Chapter 1

Putting a Name to Linear Algebra

In This Chapter

- Aligning the algebra part of linear algebra with systems of equations
- Making waves with matrices and determinants
- Vindicating yourself with vectors
- Keeping an eye on eigenvalues and eigenvectors

The words *linear* and *algebra* don't always appear together. The word *linear* is an adjective used in many settings: *linear equations, linear regression, linear programming, linear technology,* and so on. The word *algebra,* of course, is familiar to all high school and most junior high students. When used together, the two words describe an area of mathematics in which some traditional algebraic symbols, operations, and manipulations are combined with vectors and matrices to create systems or structures that are used to branch out into further mathematical study or to use in practical applications in various fields of science and business.

The main elements of linear algebra are systems of linear equations, vectors and matrices, linear transformations, determinants, and vector spaces. Each of these topics takes on a life of its own, branching into its own special emphases and coming back full circle. And each of the main topics or areas is entwined with the others; it's a bit of a symbiotic relationship — the best of all worlds.

You can find the systems of linear equations in Chapter 4, vectors in Chapter 2, and matrices in Chapter 3. Of course, that's just the starting point for these topics. The uses and applications of these topics continue throughout the book. In Chapter 8, you get the big picture as far as linear transformations; determinants begin in Chapter 10, and vector spaces are launched in Chapter 13.

Solving Systems of Equations in Every Which Way but Loose

A *system of equations* is a grouping or listing of mathematical statements that are tied together for some reason. Equations may associate with one another because the equations all describe the relationships between two or more variables or unknowns. When studying systems of equations (see Chapter 4), you try to determine if the different equations or statements have any common solutions — sets of replacement values for the variables that make all the equations have the value of truth at the same time.

For example, the system of equations shown here consists of three different equations that are all true (the one side is equal to the other side) when $x = 1$ and $y = 2$.

$$\begin{cases} 2x + y = 4 \\ y^2 - x = 3 \\ \sqrt{x+3} = y \end{cases}$$

The only problem with the set of equations I've just shown you, as far as *linear* algebra is concerned, is that the second and third equations in the system are *not* linear.

A *linear equation* has the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = k$, where a_i is a real number, x_i is a variable, and k is some real constant.

Note that, in a linear equation, each of the variables has an exponent of exactly 1. Yes, I know that you don't see any exponents on the *x*s, but that's standard procedure — the 1s are assumed. In the system of equations I show you earlier, I used x and y for the variables instead of the subscripted *x*s. It's easier to write (or type) x, y, z, and so on when working with smaller systems than to use the subscripts on a single letter.

I next show you a system of *linear* equations. I'll use x, y, z, and w for the variables instead of x_1, x_2, x_3, and x_4.

$$\begin{cases} 2x + y - 3z + 4w = 11 \\ x + 3y - 2z + w = 5 \\ 3x + y + 2w = 13 \\ 4x + y + 5z - w = 17 \end{cases}$$

The system of four linear equations with four variables or unknowns *does* have a single solution. Each equation is true when $x = 1$, $y = 2$, $z = 3$, and $w = 4$. Now a caution: Not every system of linear equations has a solution. Some systems of equations have no solutions, and others have many or infinitely many solutions. What you find in Chapter 4 is how to determine which situation you have: none, one, or many solutions.

Systems of linear equations are used to describe the relationship between various entities. For example, you might own a candy store and want to create different selections or packages of candy. You want to set up a 1-pound box, a 2-pound box, a 3-pound box, and a diet-spoiler 4-pound box. Next I'm going to describe the contents of the different boxes. After reading through all the descriptions, you're going to have a greater appreciation for how nice and neat the corresponding equations are.

The four types of pieces of candy you're going to use are a nougat, a cream, a nut swirl, and a caramel. The 1-pounder is to contain three nougats, one cream, one nut swirl, and two caramels; the 2-pounder has three nougats, two creams, three nut swirls, and four caramels; the 3-pounder has four nougats, two creams, eight nut swirls, and four caramels; and the 4-pounder contains six nougats, five creams, eight nut swirls, and six caramels. What does each of these candies weigh?

Letting the weight of nougats be represented by x_1, the weight of creams be represented by x_2, the weight of nut swirls be represented by x_3, and the weight of caramels be represented by x_4, you have a system of equations looking like this:

$$\begin{cases} 3x_1 + 1x_2 + 1x_3 + 2x_4 = 16 \\ 3x_1 + 2x_2 + 3x_3 + 4x_4 = 32 \\ 4x_1 + 2x_2 + 8x_3 + 4x_4 = 48 \\ 6x_1 + 5x_2 + 8x_3 + 6x_4 = 64 \end{cases}$$

The pounds are turned to ounces in each case, and the solution of the system of linear equations is that $x_1 = 1$ ounce, $x_2 = 2$ ounces, $x_3 = 3$ ounces, and $x_4 = 4$ ounces. Yes, this is a very simplistic representation of a candy business, but it serves to show you how systems of linear equations are set up and how they work to solve complex problems. You solve such a system using algebraic methods or matrices. Refer to Chapter 4 if you want more information on how to deal with such a situation.

Systems of equations don't always have solutions. In fact, a single equation, all by itself, can have an infinite number of solutions. Consider the equation $2x + 3y = 8$. Using ordered pairs, (x,y), to represent the numbers you want, some of the solutions of the system are (1,2), (4,0), (–8,8), and (10,–4). But none of the solutions of the equation $2x + 3y = 8$ is also a solution of the equation $4x + 6y = 10$. You can try to find some matches, but there just aren't any. Some solutions of $4x + 6y = 10$ are (1,1), (4,–1), and (10,–5). Each equation has an infinite number of solutions, but no pairs of solutions match. So the system has no solution.

$$\begin{cases} 2x + 3y = 8 \\ 4x + 6y = 10 \end{cases}$$

Knowing that you don't have a solution is a very important bit of information, too.

Matchmaking by Arranging Data in Matrices

A matrix is a rectangular arrangement of numbers. Yes, all you see is a bunch of numbers — lined up row after row and column after column. Matrices are tools that eliminate all the fluff (such as those pesky variables) and set all the pertinent information in an organized logical order. (Matrices are introduced in Chapter 3, but you use them to solve systems of equations in Chapter 4.) When matrices are used for solving systems of equations, you find the coefficients of the variables included in a matrix and the variables left out. So how do you know what is what? You get organized, that's how.

Here's a system of four linear equations:

$$\begin{cases} x_1 - 2x_2 + 3x_3 - 6x_4 = 8 \\ x_1 \qquad\quad - 10x_3 + x_4 = 11 \\ 4x_1 + x_2 \qquad\quad - 4x_4 = 8 \\ 3x_1 + 5x_2 - 7x_3 \qquad\quad = 6 \end{cases}$$

When working with this system of equations, you may use one matrix to represent all the coefficients of the variables.

$$A = \begin{bmatrix} 1 & -2 & 3 & -6 \\ 1 & 0 & -10 & 1 \\ 4 & 1 & 0 & -4 \\ 3 & 5 & -7 & 0 \end{bmatrix}$$

Notice that I placed a 0 where there was a missing term in an equation. If you're going to write down the coefficients only, you have to keep the terms in order according to the variable that they multiply and use *markers* or placeholders for missing terms. The coefficient matrix is so much easier to look at than the equation. But you have to follow the rules of order. And I named the matrix — nothing glamorous like *Angelina,* but something simple, like A.

When using coefficient matrices, you usually have them accompanied by two vectors. (A *vector* is just a one-dimensional matrix; it has one column and many rows or one row and many columns. See Chapters 2 and 3 for more on vectors.)

The vectors that correspond to this same system of equations are the vector of variables and the vector of constants. I name the vectors X and C.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad C = \begin{bmatrix} 8 \\ 11 \\ 8 \\ 6 \end{bmatrix}$$

Once in matrix and vector form, you can perform operations on the matrices and vectors individually or perform operations involving one operating on the other. All that good stuff is found beginning in Chapter 2.

Let me show you, though, a more practical application of matrices and why putting the numbers (coefficients) into a matrix is so handy. Consider an insurance agency that keeps track of the number of policies sold by the different agents each month. In my example, I'll keep the number of agents and policies small, and let you imagine how massive the matrices become with a large number of agents and different variations on policies.

At Pay-Off Insurance Agency, the agents are Amanda, Betty, Clark, and Dennis. In January, Amanda sold 15 auto insurance policies, 10 dwelling/home insurance policies, 5 whole-life insurance policies, 9 tenant insurance policies, and 1 health insurance policy. Betty sold . . . okay, this is already getting drawn out. I'm putting all the policies that the agents sold in January into a matrix.

$$\begin{array}{r c} & \begin{array}{ccccc} A & D & W & T & H \end{array} \\ \begin{array}{r} \text{Amanda} \\ \text{Betty} \\ \text{Clark} \\ \text{Dennis} \end{array} & \begin{bmatrix} 15 & 10 & 5 & 9 & 1 \\ 10 & 9 & 4 & 9 & 2 \\ 20 & 0 & 0 & 23 & 1 \\ 15 & 6 & 10 & 6 & 5 \end{bmatrix} \end{array}$$

If you were to put the number of policies from January, February, March, and so on in matrices, it's a simple task to perform *matrix addition* and get totals for the year. Also, the commissions to agents can be computed by performing *matrix multiplication.* For example, if the commissions on these policies are flat rates — say \$110, \$200, \$600, \$60, and \$100, respectively, then you create a vector of the payouts and multiply.

$$\begin{array}{c} \\ \text{Amanda} \\ \text{Betty} \\ \text{Clark} \\ \text{Dennis} \end{array}\begin{array}{c} \begin{array}{ccccc} \text{A} & \text{D} & \text{W} & \text{T} & \text{H} \end{array} \\ \begin{bmatrix} 15 & 10 & 5 & 9 & 1 \\ 10 & 9 & 4 & 9 & 2 \\ 20 & 0 & 0 & 23 & 1 \\ 15 & 6 & 10 & 6 & 5 \end{bmatrix} \end{array} * \begin{bmatrix} 110 \\ 200 \\ 600 \\ 60 \\ 100 \end{bmatrix} = \begin{bmatrix} 7{,}290 \\ 6{,}040 \\ 3{,}680 \\ 9{,}710 \end{bmatrix}\begin{array}{c} \text{Amanda} \\ \text{Betty} \\ \text{Clark} \\ \text{Dennis} \end{array}$$

This matrix addition and matrix multiplication business is found in Chapter 3. Other processes for the insurance company that could be performed using matrices are figuring the percent increases or decreases of sales (of the whole company or individual salespersons) by performing operations on summary vectors, determining commissions by multiplying totals by their respective rates, setting percent increase goals, and so on. The possibilities are limited only by your lack of imagination, determination, or need.

Valuating Vector Spaces

In Part IV of this book, you find all sorts of good information and interesting mathematics all homing in on the topic of vector spaces. In other chapters, I describe and work with vectors. Sorry, but there's not really any separate chapter on *spaces* or space — I leave that to the astronomers. But the words *vector space* are really just a mathematical expression used to define a particular group of elements that exist under a particular set of conditions. (You can find information on the properties of vector spaces in Chapter 13.)

Think of a vector space in terms of a game of billiards. You have all the elements (the billiards balls) that are confined to the top of the table (well, they stay there if hit properly). Even when the billiard balls interact (bounce off one another), they stay somewhere on the tabletop. So the billiard balls are the elements of the vector space and the table top is that vector space. You have operations that cause actions on the table — hitting a ball with a cue stick or a ball being hit by another ball. And you have rules that govern how all the actions can occur. The actions keep the billiard balls on the table (in the vector space). Of course, a billiards game isn't nearly as exciting as a vector space, but I wanted to relate some real-life action to the confinement of elements and rules.

A vector space is linear algebra's version of a type of classification plan or design. Other areas in mathematics have similar entities (classifications and designs). The common theme of such designs is that they contain a set or grouping of objects that all have something in common. Certain properties are attached to the plan — properties that apply to all the members of the grouping. If all the members must abide by the rules, then you can make judgments or conclusions based on just a few of the members rather than having to investigate every single member (if that's even possible).

Vector spaces contain vectors, which really take on many different forms. The easiest form to show you is an actual vector, but the vectors may actually be matrices or polynomials. As long as these different forms follow the rules, then you have a vector space. (In Chapter 14, you see the rules when investigating the subspaces of vector spaces.)

The rules regulating a vector space are highly dependent on the operations that belong to that vector space. You find some new twists to some familiar operation notation. Instead of a simple plus sign, +, you find $\oplus$. And the multiplication symbol, $\times$, is replaced with $\otimes$. The new, revised symbols are used to alert you to the fact that you're not in Kansas anymore. With vector spaces, the operation of addition may be defined in a completely different way. For example, you may define the vector addition of two elements, x and y, to be $x \oplus y = 2x + y$. Does that rule work in a vector space? That's what you need to determine when studying vector spaces.

Determining Values with Determinants

A *determinant* is tied to a matrix, as you see in Chapter 10. You can think of a determinant as being an operation that's performed on a matrix. The determinant incorporates all the elements of a matrix into its grand plan. You have a few qualifications to meet, though, before performing the operation *determinant.*

Square matrices are the only candidates for having a determinant. Let me show you just a few examples of matrices and their determinants. The matrix A has a determinant $|A|$ — which is also denoted det (A) — and so do matrices B and C.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -3 & 8 & 0 \\ 2 & -1 & 1 \end{bmatrix}, \det(A) = |A| = -51$$

$$B = \begin{bmatrix} -2 & 3 \\ 5 & -8 \end{bmatrix}, \det(B) = |B| = 1$$

$$C = \begin{bmatrix} 4 \end{bmatrix}, \det(C) = |C| = 4$$

The matrices A, B, and C go from a 3×3 matrix to a 2×2 matrix to a 1×1 matrix. The determinants of the respective matrices go from complicated to simple to compute. I give you all the gory details on computing determinants in Chapter 10, so I won't go into any of the computations here, but I do want to introduce you to the fact that these square matrices are connected, by a particular function, to single numbers.

All square matrices have determinants, but some of these determinants don't amount to much (the determinant equals 0). Having a determinant of 0 isn't a big problem to the matrix, but the value 0 causes problems with some of the applications of matrices and determinants. A common property that all these 0-determinant matrices have is that they don't have a multiplicative inverse.

For example, the matrix D, that I show you here, has a determinant of 0 and, consequently, no inverse.

$$D = \begin{bmatrix} 1 & 2 & 5 \\ -3 & 8 & -4 \\ -2 & 10 & 1 \end{bmatrix}, \det(D) = |D| = 0$$

Matrix D looks perfectly respectable on the surface, but, lurking beneath the surface, you have what could be a big problem when using the matrix to solve problems. You need to be aware of the consequences of the determinant being 0 and make arrangements or adjustments that allow you to proceed with the solution.

For example, determinants are used in Chapter 12 with Cramer's rule (for solving systems of equations). The values of the variables are ratios of different determinants computed from the coefficients in the equations. If the determinant in the denominator of the ratio is zero, then you're out of luck, and you need to pursue the solution using an alternate method.

Zeroing In on Eigenvalues and Eigenvectors

In Chapter 16, you see how eigenvalues and eigenvectors correspond to one another in terms of a particular matrix. Each eigenvalue has its related eigenvector. So what are these eigen-things?

First, the German word *eigen* means *own.* The word *own* is somewhat descriptive of what's going on with eigenvalues and eigenvectors. An eigenvalue is a number, called a *scalar* in this linear algebra setting. And an eigenvector is an $n \times 1$ vector. An eigenvalue and eigenvector are related to a particular $n \times n$ matrix.

For example, let me reach into the air and pluck out the number 13. Next, I take that number 13 and multiply it times a 2×1 vector. You'll see in Chapter 2 that multiplying a vector by a scalar just means to multiply each element in the vector by that number. For now, just trust me on this.

$$13 * \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 39 \\ 52 \end{bmatrix}$$

That didn't seem too exciting, so let me up the ante and see if this next step does more for you. Again, though, you'll have to take my word for the multiplication step. I'm now going to multiply the same vector that just got multiplied by 13 by a matrix.

$$\begin{bmatrix} 1 & 9 \\ 12 & 4 \end{bmatrix} * \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 39 \\ 52 \end{bmatrix}$$

The resulting vector is the same whether I multiply the vector by 13 or by the matrix. (You can find the hocus-pocus needed to do the multiplication in Chapter 3.) I just want to make a point here: Sometimes you can find a single number that will do the same job as a complete matrix. You can't just pluck the numbers out of the air the way I did. (I actually peeked.) Every matrix has its *own* set of *eigenvalues* (the numbers) and *eigenvectors* (that get multiplied by the eigenvalues). In Chapter 16, you see the full treatment — all the steps and procedures needed to discover these elusive entities.