PART I

The Single Component Case

Optimal f

CHAPTER 1

The General History of Geometric Mean Maximization

Geometric mean maximization, or "growth-optimality," is the idea of maximizing the size of a stake when the amount you have to wager is a function of the outcome of the wagers up to that point. You are trying to maximize the value of a stake over a series of plays wherein you do not add or remove amounts from the stake.

The lineage of reasoning of geometric mean maximization is crucial, for it is important to know how we got here. I will illustrate, in broad strokes, the history of geometric mean maximization because this story is about to take a very sharp turn in Part III, in the reasoning of how we utilize geometric mean maximization. To this point in time, the notion of geometric mean maximization has been a *criterion* (just as being at the growth-optimal point, maximizing growth, has been the criterion before we examine the nature of the curve itself).

We will see later in this text that it is, instead, a framework (something much greater than the antiquated notion of "portfolio models"). This is an unavoidable perspective that gives *context* to our actions, but our criterion is rarely growth optimality. Yet growth optimality is the criterion that is solved mathematically. Mathematics, devoid of human propensities, proclivities, and predilections, can readily arrive at a clean, "optimal" point. As such, it provides a framework for us to satisfy our seemingly labyrinthine appetites.

On the ninth of September 1713, Swiss mathematician Nicolaus Bernoulli, whose fascination with difference equations had him corresponding with French mathematician Pierre Raymond de Montmort, whose fascination was finite differences, wrote to Montmort about a paradox that involved the intersection of their interests.

Bernoulli described a game of heads and tails, a coin toss in which you essentially pay a cover charge to play. A coin is tossed. If the coin comes up heads, it is tossed again repeatedly until it comes up tails. The pot starts at one unit and doubles every time the coin comes up heads. You win whatever is in the pot when the game ends. So, if you win on the first toss, you get your one unit back. If tails doesn't appear until the second toss, you get two units back. On the third toss, a tails will get you four units back, *ad infinitum*.

Thus, you win 2^{q-1} units if tails appears on the qth toss.

The question is "What *should* you pay to enter this game, in order for it to be a 'fair' game based on Mathematical Expectation?"

Suppose you win one unit with probability .5, two units with probability .25, four units with probability .125, *ad infinitum*. The Mathematical Expectation is therefore:

$$ME = 2^0 * \frac{1}{2^1} + 2^1 * \frac{1}{2^2} + 2^2 * \frac{1}{2^3} \cdots \tag{1.01}$$
$$ME = .5 + .5 + .5 \ldots$$
$$ME = \sum_{q=1}^{\infty} .5 = \infty$$

The expected result for a player in such a game is to win an infinite amount. So just what is a fair cover charge, then, to enter such a game?[1] This is quite the paradox indeed, and one that shall rendezvous with us in the sequel in Part III.

The cognates of geometric mean maximization begin with Nicolaus Bernoulli's cousin, Daniel Bernoulli.[2,3] In 1738, 18 years before the birth

[1]A cover charge would be consistent with the human experience here. After all, it takes money to make money (though, it doesn't take money to lose money).

[2]Daniel was one of eight members of this family of at least eight famous mathematicians of the late seventeenth through the late eighteenth century. Daniel was cousin to Nicolaus, referred to here, whose father and grandfather bore the same name. The grandson, Daniel's cousin, is often referred to Nicolaus I, and as the nephew of Jakob and Johann Bernoulli, the latter being Daniel's father. As an aside, one of Daniel's two brothers was also named Nicolaus, and he is known as Nicolaus II, who would thus be cousin as well to Nicolaus I, whose father was named Nicolaus as well as his grandfather (the grandfather thus to not only Nicolaus I, but to Daniel and his brothers, including Nicolaus II).

[3]Though in our context we look upon Daniel Bernoulli in the context of his pioneering work in probability, he is primarily famous for his applications of mathematics

of Mozart, Daniel made the first known reference to what is known as "geometric mean maximization." Arguably, his paper drew upon the thoughts and intellectual backdrop of his era, the Enlightenment, the Age of Reason. Although we may credit Daniel Bernoulli here as the first cognate of geometric mean maximization (as he is similarly credited as the father of utility theory by the very same work), he, too, was a product of his time. The incubator for his ideas began in the 1600s in the belching mathematical cauldron of the era.

Prior to that time, there is no known mention in any language of even generalized optimal reinvestment strategies. Merchants and traders, in any of the developing parts of the earth, evidently never formally codified the concept. If it was contemplated by anyone, it was not recorded.

As for what we know of Bernoulli's 1738 paper (originally published in Latin), according to Bernstein (1996), we find a German translation appearing in 1896, and we find a reference to it in John Maynard Keynes' 1921 *Treatise on Probability.*

In 1936, we find an article in *The Quarterly Journal of Economics* called "Speculation and the carryover" by John Burr Williams that pertained to trading in cotton. Williams posited that one should bet on a representative price and that if profits and losses are reinvested, the method of calculating this price is to select the geometric mean of all of the possible prices.

Interesting stuff.

By 1954, we find Daniel Bernoulli's 1738 paper finally translated into English in *Econometrica.*

When so-called game theory came along in the 1950s, concepts were being widely examined by numerous economists, mathematicians, and academicians, and this fecund backdrop is where we find, in 1956, John L. Kelly Jr.'s paper, "A new interpretation of information rate." Kelly demonstrated therein that to achieve maximum wealth, a gambler should maximize the expected value of the logarithm of his capital. This is so because the logarithm is additive in repeated bets and to which the law of large numbers applies. (Maximizing the sum of the logs is akin to maximizing the product of holding period returns, that is, the "Terminal Wealth Relative.")

In his 1956 paper in the *Bell System Technical Journal,* Kelly showed how Shannon's "Information Theory" (Shannon 1948) could be applied to the problem of a gambler who has inside information in determining his growth-optimal bet size.

When one seeks to maximize the expected value of the stake after n trials, one is said to be employing "The Kelly criterion."

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to mechanics and in particular to fluid mechanics, particularly for his most famous work, *Hydrodynamique* (1738), which was published the very year of the paper of his we are referring to here!

The Kelly criterion states that we should bet that fixed fraction of our stake (f) that maximizes the growth function G(f):

$$G(f) = P * \ln(1 + B * f) + (1 - P) * \ln(1 - f) \tag{1.02}$$

where: f = the optimal fixed fraction
P = the probability of a winning bet/trade
B = the ratio of amount won on a winning bet to amount lost on a losing bet
ln( ) = the natural logarithm function

Betting on a fixed fractional basis such as that which satisfies the Kelly criterion is a type of wagering known as a Markov betting system. These are types of betting systems wherein the quantity wagered is not a function of the previous history, but rather, depends only upon the parameters of the wager at hand.

If we satisfy the Kelly criterion, we will be growth optimal in the long-run sense. That is, we will have found an optimal value for f (as the optimal f is the value for f that satisfies the Kelly criterion).

In the following decades, there was an efflorescence of papers that pertained to this concept, and the idea began to embed itself into the world of capital markets, at least in terms of academic discourse, and these ideas were put forth by numerous researchers, notably Bellman and Kalaba (1957), Breiman (1961), Latane (1959), Latane and Tuttle (1967), and many others.

Edward O. Thorp, a colleague of Claude Shannon, and whose work deserves particular mention in this discussion, is perhaps best known for his 1962 book, *Beat the Dealer* (proving blackjack *could* be beaten). In 1966, Thorp developed a winning strategy for side bets in baccarat that employed the Kelly criterion. Thorp has presented formulas to determine the value for f that satisfies the Kelly criterion.

Specifically:

If the amount won is equal to the amount lost:

$$f = 2 * P - 1 \tag{1.03}$$

which can also be expressed as:

$$f = P - Q \tag{1.03a}$$

where: f = the optimal fixed fraction
P = the probability of a winning bet/trade
Q = The probability of a loss, or the complement of P, equal to 1 $- P$

Both forms of the equation are equivalent

This will yield the correct answer for the optimal f value provided the quantities are the same regardless of whether a win or a loss. As an example, consider the following stream of bets:

$$-1,\ +1,\ +1,\ -1,\ -1,\ +1,\ +1,\ +1,\ +1,\ -1$$

There are 10 bets, 6 winners, hence:

$$\begin{aligned} f &= 2 * .6 - 1 \\ &= 1.2 - 1 \\ &= .2 \end{aligned}$$

If all of the winners and losers were not for the same size, then this formula would not yield the correct answer. Reconsider our 2:1 coin toss example wherein we toss a coin and if heads comes up, we win two units and if tails we lose one unit. For such situations the Kelly formula is:

$$f = ((B + 1) * P - 1)/B \tag{1.04}$$

where: f = the optimal fixed fraction
P = the probability of a winning bet/trade
B = the ratio of amount won on a winning bet to amount lost on a losing bet

In our 2:1 coin toss example:

$$\begin{aligned} f &= ((2 + 1).5 - 1)/2 \\ &= (3 * .5 - 1)/2 \\ &= (1.5 - 1)/2 \\ &= .5/2 \\ &= .25 \end{aligned}$$

This formula yields the correct answer for optimal f provided all wins are always for the same amount and all losses are always for the same amount (that is, most gambling situations). If this is not so, then this formula does *not* yield the correct answer.

Notice that the numerator in this formula equals the Mathematical Expectation for an event with two possible outcomes. Therefore, we can say that as long as all wins are for the same amount, and all losses are for the same amount (regardless of whether the amount that can be won equals the amount that can be lost), the f that is optimal is:

$$f = \text{Mathematical Expectation}/B \tag{1.05}$$

The concept of geometric mean maximization did not go unchallenged in subsequent decades. Notables such as Samuelson (1971, 1979), Goldman (1974), Merton and Samuelson (1972), and others posited various and compelling arguments to not accept geometric mean maximization as the criterion for investors.

By the late 1950s and in subsequent decades there was a different, albeit similar, discussion that is separate and apart from geometric mean maximization. This is the discussion of portfolio optimization. This parallel line of reasoning, that of maximizing returns vis-à-vis "risk," absent the effects of reinvestment, would gain widespread acceptance in the financial community and relegate geometric mean maximization to the back seat in the coming decades, in terms of a tool for relative allocations.

Markowitz's 1952 *Portfolio Selection* laid the foundations for what would become known as "Modern Portfolio Theory." A host of others, such as William Sharpe, added to the collective knowledge of this burgeoning discipline.

Apart from geometric mean maximization, there were points of overlap. In 1969 Thorp presented the notion that the Kelly criterion should replace the Markowitz criterion in portfolio selection. By 1971 Thorp had applied the Kelly criterion to portfolio selection. In 1976, Markowitz too would join in the debate of geometric growth optimization. I illustrated how the notions of Modern Portfolio Theory and Geometric Mean Optimization could overlap in 1992 via the Pythagorean relationship of the arithmetic returns and the standard deviation in those returns.

The reason that this similar, overlapping discussion of Modern Portfolio Theory is presented is because it *has* seen such widespread acceptance. Yet, according to Thorp, as well as this author (Vince 1995, 2007), it is trumped by geometric mean maximization in terms of portfolio selection.

It was Thorp who presented the "Kelly Formulas," which satisfy the Kelly criterion (which "seeks to maximize the expected value of the stake after n trials"). This was first presented in the context of two possible gambling outcomes, a winning outcome and a losing outcome. Understand that the Kelly formulas presented by Thorp caught hold, and people were trying to implement them in the trading community.

In 1983, Fred Gehm referred to the notion of using Thorp's Kelly Formulas, and pointed out they are only applicable when the probability of a win, the amount won and the amount lost, "completely describe the distribution of potential profits and losses." Gehm concedes that "this is not the case" (in trading). Gehm's book, *Commodity Market Money Management*, was written in 1983, and thus he concluded (regarding determining the optimal fraction to bet in trading) "there is no alternative except to use complicated and expensive Monte Carlo techniques." (Gehm 1983, p. 108)

In 1987, the Pension Research Institute at San Francisco State University put forth some mathematical algorithms to amend the concepts of Modern Portfolio Theory to account for the differing sentiments investors had pertaining to upside variance versus downside variance. This approach was coined "Postmodern Portfolio Theory."

The list of names in this story of mathematical twists and turns is nowhere near complete. There were many others in the past three centuries, particularly in recent decades, who added much to this discussion, whose names are not even mentioned here.

I am not seeking to interject myself among these august names. Rather, I am trying to show the lineage of reasoning that leads to the ideas presented in this book, which necessarily requires the addition of ideas I have previously written about. As I said, a very sharp turn is about to occur for two notions—the notion of geometric mean maximization as a criterion, and the notion of the value of "portfolio models." Those seemingly parallel lines of thought are about to change.

In September 2007, I gave a talk in Stockholm on the Leverage Space Model, the maximization for multiple, simultaneous positions, and juxtaposed it to a quantification of the probability of a given drawdown. Near the end of the talk, one supercilious character snidely asked, "So what's new? I don't see anything new in what you've presented." Without accusing me outright, he seemed to imply that I was presenting, in effect, Kelly's 1956 paper with a certain elision toward it.

This has been furtively volleyed up to me on more than one occasion: the intimation that I somehow repackaged the Kelly paper and, further, that what I have presented was already presented in Kelly. Those who believe this are conflating Kelly's paper with what I have written, and they are often ignorant of just what the Kelly paper does contain, and where it first appears.

In fact, I have tried to use the same mathematical nomenclature as Thorp, Kelly, and others, including the use of "f" and "G"[4] solely to provide continuity for those who want the full story, how it connects, and out of

[4] In this text, however, we will refer to the geometric mean HPR as GHPR, as opposed to G, which is how I, as well as the others, have previously referred to it. I am using this nomenclature to be consistent with the variable we will be referring to later, AHPR, as the arithmetic mean HPR.

respect for these pioneering, soaring minds. I have not claimed to be the eponym for anything I have uncovered or added to this discussion.

Whether known by Kelly or not, the cognates to his paper are from Daniel Bernoulli. It is very likely that Bernoulli was not the originator of the idea, either. In fairness to Kelly, the paper was presented as a solution to a technological problem that did not exist in Daniel Bernoulli's day.

As for the Kelly paper, it merely tells us, for at least the second time, that there is an optimal fraction in terms of quantity when what we have to work with on subsequent periods is a function of what happens during this period.

Yes, the idea is monumental. Its application, I found, left me with a great deal of work to be done. Fortunately, the predecessors in this nearly three-centuries-old story to these lines of thought memorialized what they had seen, what they found to be true about it.

I was introduced to the notion of geometric mean maximization by Larry Williams, who showed me Thorp's "Kelly Formulas," which he sought to apply to the markets (because he has the nerve for it).

Seeing that this was no mere nostrum and that there was some inherent problem with it (in applying those formulas to the markets, as they mathematically solve for a "2 possible outcome" gambling situation), I sought a means of applying the concept to a stream of trades. Nothing up to that point provided me with the tools to do so. Yes, it is geometric mean maximization, or "maximizing the sum of the logs," but it's not in a gambling situation. If I followed that path without amendment, I would end up with a "number" between 0 and X. It tells me neither what my "risk" is (as a percentage of account equity) nor how many contracts or shares to put on.

Because I wanted to apply the concept of geometric mean maximization to trading, I had to discern my own formulas, because this was not a gambling situation (nor was it bound between 0 and 1), to represent the fraction of our stake at risk, just as the gambling situation innately bounds f between 0 and 1.

In 1990, I provided my equations to do just that. To find the optimal f (for "fraction," thus implying a number $0 <= f <= 1$), given a stream of trades (or, of periodic profits and losses; for example, the daily, or monthly, or quarterly, or annual profit/loss), we must first convert them into a "Holding Period Return," remaining within the nomenclature of those before me, for a given f value, or "HPR(f)." This is simply 1 + the rate of return, and is given as:

$$HPR(f) = 1 + f * \frac{-trade}{BiggestLoss} \tag{1.06}$$

where: f = the value we are using for f
$-trade$ = the profit or loss on a trade with the sign reversed so that losses are positive numbers and profits are negative
$BiggestLoss$ = the P&L over the entire stream that resulted in the biggest loss. (This should always be a negative number.)

Thus, a gain of 5 percent would see an HPR(f) of 1.05. A loss of 5 percent would see an HPR(f) of .95.

By multiplying together all of the HPR(f)s, we obtain the "Terminal Wealth Relative," or "TWR(f)." This is simply the geometric product of the HPR(f)s, and it represents the multiple made on our starting stake at the end of the stream of profits and losses:

$$TWR(f) = \prod_{i=1}^{n} HPR(f)_i \tag{1.07}$$

or:

$$TWR(f) = \prod_{i=1}^{n} \left(1 + f * \frac{-trade_i}{BiggestLoss}\right) \tag{1.07a}$$

and geometric mean (GHPR(f)) is simply the n^{th} root of the TWR(f). GHPR(f) represents the multiple you made on your stake, on average, per HPR(f):

$$GHPR(f) = \sqrt[n]{\prod_{i=1}^{n} HPR(f)_i} = \left(\prod_{i=1}^{n} HPR(f)_i\right)^{1/n} \tag{1.08}$$

or:

$$GHPR(f) = \sqrt[n]{\prod_{i=1}^{n} \left(1 + f * \frac{-trade_i}{BiggestLoss}\right)}$$
$$= \left(\prod_{i=1}^{n} \left(1 + f * \frac{-trade_i}{BiggestLoss}\right)\right)^{1/n} \tag{1.08a}$$

where: f = the value we are using for f
$-trade_i$ = the profit or loss on the i^{th} trade with the sign reversed so that losses are positive numbers and profits are negative
$BiggestLoss$ = the P&L that resulted in the biggest loss. (This should always be a negative number.)
n = the total number of trades
GHPR(f) = the geometric mean of the HPR(f)s

The value for $f(0 <= f <= 1)$ that maximizes GHPR(f) (or TWR(f), as both are maximized at the same value for f) is the optimal f. It is an optimization problem: Simply optimize f for greatest GHPR(f) or TWR(f). The value for the optimal f is irrespective of the order the HPR(f)s occur in; all permutations of a stream of HPR(f)s result in the same optimal f value.

These equations would give you the same answer for the 2:1 coin toss as the Kelly formula answer of $f = .25$. So, these formulas can be used in lieu of the Kelly formulas. What's more, these formulas work when there are more than two possible outcomes.

Furthermore, the f derived from the 1990 procedure detailed here can then be converted into a number of "units" to put on (number of shares or contracts). Since the inputs in terms of $trade_i$ and biggest loss must be determined from a particular trading size, be it 100 shares or one contract, it can be any arbitrary, though consistent, amount you choose (which we will call a "unit"). Thus, once an optimal f is determined, based on the results of trading in one unit, we can determine how many units we should have on for a given trade or period (depending upon whether the stream of HPR(f)s was derived by using trades or periods) as:

$$f\$ = -BiggestLoss/f \tag{1.09}$$

$f\$$ represents how much to capitalize each unit for a given trade or period by. To then determine how many units to have on:

$$\text{Number of Units to Assume} = \text{Account Equity}/f\$ \tag{1.10}$$

For example, if we have a stake of 100 units, a biggest loss of -1 units (using our 2:1 coin toss game here) and an f of .25, we would thus have:

$$f\$ = -BiggestLoss/f$$
$$f\$ = -(-1)/.25$$
$$f\$ = 1/.25$$
$$f\$ = 4$$

(So, if one unit is one wager in this game, we make one wager for every 4 units in our stake):

$$\text{Number of Units to Assume} = 100/4 = 25$$

Thus, we make 25 wagers, which in this case correspond to a 25 percent fraction of our stake risked.

If we were trading and we had an optimal $f\$$ of .25, and our biggest loss per unit was 10,000 units, we would have an $f\$$ of 40,000 units and would thus trade one unit per every 40,000 units in our account equity. Such a position sizing would represent having 25 percent of our account at risk.

Do not be dissuaded by margin requirements. They have nothing to do with what is the mathematically optimal amount to finance a trade by (often, margin requirements will be more than $f\$$).

Do not be dissuaded by having the variable *BiggestLoss* in the equations. This will be addressed in the following chapter.

So these equations can be used in lieu of the Kelly formulas for trading situations, but they are applicable to trading only one game, only one component, at a given time.

However, I was interested in multiple, simultaneous games that were not simply gambling games. I was interested in portfolios of tradable components and thus had to determine my own equations for dealing with multiple, simultaneous positions, because again, this is a trading situation. Kelly and others intimated that such an approach could be worked out for trading, and in my search for answers to these problems I had encountered Mike Pascual, a brilliant fellow, who had worked it out for gambling situations (taking the Kelly Formulas to the next level—multiple, simultaneous wagers). Yet even so I was left in a dead end in terms of applying this approach to market outcomes for the very same reasons that I could not apply the Kelly Formulas to market outcomes of a single component; the distribution of market outcomes is more complex than for gambling outcomes. (I will not attempt to cover all that Pascual has covered; interested readers are referred to Pascual [1987]).

I had to work out the formulas for geometric mean maximization for market-related situations (and bounding the result, f, between 0 and 1) as opposed to simpler, gambling illustrations, and I had to work it out for multiple, simultaneous "plays," that is, "portfolios." The equations for such will be provided later in this story, when we get to the discussion of Leverage Space Part II.

Most important, where the predecessors (including Kelly) of this "geometric mean maximization" notion came up short for me, in terms of market application, was that they only alluded to the fact that there *was* an optimal point.

Having an optimal point implies a curved function, and it is the dynamics of the curve itself (as bound between 0 and 1, to put in context and give meaning to being on the curve!) that we use to discern the information about our actions in the marketplace. There are a great deal of information, payoffs, and consequences to being at different points along this curve (which, because we are oblivious to them, we are likely migrating along with each trade, and hence, assuming different potential payoffs and consequences from trade to trade!).

Furthermore, as I pursued my passion in this vein, I discovered what really was an entire domain to this netherworld that had been heretofore undiscovered: the very nature of the curve. Prior to my illumination of the character of these curves (and the fact that they are at work on us whether we acknowledge them or not), people, very smart guys in fact, would talk about things like betting "half Kelly," or other arbitrary, ad hoc things like this. I could see that no one had explored the dynamics of the curve. That is where the real story is here. (Because "half kelly" is an arbitrary point in terms of the dynamics of the curve and shows a common perspective that is oblivious to the dynamics of the curve, and hence, the tradeoffs of the curve and the mathematically significant points moving and migrating along it.)

It is the character of the curve whereupon the optimal point resides that *is* what is the netherworld, it *is* leverage space. The nature of the curve itself—that *is* where we find information about our actions, and therefore what we shall discuss in forthcoming chapters.

I had unwittingly stumbled into what was an entire domain, found myself in a place alive with geometric relationships, this place I call this netherworld of "leverage space." Things, the good predecessors in this line of reasoning evidently never saw, which I have had the pleasure of being utterly fascinated by.

Had others in this nearly three-centuries-old story seen this, they would have memorialized in writing what they had seen regarding these things, as I have tried to do over the decades.

I contend you are somewhere on the curve, ineluctably, and that there are characteristics to being at different points on that curve that have not been identified. Further, unless you are risking a certain, fixed, percentage of your "stake" on each "play," you are ineluctably migrating about that curve, and the characteristics of those heretofore-unidentified points on that curve apply to you, but unbeknownst to you.