. P. and Basu, A. P. (1981). Weibull accelerated life tests when there are competing causes of failure. *Communications in Statistical Theory and Methods*, **10**, 2073–2100.
 23. Klein, J. P. and Basu, A. P. (1982). Accelerated life testing under competing exponential failure distributions. *IAPQR Trans.*, **7**, 1–20.
 24. Lawless, J. F. (2002). *Statistical Models and Methods for Lifetime Data*, 2nd ed. Wiley, New York.
 25. LuValle, M. (2000). A theoretical framework for accelerated testing. In *Recent Advances in Reliability Theory. Methodology, Practice and Inference*, N. Limnios and M. Nikulin, eds., pp. 419–434. Birkhauser, Boston, MA.
 26. Miner, M. A. (1945). Cumulative damage in fatigue. *J. Appl. Mechanics*, **12**, A159–A164.
 27. Meeker, Jr. W. Q. (1984). A comparison of accelerated life test plans for Weibull and lognormal distributions and type I censoring. *Technometrics*, **26**, 157–172.
 28. Meeker, Jr. W. Q. and Escobar, L. A. (1993). A review of recent research and current issues in accelerating testing. *International Statistical Review*, **61**, 147–168.
 29. Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

30. Meeter, C. A. and Meeker, Jr. W. Q. (1994). Optimum accelerated life tests with a nonconstant scale parameter. *Technometrics*, **36**, 71–83.
31. Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*. Wiley, New York.
32. Nelson, W. and Meeker, W. (1991). Accelerated testing: statistical models, test plans, and data analysis. *Technometrics*, **33**, 236–238.
33. Rodrigues, J., Bolfarine, H., and Lourada-Neto, F. (1993). Comparing several accelerated life models. *Communications in Statistical Methods and Theory*, **22**, 2297–2308.
34. Schmoeyer, R. (1986). An exact distribution-free analysis for accelerated life testing at several levels of a single stress. *Technometrics*, **28**, 165–175.
35. Schmoeyer, R. (1991). Nonparametric analyzes for two-level single-stress accelerated life tests. *Technometrics*, **33**, 175–186.
36. Sedyakin, N. M. (1966). On one physical principle in reliability theory. (In Russian). *Techn. Cybernetics*, **3**, 80–87.
37. Sethuraman, J. and Singpurwalla, N. D. (1982). Testing of hypotheses for distributions in accelerated life tests. *JASA*, **77**, 204–208.
38. Shaked, M. and Singpurwalla, N. D. (1983). Inference for step-stress accelerated life tests. *J. Statist. Plann. Inference*, **7**, 295–306.
39. Singpurwalla, N. D. (1971). Inference from accelerated life tests when observations are obtained from censored samples. *Technometrics*, **13**, 161–170.
40. Singpurwalla, N. D., Castellino, V. C., and Goldschen, D. Y. (1975). Inference from accelerated life tests using Eyring type re-parameterization. *Naval Research Logistics Quarterly*, **22**, 289–296.
41. Singpurwalla, N. (1995). Survival in dynamic environments. *Statistical Sciences*, **1**, 86–103.
42. Viertl, R. (1988). *Statistical Methods in Accelerated Life Testing*. Vandenhoeck and Ruprecht, Göttingen.
43. Bagdonavičius, V. and Nikulin, M. (2004). Semiparametric analysis of degradation and failure time data with covariates. In *Parametric and Semiparametric Models with Applications to Reliability, Survival Analysis and Quality of Life*, M. S. Nikulin, N. Balakrishnan, N. Limnios, and M. Mesbah, eds., pp. 41–65. Birkhauser, Boston.
44. Mann, N. R., Schafer, R. E., and Singpurwalla, N. (1974). *Methods for Statistical Analysis of Reliability and Life Data*. Wiley, New York.
45. Pieruschka, E. (1961). Relation between lifetime distribution and the stress level causing failures. LMSD-800440, Lockheed Missiles and Space Division, Sunnyvale, California.