

Module 1

Modeling with Differential Equations

Prerequisites

The minimum prerequisites for Module 1, “Modeling with Differential Equations,” are

- Single variable differentiation and integration.
- Experience with modeling through related rates or optimization is helpful.

Learning Objectives

After successfully completing this module, the reader will be able to

- Verify if a function is a solution of a differential equation, by hand or with a computer algebra system (CAS).
- Model physical phenomena using differential equations.
- State the differential equations that govern the phenomena discussed in this module.

Introduction

Differential equations are fundamental to mathematics, science and engineering. Many mathematical models lead to equations that involve a quantity and its rates of change. After introducing terminology in Section 1.1, this module and its relatives “Before Module 3” (pp. 37–50), “Before Module 6” (pp. 129–132) and “Before Module 7” (pp. 191–198) showcase the wide variety of models involving differential equations.

1.1 Terminology

We start with the terminology, because it is always easier to talk about a subject once the specialized vocabulary is available. The formal definitions may be a bit technical, but the examples will show that the ideas are straightforward.

Definition 1.1 Informally, a **differential equation** is an equation that involves a function and its derivatives. The **order** of the differential equation is the highest order of a derivative that occurs in the equation.

Formally, if $F(u, v_0, v_1, \dots, v_n)$ is a function of $n + 2$ variables that explicitly depends on v_n , then the statement $F(x, y, y', \dots, y^{(n)}) = 0$ is a **differential equation of order n** . The formal definition of “ F depends explicitly on v_n ” means that there are u, v_0, \dots, v_{n-1} and w_n so that $F(u, v_0, \dots, v_{n-1}, v_n) \neq F(u, v_0, \dots, v_{n-1}, w_n)$. This condition assures that the variable v_n is actually used in whichever way F is defined.

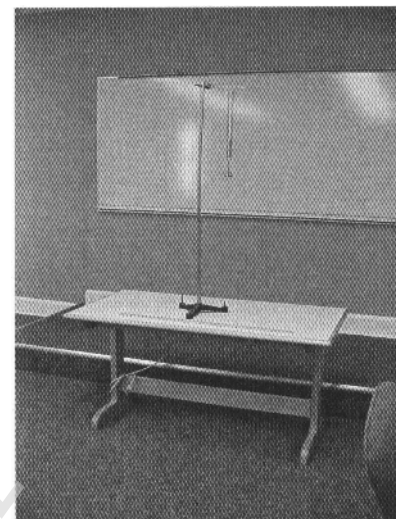


Figure 1.1: Differential equations can be used to model oscillating systems like spring-mass systems (see Section B3.1), ...

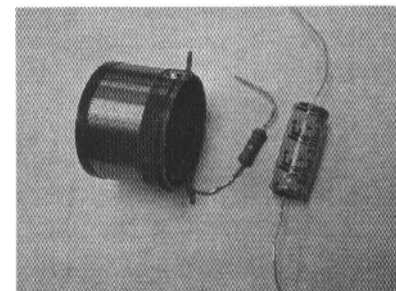


Figure 1.2: ... circuits consisting of inductors, resistors and capacitors (see Section B3.2), ...

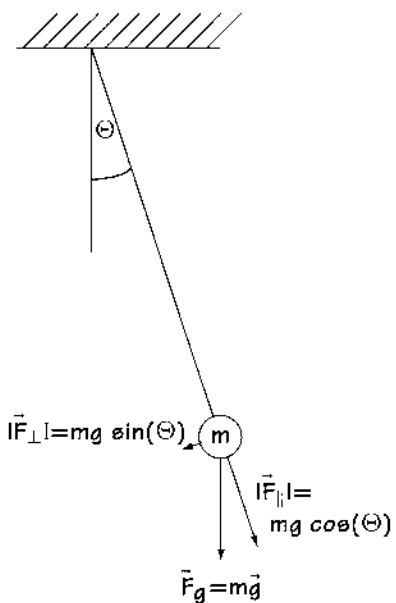


Figure 1.3: ... or the simple pendulum (see Section B3.3).

For the purpose of this text, the informal definition will be sufficient, and we will not work with the formal definition. As always in mathematics, a solution is an entity that, when substituted into the equation, leads to a true statement.

So the “big idea” for solutions of differential equations is that the equation must work out when you put in the solution.

Definition 1.2 A function is a **solution** of a differential equation if and only if we obtain a true statement upon appropriately substituting the function and its derivatives into the differential equation.

Formally, $f(x)$ is a solution of $F(x, y, y', \dots, y^{(n)}) = 0$ if and only if for all x in the domain of f we have $F(x, f(x), f'(x), \dots, f^{(n)}(x)) = 0$.

So checking if a function solves a differential equation is an exercise in differential calculus. In this module, we will only *check* if given functions solve given differential equations. In the remaining modules we explore how to *find* solutions.

Example 1.3 The equation $y' = y$ is a differential equation of order 1, or a first order differential equation, because the highest order of a derivative in the equation is 1.

The function $f(x) = e^x$ is a solution, because $f'(x) = e^x = f(x)$. \square

Can you find other solutions of the differential equation in Example 1.3?

Example 1.4 The equation

$$y' + \sin(x)y = 0$$

is a first order differential equation. We claim that the function $f(x) = e^{\cos(x)}$ is a solution of the differential equation.

To verify this claim, we compute $f'(x) = -e^{\cos(x)} \sin(x)$ and substitute f and f' into the equation. We check

$$\begin{aligned} f'(x) + \sin(x)f(x) &\stackrel{?}{=} 0 \\ -e^{\cos(x)} \sin(x) + \sin(x)e^{\cos(x)} &\stackrel{?}{=} 0 \\ 0 &= 0. \quad \checkmark \end{aligned}$$

Similarly, we can show that every function in the family $f_c(x) = ce^{\cos(x)}$ is a solution of this differential equation. See Figure 1.4 for a check with a computer algebra system (CAS). The letter c denotes an arbitrary constant. \square

$$f(c, x) := c \cdot e^{\cos(x)}$$

$$\frac{d}{dx} f(c, x) + \sin(x) f(c, x) \rightarrow 0$$

Figure 1.4: Using a CAS to check if every function $f_c(x) = ce^{\cos(x)}$ solves the differential equation in Example 1.4. Of course, a CAS is not needed for this computation, but it demonstrates how to check solutions with a CAS. *Note.* All sample CAS computations in this text are done in **MathCAD**.

Example 1.4 shows that differential equations can have many solutions. We have encountered this phenomenon before. Every time we solved an indefinite integral $\int h(x) dx$, we solved a first order differential equation of the form $y' = h(x)$. Every possible choice of the integration constant gives a different solution.

If, by some mistake, we try to check a function that is not a solution, we will get a contradiction. This is how we can safeguard against computational mistakes as we solve differential equations: If the check does not work out, the function is not a solution.

Example 1.5 Determine if the function $f(x) = e^x$ is a solution of the differential equation $y' + \sin(x)y = 0$. Justify your answer.

Substituting f and its derivative into the differential equation we obtain

$$e^x + \sin(x)e^x = 0 \quad \text{or} \quad e^x(1 + \sin(x)) = 0.$$

Because the left side is *not identical to zero*, this function is not a solution of the given differential equation. Specifically, it does not matter that $1 + \sin(x) = 0$ for all x of the form $\frac{3\pi}{2} + n \cdot 2\pi$, where n is any integer. As soon as the equation is violated at one point, the function is not a solution of the differential equation. \square

For antiderivatives, we were able to obtain unique solutions by specifying an initial point $y_0 = f(x_0)$ for the function. Moreover, if a function is integrated twice, there are two

integration constants. In this case we would need to specify the initial point and the derivative at the initial point to identify a unique solution. This idea is central to the definition of an initial value problem.

Definition 1.6 An initial value problem consists of an n^{th} order differential equation $F(x, y, y', \dots, y^{(n)}) = 0$ and of specified initial values (or **initial conditions**)

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

for the first $n - 1$ derivatives of the function. (The function itself counts as the zeroth derivative.) A function that satisfies the differential equation and the initial conditions at x_0 is called a **solution** of the initial value problem.

The models in the following sections will provide a physical context for initial conditions in specific situations. For now, we simply note that we can check if a function satisfies an initial value problem.

Example 1.7 Determine if the function $f(x) = 5e^{3x}$ solves the initial value problem $y' = 3y$, $y(0) = 5$. Justify your answer.

We must check the differential equation and the initial condition. The function solves the initial value problem, because $f'(x) = 15e^{3x} = 3f(x)$ and $f(0) = 5e^0 = 5$. \square

If a function does not satisfy the differential equation or if the function does not have the right initial values (or both), then the function is not a solution of the initial value problem.

Example 1.8 Determine if the function $f(x) = e^{3x}$ satisfies the initial value problem $y' - 3y = 1$, $y(0) = 1$. Justify your answer.

Although the function assumes the value 1 at $x_0 = 0$, the function does not satisfy the whole initial value problem, because $f'(x) - 3f(x) = 3e^{3x} - 3e^{3x} = 0 \neq 1$. \square

Under mild conditions on the differential equation each initial value problem has a unique solution. That is, there will be exactly one function that satisfies the differential equation and the initial conditions. Nonetheless, there are initial value problems for which the solutions are not unique. We will not encounter such situations in this text, but Exercise 4 in Section 5.1 serves as a warning that it *might* happen.

Convention 1.9 As long as every initial value problem has a unique solution, we call a family of solutions with as many constants as the order of the differential equation the **general solution** of the differential equation. This simple idea will suffice until Example 5.16, which motivates a more formal approach.

In some models it is not possible or not appropriate to specify enough derivatives at one point. Instead, the model might provide values at several (typically two) points. If that is the case, we speak of a boundary value problem.

Definition 1.10 A boundary condition for an ordinary differential equation specifies a condition on the solution for a value of the independent variable x . A **boundary value problem** consists of a differential equation and a set of boundary conditions at two or more distinct values of the independent variable x . A function is a **solution** of a boundary value problem if and only if it satisfies the differential equation and all boundary conditions.

Again, in this module we will only check if certain functions solve given boundary value problems.

So, for a first order differential equation we specify the value of the function at x_0 . For a second order differential equation we specify the value of the function and the value of its derivative at x_0 . For a third order differential equation we specify the value of the function, the value of its derivative and the value of its second derivative at x_0 , and so on.

Remember that a solution must satisfy all parts of the initial value problem.

Example 1.11 Determine if the function $f(x) = x \cos(x)$ satisfies the boundary value problem

$$xy' - y = -x^2 \sin(x), \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$$

Justify your answer.

We note that $f'(x) = \cos(x) - x \sin(x)$, so that

$$xf'(x) - f(x) = x(\cos(x) - x \sin(x)) - x \cos(x) = -x^2 \sin(x).$$

Moreover, $f(0) = 0 \cos(0) = 0$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) = 0$. Thus the function solves the boundary value problem. \square

As for initial value problems, even if only *one* condition fails, the boundary value problem is *not* satisfied.

Example 1.12 Determine if the function $f(x) = x^2$ satisfies the boundary value problem

$$x^2 y'' - 2y = 0, \quad y(-2) = y(2) = 4, \quad y(0) = 1.$$

Verify that f satisfies all other conditions. Justify your answer.

Because $f(0) = 0 \neq 1$, the function does not satisfy the boundary value problem. This is despite the fact that the function satisfies all other conditions. \square

Exercises

In each of the following exercises, determine if the given function f is a solution of the differential equation. Justify your answer.

- $f(x) = xe^x$, $y'' - y' + \frac{y}{x} = 0$
- $f(x) = x \sin(x)$, $y'' + y = 2 \cos(x)$
- $f(x) = x^3$, $x^2 y'' + xy' + y = 10x^3$
- $f(x) = x^2$, $y'' = 2$
- $f(x) = \sin(2x)$, $y'' - y = 0$
- $f(x) = 3x + 1$, $y'' - y' + 3y = 9x$
- $f(x) = \cos^2(x)$, $y'' + 4y = 2$

8. Match each of the following functions with the differential equation(s) it solves (if any).

Functions:

- $f(x) = \frac{1}{1-x^2}$
- $f(x) = \sqrt{1-x^2}$
- $f(x) = e^x (x^5 + x^4)$
- $f(x) = -\frac{1}{x^2}$

Equations:

- $y' = \frac{x}{y}$
- $xy' + y = x^2 y^2$
- $2xy + (x^2 + 1)y' = 0$
- $xy' - 4y = x^6 e^x$

In each of the following exercises, determine if the given function f solves the given initial value problem. Justify your answer.

- $f(x) = e^{2x}$, $y' = 2y$, $y(0) = 1$
- $f(x) = x^3$, $y' = 3x^2$, $y(0) = 2$

- $f(x) = \arctan(x)$, $y'(1+x^2) + y = 1$, $y(0) = 0$
- $f(x) = x \ln(x) - x$, $xy' - x = y$, $y(1) = -1$
- Determine which of the functions $f(x) = e^{\sin(x)}$ and $g(x) = e^{\cos(x)}$ satisfies the differential equation $y' - y \cos(x) = 0$. Show your work.
- Decide if the function $f(x) = x \cos(x)$ satisfies either of the following two initial value problems.
 - $xy'' + 2y' + xy = 2 \cos(x)$, $y(0) = 0$, $y'(0) = 1$
 - $xy'' + 2y' - xy = 2 \cos(x)$, $y(0) = 1$, $y'(0) = 0$

In each of the following, determine if the given function f solves the given boundary value problem. Justify your answer.

- $f(x) = \frac{e^x + e^{-x}}{2}$,
 $y'' - y = 0$, $y(0) = 1$, $y(1) = y(-1) = \frac{e^2 + 1}{2e}$
- $f(x) = \frac{e^{2x} + e^{-2x}}{2}$,
 $y'' - y = 0$, $y(0) = 1$, $y(1) + y(-1) = 0$
- $f(x) = x^2 + 4x + 1$,
 $y'' = 2$, $y(0) = 1$, $y'(0) = 4$, $y(-4) = 1$, $y'(-4) = -4$
- $f(x) = \sin(x)$,
 $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$, $y(\pi) = 0$, $y'(\pi) = -1$
- $f(x) = \sin(4x)$,
 $y'' + y = 0$, $y(0) = y(\pi) = 0$, $y'(0) = y'(\pi) = 16$

20. For the initial value problem

$$\begin{aligned} y' + 4y &= 10 \\ y(0) &= 6 \end{aligned}$$

assume that a solution of the form

$$y(x) = Ae^{cx} + B$$

exists. Find parameters A , B , c such that the above initial value problem is satisfied.

Hint. Substitute y into the differential equation and initial condition to obtain a system of equations for A , B and c .

As solution methods are introduced in later modules, we will frequently refer to specific models. When first reading the following sections, the reader should concentrate on the modeling process, on verifying that the indicated functions really are solutions of the differential equations and on the physical interpretation of the results. There will be plenty of time to think about solution methods in the remainder of the book. Frequently revisiting this module and its relatives “Before Module 3,” pp. 37–50, “Before Module 6,” pp. 129–132 and “Before Module 7,” pp. 191–198 will make the reader very familiar with the models.

Because modeling is a universal activity for which too few good “practice problems” exist, and because there can be surprising connections between seemingly dissimilar systems (see the end of Section B3.2), readers should strive to understand all models, regardless of what their individual background is.

No need to read all modeling sections right away. But read at least Sections 1.2–1.4 before moving on to Module 2. Then, at the start of each module, read the modeling sections that are referenced for motivation and examples for the module.

1.2 Differential Equations Describing Populations

We start with a very general **task**. *Set up a mathematical model that describes the growth of a population of simple organisms, such as bacteria.*

Bacteria populations are usually assumed to grow at a rate that is proportional to the size N of the population. The growth rate is the derivative of the population function with respect to time. Assuming unlimited resources (food, space) we obtain the following differential equation.

Mathematical Model 1.13 *The differential equation*

$$\frac{dN}{dt} = kN$$

with $k > 0$ is called the **differential equation for exponential growth**. This differential equation is a first, simple model for population growth under the assumption that resources are unlimited and that the growth rate is proportional to the size of the population. In such a model, the usual initial condition is the population at the start of the observation. The constant k depends on the species under investigation.

The name of the equation is easily explained by the following.

Example 1.14 *Determine if $N(t) = e^{2t}$ is a solution of the differential equation for exponential growth $\frac{d}{dt}N = 2N$. Justify your answer.*

Clearly, $\frac{d}{dt}N(t) = \frac{d}{dt}e^{2t} = 2e^{2t} = 2N(t)$. □

We will ultimately show that all solutions of the differential equation for exponential growth with $k > 0$ and positive starting value are exponential functions (see Exercise 36 in Section 2.1).

Physical Interpretation 1.15 Physically, the above says that if the growth rate is proportional to the size of the population, then the population grows exponentially. Because exponential growth is so fast, it cannot be sustained indefinitely. Therefore our model ultimately leads to unrealistic results. This is because the assumption of unlimited resources is not realistic. Consequently, this simple model will only be valid for short time periods of plentiful resources. For longer time periods, the model must be adjusted. □

To refine the model we can introduce a carrying capacity K for the population. The carrying capacity should be so that growth slows as the population size approaches the carrying capacity from below. Growth should be negative, that is, a decline, if the carrying capacity was exceeded (also see Figure 1.5). The simplest way to model this idea is to multiply the right-hand side by a term $(1 - N/K)$. This term is positive for $N < K$,

What does “proportional” mean?

(If anything in a technical text is unclear and you can’t resolve the problem right away, put a question in the margin and answer it later, as done below.)

“A quantity a is proportional to a quantity b if and only if there is a $k > 0$ so that $a = kb$.”

Remember that we have the whole remainder of the text to discuss solution methods. Here we simply verify that certain functions are solutions, leaving details about how to find them for later.

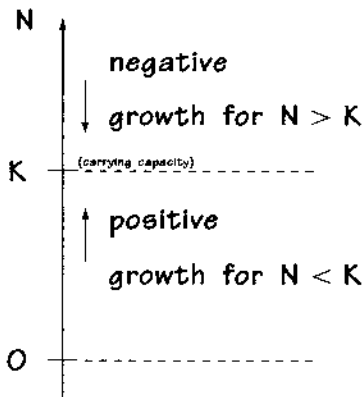


Figure 1.5: Growth phenomena with carrying capacity.

it shrinks as N approaches K and it is negative for $N > K$. The resulting differential equation is the following.

Mathematical Model 1.16 *The differential equation*

$$\frac{dN}{dt} = kN \left(1 - \frac{N}{K} \right)$$

with $k > 0$ is called the **logistic equation**. The logistic equation models growth phenomena that ultimately encounter a threshold but in which the rate of change is initially proportional to the dependent variable N . The usual initial condition is the population size at the start of the observation.

In many models that involve differential equations, the independent variable is time and initial conditions are specified at a certain time. To make computations easier, initial conditions are typically given for $t = 0$. This corresponds, in real life, to starting a clock at the start of an experiment or an observation.

Exercises

For each of the following exercises, set up (do not solve) a differential equation and an initial value problem that governs the phenomenon.

1. In a model for economic growth, let the gross national product of a country be given by p . Find a differential equation that models a 2% growth every year.
2. In a model for economic growth, let the gross national product of a country be given by p . Find a differential equation that models growth by a factor k every year. Then give a range of reasonable values for k .
3. In radioactive decay, during each time unit a certain (assumed to be constant) fraction of the atoms of the radioactive substance decays. Find a differential equation that models radioactive decay.
4. Show that the inflection point in the solution of a logistic equation $\frac{dy}{dt} = ky(M - y)$ occurs at $y = \frac{M}{2}$. What can be deduced about a solution if the initial value y_0 is greater than $\frac{M}{2}$?

Hint. Differentiate both sides of the logistic equation. The logistic equation can also be used to determine for which values of y we have $y' = 0$.

5. The general solution of the differential equation for exponential growth can basically be guessed once we recall that the exponential function is its own derivative. This type of pattern recognition will be important when we discuss elementary ways to solve linear differential equations with constant coefficients in Module 3.

To practice pattern recognition, for the following types of functions decide if their derivatives are the same *type* of function.

- (a) Polynomials
- (b) Rational functions
- (c) Logarithms
- (d) Exponential functions
- (e) Sine functions
- (f) Cosine functions
- (g) Tangent functions

1.3 Remarks on Modeling with Differential Equations

At this stage, it is a good idea to think about modeling in general. Mathematical modeling always starts with a question about a natural phenomenon or an engineering application. In the modeling sections of this text, we simply designate this question as a task. Once the task is given, mathematical modeling involves the same guiding principles as problem solving. These guiding principles are often ascribed to **G. Pólya** (see [21]).

Steps in problem solving.

1. Define the problem.
2. Devise a plan to solve the problem.
3. Carry out the plan.
4. Test and evaluate the results.

These steps actually need to be as vague as given above, because they apply to *any* mathematical modeling, and indeed to any problem. Anything more specific would immediately limit the context. Because the beauty of mathematics is in its broad applicability, any such limitation is inappropriate. So, because a more specific “modeling algorithm” is

not available, we must acquire the right skills so that the above steps will lead us to a valid model.

Let us analyze how we set up the differential equations in Section 1.2. The initial task did not specify that we were supposed to use differential equations. It just turned out that the proportionality between growth rate and population size led to the differential equation for exponential growth. This differential equation defines the problem, because it provides something mathematical that can be solved.

Because finding mathematical models is challenging, let us analyze the instruction “define the problem” from a more general point-of-view. Defining the problem will always require some knowledge of the context. In this context, there will be certain variable quantities (like the population and the growth rate), there can be unknown constant quantities (such as the constant of proportionality k) and there will be an independent quantity on which the other variable quantities depend (time in this case). Overall, the following is important as we take stock of the situation.

Remember that the first model was $\frac{dN}{dt} = kN$.

- Always remember what the underlying independent variable is (for differential equations the independent variable often is time).
- Be aware which quantities in the model are constant and which quantities can change. (Not all letters denote variable quantities.)

Once we have familiarized ourselves with the variables and constants involved in the situation, we need to determine how the quantities relate to each other. Here is where background knowledge from the field in which the problem arises is helpful.

To relate the quantities to each other we can try to

- Use reasonable working hypotheses (for example, we can assume that the growth rate of a bacteria colony is proportional to its size).
- Visualize (for example, with force diagrams or sketches of circuits).
- Use known laws (for example, Newton’s laws of motion or Kirchhoff’s laws).

Once we have defined the problem, we can devise a plan to solve it. With the way we split up the steps so far, we need a plan to solve the differential equation. After we have a plan, we execute it, that is, we solve the differential equation. These two steps will occupy the other modules in this text.

But we are not done once we have found a solution. The last step is to test and evaluate the results. Remember that Physical Interpretation 1.15 showed that, for long periods of observation, the differential equation for exponential growth does not lead to a realistic model. Evaluation of the results is a common step in science, especially when a phenomenon is not yet fully expressed with reliable models. When the results obtained from a model differ from reality, we need to determine the problem (in this case, the model differs from reality after long time periods) and make adjustments. These adjustments lead to a new model, the logistic equation in our case, which is then solved and evaluated.

For the logistic equation, we can give a qualitative analysis. Because populations are never negative (and because the future development of populations of size zero is not very interesting), we concentrate on positive values of N . For $0 < N < K$ the right side of the logistic equation is positive. Therefore, for $0 < N < K$ the population size has a positive derivative, that is, the population is increasing. This property corresponds to the observable fact that a population will grow in a favorable environment. For $N > K$ the population size has a negative derivative, that is, the population is decreasing. This property corresponds to the observable fact that if a population requires more resources than the environment can provide, then the population will shrink. Overall, we can say that the logistic equation should give results that qualitatively behave the right way.

What was the logistic equation again?

Throughout this text we will ask ourselves if the solutions for mathematical models are

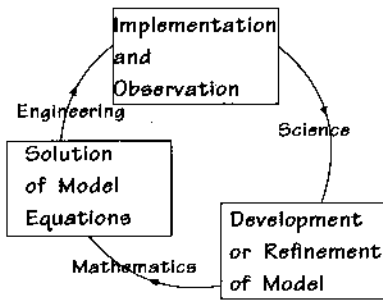


Figure 1.6: Visualization of the modeling cycle.

actually physically sensible. One impressive fact about differential equations is that the models will give sensible results *for the situation for which they were designed*. We should note that even though the differential equation for exponential growth is not a good model for bacteria growth in all situations, it is a very good model as long as nourishment and space are plentiful. A small bacteria colony on a petri dish will grow exponentially for a while. So the model is valid until the underlying assumptions no longer hold, that is, until nourishment and space are no longer plentiful.

In general, the modeling cycle (see Figure 1.6) begins with observations, in nature or in experiments, and ends with verification in experiments and in nature. It can be, and often is, traversed multiple times as the model is refined to give better results. Although all disciplines contribute to all stages, setting up a model based on observations is mainly considered the domain of science, solving the model equations is mainly considered the domain of mathematics and the implementation of the solution in devices and new experiments is mainly considered the domain of engineering.

Exercise

1. Comment on the following statement about a (fictitious) theory: “Confidence in the validity of this theory is so high that, if an experiment leads to data contrary to the theoretical predictions, experimenters look for and find the flaw in the experiment and can thus correct their measurements. In this fashion excellent agreement between theoretical and experimental values is achieved.”

Does this statement describe a sound scientific/engineering approach? Would you trust a theory that is justified as above? Explain why or why not.

1.4 Newton’s Law of Cooling



Figure 1.7: Newton’s Law of Cooling governs heat exchange in general. So it governs the cooling of warm objects (right) as well as the warming of cool objects (left).

Task. Set up a mathematical model that describes how the temperature of an object changes if the object is surrounded by a medium whose temperature does not change.

Although this situation sounds abstract, it is encountered every day. A hot cup of coffee (our object, also see Figure 1.7) left sitting on a table is surrounded by air (a medium) whose temperature we can assume to be constant overall. We observe that the cup of coffee will cool off until it reaches the temperature of the surrounding air (the “room temperature”). Similarly, if we leave a cold drink sitting on a table, the drink will warm up until it reaches room temperature.

It seems reasonable to assume that the rate at which an object cools off or warms up (that is, the derivative of its temperature) is proportional to the difference between the object’s temperature T and the temperature T_s of the surrounding space.

Mathematical Model 1.17 With T being the temperature of an object, T_s being the constant temperature of the surrounding environment and $k > 0$ being a proportionality constant, the differential equation

$$\frac{d}{dt}T = k(T_s - T)$$

is called **Newton’s Law of Cooling**. The usual initial condition is the temperature of the object at the start of the observation.

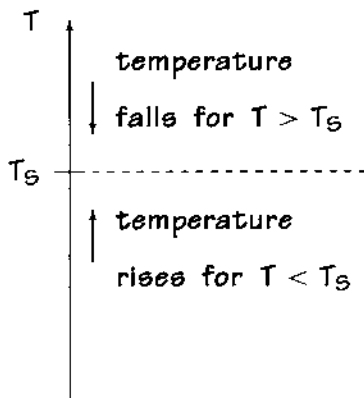


Figure 1.8: Visualizing Newton’s Law of Cooling.

To at least qualitatively evaluate the model, note that for $T > T_s$ the right side of the equation is negative. This means that for $T > T_s$ the derivative of T is negative and hence T is decreasing. This property reflects our everyday observation that warm objects cool off until they reach the temperature of the surrounding space. Similarly, for $T < T_s$ the derivative of T is positive and T is increasing. This property reflects our everyday observation that cold objects warm up until they reach the temperature of the surrounding space. Figure 1.8 gives a graphical representation of what is said above.

Overall, we can say that, at least qualitatively, solutions of Newton’s Law of Cooling should behave the right way. A quantitative evaluation would require the comparison of measurements with solutions of Newton’s Law of Cooling.

Exercise

1. Is $T_s = 293$ a sensible value for Newton's Law of Cooling when we consider the cooling of a cup of coffee?

1.5 Loaded Horizontal Beams

Note. Despite its simplicity, the equation in Mathematical Model 1.20 requires sophisticated results from mechanics in its derivation. The interested reader should consider a mechanics text for details. The example was included in the text without the derivation, because it is one of the best ways to visualize boundary conditions for differential equations.

Task. Set up a mathematical model that describes how a horizontal beam bends when a load is placed upon it.

Houses, bridges, and so on, are made up of beams that support the rest of the structure. Because no material is absolutely rigid, any horizontal beam bends once a load is applied. For example, when a piece of furniture is moved into a house, the support beams in the floor will bend. It is important to assure that this displacement from the original shape remains within safe margins. This is achieved by choosing the right materials and dimensions for the beam and the appropriate way to install it.

For the model, we envision the loaded beam as a large number of thin surfaces that are laminated together. As the beam is bent by a load, the surfaces on the inside of the bend will be compressed, while the surfaces on the outside will be stretched. If we assume that the beam is a continuum of surfaces of differential thickness, then there must be one surface which is neither stretched nor compressed.

Definition 1.18 The **neutral surface** of a beam under a given load (see Figure 1.9) is the unique surface that is neither stretched nor compressed.

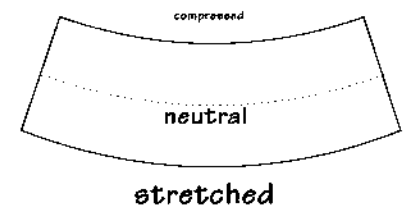
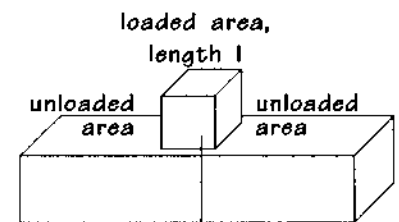


Figure 1.9: The neutral surface in a loaded beam is neither compressed nor stretched.

To simplify the model, we will assume that the load does not vary along the width of the beam. Therefore, the displacement from equilibrium for each point on each surface will only depend on how far away the point is from the ends of the beam. Viewing the beam from the side (see Figure 1.9) shows that the surfaces, and in particular the neutral surface, can be modeled as curves. In particular, we can model the load one-dimensionally.

Definition 1.19 The **load** on a beam (see Figure 1.10) is the function $w(x)$ that gives the force per unit distance that acts on the beam. In particular, the integral of the load over the length of the beam gives the total force acting on the beam. We will only consider vertical loading. That is, all our loads will have the same direction.



force: given by the weight load: only on loaded part, equal to force/length on the loaded part, 0 otherwise, if loading is uniform

Figure 1.10: Vertical loading of a horizontal beam.

A rather intricate analysis of the mechanics involved now leads to the following model.

Mathematical Model 1.20 The differential equation

$$EI \frac{d^4}{dx^4} y = w(x)$$

describes the **elastic curve**, that is, the shape of the neutral surface of a horizontal beam under a load $w(x)$. In this equation, y is the neutral surface's displacement from equilibrium, where equilibrium is assumed to be at 0, E is Young's modulus of elasticity and I is the moment of inertia of a cross section. The product EI is also referred to as the beam's **flexural rigidity**.

Beams are typically supported in *two* places. Therefore we must specify conditions in two places, and we encounter boundary value problems in a very natural way. For supported

beams, we usually specify boundary conditions at $x = 0$ and at $x = L$. Different designs lead to different boundary conditions.

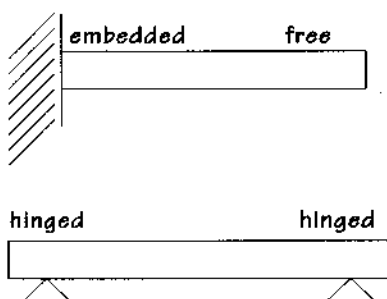


Figure 1.11: The three boundary conditions for beams are embedded (top left), free (top right) and hinged (bottom, both ends). As shown, they can be combined with either end satisfying any of the conditions (also see Exercise 3).

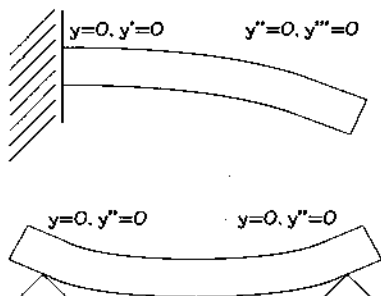


Figure 1.12: Boundary conditions when the beam is loaded.

Definition 1.21 The end of a beam can be **embedded** (also referred to as **clamped**), **free** or **simply supported** (also referred to as **pin supported**, **fulcrum supported** or **hinged**). This leads to the following boundary conditions at the ends (also see Figure 1.11).

End of beam is	Boundary Condition at that end
embedded	$y = 0, y' = 0$
free	$y'' = 0, y''' = 0$
simply supported	$y = 0, y'' = 0$

A beam can have any of the boundary conditions at either end. (Some combinations will have unrealistic physical counterparts, see Exercise 3.) The only combination with a special name appears to be a **cantilever beam**, which is embedded at one end and free on the other end.

The boundary conditions can be understood as follows (see Figure 1.12).

- At a horizontally embedded end (see Figure 1.12, top left), the neutral surface will be horizontal and in the equilibrium position. Hence $y = 0$ and $y' = 0$.
- At a free end (see Figure 1.12, top right) the neutral surface can be anywhere and at any slope. Hence there are no conditions on y and y' . However, because there are no forces to bend a free end, we have $y'' = 0$ and $y''' = 0$.
- A simply supported end (see Figure 1.12, bottom) sits on top of a support. Hence $y = 0$. The support does not force any direction on the beam. Hence there is no condition on y' . However, if the support does not force any direction on the beam, it also cannot bend the beam at the end. Hence $y'' = 0$.

With Definition 1.21 we can set up the boundary conditions for any supported beam. Because solving the differential equation only involves integration, we present some solutions here.

Example 1.22 Solve the boundary value problem

$$y'''' = 24, \quad y(0) = y(1) = 0, \quad y'(0) = -1, \quad y'(1) = 1.$$

Integrating four times gives the general solution of the differential equation as $y(x) = x^4 + ax^3 + bx^2 + cx + d$. The boundary conditions provide the following equations for the coefficients.

$$d = 0 \quad \boxed{\text{used } y(0) = 0}$$

$$1 + a + b + c + d = 0 \quad \boxed{\text{used } y(1) = 0}$$

$$c = -1 \quad \boxed{\text{used } y'(0) = -1}$$

$$4 + 3a + 2b + c = 1 \quad \boxed{\text{used } y'(1) = 1}$$

Now use the two known coefficients.

$$a + b = 0$$

$$3a + 2b = -2$$

Now solve the system.

$$a = -2, \quad b = 2.$$

Check.

It is easy to see that the differential equation is solved and that $y(0) = y(1) = 0$.

For the conditions on $y'(x) = 4x^3 - 6x^2 + 4x - 1$, we see that $y'(0) = -1$ and $y'(1) = 4 - 6 + 4 - 1 = 1$.

The solution of the boundary value problem is $y(x) = x^4 - 2x^3 + 2x^2 - x$. Double checking is always a good idea (see margin). \square

Example 1.23 State and solve the boundary value problem for a beam of length L that is embedded at both ends and under a constant load $w(x) = w_0$.

By Mathematical Model 1.20 the differential equation is $EIy'''' = w_0$. The general solution of this equation, since E , I and w_0 are constants, is

$$y(x) = \frac{w_0}{24EI}x^4 + ax^3 + bx^2 + cx + d.$$

By Definition 1.21 the boundary conditions are

$$y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0.$$

From $y(0) = 0$ and $y'(0) = 0$ we obtain $c = d = 0$. The remaining conditions yield

$$\frac{w_0}{24EI}L^4 + aL^3 + bL^2 = y(L) = 0,$$

$$4\frac{w_0}{24EI}L^3 + 3aL^2 + 2bL = y'(L) = 0,$$

$$\frac{w_0}{24EI}L^2 + aL + b = 0,$$

$$\frac{w_0}{6EI}L^2 + 3aL + 2b = 0,$$

$$\frac{w_0}{12EI}L^2 + aL \stackrel{(m)}{=} 0,$$

$$a = -\frac{w_0}{12EI}L,$$

$$b = -\frac{w_0}{24EI}L^2 - aL = \frac{w_0}{24EI}L^2.$$

Subtract twice the first equation from the second.

In advanced texts, steps that used to be a whole separate problem in calculus, like the integration by parts above, may be skipped. The reader is expected to know calculus and to fill in the gaps. In this text, the symbol **(m)** indicates that some skipped steps are explained more fully in the margin or that the reader should fill in these steps. Readers of advanced texts should always make sure they understand the transition from one line to the next. Appropriate notes in the margin help with this task.

Thus the general solution of the boundary value problem is

$$y(x) = \frac{w_0}{24EI}x^4 - \frac{w_0}{12EI}Lx^3 + \frac{w_0}{24EI}L^2x^2.$$

For a double check, consider the margin. □

To avoid breaking beams, engineers need to compute the forces and moments in the beam. If the bending moment $|M| \approx EIy''$ is too large at one point, the beam will break. Hence, it is important to know where the maximum bending moment is located and how large it is. After computing the maximum bending moment, materials and designs can be chosen to assure that the beam will support the desired load.

Example 1.24 For the beam in Example 1.23 find the maximum of the bending moment.

$$\begin{aligned} M &\approx -EIy'' = -\frac{d^2}{dx^2} \left(\frac{w_0}{24}x^4 - \frac{w_0}{12}Lx^3 + \frac{w_0}{24}L^2x^2 \right) \\ &= -\frac{w_0}{2}x^2 + \frac{w_0}{2}Lx - \frac{w_0}{12}L^2. \end{aligned}$$

The vertex of this parabola is at $x = L/2$, so the maximum bending moment occurs in the middle of the beam. It is $M_{\max} \approx w_0L^2/24$. □

Check. $y(0)=y'(0)=0 \checkmark$

$$\begin{aligned} y(L) &= \frac{w_0}{24EI}L^4 - \frac{w_0}{12EI}L^4 \\ &\quad + \frac{w_0}{24EI}L^4 \\ &= 0 \checkmark \end{aligned}$$

Finish the check.

$$y'(x) =$$

$$y'(L) =$$

$$y''''(x) =$$

This seems to make sense. Embedded beams should bend worst in the center.

Exercises

1. State the boundary value problem for a horizontal beam of length L under a constant load $w(x) = w_0$ such that
 - (a) One end is embedded, the other is free (cantilever beam).
 - (b) Both ends are simply supported.
 - (c) One end is embedded, the other is simply supported.
2. Interpret Example 1.22 as the boundary value problem for a loaded beam and describe how the beam is supported at either end. (The support is not among the standard types given in Definition 1.21.) Sketch the support if necessary. Is this a realistic way to install a support beam?
3. There are nine possible combinations of boundary conditions for the ends of a beam (see Definition 1.21).
 - (a) Show that, taking symmetry into account, there are only six combinations that actually differ from each other.
 - (b) Identify the four combinations that reflect a realistic situation.

This module has given a brief introduction to modeling with differential equations and it has established the fundamental terminology. We will return to modeling Before Module 3, pp. 37–50, Before Module 6, pp. 129–132 and Before Module 7, pp. 191–198 to introduce the reader to further models. These models are the reason why we investigate certain types of equations. The presentation of the models will be similar to what was established here: We set a task and we use physics to establish the modeling equation(s). In this fashion the reader will be exposed to modeling multiple times and the individual sections are less overwhelming than if all models were presented in this module.