1 Alternatives to Black–Scholes Formulation in Finance

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1.1 Introduction

The Black–Scholes formula was first proposed in the seminal papers by Black and Scholes [2] and Merton [30] in order to price European options.[1] It is widely recognized that the Black–Scholes formula is the first step for both practitioners and academic researchers to understand the pricing of derivatives and their hedging strategies under the continuous-time setting. See Kijima and Muromachi [28] for details of the Black–Scholes formula.

However, there are apparent drawbacks in the Black–Scholes formula due to the strong assumptions made in the Black-Scholes setting. Namely, they assume that (i) the stock return is normally distributed, (ii) the instantaneous interest rate is constant over time, and (iii) the volatility is also a constant. In contrast, real markets often exhibit asymmetric and leptokurtic return distributions, and both interest rates and volatility are considered to fluctuate stochastically in time. In particular, implied volatility smile and volatility clustering are observed.

[1] Because of this reason, the pricing formula is often called the Black–Merton–Scholes formula. However, in this chapter, we adopt the market convention and use the term "Black–Scholes" formula.

Implied volatility smile is a phenomenon that shows a U-shaped implied volatility curve across strike prices. On the other hand, volatility clustering implies that volatilities are autocorrelated while the returns themselves have no autocorrelation. It is apparent that these phenomena are not consistent with the assumption of constant volatility. Hence, it is important to incorporate these features to pricing models.

Many researchers have tried to extend the Black–Scholes model in order to capture these phenomena observed in actual markets by introducing jumps, stochastic interest rates, and other factors. Among them, constant elasticity models [9,11], stochastic volatility models, normal jump models [31], affine stochastic-volatility, and affine jump-diffusion models [24,15], Lévy process models [22] have been developed to resolve the volatility smile in option pricing.[2]

As a model becomes more realistic, it would become more difficult to derive an-

[2] A Lévy process alone cannot represent such a volatility clustering (or volatility persistence); however, it can incorporate the volatility clustering effect if combined with other processes. See some discussions in Section 1.6 below.

alytical solutions for derivative prices.[3] Hence, we need to develop pricing methods to evaluate derivative prices for such advanced models. In the finance literature, such pricing techniques have also been developed. Such methods include the change of measure techniques, Monte Carlo simulation, approximations, and transform techniques. In the following, we not only demonstrate the features of advanced models but also present some of these pricing methods and techniques.

This chapter is organized as follows. The next section discusses the motivation of alternative models more explicitly. Section 1.3 formally describes the valuation techniques for the reader's convenience. While Section 1.4 introduces stochastic interest-rate models for option pricing, Section 1.5 discusses stochastic volatility models. Finally, in Section 1.6, we explain several facts on models with Lévy processes.

Throughout this chapter, we fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with filtration $\{\mathcal{F}_t;\, 0 \le t \le T\}$, where $\mathcal{F}_t$ denotes the information about asset prices available in the market at time t.

1.2 Motivation for Alternative Models

In this section, we first describe the Black–Scholes model and explain briefly the drawbacks of the Black–Scholes formula.

Consider a financial market in which there are one risky asset (stock) and one risk-free asset (bank account). Assume throughout this chapter that (i) the market is frictionless, i.e., there are no transaction costs or taxes, (ii) all the assets are perfectly divisible, (iii) short sales of all assets are allowed without restriction, (iv) the borrowing and lending rates of the risk-free asset are the same, and (v) all investors are price-takers. Also, we assume that the assets pay no dividends.

[3]The popularity of the Black–Scholes model for practitioners depends on its closed-form solution.

Under the probability measure P (objective measure), suppose that the present time is $t = 0$, and the time-t price of the bank account (risk-free asset), denoted by $B(t)$, follows the ordinary differential equation (ODE for short)

$$\frac{dB(t)}{B(t)} = r\,dt, \qquad t \ge 0, \quad B(0) = 1, \quad (1)$$

where the riskfree interest rate r is constant. On the other hand, the time-t price of the stock (risky asset), denoted by $S(t)$, follows the stochastic differential equation (SDE for short)

$$\frac{dS(t)}{S(t)} = \mu\,dt + \sigma\,dz(t), \qquad t \ge 0, \quad S(0) = S, \quad (2)$$

where $z(t)$ is the standard Brownian motion and the volatility σ is a constant. This model is called the *Black–Scholes model* in the finance literature.

The solutions of (1) and (2) are given by

$$B(t) = e^{rt}, \qquad t \ge 0, \quad (3)$$
$$S(t) = S\,e^{\nu t + \sigma z(t)}, \qquad t \ge 0, \quad (4)$$

respectively, where $\nu = \mu - \sigma^2/2$. The solution (4) is derived by applying Ito's formula (see Kijima and Muromachi [28]). While the bank account $B(t)$ is a deterministic function of time t, $\log S(t)$ is normally distributed with mean $\log S + \nu t$ and variance $\sigma^2 t$. Hence, the stock price $S(t)$ follows a lognormal distribution in the Black–Scholes model.

Merton [30] shows that the price of a call option[4] with strike price K and maturity

[4]Options considered in this chapter are European type, unless stated otherwise. A brief discussion about American options is given in Kijima and Muromachi [28].

T is given by

$$\begin{aligned} C_{BS}(S,T,K,\sigma^2,r) \\ &= SN(d_1) - Ke^{-rT}N(d_2), \quad (5) \\ d_1 &= \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}, \end{aligned}$$

where $N(\cdot)$ denotes the standard normal cumulative distribution function. See Kijima and Muromachi [28] for the derivation of the *Black–Scholes formula* (5).

When a call option is quoted as $c_{obs}(K)$ for each strike price K in the market, the constant $\sigma_{imp}(K)$ satisfying

$$c_{obs}(K) = C_{BS}(S,T,K,\sigma^2_{imp}(K),r) \quad (6)$$

is called the *implied volatility.* Implied volatilities are important for market participants for the pricing of other derivatives and risk management of derivative transactions. In fact, all of foreign exchange option transactions between professional banks are quoted in terms of implied volatilities, since the Black–Scholes formula is their "common knowledge." On the other hand, option premiums are shown to their customers who are less professional in option trading.

Recall that, in the Black–Scholes setting, the risk-free interest rate and volatility are constant and uncertainty is described by the standard Brownian motion. Because of these assumptions, the stock return distribution turns out to be a normal. However, the return distribution often exhibits an asymmetric and fat-tailed feature, yielding the *volatility smile.*[5] That is, the implied volatility curve across strike prices often shows a U-shaped pattern, whence contradicting the Black–Scholes setting.

In order to overcome this deficiency of the Black–Scholes formula, we need to modify the Black–Scholes model so as to construct an asymmetric and fat-tailed return distribution. Also, in reality, interest rates as well as volatility fluctuate randomly in time, and actual returns often exhibit jumps. Hence, not only to make the model more realistic, but also to overcome the deficiency of the option pricing formula, there have been made many attempts to construct a "good" model for the pricing of derivatives in the finance literature. These include stochastic interest-rate models, stochastic volatility models and models with Lévy processes. In the following, we explain such models in detail. Before proceeding, however, we briefly discuss valuation methods of derivatives in the next section for the reader's convenience.

[5] When the implied volatility curve across strike prices shows a decreasing pattern, it is called a *volatility smirk.*

1.3 Methods of Valuation

In this section, we consider a more general model for the price processes $B(t)$, $S(t)$ than the Black–Scholes model (1)–(2), respectively. Namely, for some finite T, suppose that

$$\begin{aligned} \frac{dB(t)}{B(t)} &= r(t)\,dt, \\ & \qquad 0 \le t \le T, \quad B(0) = 1, \quad (7) \\ \frac{dS(t)}{S(t)} &= \mu(t)\,dt + \sigma(t)\,dz(t), \\ & \qquad 0 \le t \le T, \quad S(0) = S, \quad (8) \end{aligned}$$

where $r(t)$, $\mu(t)$, $\sigma(t)$ are adapted to the filtration. Note that the distribution of log-price $\log S(t)$ is difficult to derive in the general setting.

Define $S^*(t) = S(t)/B(t)$, the *relative price* of $S(t)$ with numéraire $B(t)$. If all the relative prices in the market are martingale under some probability measure Q, the measure Q is called a *risk-neutral measure.* If the assets pay no dividends, the existence of such a probability measure is guaranteed by the following crucial result, called the *fundamental theorem of asset pricing.*[6]

Theorem 1.3.1 (Asset Pricing Theorem) *There are no arbitrage opportunities if and only if there exists a risk-neutral probability measure. If this is the case, the price of a contingent claim X is given by $\pi(X) = E^*[X/B(T)]$.*

We want to construct a risk-neutral measure Q. By the Girsanov theorem, the process defined by

$$dz^Q(t) = dz(t) + \lambda(t)\, dt \tag{9}$$

is a standard Brownian motion under Q, where the process $\lambda(t)$ is called the *market price of risk.* From (8) and (9), the SDE for $S(t)$ under Q is given by

$$dS(t) = (\mu(t) - \sigma(t)\lambda(t))\, dt + \sigma(t)\, dz^Q(t).$$

Since the relative price $S(t)/B(t)$ is a martingale under the risk-neutral measure Q, the drift term in $S(t)$ under Q must be equal to $r(t)S(t)\, dt$. It follows that the market price of risk is uniquely determined as

$$\lambda(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \tag{10}$$

and the market is *complete.*[7] From (8)–(10), the price process satisfies the SDE

$$\frac{dS(t)}{S(t)} = r(t)\, dt + \sigma(t)\, dz^Q(t), \quad 0 \le t \le T, \quad S(0) = S$$

under the risk-neutral measure Q. If, in particular, the adapted processes $r(t), \sigma(t)$ are assumed to be $r(t) = r(S(t), t)$ and $\sigma(t) = \sigma(S(t), t)$, respectively, we obtain the SDE

$$\frac{dS(t)}{S(t)} = r(S(t), t)\, dt + \sigma(S(t), t)\, dz^Q(t), \quad 0 \le t \le T, \quad S(0) = S, \tag{11}$$

as the price process under Q.

The desired risk-neutral measure Q is now defined by the Radon–Nikodym derivative

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t \lambda(u)\, dz(u) + \frac{1}{2}\int_0^t \lambda^2(u)\, du\right\}, \tag{12}$$

where $\lambda(t)$ is the market price of risk. Note that Q is equivalent to P, i.e., $Q(A) > 0$ if and only if $P(A) > 0$. Since the relative price $S^*(t)$ is a martingale under Q which is equivalent to P, the probability measure Q is often called an *equivalent martingale measure.*

1.3.1 Equivalent Martingale Measure

Any asset whose price process is strictly positive can be treated as a numéraire. Formally, an equivalent martingale measure (EMM) with respect to a numéraire is defined as an equivalent probability measure under which the relative price of any asset with respect to the numéraire asset is a martingale. As we have already explained, if the bank account is taken as the numéraire, the EMM is the risk-neutral measure Q. The EMM with respect to a zero-coupon bond price with maturity T is called the *T-forward measure*, which we denote by Q^T throughout this chapter.

By the definition of the risk-neutral measure, the time-t price $X(t)$ of a contingent

[6] The theorem fails to hold when the time horizon is infinite. See, e.g., Pliska [38] for details.

[7] The market is complete if and only if any contingent claim can be replicated by a portfolio process of the underlying assets, and the price of the contingent claim is uniquely determined as the initial hedging cost. It implies that the risk-neutral measure is uniquely determined. If the market is incomplete (with maintaining no-arbitrage opportunity), on the other hand, such a price system must be supported by a risk-neutral measure; however, the risk-neutral measure may not be unique.

claim with payoff $X(T)$ at time T must satisfy the martingale condition

$$\frac{X(t)}{B(t)} = E_t^Q\left[\frac{X(T)}{B(T)}\right], \tag{13}$$

where $E_t^Q[\cdot] = E^Q[\cdot|\mathcal{F}_t]$ denotes the expectation operator under Q conditional on $\mathcal{F}_t$. As a special case, the zero-coupon bond price $P(t,T)$ with maturity T is given by

$$P(t,T) = E_t^Q\left[\frac{B(t)}{B(T)}\right]. \tag{14}$$

Of course, if the bank account $B(t)$ is deterministic, the zero-coupon bond price is given by $P(t,T) = B(t)/B(T) = e^{-\int_t^T r(u)\,du}$. Hence, if the interest rates are deterministic, the risk-neutral measure Q and the T-forward measure Q^T coincide each other.

Similarly, consider a call option written on the underlying asset $S(t)$ with strike price K and maturity T. Since the payoff of the call option at the maturity is given by $(S(T)-K)_+$, where $(x)_+ = \max\{x, 0\}$, the call option price $C(t)$ is evaluated under the risk-neutral measure Q as

$$C(S(t),t) = B(t)E_t^Q\left[\frac{(S(T)-K)_+}{B(T)}\right], \tag{15}$$

These formulas hold once a risk-neutral measure Q is appropriately built. It does not matter whether the market is complete or not.

Equivalent martingale measures are linked each other by the technique of the *change of numéraire*. Let us denote by Q^S the EMM with respect to asset price S. Then, the relative prices $S(t)/B(t)$ and $P(t,T)/B(t)$, each of which is a martingale under Q, define the change of measures to Q^S and Q^T from Q by

$$\begin{aligned}\left.\frac{dQ^S}{dQ}\right|_{\mathcal{F}_t} &= L^S(t) \equiv \frac{S(t)}{B(t)}\frac{B(0)}{S(0)},\\ \left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t} &= L^T(t) \equiv \frac{P(t,T)}{B(t)}\frac{B(0)}{P(0,T)},\end{aligned}$$

respectively.

Note from (15) that

$$C(S(t),t) = B(t)E_t^Q\left[\frac{S(T)1_A - K1_A}{B(T)}\right],$$

where $A = \{S(T) > K\}$ is the event that the stock price ends up *in-the-money* at the maturity, and 1_A denotes the indicator function, meaning that $1_A = 1$ if A is true and $1_A = 0$ otherwise. Using the change of measure technique, one can rewrite the conditional expectation as

$$\begin{aligned} &C(S(t),t)\\ &\quad= B(t)E_t^Q\left[\frac{L^S(T)}{L^S(t)}\frac{S(t)}{B(t)}1_A\right]\\ &\qquad -B(t)E_t^Q\left[\frac{L^T(T)}{L^T(t)}\frac{P(t,T)}{B(t)}K1_A\right]\\ &\quad= S(t)E_t^S[1_A]\\ &\qquad -KP(t,T)E_t^T[1_A]. \end{aligned} \tag{16}$$

Here, $\Pi_1 = E_t^S[1_A]$ and $\Pi_2 = E_t^T[1_A]$ are the probabilities that the option contract ends up in-the-money under Q^S and Q^T, respectively. Hence, for the pricing of options, it is enough to know the distribution of $\log\frac{P(T,T)}{S(T)}$ under Q^S and that of $\log\frac{S(T)}{P(T,T)}$ under Q^T, once the underlying stock price model is specified. This means that an option pricing formula can be derived independently of the underlying model, to some extent, by the change of numéraire technique.

1.3.2 Partial Differential Equation

By the Markov property, the call option price is a function of stock price $S(t)$ and time t. If we denote the price by $C(t) = C(S(t),t)$, it follows from (11) and Ito's formula that the price process satisfies the SDE

$$\begin{aligned} dC(t) &= \mathcal{L}C(S(t),t)\,dt\\ &\quad +C_S(S(t),t)\sigma(t)S(t)\,dz^Q(t), \end{aligned} \tag{17}$$

where $\mathfrak{L}$ is the infinitesimal generator defined by

$$\begin{aligned}\mathfrak{L}f(S,t) &= f_t(S,t)+r(S,t)Sf_S(S,t)\\ &\quad +\frac{1}{2}\sigma^2(S,t)S^2 f_{SS}(S,t).\end{aligned} \tag{18}$$

Since the relative price $C(S(t),t)/B(t)$ is a Q-martingale, we must have the equation

$$\mathfrak{L}C(S,t)=r(S,t)C(S,t).$$

More precisely, the price function $C(S,t)$ satisfies the partial differential equation (PDE for short)

$$\begin{aligned}&C_t(S,t)+r(S,t)SC_S(S,t)\\ &\quad+\frac{1}{2}\sigma^2(S,t)S^2C_{SS}(S,t)\\ &\quad-r(S,t)C(S,t)=0\end{aligned} \tag{19}$$

with the boundary condition

$$C(S,T)=(S-K)_+$$

for the case of a call option with strike price K and maturity T. The call option price is obtained by solving the above PDE analytically or numerically.

1.3.3 Characteristic Function and the Fourier Transform

Fix $t<T$, and let $q(s)$ be the density function of the random variable $\log S(T)$ under the risk-neutral measure Q conditional on $\mathcal{F}_t$. The characteristic function of $q(s)$ is denoted by

$$\begin{aligned}\phi(u) &\equiv E_t^Q\left[\exp\{iu\log S(T)\}\right]\\ &= \int_{-\infty}^{\infty} e^{ius}q(s)\,ds.\end{aligned}$$

Many authors, for example, Bakshi and Madan [1], show that the call option price with strike K and maturity T is written as (cf. (16))

$$C(S(t),t)=S(t)\Pi_1-KP(t,T)\Pi_2,$$

where the probabilities Π_1, Π_2 are calculated numerically by

$$\begin{aligned}\Pi_1 &= \frac{1}{2}+\frac{1}{\pi}\\ &\quad\times\int_0^\infty \mathrm{Re}\left(\frac{\exp\{-iu\log K\}\phi(u-i)}{iu\phi(-i)}\right)du,\end{aligned} \tag{20}$$

$$\begin{aligned}\Pi_2 &= \frac{1}{2}+\frac{1}{\pi}\\ &\quad\times\int_0^\infty \mathrm{Re}\left(\frac{\exp\{-iu\log K\}\phi(u)}{iu}\right)du,\end{aligned} \tag{21}$$

respectively. Here, $\mathrm{Re}(x)$ denotes the real part of complex number x.

Let $C(k)$ be the call option price with strike $\exp(k)$, i.e.

$$C(k)\equiv\int_k^\infty e^{-r(T-t)}(e^s-e^k)q(s)\,ds,$$

and let $\psi(v)$ be the Fourier transform of $\exp(\alpha k)C(k)$. Then,

$$\begin{aligned}\psi(v) &\equiv \int_{-\infty}^{\infty} e^{ivk+\alpha k}C(k)\,dk\\ &= \int_{-\infty}^{\infty} e^{ivk}\int_k^\infty e^{\alpha k}e^{-r(T-t)}\\ &\quad\times(e^s-e^k)q(s)\,dsdk\\ &= \int_{-\infty}^{\infty} e^{-r(T-t)}q(s)\\ &\quad\times\left(\frac{e^{(\alpha+1+iv)s}}{\alpha+iv}-\frac{e^{(\alpha+1+iv)s}}{\alpha+iv}\right)ds\\ &= \frac{e^{-r(T-t)}\phi(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}.\end{aligned}$$

Hence, once we obtain the characteristic function $\phi(u)$, the call option price is calculated by the Fourier inverse transform as

$$C(k)=\frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{\infty}e^{-ivk}\psi(v)\,dv.$$

Note that the point $u=0$ is singular in the integrands of (20) and (21). This fact is inherited to the Fourier transform $\psi(v)$, which includes the singularity at $(\alpha, v) = (0, 0)$. For an appropriately chosen α, however, $\psi(v)$ is not singular so that the following fast Fourier transform (FFT) is applicable: The call option price can be approximated as

$$C(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi(v_j)\eta, \quad v_j = \eta(j-1),$$

for sufficiently large N. The FFT is an efficient algorithm to calculate the above integral, when $\psi(v)$ is not singular for an appropriately chosen α. See, e.g., Carr and Madan [6] for details.

1.4 Stochastic Interest-Rate Models

In this section, we consider the Black–Scholes model under a stochastic interest-rate economy. That is, the risk-free interest rate in (1) is no longer a constant, but follows a stochastic process. More precisely, suppose that $(r(t), S(t))$ satisfies the SDE

$$dr(t) = \mu_r(t)\,dt + \sigma_r(t)\cdot dz^Q(t), \quad t \geq 0, \tag{22}$$

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma_S(t)\cdot dz^Q(t), \quad t \geq 0, \tag{23}$$

under the risk-neutral measure Q, where $\mu_r(t)$ is adapted to the filtration, $\sigma_r(t), \sigma_S(t)$ are vectors of volatility coefficients adapted to the filtration, and $\cdot$ denotes the inner product. Note the difference between this model and the model (11). While the risk-free interest rate $r(t)$ in (11) is locally deterministic, $r(t)$ in (22) is stochastic due to the term $\sigma_r(t)\cdot dz^Q(t)$. The market can be complete under certain conditions imposed on the coefficients.

Even in the stochastic interest-rate model, the valuation methods explained in the previous section can be applied. To be more specific, we consider the model

$$dr(t) = \mu_r(r(t), t)\,dt + \sigma_r(r(t), S(t), t)\cdot dz^Q(t), \tag{24}$$

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma_S(r(t), S(t), t)\cdot dz^Q(t). \tag{25}$$

Then, by Ito's formula, the process of the call option price $C(t) = C(r(t), S(t), t)$ written on $S(t)$ follows the SDE

$$dC(t) = \mathfrak{L}C(r(t), S(t), t)\,dt + \nabla C(r(t), S(t), t)\cdot dz(t),$$

where $\mathfrak{L}$ is the infinitesimal generator given by

$$\mathfrak{L}f(r,s,t) = f_t(r,s,t) + \mu_r(r,t)f_r(r,s,t) + rsf_s(r,s,t) + \frac{1}{2}\sum_{i,j=r,s} s\sigma_i(r,s,t)\sigma_j(r,s,t) \times f_{ij}(r,s,t) \tag{26}$$

and $\nabla f(r,s,t)$ is the vector defined by

$$\nabla f(r,s,t) = \sigma_r(r,s,t)f_r(r,s,t) + s\sigma_S(r,s,t)f_s(r,s,t).$$

Since the relative process $C(t)/B(t)$ is a martingale under Q, the price function $C(r, S, t)$ must satisfy the PDE

$$\mathfrak{L}C(r,S,t) - rC(r,S,t) = 0 \tag{27}$$

with the boundary condition $C(r, S, T) = (S-K)_+$ as before. It is not straightforward to solve the PDE (27) due to the multidimensional feature.

1.4.1 T-Forward Measure

We have already mentioned that the EMM with respect to the zero-coupon bond price with maturity T is the *T-forward measure* Q^T. The T-forward measure is particularly useful for the pricing and hedging of derivatives under the stochastic interest-rate economy.

To see this, consider a call option written on $S(t)$ with strike price K and maturity T. Recall from (15) that the call option price $C(t)$ is evaluated under the risk-neutral measure Q as

$$\frac{C(S(t),t)}{B(t)} = E_t^Q\left[\frac{(S(T)-K)_+}{B(T)}\right],$$

because the relative price $C(t)/B(t)$ is a martingale under Q. Similarly, since the relative price $C(t)/P(t,T)$ is a martingale under Q^T, we have

$$\frac{C(S(t),t)}{P(t,T)} = E_t^T\left[\frac{(S(T)-K)_+}{P(T,T)}\right].$$

Since $P(T,T)=1$, it follows that

$$C(S(t),t) = P(t,T)E_t^T\left[(S(T)-K)_+\right], \tag{28}$$

where $E_t^T[\cdot]$ denotes the conditional expectation operator under Q^T.

The forward method (28) has an apparent advantage over the risk-neutral method (15) for stochastic interest-rate models, because it is enough to know the (one-dimensional) distribution of $S(T)$ under the T-forward measure Q^T. In contrast, from (15), we need to know the two-dimensional distribution of $(B(T),S(T))$ under the risk-neutral measure Q, when the interest rates are stochastic.

1.4.2 Affine Term-Structure Models

We have already seen that the zero-coupon bond price is given by (14). In this section, we explain affine term-structure models (ATSMs) that include, as a special case, popular short rate (1-factor) models such as the Vasicek model [36] and Cox–Ingersoll–Ross (CIR) model [10]. ATSMs extend them to multi-factor models with ease. Also, in any ATSM, the zero-coupon bond price is expressed as an exponentially affine function of state variables with time-dependent coefficients which can be easily calculated analytically or numerically. Because of the generality and tractability, ATSMs are very attractive for practitioners. For detailed discussions of ATSMs, the reader is referred to Duffie and Kan [14].

Suppose that the state of the economy is described by a vector of factors (or state variables), $X=(X_1,\dots,X_J)^\top$ say, which follows the SDE

$$dX(t) = \mu_X(X(t))\,dt + \sigma_X(X(t))\,dz^Q(t)$$

under Q. We are interested in a class of economy in which the zero-coupon bond price is represented in the form of an exponentially affine function of the state vector. That is,

$$P(t,T) = \exp\{A(t,T) + B(t,T)^\top X(t)\}. \tag{29}$$

According to Duffie and Kan [14], under certain conditions, the bond price $P(t,T)$ is expressed as (29) if and only if the short rate $r(t)$ is an affine function of $X(t)$ with $\mu_X(x) = K(\theta - x)$ and $\sigma_X(x) = \Sigma D(x)$, where $\theta \in \mathbb{R}^J$, $K,\Sigma \in \mathbb{R}^{J\times J}$ and $D(x)$ is a diagonal matrix in the form of

$$\begin{aligned} D(x) &= \operatorname{diag}(\sqrt{\alpha_1 + \beta_1 \cdot x}, \\ &\quad \cdots, \sqrt{\alpha_n + \beta_n \cdot x} \end{aligned}$$

with $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}^J$ for $i=1,2,\dots,J$. Hence, it is sufficient to restrict our attention to the case that

$$\begin{aligned} r(t) &= \delta_0 + \delta_X^\top X(t), &(30)\\ dX(t) &= K(\theta - X(t))\,dt \\ &\quad + \Sigma D(X(t))\,dz^Q(t), &(31) \end{aligned}$$

under the risk-neutral measure Q, where $\delta_0 \in \mathbb{R}$, $\delta_X \in \mathbb{R}^J$ and $\Sigma \in \mathbb{R}^{J\times J}$ is a matrix such that $\Sigma\Sigma^\top$ is a covariance matrix.

The Feynman–Kac formula yields the following system of ODEs:

$$\begin{aligned}\frac{\partial}{\partial t}A(t,T) &= -(K\theta)^\top B(t,T) \\ &\quad -\frac{1}{2}\sum_{j=1}^{J}(\Sigma^\top B(t,T))_j{}^2\alpha_j + \delta_0, \\ &\quad A(T,T) = 0, \\ \frac{\partial}{\partial t}B(t,T) &= K^\top B(t,T) \\ &\quad -\frac{1}{2}\sum_{j=1}^{J}(\Sigma^\top B(t,T))_j{}^2\beta_j + \delta_X, \\ &\quad B(T,T) = 0.\end{aligned}$$

This system of ODEs can be solved numerically (in a closed form for special cases) relatively easily. With the solutions $A(t,T)$, $B(t,T)$ at hand, the zero-coupon bond price is now obtained from (29).

The SDE of the zero-coupon bond price is given by

$$\begin{aligned}\frac{dP(t,T)}{P(t,T)} &= r(t)\,dt \\ &\quad + B(t,T)^\top \Sigma D(X(t))\,dz^Q(t).\end{aligned}$$

By carrying out the change of measure to the forward measure Q^T, it is observed that the factor $X(t)$ has an affine type SDE (31) with time-dependent coefficients under the new measure Q^T.

1.4.3 Gaussian Short Rate Models

A simple Gaussian short rate model allows us to obtain option formulas in closed form in a straightforward fashion. To demonstrate this, we assume that the short rate fluctuates randomly with correlation to the stock price under Q. That is, we assume that

$$dr(t) = \sigma_r\,dz_1^Q(t),$$

$$\begin{aligned}\frac{dS(t)}{S(t)} &= r(t)\,dt + \sigma_S(\rho\,dz_1^Q(t) \\ &\quad + \sqrt{1-\rho^2}\,dz_2^Q(t)),\end{aligned}$$

where z_1^Q, z_2^Q are independent standard Brownian motions under the risk-neutral measure Q.[8]

Because the model (32) has the simplest affine form (31) with $r(t) = X(t)$, we have the system of ODEs

$$\begin{aligned}\frac{\partial}{\partial t}A(t,T) &= -\frac{1}{2}\sigma_r^2 B^2(t,T), \\ &\quad A(T,T) = 0, \\ \frac{\partial}{\partial t}B(t,T) &= 1, \qquad B(T,T) = 0.\end{aligned}$$

This system of ODEs can be solved easily to yield

$$B(t,T) = -(T-t), \quad A(t,T) = \frac{\sigma_r^2}{6}(T-t)^3.$$

It follows that the zero-coupon bond price is given by

$$\begin{aligned}P(t,T) &\\ &= \exp\left\{-(T-t)r(t) + \frac{\sigma_r^2}{6}(T-t)^3\right\}.\end{aligned} \tag{32}$$

Now, by Ito's formula, the SDEs of the relative prices with respect to the zero-coupon bond are obtained as

$$\begin{aligned}d\left(\frac{B(t)}{P(t,T)}\right) &\\ &= \sigma_r(T-t)\frac{B(t)}{P(t,T)} \\ &\quad \times\left(\sigma_r(T-t)\,dt + dz_1^Q(t)\right),\end{aligned}$$

[8] Gaussian models allow the interest rates to become negative with positive probability. In order to overcome this drawback, positive interest-rate models have been developed. See, e.g., Flesaker and Hughston [20] and Jin and Glasserman [26] for details.

$$\begin{aligned} d\left(\frac{S(t)}{P(t,T)}\right) &= \frac{S(t)}{P(t,T)}\Big\{(\sigma_r(T-t)+\rho\sigma_S) \\ &\quad \times \left(\sigma_r(T-t)\,dt + dz_1^Q(t)\right) \\ &\quad + \sqrt{1-\rho^2}\sigma_S\, dz_2^Q(t)\Big\}. \end{aligned}$$

Since the relative prices must be martingales under Q^T, the process $(z_1^T(t),\, z_2^T(t))$ defined by

$$\begin{aligned} dz_1^T(t) &= dz_1^Q(t) + \sigma_r(T-t)\,dt \\ dz_2^T(t) &= dz_2^Q(t) \end{aligned}$$

is a two-dimensional standard Brownian motion under Q^T by the Girsanov theorem. It follows that

$$\begin{aligned} \log\frac{S(T)}{P(T,T)} &= \log\frac{S(t)}{P(t,T)} - \frac{1}{2}\int_t^T (\sigma_r^2(T-u)^2 \\ &\quad +2\sigma_r(T-u)\rho\sigma_S + \sigma_S^2)\,du \\ &\quad + \int_t^T (\sigma_r(T-u)+\rho\sigma_S)\,dz_1^T(u) \\ &\quad + \int_t^T \sqrt{1-\rho^2}\sigma_S\,dz_2^T(u) \end{aligned}$$

under the T-forward measure Q^T. Hence, noting $P(T,T)=1$, the random variable $\log S(T)$ is normally distributed under Q^T with mean $\mu_2(t,T)$ and variance $V_2(t,T)$, where

$$\mu_2(t,T) = \log\frac{S(t)}{P(t,T)} - \frac{1}{2}V_2(t,T)$$

and

$$\begin{aligned} V_2(t,T) &= \int_t^T (\sigma_r(T-u)+\rho\sigma_S)^2\,du \\ &\quad + \int_t^T (\sqrt{1-\rho^2}\sigma_S)^2\,du \\ &= \frac{1}{3}\sigma_r^2(T-t)^3 + \sigma_r(T-t)^2\rho\sigma_S \\ &\quad +\sigma_S^2(T-t). \end{aligned}$$

It follows from (16) that the probability of ending up in-the-money under Q^T is obtained as

$$\begin{aligned} E_t^T[1_A] &= Q_t^T\{\log S(T) > \log K\} \\ &= N\Bigg(\frac{\log(S(t)/(KP(t,T)))}{\sqrt{V_2(t,T)}} \\ &\quad -\frac{1}{2}\sqrt{V_2(t,T)}\Bigg). \end{aligned} \tag{33}$$

Next, we shall find the EMM Q^S with respect to the stock price. To this end, a similar calculation yields

$$\begin{aligned} d\left(\frac{B(t)}{S(t)}\right) &= \frac{B(t)}{S(t)}\Big(\sigma_S^2\,dt - \sigma_S\Big(\rho\,dz_1^Q(t) \\ &\quad + \sqrt{1-\rho^2}\,dz_2^Q(t)\Big)\Big), \end{aligned}$$

$$\begin{aligned} d\left(\frac{P(t,T)}{S(t)}\right) &= \left(\frac{P(t,T)}{S(t)}\right) \\ &\quad \times\Big\{(\rho\sigma_r\sigma_S+\sigma_S^2)\,dt - \Big(\sigma_r\,dz_1^Q(t) \\ &\quad -\sigma_S\left(\rho\,dz_1^Q(t)+\sqrt{1-\rho^2}\,dz_2^Q(t)\right)\Big)\Big\}. \end{aligned}$$

Then, the process $(z_1^S(t),\, z_2^S(t))$ defined by

$$\begin{aligned} dz_1^S(t) &= dz_1^Q(t) - \rho\sigma_S\,dt \\ dz_2^S(t) &= dz_2^Q(t) - \sqrt{1-\rho^2}\sigma_S\,dt \end{aligned}$$

forms a two-dimensional standard Brownian motion under Q^S. It follows that

$$\begin{aligned} \log\frac{P(T,T)}{S(T)} &= \log\frac{P(t,T)}{S(t)} \\ &\quad -\frac{1}{2}(\sigma_r^2 + 2\rho\sigma_r\sigma_S + \sigma_S^2)(T-t) \\ &\quad -(\sigma_r+\rho\sigma_S)(z_1^S(T) - z_1^S(t)) \\ &\quad -\sqrt{1-\rho^2}\sigma_S(z_2^S(T) - z_2^S(t)) \end{aligned}$$

is normally distributed with mean $\mu_1(t,T)$ and variance $V_1(t,T)$, where

$$\begin{aligned}\mu_1(t,T) &= \log\frac{S(t)}{P(t,T)} - \frac{1}{2}V_2(t,T),\\ V_1(t,T) &= \left(\sigma_r^2 + 2\rho\sigma_r\sigma_S + \sigma_S^2\right)(T-t).\end{aligned}$$

Hence, the probability of ending up in-the-money under Q^S is obtained as

$$\begin{aligned}E_t^S[1_F] &= Q^S\{-\log S(T) < -\log K\}\\ &= N\left(\frac{\log(S(t)/(KP(t,T)))}{\sqrt{V_1(t,T)}} + \frac{1}{2}\sqrt{V_1(t,T)}\right). \qquad (34)\end{aligned}$$

Combining the results (33) and (34), the option price is given by (16). Other Gaussian short rates can be applied similarly. See Kijima and Muromachi [28] for more general Gaussian term-structure model.

1.5 Stochastic Volatility Models

The smile effect suggests that the volatility of asset return is not a constant but may be dependent on the underlying price or fluctuate randomly. In this section, while the Black–Scholes model assumes a constant volatility in (2), we treat the volatility to be either a function of the underlying stock price (or forward price) or itself stochastic.

In the former case, we assume that the price process follows the SDE (11) with the riskfree interest rate being a constant r. That is, the price process satisfies the SDE

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma(S(t),t)\,dz^Q(t), \quad 0 \le t \le T, \quad S(0) = S, \qquad (35)$$

under Q. On the other hand, the latter case assumes that

$$\frac{dS(t)}{S(t)} = r\,dt + \sqrt{v(t)}\,dz_1^Q(t), \qquad (36)$$

$$dv(t) = \mu_v(t)\,dt + \sigma_v(t)\,dz_2^Q(t), \qquad (37)$$

where $\mu_v(t), \sigma_v(t)$ are adapted to the filtration and $dz_1(t)\,dz_2(t) = \rho\,dt$. Note the difference between the model (35) and that of (36)–(37). That is, while the volatility $\sigma(S(t),t)$ in (35) is locally deterministic, $v(t)$ in (37) is stochastic due to the term $\sigma_v(t)\,dz_2^Q(t)$. Note that, in the latter model, the market cannot be complete because volatility is not a tradeable asset.

1.5.1 Local Volatility Models

The smile effect of implied volatility $\sigma_{imp}(K)$ for strike price K implies that the constant volatility assumption is inconsistent so that calculation of Greeks of derivatives and the pricing of path-dependent options could not make sense. Local volatility models are built based on the assumption that the volatility is a function of the underlying asset price.

Consider a European option with maturity T and strike price K. The forward price $F(t)$ of the underlying asset settled at time T is calculated as

$$F(t) = E_t^T[S(T)],$$

which is a martingale under the forward measure Q^T. It follows that it satisfies an SDE in the form of

$$dF(t) = C(t)\,dz^T(t), \qquad F(0) = f,$$

where $C(t)$ is the diffusion coefficient.

Local volatility (LV) models were developed by Dupire [16,17] and Derman and Kani [12,13]. The LV model assumes that the volatility term is a piecewise constant function of time t, say $\sigma_{LV}(t,F)$. The forward price is assumed to follow the SDE

$$dF(t) = \sigma_{LV}(F(t))\,dz^T(t)$$

under the forward measure Q^T. One can calibrate the volatility function $\sigma_{LV}(F)$ from market data. LV models are self-consistent, arbitrage-free, and can be calibrated precisely to match the observed market smiles and skews.

The implied volatility curve in the calibrated LV model is given by

$$\begin{aligned}\sigma_{IV}&(K,F)\\ &= \sigma_{LV}\left(\frac{1}{2}(F+K)\right)\\ &\quad\times\left\{1+\frac{1}{24}\frac{\sigma''_{LV}\left(\frac{1}{2}(F+K)\right)}{\sigma_{LV}\left(\frac{1}{2}(F+K)\right)}\right.\\ &\quad\left.\times(F-K)^2+\cdots\right\}.\end{aligned}$$

Note that the implied volatility changes when the underlying asset price changes. The drawback of the LV model is that the direction of the implied volatility curve is the opposite to the direction that is usually observed in the market. It suggests that the *delta risk* is miscalculated in the LV model.

1.5.2 The CEV Model

The CEV (constant elasticity of variance) model is one of the local volatility models. The CEV process is defined by the SDE

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma S^{\beta}(t)\,dz^{Q}(t), \qquad (38)$$

and first considered by Cox [8]. The parameter β is the elasticity of the local volatility function $\sigma(S) = \sigma S^{\beta}$. In Davydov and Linetsky [11] it is reported that option prices in the S & P 500 stock index imply that β is negative and a typical value is as low as $\beta = -4$.

In the case of $r = 0$,[9] it is well known (see, e.g., Borodin and Salminen [4]) that the transformed process $S^{-\beta}(t)/(\sigma|\beta|)$ follows a standard Bessel process of order $\nu = 1/(2|\beta|)$. By making use of this fact, Davydov and Linetsky [11] showed that the density function of the CEV process (38) is given by

$$\begin{aligned}p_r(T;&\,S,S(T))\\ &= e^{-rT}p_0(\tau(T);S,e^{-rT}S(T)),\end{aligned}$$

where

$$\begin{aligned}\tau(T) &= \frac{1}{2r\beta}(e^{2r\beta T}-1),\\ p_0(t,x,y) &= \frac{y^{-2\beta-\frac{3}{2}}x^{\frac{1}{2}}}{\sigma^2|\beta|t}\\ &\quad\times\exp\left\{-\frac{x^{-2\beta}+y^{-2\beta}}{2\sigma^2\beta^2t}\right\}\\ &\quad\times I_\nu\left(\frac{x^{-\beta}y^{-\beta}}{\sigma^2\beta^2t}\right).\end{aligned}$$

Here I_ν is the modified Bessel function of the *first kind* with order ν.

Using the above density function of $S(T)$ under Q, the price of a call option with strike price K and maturity T is calculated as

$$C(S;K,T) = \begin{cases} SQ(\zeta;n-2,y_0) \\ \quad -e^{-rT}K(1-Q(y_0;n-2,\zeta)), & \beta>0,\\ SQ(y_0;n-2,\zeta) \\ \quad -e^{-rT}K(1-Q(\zeta;n-2,y_0)), & \beta<0,\end{cases}$$

where

$$\begin{aligned}n &= 2+\frac{1}{|\beta|},\\ \zeta &= \frac{2rS^{-2\beta}}{\sigma^2\beta(e^{2r\beta T}-1)},\\ y_0 &= \frac{-2rK^{-2\beta}}{\sigma^2\beta(e^{2r\beta T}-1)},\end{aligned}$$

and $Q(x;u,v)$ is the complementary non-central chi-square distribution function

[9] This assumption does not lack the generality, because we can consider the relative process $S^*(t) = S(t)/B(t)$ rather than the price process from the beginning.

with u degrees of freedom and non-centrality parameter v.

For detailed analyses of the CEV process, we refer to Davydov and Linetsky [11] and references therein. They investigated the effect of β on the call option price and implied volatility. Since β produces the volatility smile, those effects tend to be significant for out-of-the money options and in-the-money options rather than at-the-money options.

1.5.3 The Heston Model

In the subsequent three sections, we consider the second type of volatility fluctuation models. That is, the volatility actually fluctuates stochastically in time and is modeled by some SDE.

Among them, Heston [24] is the first to construct such a model. Suppose that the stock price and the volatility follow the SDE

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \mu\, dt + \sqrt{v(t)}\, dz_1(t), \\ dv(t) &= \kappa(\theta - v(t))\, dt + \sigma\sqrt{v(t)}\, dz_2(t), \end{aligned}$$

where $dz_1(t)dz_2(t) = \rho\, dt$ and the parameters $\mu, \theta, \kappa, \sigma, \rho$ are some constants. Note that, in the Heston model, $\sqrt{v}$ is the volatility and v follows the CIR [10] type process. The correlation between the price and volatility produces the return skew and strike-price biases in the Black–Scholes model.

Suppose that the riskfree interest rate is r and the price of volatility risk is given by λv, where r and λ are some constants. Let $U(S, v, t)$ be the time-t price of any derivative written on $S(t)$. Then, by the standard no-arbitrage arguments, one can obtain the following PDE for the value function $U(S, v, t)$:

$$\begin{aligned} \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v S\frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} \\ + rS\frac{\partial U}{\partial S} + (\kappa(\theta - v) - \lambda v)\frac{\partial U}{\partial v} \\ -rU + \frac{\partial U}{\partial t} = 0 \end{aligned} \qquad (39)$$

with an appropriate boundary condition.

As before, thanks to the change of numéraire technique, the price of a call option with strike K and maturity T must be given by

$$C(S, v, t) = SP_1(S, v) - KP(t, T)P_2(S, v), \qquad (40)$$

where $P_j(x, v)$, $j = 1, 2$, with $x = \log S$ is the probability that the option ends up in-the-money under an equivalent martingale measure with respect to the numéraire of either $S(t)$ or $P(t, T)$. That is, we define

$$\begin{aligned} & P_1(x, v, t; K) \\ &= Q^S\{\log S(T) \geq \log K \,|\, \log S(t) = x, \\ & \qquad\qquad v(t) = v\}, \\ & P_2(x, v, t; K) \\ &= Q^T\{\log S(T) \geq \log K \,|\, \log S(t) = x, \\ & \qquad\qquad v(t) = v\}. \end{aligned}$$

In the following, we obtain the Fourier transform of these probabilities.

By plugging (40) into (39), we obtain the PDE for each P_j, $j = 1, 2$, as

$$\begin{aligned} \frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} &+ \rho\sigma v\frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} \\ &+ (r + u_j v)\frac{\partial P_j}{\partial x} + (a_j - b_j v)\frac{\partial P_j}{\partial v} \\ &+ \frac{\partial P_j}{\partial t} = 0, \end{aligned} \qquad (41)$$

where

$$\begin{aligned} u_1 &= \frac{1}{2}, \qquad u_2 = -\frac{1}{2}, \qquad a = \kappa\theta, \\ b_1 &= \kappa + \lambda - \rho\theta, \qquad b_2 = \kappa + \lambda \end{aligned}$$

with the boundary condition

$$P_j(x, v, T; \log K) = 1_{\{x \geq \log K\}}.$$

In order to solve the PDE (41), we consider the process $(x(t),\ v(t))$ satisfying

$$\begin{aligned} dx(t) &= (r + u_j v(t))\,dt \\ &\quad + \sqrt{v(t)}\,dz_1(t), \\ dv(t) &= (a_j - b_j v(t))\,dt \\ &\quad + \sigma\sqrt{v(t)}\,dz_2(t). \end{aligned} \tag{42}$$

Note that P_j coincides with the probability

$$\begin{aligned} &P_j(x, v, t; K) \\ &\quad = \Pr\{x(T) \geq \log K | x(t) = x, \\ &\qquad v(t) = v\}, \qquad j = 1, 2. \end{aligned}$$

Now, for each j, define the characteristic function of $x(T)$ by

$$\begin{aligned} &f_j(x, v, t; \phi) \\ &\quad = E[e^{i\phi x(T)} \mid x(t) = x, v(t) = v], \\ &\qquad\qquad j = 1, 2. \end{aligned}$$

Of course, f_j satisfies the PDE (41) with the boundary condition

$$f_j(x, v, T; \phi) = e^{i\phi x}.$$

Since the model (42) is of an affine type, it is plausible to guess the solution as

$$\begin{aligned} &f_j(x, v, t; \phi) \\ &= \exp\{C(T - t; \phi) + D(T - t; \phi)v + i\phi x\}. \end{aligned}$$

Using the same technique as described in Section 1.3.3, we can derive a system of ODEs for C and D, which can be solved as

$$\begin{aligned} C(\tau; \phi) &= r\phi i\tau + \frac{a}{\sigma^2}\Bigg((b_j - \rho\sigma\phi i + d)\tau \\ &\quad -2\log\left(\frac{1 - ge^{d\tau}}{1 - g}\right)\Bigg), \\ D(\tau; \phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}, \end{aligned}$$

where

$$\begin{aligned} g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\ d &= \sqrt{(b_j - \rho\sigma\phi i)^2 - \sigma^2(2u_j\phi i - \phi^2)}. \end{aligned}$$

The desired probability is calculated by the Fourier inversion formula

$$\begin{aligned} &P_j(x, v, t; K) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{Ke^{-i\phi}f_j(x, v, t; \phi)}{i\phi}\right] d\phi. \end{aligned}$$

We note that, as pointed out in Section 1.3.3, the FFT is not applicable for the Fourier inversion because of its singularity. Carr and Madan [6] proposed application of the FFT to the option payoff instead of the probability.

1.5.4 SABR Model

The SABR (stochastic α, β, ρ) model extends the local volatility (LV) model to incorporate the dynamic behavior of the volatility smile. Recall that the LV models are self-consistent, arbitrage-free, and can be calibrated so as to match observed market smiles and skews precisely. However, the dynamic behavior of smiles implied by the LV models is exactly the opposite to the behavior observed in the market. To resolve the problem, the SABR model is built within the framework of stochastic volatility models.

We follow Hagan et al. [23] to introduce the SABR model in this section. Consider the forward price $F(t)$ under the T-forward measure Q^T whose dynamics is described by the SDE

$$\begin{aligned} dF(t) &= \hat{\alpha}(t)F(t)^\beta\,dz_1(t), \quad F(0) = f, \\ d\hat{\alpha}(t) &= \nu\hat{\alpha}(t)\,dz_2(t), \qquad \hat{\alpha}(0) = \alpha, \end{aligned}$$

where $dz_1(t)dz_2(t) = \rho\,dt$. Here, the parameters α, β, ρ are some constants.

By using a singular perturbation technique, Hagan et al. [23] derived the price of a call option with strike K and maturity T as

$$\begin{aligned} V_{call}(0) &= P(0, T)\left(fN(d_1) - KN(d_2)\right), \\ d_{1,2} &= \frac{\log(f/K) \pm \frac{1}{2}\sigma_B^2(K, f)T}{\sigma_B(K, f)\sqrt{T}}, \end{aligned}$$

where the implied volatility is given by

$$\begin{aligned}\sigma_B(K,f) \\ \approx \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}(\log(f/K))^2+\cdots\right\}} \\ \times\left(\frac{\xi}{x(\xi)}\right) \\ \times\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}\right.\right. \\ \left.\left.+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T+\cdots\right\},\end{aligned}$$

with

$$\begin{aligned}\xi &= \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log(f/K), \\ x(\xi) &= \log\left\{\frac{\sqrt{1-2\rho\xi+\xi^2}+\xi-\rho}{1-\rho}\right\}.\end{aligned}$$

In particular, for the at-the-money (ATM) option, the implied volatility is reduced to

$$\begin{aligned}\sigma_{ATM} &\equiv \sigma_B(f,f) \\ &= \frac{\alpha}{f^{1-\beta}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}\right.\right. \\ &\quad \left.+\frac{1}{4}\frac{\rho\beta\nu\alpha}{f^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2\right]T \\ &\quad \left.+\cdots\right\}.\end{aligned}$$

Recall that the aim of the SARB model is to manage the smile and skew effects that are represented by the parameters β, ρ, ν. For the management of smile risk, the implied volatility is approximated as

$$\begin{aligned}\sigma_B(K,f) &= \frac{\alpha}{f^{1-\beta}}\left(1-\frac{1-\beta-\rho\lambda}{2}\right. \\ &\quad \times\log\frac{K}{f}+\frac{1}{12}((1-\beta)^2 \\ &\quad +(2-3\rho^2)\lambda^2)\left(\log\frac{K}{f}\right)^2 \\ &\quad \left.+\cdots\right), \qquad \lambda=\frac{\nu}{\alpha}f^{1-\beta},\end{aligned} \tag{43}$$

assuming that the strike K is not so far from the forward price $F(0) = f$.

The approximated implied volatility (43) is decomposed to three parts for risk management purpose:

1. The term $\alpha/f^{1-\beta}$ is an approximation of the ATM implied volatility $\sigma_B(f,f)$, which is called the *backbone*. The backbone of the SABR model is downward sloping when $\beta = 0$ (normal model) and flat when $\beta = 1$(SV model).
2. The first term $-\frac{1-\beta-\rho\lambda}{2}\log(K/f)$ in the parentheses is the skew, which is a decreasing function of strike price K when $0 \le \beta \le 1$.
3. The second term in the parenthesis is the smile. It is found that the movement of the implied volatility curve associated with the change of f is consistent with the observation in the market as desired.

1.5.5 Regime-Switching Model

It is well known that the up-trend volatility of a stock price tends to be smaller than its down-trend volatility. Regime-switching models have a feature that the parameters switch randomly in time among a finite number of states $\mathcal{M} = \{1, 2, \ldots, m\}$. For example, the volatility coefficient (or the drift coefficient) of the SDE of a stock price takes a high value or a low value, depending on the market mode. The regime may be reflected by the state of the economy, the general sentiment of investors in the market, and other factors. See, e.g., Elliott et al. [18] for details of regime-switching models in finance.

We denote the regime at time t by $\alpha(t) \in \mathcal{M}$, and suppose that $\alpha(t)$ follows a continuous-time Markov chain. The stock price process $S(t)$ is assumed to satisfy the SDE

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma(\alpha(t))\,dz(t), \quad t \ge 0,$$

under P, where $z(t)$ is a standard Brownian motion, independent of the Markov chain $\alpha(t)$, and $\mu(t)$ is adapted to the filtration.

Let $\mathbf{Q} = (q_{ij})_{m\times m}$ be the generator of $\alpha(t)$ with

$$q_{ij} \geq 0 \quad \text{for} \quad i \neq j,$$
$$\sum_{j=1}^{m} q_{ij} = 0 \quad \text{for} \quad \forall i \in \mathcal{M}.$$

Assuming that the market price of risk for regime switches is zero, the risk-neutral measure Q can be found by the Radon–Nikodym derivative (12), i.e.,

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \exp\left\{\int_0^T \beta(t)dz(t) - \frac{1}{2}\int_0^T \beta^2(t)\,dt\right\},$$
$$\beta(t) = \frac{r-\mu(\alpha(t))}{\sigma(\alpha(t))}.$$

By the Girsanov theorem, the process defined by

$$z^Q(t) = z(t) - \int_0^t \beta(u)\,du$$

is a standard Brownian motion and independent of $\alpha(t)$ under Q. It follows that the price of a derivative security with payoff $h(S)$ is given by

$$\begin{aligned} C(t,S,i) &= E^Q[e^{-r(T-t)}h(S(T)) \mid S(t)=S, \\ &\qquad\qquad \alpha(t)=i]. \end{aligned}$$

For more general treatments of the risk-neutral valuation, the reader is referred to Elliott and Kopp [19].

For each regime i, the PDE for the price function is obtained in the same way as before. That is,

$$\begin{aligned} \mathcal{L}C(t,S,i) &= rC(t,S,i), \\ &\quad i \in i = 1,2,\ldots,m, \end{aligned}$$

with the boundary condition $C(T,S,i) = h(S)$, where

$$\begin{aligned} \mathcal{L}F(t,x,i) &= \frac{\partial}{\partial t}F(t,x,i) \\ &\quad + \frac{1}{2}x^2\sigma^2(i)\frac{\partial^2}{\partial x^2}F(t,x,i) \\ &\quad + rx\frac{\partial}{\partial x}F(t,x,i) \\ &\quad + \mathbf{Q}F(t,x,\cdot)(i) \end{aligned}$$

with

$$\mathbf{Q}F(t,x,\cdot)(i) = \sum_{j=1}^{m} q_{ij}F(t,x,i).$$

The price function is calculated numerically by solving the system of PDEs.

Note that, when the payoff $h(S)$ is not a smooth function like call options, the solution may not be unique (may not be differentiable) due to the non-smooth boundary conditions. To resolve this problem, Yao et al. [37] proposed to approximate $h(S)$ with a smooth function $h_\delta(S)$ that converges to $h(S)$ as $\delta \to 0$. Under an appropriate choice of $h_\delta(S)$, it is shown that the smoothed solution converges to the original solution.

1.6 Models with Lévy Processes

In the finance literature, Lévy processes are often used to represent jumps of stock prices. They belong to a wider class of processes with rich features, including Brownian motions and Poisson processes.

An $\mathbb{R}^d$-valued process $X(t)$ is called a (d-dimensional) Lévy process if the following conditions are satisfied [35]:

1. It has independent increments. The random variables $X(t_0)$, $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, $\cdots, X(t_n) - X(t_{n-1})$ are mutually independent for all $n \geq 1$, $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$.
2. It starts at zero ($X(0) = 0$ a.s.).

3. It is homogeneous in time. The distribution of $\{X(s+t) - X(t) : t \geq 0\}$ does not depend on s.

4. It is stochastically continuous. $\forall \varepsilon > 0$, $\lim_{t\to 0} P[|X(s+t) - X(s)| > \varepsilon] = 0$.

5. As a function of t, $X(t, \omega)$ is right-continuous with left limits a.s.

A Lévy processes $X(t)$ is characterized by the triplet (μ, Σ, π), called the *Lévy characteristics*. The first parameter μ is a constant drift, the second Σ is a constant covariance matrix of the continuous part, and the third one π, called the *Lévy measure*, describes the arrival rate for jumps of every possible size. By the Lévy–Khintchine formula, the Lévy process $X(t)$ with the Lévy characteristics has the characteristic function given by

$$\phi_{X(t)}(\theta) \equiv E\left[e^{i\theta^\top X(t)}\right] = e^{-t\Psi_X(\theta)},$$

where the function Ψ_X is the *characteristic exponent* [7] defined by

$$\begin{aligned}\Psi_X(\theta) \\ &= -i\mu^\top\theta + \frac{1}{2}\theta^\top\Sigma\theta \\ &\quad + \int_{\mathbb{R}^d}\left(1 - e^{i\theta^\top x} + i\theta^\top x \mathbf{1}_{|x|<1}\right)\pi(dx).\end{aligned} \tag{44}$$

A Brownian motion with drift, $X(t) = \mu t + \sigma z(t)$ say, is an example of Lévy process with no jumps, i.e., $\pi = 0$.

If the number of jumps of a Lévy process in every time interval is finite, so that the jumps are considered as "rare events," the model is called a *jump-diffusion model* (or a finite activity model). One of such examples is a combination of a Brownian motion with drift and a compound Poisson process considered in, e.g., Merton [31] and Kou [29]. On the other hand, if jumps arrive infinitely often in every time interval, then the model is called an *infinite activity model*. In some cases, an infinite activity model is formulated as a Brownian subordination, which makes the analysis tractable. The subordinator can be interpreted as a "business time" instead of a "clock time." The variance gamma model (explained below) falls in this category. Table 1 summarizes the Lévy measure and the characteristic exponent of Lévy models discussed in this section.

The infinitesimal generator of a Lévy process $X(t)$ with characteristic triplet (μ, Σ, π) is given by

$$\begin{aligned}&\mathcal{L}^X f(x) \\ &= \sum_{j=1}^{d}\mu_j\frac{\partial f}{\partial x_j}(x) \\ &\quad + \frac{1}{2}\sum_{j,k=1}^{d}\Sigma_{jk}\frac{\partial^2 f}{\partial x_j \partial x_k}(x) \\ &\quad + \int_{\mathbb{R}^d}\Bigg(f(x+y) - f(x) \\ &\quad - \sum_{j=1}^{d} y_j\frac{\partial f}{\partial x_j}(x)\mathbf{1}_{\{|y|<1\}}\Bigg)\pi(dy).\end{aligned}$$

1.6.1 The Merton Model

Merton [31] considered a jump-diffusion model for a stock price under a risk-neutral measure Q. The stock price $S(t)$ is driven by a Brownian motion $z(t)$ and jumps under Q. Jumps in the log-price $\log S(t)$ occur subject to a Poisson process $N(t)$ with intensity λ, which means that the number of jumps up to time t follows a Poisson distribution with mean λt. The size of the ith jump is represented by Y_i, which are identical and normally distributed with mean m and variance δ^2. The processes $N(t)$, $z(t)$ and random variables Y_i are independent of each other. The risk-free interest rate and volatility are constant and denoted by r and σ, respectively.

Under these assumptions, the stock price process under the risk-neutral measure Q

Table 1: Lévy measure and characteristic exponent.

Model	Lévy measure $\pi(dx)/dx$	Characteristic exponent $\Psi(\theta)$
Pure continuous Lévy component		
$\mu t + \sigma z(t)$	none	$-i\mu\theta + \frac{1}{2}\sigma^2\theta^2$
Finite-activity pure jump Lévy components		
Merton (1976)	$\lambda\frac{1}{\sqrt{2\pi\sigma_J^2}}\exp\left\{-\frac{(x-\alpha)^2}{2\sigma_J^2}\right\}$	$\lambda\left(1-e^{i\theta\alpha-\frac{1}{2}\sigma_J^2\theta^2}\right)$
Kou (2002)	$\lambda\frac{1}{2\eta}\exp\left(-\frac{\lvert x-k\rvert}{\eta}\right)$	$\lambda\left(1-e^{i\theta k}\frac{1-\eta^2}{1+\theta^2\eta^2}\right)$
Infinite-activity pure jump Lévy components		
VG	$\frac{\mu_\pm^2}{v_\pm}\frac{\exp\{-\frac{\mu_\pm}{v_\pm}\lvert x\rvert\}}{\lvert x\rvert}$ $\mu_\pm = \frac{1}{2}\lambda\left(\sqrt{\alpha^2+2\sigma_J^2}\pm\alpha\right)$ $v_\pm = \mu_\pm^2/\lambda$	$\lambda\log\left(1-i\theta\alpha+\frac{1}{2}\sigma_J^2\theta^2\right)$

is represented as

$$S(t) = S(0)\exp\left\{\mu^Q t + \sigma z(t) + \sum_{i=1}^{N(t)} Y_i\right\}, \tag{45}$$

where

$$\begin{aligned}\mu^Q &= r - \frac{1}{2}\sigma^2 - \lambda E[e^{Y_i}-1] \\ &= r - \frac{1}{2}\sigma^2 - \lambda(e^{m+\delta^2/2}-1).\end{aligned}$$

The drift coefficient μ^Q is determined so that the discounted stock price $e^{-rt}S(t)$ becomes a martingale under Q.

The no-arbitrage price of a call option with strike K and maturity T is obtained by taking the expectation of the discounted cashflow under the risk-neutral measure Q. By using the function $H(x) = (x-K)_+$, the call price is calculated as follows:

$$\begin{aligned}C_M(t,S) &\equiv e^{-r(T-t)}E^Q\left[(S(T)-K)_+ \mid S(t)=S\right] \\ &= e^{-r(T-t)}E^Q\Big[H\Big(S\exp\Big\{\mu^Q(T-t) \\ &\qquad +\sigma z(T-t) + \sum_{i=1}^{N(T-t)} Y_i\Big\}\Big)\Big] \\ &= e^{-r(T-t)}\sum_{n=0}^{\infty} Q\{N(T-t)=n\} \\ &\qquad \times E^Q\Big[H\Big(S\exp\Big\{\mu^Q(T-t) \\ &\qquad +\sigma z(T-t) + \sum_{i=1}^{n} Y_i\Big\}\Big)\Big].\end{aligned}$$

Since $\sum_{i=1}^n Y_i$ is normally distributed with mean nm and variance $n\delta^2$, it is rewritten as $nm + \sqrt{n\delta^2}W$ in law, where W follows a standard normal distribution that is independent of other variables. Thus, $\sigma z(T-t) + \sqrt{n\delta^2}W$ is normally distributed with zero mean and standard deviation of

$\sigma_n\sqrt{T-t}$ where $\sigma_n^2 = \sigma^2 + n\delta^2/(T-t)$. It follows that

$$\begin{aligned}
&C_M(t,S)\\
&= e^{-r(T-t)}\sum_{n=0}^{\infty} Q\{N(T-t)=n\}\\
&\quad \times E^Q[H(S\exp\{\mu^Q(T-t)\\
&\quad +\sigma z(T-t)+nm+\sqrt{n\delta^2}W\})]\\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}\\
&\quad \times C_{BS}(S_n, T-t;\sigma_n),
\end{aligned}$$

where $C_{BS}(S,T;\sigma)$ denotes the call option price calculated by the Black–Scholes formula (5), and

$$\begin{aligned}
S_n &= S\exp\{nm + n\delta^2/2\\
&\quad -\lambda(e^{m+\delta^2/2}-1)(T-t)\}.
\end{aligned}$$

This formula is instructive, because the option price is the averaged price of the Black–Scholes call option prices with adjusted parameters S_n and σ_n, where S_n is the stock price adjusted by anticipated n jumps prior to the maturity and σ_n is the adjusted volatility.

For the jump-diffusion model (45), the price function $C(S,t)$ satisfies the following partial integro-differential equation (PIDE), not the ordinary PDE. That is,

$$\begin{aligned}
&\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC\\
&\quad + \int\left[C(t,Se^y) - C(t,S) - S(e^y-1)\frac{\partial C}{\partial S}\right]\\
&\qquad \times \nu(y)\,dy = 0,
\end{aligned}$$

where $\nu(y)$ is the density function of jumps Y_i. The integral part appears due to the possibility of jumps.

Finally, for the reader's convenience, we present the change of measure formula for jump-diffusion models. Suppose that $z(t)$ is a Brownian motion defined on $(\Omega, \mathcal{F}, P)$. The size of the ith jump is represented by Y_i with density function $\nu(y)$, and $N(t)$ is a Poisson process with intensity λ. These random components are independent of each other. Then, the process $\xi(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process with intensity λ and jump size density $\nu(y)$ under P.

Suppose that we are given another intensity $\tilde{\lambda}$, density function $\tilde{\nu}(y)$, and an adapted process $\theta(t)$. Then, we can define a new probability measure Q via the Radon–Nikodym process $L(t)$ as

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = L(t) \equiv L_1(t)L_2(t),$$

where

$$\begin{aligned}
L_1(t) &= \exp\left\{-\int_0^t \theta(u)\,dz(u)\right.\\
&\qquad \left. -\frac{1}{2}\int_0^t \theta^2(u)\,du\right\},\\
L_2(t) &= e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{\nu}(Y_i)}{\lambda\nu(Y_i)}.
\end{aligned}$$

By the Girsanov theorem, under the new measure Q, $\tilde{z}(t) = z(t) + \int_0^t \theta(s)\,ds$ is a standard Brownian motion, $\tilde{\xi}(t) = \sum_{i=1}^{\tilde{N}(t)} \tilde{Y}_i$ is a compound Poisson process with intensity $\tilde{\lambda}$ and jump size density $\tilde{\nu}(y)$, and these random components are independent of each other.

1.6.2 The Kou Model

In the Merton model, because jump sizes are normally distributed, the return process has a symmetric feature. In order to model an asymmetric return with keeping analytical tractability, Kou [29] developed a model in which Y_i follows an asymmetric double-exponential distribution with density function

$$\nu(y) = p\eta_1 e^{-\eta_1 y}1_{\{y\geq 0\}} + q\eta_2 e^{-\eta_2 y}1_{\{y<0\}},$$

where $0 \leq p \leq 1$, $q = 1-p$, $\eta_1 > 1$, $\eta_2 > 0$. The condition $\eta_1 > 1$ is required to ensure $E[e^Y] < \infty$ and $E[S(t)] < \infty$.

The stock price jumps upwards with probability p and the jump size is exponentially distributed with mean $1/\eta_1$, whereas it jumps downwards with probability q and the jump size is exponentially distributed with mean $1/\eta_2$. The mean and variance of the jump size are given by

$$E[Y] = \frac{p}{\eta_1} - \frac{q}{\eta_2},$$
$$V[Y] = pq\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^2 + \left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}\right).$$

Note also that

$$E[e^Y] = p\frac{\eta_1}{\eta_1 - 1} + q\frac{\eta_2}{\eta_2 + 1}.$$

Using the *memoryless* property of exponential distributions, Kou [29] derived the following option price formula:

$$C(0,S) = S\Upsilon\Big(r + \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \log(K/S), T\Big) - Ke^{-rT}\Upsilon\Big(r - \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \lambda, p, \eta_1, \eta_2; \log(K/S), T\Big),$$

where $\tilde{\lambda} = \lambda(\eta+1)$, $\tilde{\eta}_1 = \eta_1 - 1$, $\tilde{\eta}_2 = \eta_2 + 1$ and

$$\zeta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1,$$
$$\tilde{p} = \frac{p}{1+\zeta}\frac{\eta_1}{\eta_1 - 1}.$$

The function $\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T)$ represents the probability $\Pr\{Z(T) \geq a\}$ for the jump-diffusion process

$$Z(t) = \mu t + \sigma z(t) + \sum_{i=1}^{N(t)} Y_i. \qquad t \geq 0.$$

The explicit functional form of Υ is found in Kou [29].

The Kou model has several advantages in financial applications. Among them, (i) it offers an explanation for the asymmetric leptokurtic return and the volatility smile, and (ii) it leads to analytical solutions for many option pricing problems including path-dependent options and interest-rate derivatives.

1.6.3 Variance Gamma Model

The variance gamma (VG) process is a process of finite variation with infinite but relatively low activity of small jumps. Thus, it accounts for high activity by having an infinite number of jumps in any interval of time.

Mathematically, the VG process can be constructed by a Brownian subordination with a gamma process. To this end, first define the Gamma distribution $\Gamma(h/\nu, \nu)$ with density function

$$f_h(g) = \frac{g^{h/\nu - 1}\exp(-g/\nu)}{\nu^{h/\nu}\Gamma(h/\nu)}, \qquad g > 0.$$

The mean and variance of $\Gamma(h/\nu, \nu)$ are h and $h\nu$, respectively. Let $G(t;\nu)$ be a gamma process with parameter ν. The gamma process is characterized by the independent and stationary increments. That is, (i) the increments $G(t+h;\nu) - G(t;\nu)$ on non-overlapping time intervals are mutually independent, and (ii) the increment $G(t+h;\nu) - G(t;\nu)$ follows the Gamma distribution $\Gamma(h/\nu, \nu)$, independent of time t.

Now, let $Y(t;\sigma,\theta)$ be a Brownian motion with drift θ and diffusion coefficient σ, i.e.

$$Y(t;\sigma,\theta) = \theta t + \sigma z(t),$$

where $z(t)$ is a standard Brownian motion. Then, the VG process $X_{VG}(t)$ is defined as a time-changed process by

$$X_{VG}(t) \equiv Y(G(t;\nu);\sigma,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu)). \tag{46}$$

It is readily shown that the characteristic function of $X_{VG}(t)$ is given by

$$\begin{aligned}\phi_{X_{VG}}(t;u) &\equiv E\left[\exp(iuX_{VG}(t))\right]\\ &= \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu},\end{aligned}$$

which can be written in the Lévy–Khintchine form. The Lévy measure of the VG process is given by

$$\begin{aligned}\pi_{VG}(x) &= \frac{1}{\nu|x|}\exp\left(Ax - B|x|\right),\\ A &= \frac{\theta}{\sigma^2},\\ B &= \sqrt{\frac{2}{\sigma^2\nu} + \frac{\theta^2}{\sigma^4}}.\end{aligned}$$

See Madan and Seneta [33] for details.

Madan et al. [32] applied the VG process to describe a stock price and developed an option pricing formula. More specifically, suppose that the stock price is given by

$$\begin{aligned}S(t) &= S(0)\exp\{rt + X_{VG}(t;\sigma,\nu,\theta)\}\\ &\quad \times \left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)^{t/\nu}, \; t \geq 0,\end{aligned}$$

under the risk-neutral measure Q. It is readily checked that the relative price $e^{-rt}S(t)$ is a martingale under Q. The price of a call option with strike price K and maturity T is obtained as

$$\begin{aligned}&c(S;K,T)\\ &= S\Psi\left(d\sqrt{\frac{1-c_1}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_1}}, \frac{T}{\nu}\right)\\ &\quad -Ke^{-rT}\Psi\left(d\sqrt{\frac{1-c_2}{\nu}}, \alpha s\sqrt{\frac{\nu}{1-c_2}}, \frac{T}{\nu}\right),\end{aligned}$$

where

$$\begin{aligned}d &= \frac{1}{s}\left(\log\frac{S}{K} + rT + \frac{T}{\nu}\log\left(\frac{1-c_1}{1-c_2}\right)\right),\\ \zeta &= \frac{\theta}{\sigma^2}, \qquad s = \frac{\sigma}{\sqrt{1 + \left(\frac{\theta}{\sigma}\right)^2\frac{\nu}{2}}},\\ \alpha &= \zeta s, \qquad c_1 = \frac{\nu(\alpha+s)^2}{2},\\ c_2 &= \frac{\nu\alpha^2}{2}.\end{aligned}$$

The function Ψ is found in Madan et al. [32].

The VG process is characterized by three parameters; σ is the volatility of the Brownian part, ν is the variance rate of the gamma time change, and θ is the drift in the Brownian part. While the parameter θ controls the skew, the parameter ν determines the excess kurtosis. According to ample empirical analyses, the hypothesis of zero skewness cannot be rejected, but the zero kurtosis is rejected. The VG model with zero skewness ($\theta = 0$) is called a symmetric VG model by Madan and Seneta [33]. In Madan et al. [32], the Black–Scholes model, the symmetric VG model and the VG model are compared for the option pricing behavior. The symmetric VG model is the best fitted among them in terms of the likelihood. The VG model performs best to account for the pricing biases.

1.6.4 Time-Changed Lévy Process

In the VG process, a gamma process plays the role of the business time on a simple Brownian motion with drift. This idea can be generalized to a nondecreasing process (subordinator) on a Lévy process, called a time-changed Lévy process. This section is written based on Carr and Wu [7].

Time-changed Lévy processes can capture the three important features simultaneously; non-normal returns, stochastic volatilities, negative correlations between asset returns and their volatilities. While a Brownian motion generates a normal return, a pure jump Lévy process does a non-normal return. A stochastic time change leads to a stochastically altering business time from the original clock and, as a result, a stochastic volatility. One may feel that the business clock runs faster for more busy days. One way to produce the correlation is that the innovations of the Lévy process are correlated with the innovations in the stochastic time change.

Let $X(t)$ be a Lévy process with characteristic exponent $\Psi_X(\theta)$ given by (44), and let $T(t)$ be a nondecreasing process. Assuming that $X(t)$ and $T(t)$ are independent of each other,[10] the characteristic function of $Y(t) = X(T(t))$ is calculated as

$$\begin{aligned} \phi_{Y(t)}(\theta) &= E[e^{i\theta Y(t)}] \\ &= E[E[e^{i\theta X(T(t))} \mid T(t)]] \\ &= E[e^{-T(t)\Psi_X(\theta)}] \\ &= \mathcal{L}_{T(t)}(\Psi_X(\theta)) \end{aligned}$$

by the tower property (iterative expectations), where $\mathcal{L}_{T(t)}(\cdot)$ is the Laplace transform of $T(t)$.

The framework of time-changed Lévy processes unifies the models of Merton [31], Heston [24] and other models in the finance literature on the option pricing. For example, let us start from the Black–Scholes model under the risk-neutral measure Q:

$$\log(S(t)/S(0)) = rt + \sigma z(t) - \frac{1}{2}\sigma^2 t, \ t \geq 0.$$

By replacing the clock t running on the continuous martingale part by the random time $T(t)$, the stock price is given by

$$\log(S(t)/S(0)) = rt + \sigma z(T(t)) - \frac{1}{2}\sigma^2 T(t).$$

[10] If $X(t)$ and $T(t)$ are correlated, the change of measure technique is a powerful tool to split the correlation between them. See, e.g., Carr and Wu [7] for details.

Suppose that the random time is absolutely continuous and represented as

$$T(t) = \int_0^t v(s)\, ds$$

with some non-negative process $v(t)$. The intensity $v(t)$ is called the *activity rate*. If the activity rate is modeled by the CIR process as

$$dv(t) = \kappa(\theta - v(t)\, dt + \sigma_v \sqrt{v(t)}\, dz_v(t),$$

the stochastic volatility model of Heston [24] is obtained, since there exists a Brownian motion $\tilde{z}(t)$ such that

$$\int_0^t \sqrt{v(t)}\, dz(t) \stackrel{d}{=} \tilde{z}(T_t),$$

where $\stackrel{d}{=}$ stands for equality in law.

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