CHAPTER 1
ELEMENTS OF GRAPH THEORY

1.1 GRAPH MODELS

The first four chapters of this book deal with graphs and their applications. A **graph** $G = (V, E)$ consists of a finite set V of **vertices** and a set E of **edges** joining different pairs of distinct vertices.* Figure 1.1a shows a depiction of a graph with $V = \{a, b, c, d\}$ and $E = \{(a, b), (a, c), (a, d), (b, d), (c, d)\}$. We represent vertices with points and edges, and lines joining the prescribed pairs of vertices. This definition of a graph does not allow two edges to join the same two vertices. Also, an edge cannot "loop" so that both ends terminate at the same vertex—an edge's end vertices must be distinct. The two ends of an undirected edge can be written in either order, (b, c) or (c, b). We say that vertices a and b are **adjacent** when there is an edge (a, b).

Sometimes the edges are ordered pairs of vertices, called **directed edges**. In a **directed graph**, all edges are directed. See the directed graph in Figure 1.1b. We write $(\overrightarrow{b, c})$ to denote a directed edge from b to c. In a directed graph, we allow one edge in each direction between a pair of vertices. See edges $(\overrightarrow{a, c})$ and $(\overrightarrow{c, a})$ in Figure 1.1b.

The combinatorial reasoning required in graph theory, and later in the enumeration part of this book, involves different types of analysis than are used in calculus and high school mathematics. There are few general rules or formulas for solving these problems. Instead, each question usually requires its own particular analysis. This analysis sometimes calls for clever model-building or creative thinking, but more often consists of breaking the problem into many cases (and subcases) that are easy enough to solve using simple logic or basic counting rules. A related line of reasoning is to solve a special case of the given problem and then to find ways to extend that

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Figure 1.1 **(*a*)** **(*b*)**

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*What this book calls a graph is referred to in many graph theory books as a *simple graph*. In general, graph theory terminology varies a little from book to book.

reasoning to all the other cases that may arise. The underlying theme here is summarized by a famous quote from the great problem-solver George Polya: "The challenging part is asking the right questions. Then the answers are easy."

In graph theory, combinatorial arguments are made a little easier by the use of pictures of the graphs. For example, a case-by-case argument is much easier to construct when one can draw a graphical depiction of each case.

Graphs have proven to be an extremely useful tool for analyzing situations involving a set of elements in which various pairs of elements are related by some property. The most obvious examples of graphs are sets with physical links, such as electrical networks, where electrical components (transistors) are the vertices and connecting wires are the edges; or telephone communication systems, where telephones and switching centers are the vertices and telephone lines are the edges. Road maps, oil pipelines, and subway systems are other examples.

Another natural form of graphs is sets with logical or hierarchical sequencing, such as computer flowcharts, where the instructions are the vertices and the logical flow from one instruction to possible successor instruction(s) defines the edges; or an organizational chart, where the people are the vertices and if person A is the immediate superior of person B, there is an edge $(\vec{A, B})$. Computer data structures, evolutionary trees in biology, and the scheduling of tasks in a complex project are other examples.

The emphasis in this book will be on problem solving, with problems about general graphs and applied graph models. Observe that we will usually not have any numbers to work with, only some vertices and edges. At first, this may seem to be highly nonmathematical. It is certainly very different from the mathematics that one learns in high school or in calculus courses. However, disciplines such as computer science and operations research contain as much graph theory as they do standard numerical mathematics.

This section consists of a collection of illustrative examples about graphs. We will solve each problem from scratch with a little logic and systematic analysis. Many of these examples will be revisited in greater depth in subsequent chapters.

The following three graph theory terms are used in the coming examples. A **path** P is a sequence of distinct vertices, written $P = x_1–x_2–\cdots–x_n$, with each pair of consecutive vertices in P joined by an edge. If in addition there is an edge (x_n, x_1), the sequence is called a **circuit**, written $x_1–x_2–\cdots–x_n–x_1$. For example, in Figure 1.1*a*, *b-d-a-c* forms a path, while *a-b-d-c-a* forms a circuit. A graph is **connected** if there is a path between every pair of vertices. The removal of certain edges or vertices from a connected graph G is said to *disconnect* the graph if the resulting graph is no longer connected—that is, if at least one pair of vertices is no longer joined by a path. The graph in Figure 1.1*a* is connected, but the removal of edges (a, b) and (b, d) will disconnect it.

Example 1: Matching

Suppose that we have five people A, B, C, D, E and five jobs a, b, c, d, e, and that various people are qualified for various jobs. The problem is to find a feasible one-to-one matching of people to jobs, or to show that no such matching can exist. We

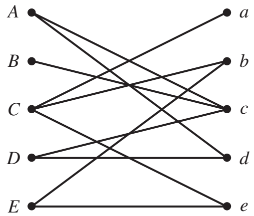


Figure 1.2

can represent this situation by a graph with vertices for each person and for each job, with edges joining people with jobs for which they are qualified. Does there exist a feasible matching of people to jobs for the graph in Figure 1.2?

The answer is no. The reason can be found by considering people *A*, *B*, and *D*. These three people as a set are collectively qualified for only two jobs, *c* and *d*. Hence there is no feasible matching possible for these three people, much less all five people. An algorithm for finding a feasible matching, if any exists, will be presented in Chapter 4. Such matching graphs in which all the edges go horizontally between two sets of vertices are called **bipartite.** Bipartite graphs are discussed further in Section 1.3. ■

Example 2: Spelling Checker

A spelling checker looks at each word X (represented in a computer as a binary number) in a document and tries to match X with some word in its dictionary, which typically contains close to 100,000 words. To understand how this checking works, we consider the simplified problem of matching an unknown letter X with one of the 26 letters in the English alphabet. In the spirit of the strategy humans use to home in on the page in a dictionary where a given word appears, the computer search procedure would first compare the unknown letter X with M, to determine whether X $\leq$ M or X $>$ M. The answer to this comparison locates X in the first 13 letters of the alphabet or the second 13 letters, thus cutting the number of possible letters for X in half. This strategy of cutting the possible matches in half can be continued with as many comparisons as needed to home in on X's letter. For example, if X $\leq$ M, then we could test whether or not X $\leq$ G; if X $>$ M, we could test whether X $\leq$ S.

This testing procedure is naturally represented by a directed graph called a **tree**. Figure 1.3 shows the first three rounds of comparisons for the letter-matching procedure. The vertices represent the different letters used in the comparisons. The left descending edge from a vertex Q points to the letter for the next comparison if X $\leq$ Q, and the right descending edge from Q points to the next letter if X $>$ Q.

For our original spelling-checker problem, a word processor would use a similar, but larger, tree of comparisons. With just 12 rounds of comparisons, it could reduce

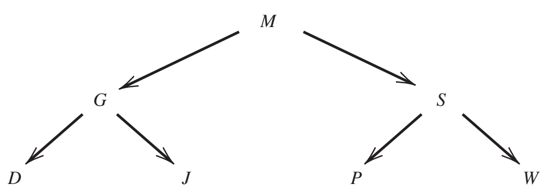


Figure 1.3

the number of possible matches for an unknown word X from 100,000 down to 25, about the number of words in a column of a page in a dictionary. (Once a list was reduced to about 25 possibilities, a computer search for X would usually run linearly down that list, just as a human would.) ■

Chapter 3 examines trees and their use in various search problems. Trees can be characterized as graphs that are connected and that have a unique path between any pair of vertices (ignoring the directions of directed edges). The next example uses trees in a very different way.

Example 3: Network Reliability

Suppose the graph in Figure 1.4 represents a network of telephone lines (or electrical transmission lines). We are interested in the network's vulnerability to accidental disruption. We want to identify sets of those lines and switching centers that must stay in service to avoid disconnecting the network.

There is no telephone line (edge) whose removal will disconnect the telephone network (graph). Similarly, there is no vertex whose removal disconnects the graph.

Is there any pair of edges whose removal disconnects the graph? There are several such pairs. For example, we see that if the two edges incident to a are removed, vertex a is isolated from the rest of the network. A more interesting disconnecting pair of edges is (b, c), (j, k). It is left to the reader as an exercise to find all disconnecting sets consisting of two edges for the graph in Figure 1.4.

Let us take a different tack. Suppose we want to find a minimal set of edges needed to link together the 11 vertices in Figure 1.4. There are several possible minimal connecting sets of edges. By inspection, we find the following one: (a, b), (b, c), (c, d), (d, h), (h, g), (h, k), (k, j), (j, f), (j, i), (i, e); the edges in this minimal connecting set are darkened in Figure 1.4. A minimal connecting set will always be a tree. One interesting general result about these sets is that if the graph G has n vertices, then a minimal connecting set for G (if any exists) always has $n - 1$ edges. ■

The number of edges incident to a vertex is called the **degree** of the vertex.

Example 4: Street Surveillance

Now suppose the graph in Figure 1.4 represents a section of a city's street map. We want to position police officers at corners (vertices) so that they can keep every block (edge) under surveillance—that is, every edge should have police officers at (at least) one of its end vertices. What is the smallest number of police officers that can do this job?

Let us try to get a lower bound on the number of police officers needed. The map has 14 blocks (edges). Corners b, c, e, f, h, and j each have degree 3, and corners a, d, g, i, and k each have degree 2. Since four vertices can be incident to at most $4 \times 3 = 12$ edges but there are 14 edges in all, we will need at least five police officers. We shall now try to find a set of five vertices incident to all the edges. If we can find such a set, we know that it is the best (smallest) solution possible.

If all five police officers were positioned at degree-3 vertices, then $5 \times 3 = 15$ edges are watched by the five police officers. Since there are only 14 edges, some

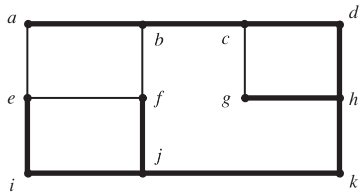


Figure 1.4

edge would be covered by police officers at both end vertices. If four police officers are at degree-3 vertices and one at a degree-2 vertex, then exactly 14 edges are watched—and no edge need be covered at both ends. (If fewer than four of the five police officers are at degree-3 vertices, we could not watch all 14 edges). With these general observations, we are ready for a systematic analysis to try to find five vertices that are collectively adjacent to all 14 edges.

Consider edge (c, d). Suppose it is watched by an officer at vertex d. Then vertex c (the other end vertex of edge (c, d)) cannot also have an officer, since we noted above that if we use a degree-2 vertex, such as d, then no edge can be watched from both end vertices. However, if vertex c cannot be used, then edge (c, g) must be watched from its other end vertex g. But now we are using two degree-2 vertices, d and g. We noted above that at most one degree-2 vertex can be used. We got into this trouble by assuming that edge (c, d) is watched from vertex d.

Now assume no officer is at vertex d. Then we must watch edge (c, d) with an officer at vertex c. Similarly, edge (d, h) can be watched only by placing an officer at vertex h. Next look at edge (h, k). It is already watched by vertex h. Then we assert that (h, k) cannot also be watched by an officer at vertex k, since k has degree-2 and we noted above that if we use a degree-2 vertex, no edge can be watched from both ends. We conclude that there cannot be an officer at vertex k. Then edge (k, j) can be watched only by placing an officer at vertex j. We now have officers required to be at vertices c, h, and j.

Similar reasoning shows that with an officer at vertex j, there cannot be an officer at vertex i; then there must be an officer at vertex e; then there cannot be an officer at vertex a; and then there must be an officer at vertex b. In sum, we have shown that we should place police officers at vertices c, h, j, e, and b. A check confirms that these five vertices do indeed watch all 14 edges. A smallest number of police officers for this surveillance problem has been found. Note that since our reasoning forced us to use exactly these five vertices, no other set of five vertices can work.

At the beginning of this example, we showed that at least five corners were needed to keep all the blocks (edges) under surveillance. Now we have produced a set of five corners that achieve such surveillance. It then follows that five is the minimum number of corners.

We conclude this example by noting that in this surveillance situation, one can also consider watching the vertices rather than the edges: How few officers are needed to watch, that is, be at or adjacent to, all the vertices? We use the same type of argument as in the block surveillance problem to get a lower bound on the number of corners needed for corner surveillance. An officer at vertex x is considered to be watching vertex x and all vertices adjacent to x. There are 11 vertices, and six of these vertices watch four vertices (themselves and three adjacent vertices). Thus three is the theoretical minimum. This minimum can be achieved. Details are left as an exercise. ■

A set C of vertices in a graph G with the property that every edge of G is incident to at least one vertex in C is called an **edge cover**. The previous example was asking for an edge cover of minimal size in Figure 1.4. The reasoning in Example 4 illustrates the kind of systematic case-by-case analysis that is common in graph theory.

The analysis in the previous example also illustrates a principle that is used over and over again in graph theory and other combinatorial settings. Namely, to show a graph has some property—in this case, the existence of a five-vertex edge cover—we assume that the property exists and deduce useful consequences of this assumption. The key consequence for the graph in Figure 1.4 was as follows:

(*) if an edge (x, y) links a 3-degree vertex x with a 2-degree vertex y then at most one of x and y can be used in a five-vertex edge cover

A subsequent consequence of (*) concerning the pair x, y is that if we want to use vertex x (and not y) in a minimal edge cover to cover (x, y), then to cover the other edge at y—call it (y, z)—vertex z would also have to be in the minimal edge cover.

We give the mnemonic name ***A****ssumptions generate helpful* ***C****onsequences*—**the AC Principle**, for short—to this strategy of assuming that a graph has a desired property in order to deduce useful consequences, consequences we use to help us show that the graph indeed has this property. The AC Principle can also be used to show that a graph does not have some property: to do so, we deduce consequences under the assumption that the graph does have the property, and then show that these consequences lead to a contradiction.

Example 5: Scheduling Meetings

Consider the following scheduling problem. A state legislature has many committees that meet for one hour each week. One wants a schedule of committee meeting times that minimizes the total number of hours of meetings—but such that two committees with overlapping membership cannot meet at the same time.

This situation can be modeled with a graph in which we create a vertex for each committee and join two vertices by an edge if they represent committees with overlapping membership. Suppose that the graph in Figure 1.4 now represents the membership overlap of 11 legislative committees. For example, vertex c's edges to vertices b, d, and g in Figure 1.4 indicate that committee c has overlapping members with committees b, d, and g.

A set of committees can all meet at the same time if there are no edges between the corresponding set of vertices. A set of vertices without an edge between any two is called an **independent set** of vertices. Our scheduling problem can now be restated as seeking a minimum number of independent sets that collectively include all vertices. This problem is discussed in depth in Section 2.3.

How many committees can meet at one time? We are asking the following graph question: What is the largest independent set of the graph? It is very hard in general

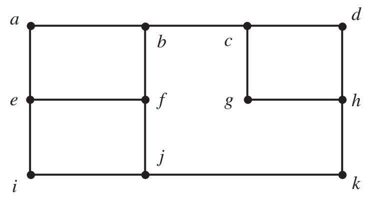


Figure 1.4

to find the largest independent set in a graph. For the graph in Figure 1.4, a little examination shows that there is one independent set of size 6, a, d, f, g, i, k. All other independent sets have five or fewer vertices.

One goal of graph theory is to find useful relationships between seemingly unrelated graph concepts that arise from different settings. Now we show that independent sets are closely related to edge covers. If V is the set of vertices in a graph G, then I will be an independent set of vertices if and only if $V - I$ is an edge cover! Why? Because if there are no edges between two vertices in I, then every edge involves (at least) one vertex not in I—that is, a vertex in $V - I$. Conversely, if C is an edge cover so that all edges have at least one end vertex in C, then there is no edge joining two vertices in $V - C$. So $V - C$ is an independent set. Check that in Figure 1.4, the vertices not in the independent set a, d, f, g, i, k form edge cover b, c, e, h, j.

A consequence of this relationship is that if I is an independent set of largest possible size in a graph, then $V - I$ will be an edge cover of smallest possible size. So finding a maximal independent set is equivalent to finding a minimal edge cover. ■

We next give an example involving directed graphs.

Example 6: Influence Model

Suppose psychological studies of a group of people determine which members of the group can influence the thinking of others in the group. We can make a graph with a vertex for each person and a directed edge $(\overrightarrow{p_1, p_2})$ whenever person p_1 influences p_2. Let the graph in Figure 1.5*a* represent a set of such influences. Now let us ask for a minimal subset of people who can spread an idea through to the whole group, either directly or by influencing someone who will influence someone else, and so forth. In graph-theoretic terms, we want a minimal subset of vertices with directed paths to all other vertices (a *directed path* from p_1 to p_k is an edge

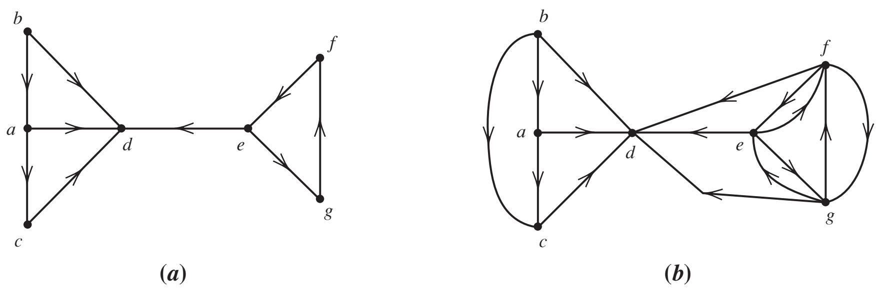


Figure 1.5

sequence $(\vec{p_1, p_2}), (\vec{p_2, p_3}) \ldots (\vec{p_{k-1}, p_k})$). Such a subset of influential vertices is called a **vertex basis**.

To aid us, we can build a *directed-path graph* for the original graph with the same vertex set and with a directed edge $(\vec{p_i, p_j})$ if there is a directed path from p_i to p_j in the original graph. Figure 1.5*b* shows the directed-path graph for the graph in Figure 1.5*a*. Now our original problem can be restated as follows: Find a minimal subset of vertices in the new graph with edges directed to all other vertices. This is just a directed-graph version of the vertex-covering problem mentioned at the end of Example 4. Observe that any vertex in Figure 1.5*b* with no incoming edges must be in this minimal subset (since no other vertices have edges to it); vertex *b* is such a vertex. Since *b* has edges to *a, c*, and *d*, then *e, f*, and *g* are all that remain to be "influenced." Either *e*, *f*, or *g* "influence" these three vertices. Then *b*, *e*, or *b*, *f*, or *b*, *g* are the desired minimal subsets of vertices. ■

1.1 EXERCISES

Summary of Exercises The first six exercises involve simple graph models. Exercises 7–24 present examples and extensions of the models presented in the examples in this section.

1. Suppose interstate highways join the six towns *A*, *B*, *C*, *D*, *E*, *F* as follows: I-77 goes from *B* through *A* to *E*; I-82 goes from *C* through *D*, then through *B* to *F*; I-85 goes from *D* through *A* to *F*; I-90 goes from *C* through *E* to *F*; and I-91 goes from *D* to *E*.

(a) Draw a graph of the network with vertices for towns and edges for segments of interstates linking neighboring towns.

(b) What is the minimum number of edges whose removal prevents travel between some pair of towns?

(c) Is it possible to take a trip starting from town *C* that goes to every town without using any interstate highway for more than one edge (the trip need not return to *C*)?

2. (a) Suppose four teams, the Aces, the Birds, the Cats, and the Dogs, play each other once. The Aces beat all three opponents except the Birds. The Birds lost to all opponents except the Aces. The Dogs beat the Cats. Represent the results of these games with a directed graph.

(b) A dominance order is a listing of teams such that the *i*th team in the order beats the $(i + 1)$ st team. Find all dominance orders for part (a).

3. (a) A schedule is to be made with five football teams. Each team is to play two other teams. Explain how to make a graph model of this problem.

(b) Show that except for interchanging names of teams, there is only one possible graph in part (a).

4. Suppose there are six people—John, Mary, Rose, Steve, Ted, and Wendy—who pass rumors among themselves. Each day John talks with Mary and Wendy; Mary talks with John, Rose, and Steve; Rose talks with Mary, Steve, and Ted; Steve talks with Mary, Rose, Ted, and Wendy; Ted talks with Rose, Steve, and Wendy;

and Wendy talks with John, Steve, and Ted. Whatever people hear one day they pass on to others the next day.

(a) Model this rumor-passing situation with a graph.

(b) How many days does it take to pass a rumor from John to Steve? Who will tell it to Steve?

(c) Is there any way that if two people stopped talking to each other, it would take three days to pass a rumor from one person to all the others?

5. **(a)** Give a direction to each edge in Figure 1.4 so that there are directed routes from any vertex to any other vertex.

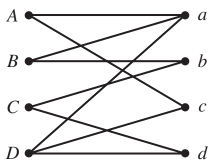
(b) Do part (a) so as to minimize the length of the longest directed path between any pair of vertices. Explain why a smaller minimum is not possible.

6. **(a)** What is the length of the longest possible path (with the most vertices) in the graph in Figure 1.3, ignoring directions of edges?

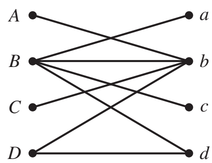
(b) What is the length of the longest possible circuit (with the most vertices) in the graph in Figure 1.4?

7. Find a matching, or explain why none exists for the following graphs:

(a)



(b)

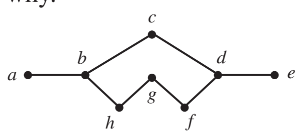


8. Give another reason why Figure 1.2 has no matching by considering the appropriate subset of jobs (showing that they cannot all be filled).

9. We generalize the idea of matching in Example 1 to arbitrary graphs by defining a matching to be a pairing off of adjacent vertices in a graph. For example, one possible matching in Figure 1.1*a* is *a-b, c-d*. Which of the following graphs have a matching? If none exists, explain why.

(a) Figure 1.4

(b)



10. **(a)** Suppose a dictionary in a computer has a "start" from which one can branch to any of the 26 letters: at any letter one can go to the preceding and succeeding letters. Model this data structure with a graph.

(b) Suppose additionally that one can return to "start" from letters *c* or *k* or *t*. Now what is the longest directed path between any two letters?

11. Build the complete testing tree in Example 2 to identify one of the 26 letters of the alphabet.

12. Repeat Example 2 using three-way comparisons (less than, greater than, or equal to) to identify one of the 26 letters.

13. Suppose eight current varieties of chipmunk evolved from a common ancestral strain through an evolutionary process in which at various stages one ancestral variety split into two varieties (none of the ancestral varieties survive when they split into two new varieties).

(a) Explain how one might model this evolutionary process with a graph.

(b) What is the total number of splits that must have occurred?

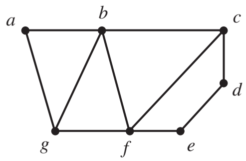
14. In Example 3, find a minimal connecting set of edges containing neither (a, b) nor (b, c).

15. **(a)** What are the other sets of two edges whose removal disconnects the graph in Figure 1.4 besides (a, b), (a, e) and (c, d), (d, h)? Either produce others or give an argument why no others exist.

(b) Find all sets of two vertices whose removal disconnects the remaining graph in Figure 1.4.

16. **(a)** For the following graph, find all sets of two vertices whose removal disconnects the graph of remaining vertices.

(b) Find all sets of two edges whose removal would disconnect the graph.



17. Find a minimal edge cover and a minimal set of vertices adjacent to all other vertices for the graph in Figure 1.2.

18. In Figure 1.4, find all sets of three vertices that are adjacent to all the other vertices. Give a careful logical analysis to justify your answer.

19. Repeat Example 4 for minimal block and corner surveillance when the network in Figure 1.4 is altered by adding edges (f, g), (g, j) and deleting (b, f).

20. Repeat Example 4 for the edge cover and minimal corner surveillance when the network is formed by a regular array of north–south and east–west streets of size:

(a) 3 streets by 3 streets **(b)** 4 streets by 4 streets **(c)** 5 streets by 5 streets

21. **(a)** A queen dominates any square on a chessboard in the same row, column, or diagonal as the queen. How few queens can dominate all squares on an 8 by 8 chessboard?

(b) Repeat this problem for bishops, which dominate only diagonals.

22. Solve the committee scheduling problem for the committee overlap graph in Figure 1.4. That is, what is the minimum number of independent sets needed to cover all vertices?

23. **(a)** Find a maximum independent set in the following graphs:

(i) Figure 1.1a **(ii)** Figure 1.2

(b) Use your result in part (a) to produce a minimal edge cover in these graphs.

24. What is the largest independent set in a circuit of length 7? Of length n?

25. **(a)** What is the largest independent set possible in a connected seven-vertex graph? Draw the graph.

(b) What is the largest independent set possible in a seven-vertex graph (need not be connected)? Draw the graph.

26. Find a vertex basis in the following directed graphs:

(a) Figure 1.1b **(b)** Figure 1.3

(c) Figure 1.4 with edges directed by alphabetical order [e.g., edge (a, e) is directed from a to e]

27. Show that the vertex basis in a directed graph is unique if there is no sequence of directed edges that forms a circuit in the graph.

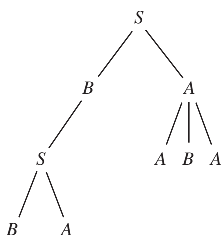
28. A game for two players starts with an empty pile. Players take turns putting one, two, or three pennies in the pile. The winner is the player who brings the value of the pile up to 16¢.

(a) Make a directed graph modeling this game.

(b) Show that the second player has a winning strategy by finding a set of four "good" pile values, including 16¢, such that the second player can always move to one of the "good" piles (when the second player moves to one of the good piles, the next move of the first player must be to a non-good pile, and from this position the second player has a move to a good pile, etc.).

29. The parsing of a sentence can be represented by a directed graph, with a vertex S (for the whole sentence) having edges to vertices Su (subject) and P (predicate), then Su and P having edges to the parts into which they are decomposed into pieces, and so on.

Consider the abstract grammar with decomposition rules: $S \to AB$, $S \to BA$, $A \to ABA$, $B \to BAS$, and $B \to S$. For example, *BAABA* can be "parsed" as shown below.



Find a parsing graph for each of the following (or explain why no parsing exists):

(a) *BABABABA* **(b)** *BBABAABA*

1.2 ISOMORPHISM

In this section we investigate some of the basic structure of graphs. We are interested in properties that distinguish one vertex in a graph from another vertex and, more generally, that distinguish one graph from another graph. We motivate this discussion with the question: how can we tell if two graphs are really the same graph, but drawn differently and with different names for the vertices? For example, are the two five-vertex graphs in Figure 1.6 different versions of the same graph?

A graph can be drawn on a sheet of paper in many different ways. Thus, it is usually possible to draw a graph in two ways that would lead a casual viewer to consider the drawings to be "different" graphs. This motivates the following definition.

> Two graphs G and G' are called **isomorphic** if there exists a one-to-one correspondence between the vertices in G and the vertices in G' such that a pair of vertices are adjacent in G if and only if the corresponding pair of vertices are adjacent in G'.

Such a one-to-one correspondence of vertices that preserves adjacency is called an **isomorphism**. A useful way to think of isomorphic graphs is as follows: the first graph can be redrawn on a transparency that can be exactly superimposed over a drawing of the second graph.

To be isomorphic, two graphs must have the same number of vertices and the same number of edges. The two graphs in Figure 1.6 pass this initial test. Both graphs have one vertex, e and 5, respectively, at the end of just one edge. Then any isomorphism of these two graphs must match e with *5*. Also, the vertices at the other ends of the edge from e and *5* must be matched; that is, d matches with *4*. (Think of superimposing one graph over the other.) The remaining three vertices in each graph are mutually adjacent (forming a triangle) and also are all adjacent to d or *4*, respectively. Thus the matching a with *1*, b with *2*, and c with *3* (or any other matching of these two subsets of three vertices) will preserve the required adjacencies. The correspondence $a - 1$, $b - 2$, $c - 3$, $d - 4$, $e - 5$ is then an isomorphism, and the two graphs are isomorphic. To visualize how they can be made to look the same, think of moving vertices *4* and *5* in the right graph upward and to the right [past edge (*1*,*3*)], so that *1, 2, 3, 4* form a quadrilateral with crossing diagonals.

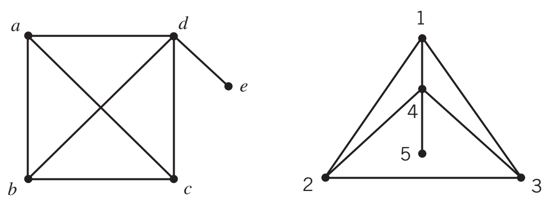


Figure 1.6

Recall that the *degree* deg(x) of a vertex is the number of edges incident to the vertex. Degrees are preserved under isomorphism—that is, two matched vertices must have the same degree. Then in Figure 1.6, e has to be matched with *5* and d matched with *4* because they are the unique vertices of degree 1 and 4 in their respective graphs. Further, two isomorphic graphs must have the same number of vertices of a given degree. For example, if they are to be isomorphic, the two graphs in Figure 1.6 must both have the same number of vertices of degree 3—they do; both have three vertices of degree 3.

A **subgraph** G' of a graph G is a graph formed by a subset of vertices and edges of G. If two graphs are isomorphic, then subgraphs formed by corresponding vertices and edges must be isomorphic. In Figure 1.6, removal of vertices e and *5* (and their incident edges) leaves two isomorphic subgraphs consisting of four mutually adjacent vertices. Once this subgraph isomorphism is noted, isomorphism of the whole graphs is easily demonstrated.

Subgraphs can be used to test for isomorphism in the following way. If a graph G has a set of six vertices forming a chordless circuit of length 6 (chordless means there are no other edges between these six vertices except the six edges forming the circuit), then any graph isomorphic to G must also have a set of six vertices forming such a chordless 6-circuit.

A graph with n vertices in which each vertex is adjacent to all the other vertices is called a **complete graph on *n* vertices**, denoted K_n. A complete graph on two vertices, K_2, is just an edge. Complete subgraphs are in a sense the building blocks of all larger graphs. For example, both graphs in Figure 1.6 consist of a K_4 and a K_2 joined at a common vertex. Conversely, every graph on n vertices is a subgraph of K_n.

Before examining other pairs of graphs for isomorphism, let us mention the practical importance of determining whether two graphs are isomorphic. Researchers working with organic compounds build up large dictionaries of compounds that they have previously analyzed. When a new compound is found, they want to know if it is already in the dictionary. Large dictionaries can have many compounds with the same molecular formula but differing in their structure as graphs (and possibly in other ways). Then one must test the new compound to see if its graph-theoretic structure is the same as the structure of one of the known compounds with the same formula (and the same in other ways)—that is, whether the new compound is graph-theoretically isomorphic to one of a set of known compounds. A similar problem arises in designing efficient integrated circuitry for a computer. If the design problem has already been solved for an isomorphic circuit (or if a piece of the new network is isomorphic to a previously designed circuit), then valuable savings in time and money are possible.

Example 1: Simple Isomorphism

Are the two graphs in Figure 1.7 isomorphic?

Both graphs have eight vertices and 10 edges. Let us examine the degrees of the different vertices. We see that *b, d, f, h* and *3, 4, 7, 8* have degree 2, while the other vertices have degree 3. Then the two graphs have the same number of vertices of degree 2 and the same number of degree 3. The respective subgraphs of the four vertices

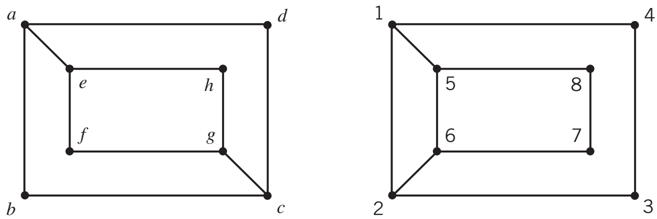


Figure 1.7

of degree 2 (and the edges between these degree-2 vertices) in each graph must be isomorphic if the whole graphs are isomorphic. However, there are no edges between any pair of *b, d, f, h*, while the other subgraph of degree-2 vertices has two edges: (*4*, *3*) and (*8*, *7*). So the subgraphs of degree-2 vertices are not isomorphic, and hence the two full graphs are not isomorphic. The reader can also check that the two subgraphs of degree-3 vertices in each graph are not isomorphic. ■

The vertices of degree 2 in the left graph in Figure 1.7 form a subgraph of mutually nonadjacent vertices. Such a subgraph is called a set of **isolated vertices**.

Let us review the reasoning used in Example 1. It is a contrapositive version of the AC Principle, ***A**ssumptions generate helpful **C**onsequences*, introduced after Example 4 in Section I.1. The contrapositive statement is that if a consequence is false, then the assumption must be false. In this case, we assume that two graphs G and G' are isomorphic. A consequence of this assumption is that G_2 and G_2' must also be isomorphic, where G_2 (G_2') is the subgraph of G (G') generated by its vertices of degree 2. For the graphs in Example 1, the contrapositive statement is that if G_2 and G_2' are not isomorphic, then the assumption that G and G′ are isomorphic must be false.

Example 2: Isomorphism in Symmetric Graphs

Are the two graphs in Figure 1.8 isomorphic?

The two graphs both have seven vertices and 14 edges. Every vertex in both graphs has degree 4. Further, both graphs exhibit all the symmetries of a regular 7-gon. With no distinctions possible among vertices within the same graph, our only option is to try to construct an isomorphism. To do this, we assume that there is an isomorphism and use the AC Principle to deduce properties of an isomorphism for these two graphs that can guide us to construct such an isomorphism. If at some point

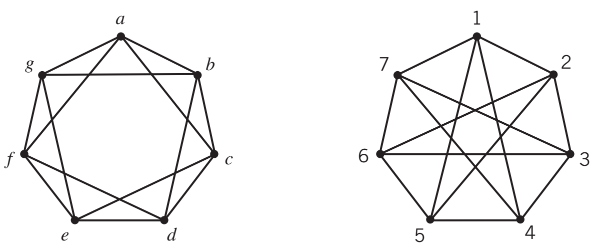


Figure 1.8

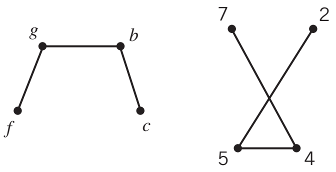


Figure 1.9

in our construction a contradiction arises, then we know our assumption was false and there is no isomorphism.

Start with vertex a in the left graph. By rotational symmetry, we can match a to any vertex in the right graph (that is, if the two graphs are isomorphic, there will exist an isomorphism with a matched to any vertex in the right graph). Let us use the match $a - 1$.

The set of neighbors of a (vertices adjacent to a) must be matched with the set of neighbors of *1*. Let us look at the subgraphs formed by these neighbors of a and *1*. See Figure 1.9. Both subgraphs are paths: one is f to g to b to c, and the other is *7* to *4* to *5* to *2*. The isomorphism must make these path subgraphs isomorphic. Thus, f and c must be matched with *7* and *2* (matching ends of the two paths). By the left-right symmetry of the graphs, it makes no difference which way f and c are matched—say $f - 7$ and $c - 2$. Then to complete the isomorphism of neighbors of a and *1*, we must match g with *4* and b with *5*. Now there remain only two unmatched vertices in each graph: d, e and *3*, *6*. Vertex g is adjacent to e but not d, and its matched vertex *4* is adjacent to *3* but not to *6*. Thus we must match e with *3* and d with *6*.

In sum, allowing for symmetries to match a with *1* and f with *7*, we conclude that if the graphs are isomorphic, one isomorphism must be $a - 1, b - 5, c - 2, d - 6, e - 3, f - 7, g - 4$. Checking edges, we see that the graphs are indeed isomorphic with this matching (if this matching were found not to be an isomorphism, then the two graphs would not be isomorphic, since the matches we made were all forced except for the symmetries involving the matches of a and f). ■

Given a graph $G = (V, E)$, its **complement** is a graph $\overline{G} = (V, \overline{E})$ with the same set of vertices but now with edges between exactly those pairs of vertices not linked in G. The union of the edges in G and $\overline{G}$ forms a complete graph. Two graphs G_1 and G_2 will be isomorphic if and only if $\overline{G}_1$ and $\overline{G}_2$ are isomorphic. The isomorphism problem in Example 2 is easy to answer using complements. Figure 1.10 shows the

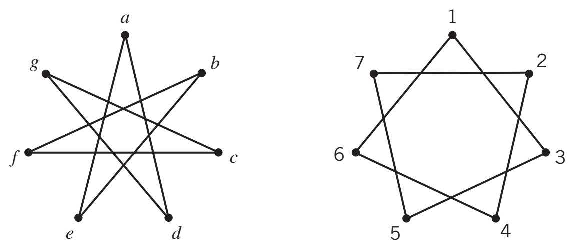


Figure 1.10

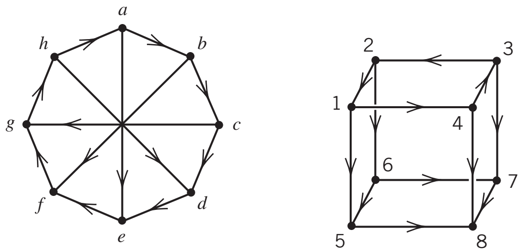


Figure 1.11

complements of the two graphs in Figure 1.8. Clearly, both these complementary graphs are just a (twisted) circuit of length 7 and hence are isomorphic.

In general, if a graph has more pairs of vertices joined by edges than pairs not joined by edges, then its complement will have fewer edges and thus will probably be simpler to analyze.

Example 3: Isomorphism of Directed Graphs

Are the two directed graphs in Figure 1.11 isomorphic?

Each graph has eight vertices and 12 edges, and each vertex has degree 3. If we break the degree of a vertex into two parts, the **in-degree** (number of edges pointed in toward the vertex) and **out-degree** (number of edges pointed out), we see that each graph has four vertices of in-degree 2 and out-degree 1, and each graph has four vertices of in-degree 1 and out-degree 2. We could try to build an isomorphism as in the previous example by starting with a match (by a symmetry argument) between *a* and 1 and then matching their neighbors (with edge directions also matched), and so forth.

However, there is a basic difference in the directed path structure of the two graphs. We will exploit this difference to prove nonisomorphism. In the left graph we can draw a directed path from any given vertex to any other vertex by going clockwise around the circle of vertices: the outer edges form a directed circuit through all the vertices in the left graph. But in the right graph, all edges between the vertex subsets $V_1 = \{1, 2, 3, 4\}$ and $V_2 = \{5, 6, 7, 8\}$ are directed from V_1 to V_2, and so there can be no directed paths from any vertex in V_2 to any vertex in V_1 (nor is there a directed circuit through all the vertices). Thus, the two graphs are not isomorphic. ■

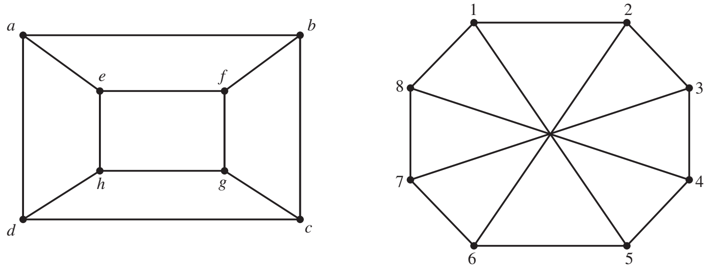
1.2 EXERCISES

1. List all nonisomorphic undirected graphs with four vertices.
2. List all nonisomorphic directed graphs with three vertices.
3. Draw two nonisomorphic graphs with
   (a) Six vertices and 10 edges
   (b) Nine vertices and 13 edges

4. If directions are ignored, are the two graphs in Figure 1.11 isomorphic?

5. Which of the following pairs of graphs are isomorphic? Explain carefully.

(a)



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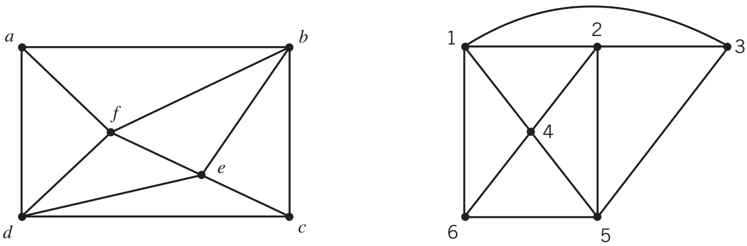

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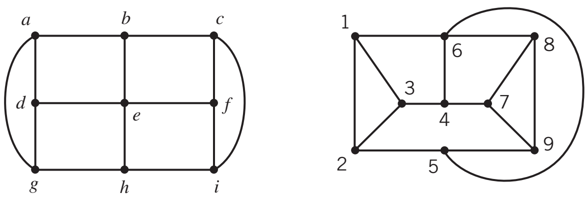

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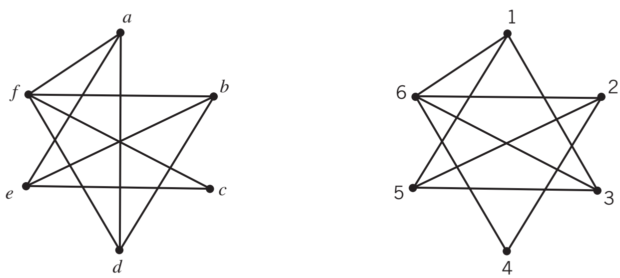

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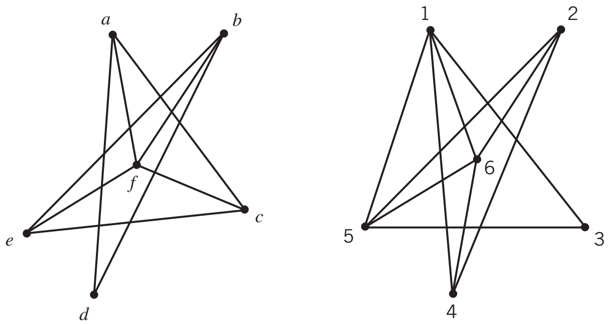
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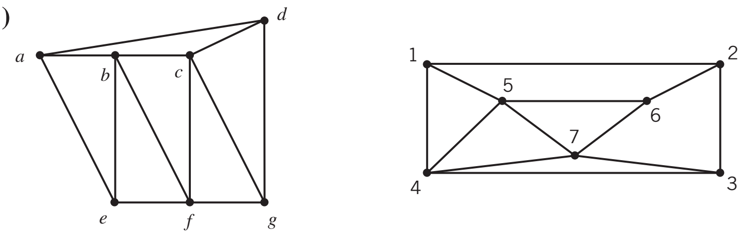
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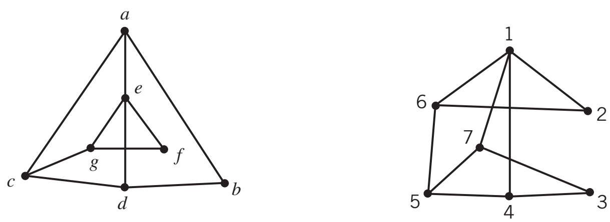
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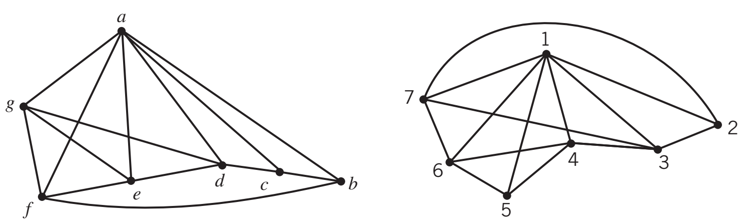
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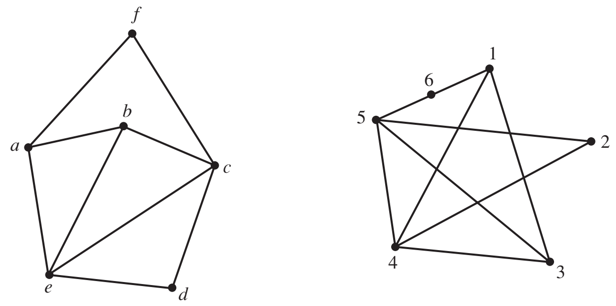
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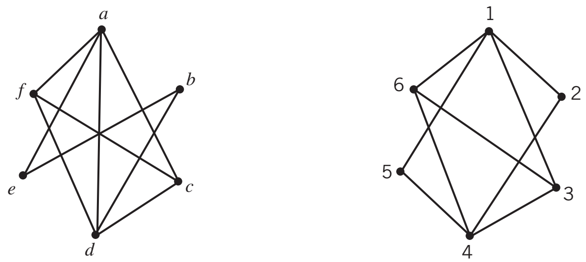
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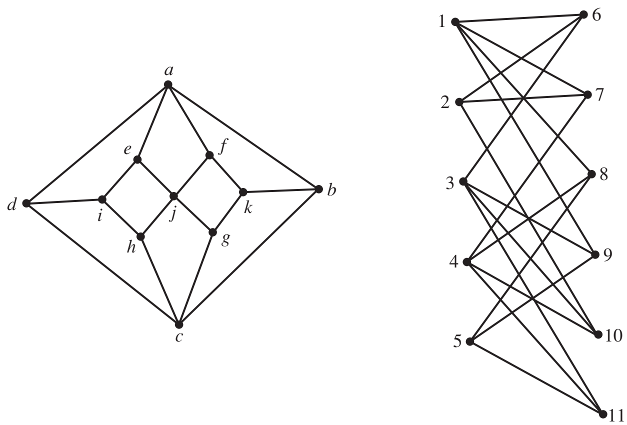
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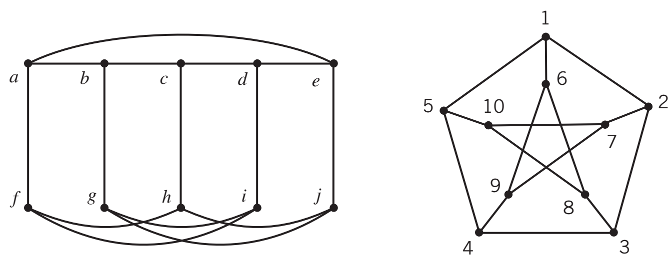
(l)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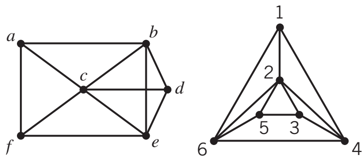
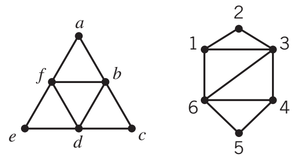
6. Which of the following pairs of graphs are isomorphic? Explain carefully.

(a)



(b)



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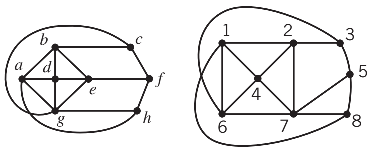

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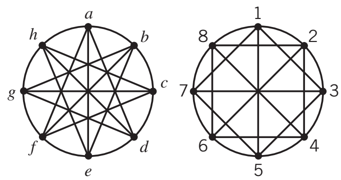

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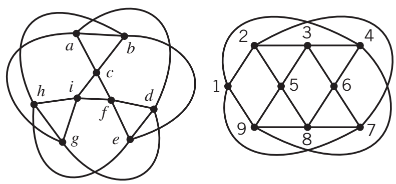

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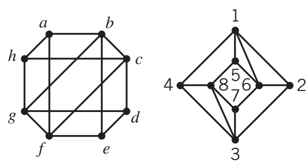

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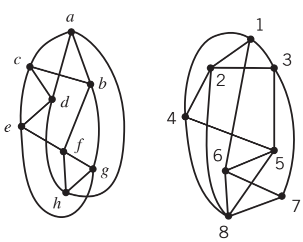

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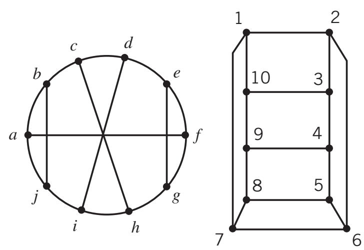
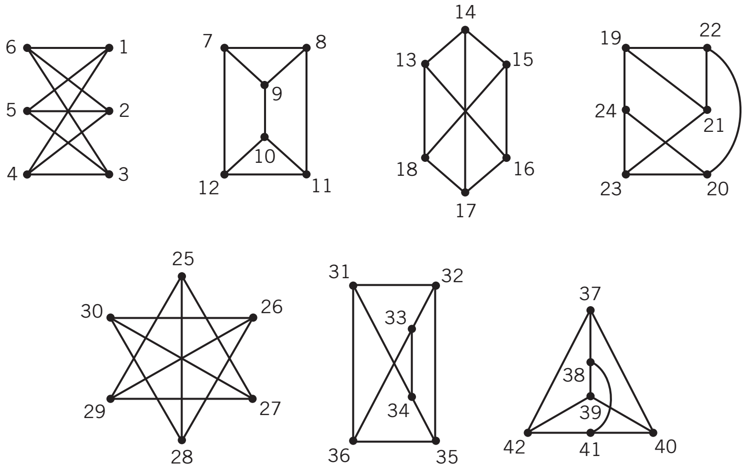
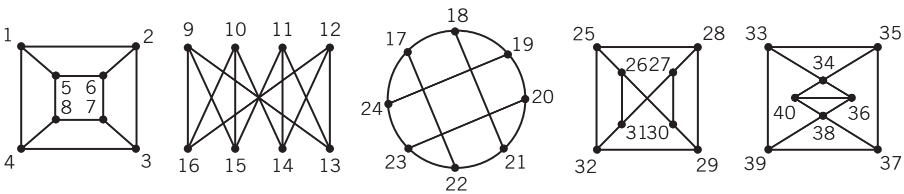


7. Which pairs of graphs in this set are isomorphic?

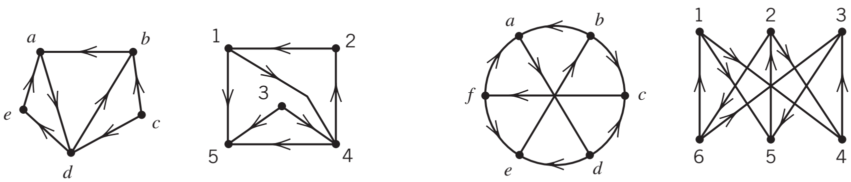


8. Which pairs of graphs in this set are isomorphic?



9. Suppose each edge in the graphs in Figure 1.8 is directed from the smaller (numerically or alphabetically) end vertex to the larger end vertex. Are the two resulting directed graphs isomorphic?

10. Are the following pairs of directed graphs isomorphic?



11. Show that all 5-vertex graphs with each vertex of degree 2 are isomorphic.

12. Are there any 6-vertex graphs with three edges incident to each vertex that are not isomorphic to one of the graphs in Exercise 7?

13. What are the sizes of the largest complete subgraphs in the two graphs in Exercise 6(g)?

14. Build 6-vertex graphs with the following degrees of vertices, if possible. If not possible, explain why not.

(a) Three vertices of degree 3 and three vertices of degree 1

(b) Vertices of degrees 1, 2, 2, 3, 4, 5

(c) Vertices of degrees 2, 2, 4, 4, 4, 4

1.3 EDGE COUNTING

There is very little in the way of general assertions that can be made about all graphs. There is one useful general theorem, a formula for counting edges.

Theorem 1

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof

Summing the degrees of all vertices counts all instances of some edge being incident at some vertex. But each edge is incident with two vertices, and so the total number of such edge–vertex incidences is simply twice the number of edges. The theorem is now proved. ◆

As an illustration of Theorem 1, consider the graph in Figure 1.12 with six vertices, three of degree 4, two of degree 3, and one of degree 2. The sum of the degrees is $4+4+4+3+3+2=20$. This sum must equal twice the number of edges. The reader can check that the number of edges in this graph is 10.

For the sum of degrees to be an even integer, there must be an even number of odd integers in the sum. Thus we obtain the following

Corollary

In any graph, the number of vertices of odd degree is even.

Let us now look at uses of this theorem and corollary.

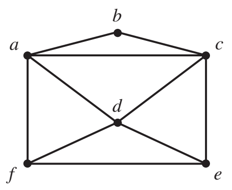


Figure 1.12

Example 1: Use of Theorem 1

Suppose we want to construct a graph with 20 edges and have every vertex of degree 4. How many vertices must the graph have?

Let $\mathbf{v}$ denote the number of vertices. The sum of the degrees of the vertices will be $4\mathbf{v}$, and by the theorem this sum must be twice the number of edges: $4\mathbf{v} = 2 \times 20 = 40$. Hence $\mathbf{v} = 10$. ■

Example 2: Edges in a Complete Graph

How many edges are there in K_n, a complete graph on n vertices?

Recall that K_n has an edge between all possible pairs of vertices. At any given vertex, there will be edges going to each of the $n - 1$ other vertices in K_n, and so each vertex has degree $n - 1$. The sum of the degrees of all n vertices in K_n will be $n(n - 1)$. Since this sum equals twice the number of edges, the number of edges is $n(n - 1)/2$. ■

Example 3: Impossible Graph

Is it possible to have a group of seven people such that each person knows exactly three other people in the group?

If we model this problem using a graph with a vertex for each person and an edge between each pair of people who know each other, then we would have a graph with seven vertices all of degree 3. But this is impossible by the Corollary—the number of vertices of odd degree must be even—and so no such set of seven people can exist. ■

Recall that a graph G is connected if every pair of vertices in G is joined by a path in G. If G is not connected, its vertices can be partitioned into connected pieces, called **components**. Formally, a component H is a connected subgraph of G such that there is no path between any vertex in H and any vertex of G not in H. The component of G containing a particular vertex x consists of x and all vertices that may be reached from x by a path in G. Because each component of G is a graph in its own right, this section's Corollary applies to each component as well as to G. We next present a puzzle that seems to have no relation to graphs.

Example 4: Mountain Climbers Puzzle

Two people start at locations A and Z at the same elevation on opposite sides of a mountain range whose summit is labeled M. See Figure 1.13*a*. We pose the following puzzle: is it possible for the people to move along the range in Figure 1.13*a* to meet at M in a fashion so that they are always at the same altitude every moment? We shall show this is possible for *any* mountain range like Figure 1.13*a*. The one assumption we make is that there is no point lower than A (or Z) and no point higher than M.

We make a *range graph* whose vertices are pairs of points (P_L, P_R) at the same altitude with P_L on the left side of the summit and P_R on the right side, such that one of the two points is a local peak or valley (the other point might also be a peak or

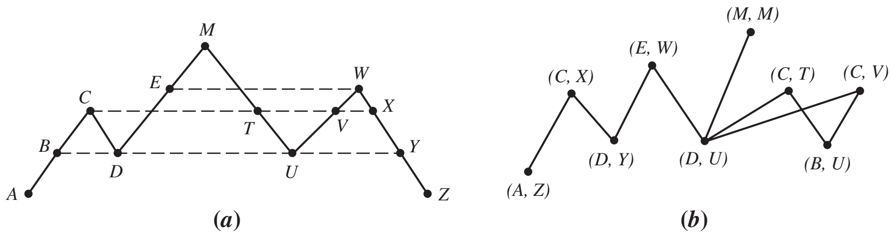


Figure 1.13

valley). The vertices for the range in Figure 1.13*a* are shown in the graph in Figure 1.13*b*. We make an edge joining vertices (P_L, P_R) and (P'_L, P'_R) if the two people can move constantly in the same direction (both going up or both going down) from point P_L to point P'_L and from P_R to P'_R, respectively. Our question now: is there a path in the range graph from the starting vertex (A, Z) to the summit vertex (M, M)? For the graph in Figure 1.13*b*, the answer is obviously yes.

We claim that vertices (A, Z) and (M, M) in any range graph have degree 1, whereas every other vertex in the range graph has degree 2 or 4. (A, Z) has degree 1 because when both people start climbing up the range from their respective sides, they have no choice initially but to climb upward until one arrives at a peak. In Figure 1.13*a*, the first peak encountered is C on the left, and so the one edge from (A, Z) goes to (C, X). A similar argument applies at (M, M). Next consider a vertex (P_L, P_R) where one point is a peak and the other point is neither peak nor valley, such as (E, W). From the peak we can go down in either direction: at W, we can go down toward Z or toward U. In either direction, the people go until one (or both) reaches a valley. At (E, W), the two edges go to (D, Y) and (D, U). Thus such a vertex has degree 2. A similar argument applies if one point (but not both) is a valley. It is left as an exercise for the reader to show that if a vertex (P_L, P_R) consists of two peaks or two valleys, such as (D, U), it will have degree 4. (A vertex consisting of a valley and a peak will have degree 0—why?)

Suppose there were no path from (A, Z) to (M, M) in the range graph. Thus, these two vertices are in different components of the range graph. We use the fact that starting vertex (A, Z) and summit vertex (M, M) are the only vertices of odd degree to obtain a contradiction. The component of the range graph consisting of (A, Z) and all the vertices that can be reached from (A, Z) would form a graph with just one vertex of odd degree, namely, (A, Z). This contradicts the Corollary, and so any range graph must have a path from (A, Z) to (M, M). ■

Many interesting properties in graph theory are dependent on certain sets of edges having even size. Euler cycles, discussed in Section 2.1, arise when all vertices have even degree.

In Example 2 of Section 1.1, we considered a matching problem involving the graph shown in Figure 1.14. The vertices on the left represented people and the vertices on the right represented jobs. An edge links a left vertex to a right vertex to indicate

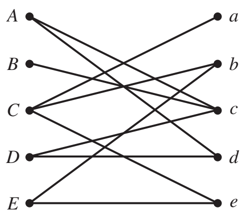


Figure 1.14

that a certain person can perform a certain job. There can never be an edge between two vertices on the left or between two vertices on the right. Such a graph is called a bipartite graph. Formally, a graph G is **bipartite** if its vertices can be partitioned into two sets V_1 and V_2 and every edge joins a vertex in V_1 with a vertex in V_2.

Bipartite graphs can be characterized by all circuits in such graphs having even length (if there are no circuits, the graph is also bipartite), where the **length** of a circuit or path is the number of edges in it.

Theorem 2

A graph G is bipartite if and only if every circuit in G has even length.

Proof

Note that it is sufficient to prove this theorem for connected bipartite graphs. We claim that if the theorem is true for each connected component of a disconnected bipartite graph G, then it is true for G (components were formally defined just above Example 4). This claim follows from G's being bipartite if and only if each of its components is bipartite, and any circuit in G's having even length if and only if any circuit in each of its components has even length.

First we show that if G is bipartite, then any circuit has an even length. If G is bipartite so that it can be drawn with all edges connecting a left vertex with a right vertex, then any circuit x_1–x_2–$x_3 \cdots$–x_n–x_1 has alternately a left vertex, then a right vertex, then a left vertex, and so on, assuming the first vertex x_1 is on the left. Odd-subscripted vertices are on the left, and even-subscripted vertices on the right. See Figure 1.15*a*. Since x_n is adjacent to x_1, x_n must be on the right, so its subscript is even. That is, there are an even number of vertices in the circuit. Any circuit has the same number of edges as vertices, and thus this circuit has even length.

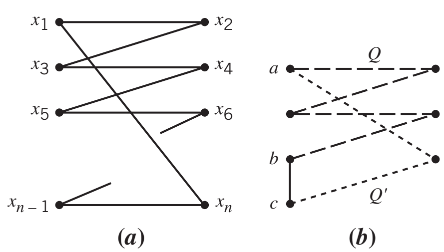


Figure 1.15

Suppose next that any circuit in G (there may be no circuits) has even length. We show how to construct a bipartite arrangement of G. We use the AC Principle and assume that a bipartition exists, and use properties of that bipartition to construct it. Take any given vertex, call it a, and put it on the left. Put all vertices adjacent to a on the right. Next put all vertices that are two edges away from a, that is, at the end of some path of length 2 from a, on the left. In general, if there is a path of odd length between a and a vertex x, put x on the right. If there is a path of even length between a and x, put x on the left.

There cannot be distinct paths P and P' between a and x of odd and of even lengths, respectively, since taking P from a to x and then returning to a on P' yields an odd-length circuit. This is impossible, since all circuits have even length. (If P and P' have a vertex q in common besides a and x, then a further argument is needed to show that there is a circuit of odd length. See Exercise 15 for details.)

Similarly, we argue that there cannot be an edge between two vertices, say, b and c, both on the left. There must exist even-length paths Q, Q' joining a with b and c, respectively (since b and c are on the left). See Figure 1.15*b*, in which Q is dashed and Q' is dotted. Observe that Q' followed by the edge (c, b) yields an odd-length path from a to b. This is impossible, since we just proved that there cannot be both an even-length path (Q) and an odd-length path (Q' plus (a, b)) from a to any other vertex in G. By similar reasoning, two vertices on the right cannot be adjacent. Thus, we have a bipartite arrangement of G. ◆

Example 5: Testing for a Bipartite Graph

Is the graph in Figure 1.16*a* bipartite?

Pick any vertex, say a, and put it on the left. We follow the approach in the second half of the proof of Theorem 2. Put vertices joined to a by an even-length path on the left and vertices joined to a by an odd-length path on the right. If all the circuits in this graph are even-length, then the reasoning in the above proof guarantees that our placement of vertices will yield a bipartite arrangement. If we end up with two vertices on the left (or on the right) being adjacent, then the graph cannot be bipartite. In this case, the construction succeeds, as shown in Figure 1.16*b*. ■

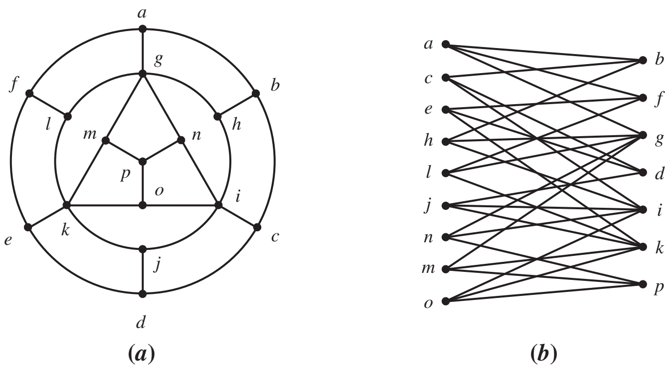
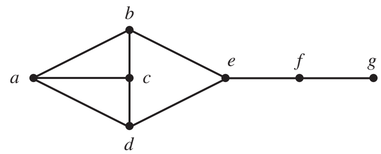


Figure 1.16

1.3 EXERCISES

1. How many vertices will the following graphs have if they contain:
   (a) 12 edges and all vertices of degree 3.
   (b) 21 edges, three vertices of degree 4, and the other vertices of degree 3.
   (c) 24 edges and all vertices of the same degree.

2. For each of the following questions, describe a graph model and then answer the question.
   (a) Must the number of people at a party who do not know an odd number of other people be even?
   (b) Must the number of people ever born who had (have) an odd number of brothers and sisters be even?
   (c) Must the number of families in Alaska with an odd number of children be even?
   (d) For each vertex x in the following graph, let $s(x)$ denote the number of vertices (including x) adjacent to at least one of x's neighbors. Must the number of vertices with $s(x)$ odd be even? Is this true in general?



3. What is the largest possible number of vertices in a graph with 19 edges and all vertices of degree at least 3?

4. Is any subgraph of a bipartite always bipartite? Prove, or give a counterexample.

5. What constraint must be placed on a bipartite graph G to guarantee that G's complement will also be bipartite?

6. If a graph G has n vertices, all of which but one have odd degree, how many vertices of odd degree are there in $\overline{G}$, the complement of G?

7. Show that a complete graph with m edges has $(1 + 8m)/2$ vertices.

8. Let G be an n-vertex graph that is isomorphic to its complement $\overline{G}$. How many edges does G have? (Hint: Use Exercise 5.)

9. Suppose all vertices of a graph G have degree p, where p is an odd number. Show that the number of edges in G is a multiple of p.

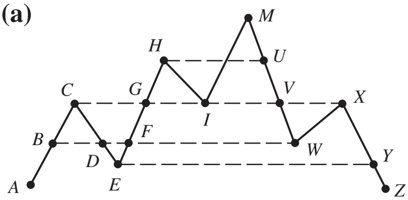
10. There used to be 26 football teams in the National Football League (NFL) with 13 teams in each of two conferences (each conference was divided into divisions, but that is irrelevant here). An NFL guideline said that each team's 14-game schedule should include exactly 11 games against teams in its own conference and three games against teams in the other conference. By considering the right

part of a graph model of this scheduling problem, show that this guideline could not be satisfied!

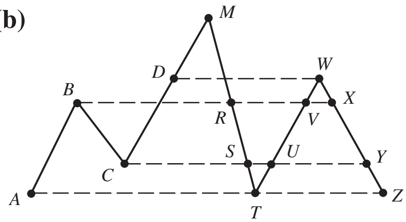
11. Prove a directed version of Theorem 1: The sum of the in-degrees of vertices in a directed graph equals the sum of the out-degrees of vertices, and further, each sum equals the number of edges.

12. Build the range graph for each of the following mountain ranges and use the graph to find a solution to the problem in Example 4.

(a)



(b)



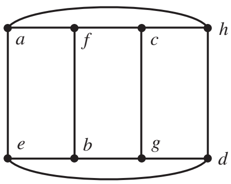
13. Prove in a range graph that if a vertex (P_L, P_R) consists of two peaks or two valleys, it will have degree 4.

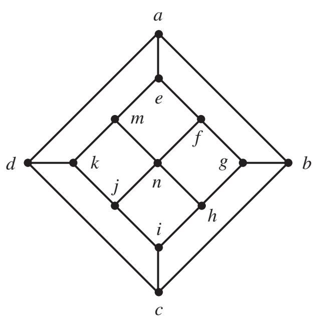
14. Prove in a range graph that if a vertex (P_L, P_R) consists of a valley and a peak, it will have degree 0.

15. Determine whether the following graphs are bipartite. If so, give the partition into left and right vertices as in Figure 1.16*b*.

(a) Figure 1.4 **(b)** Figure 1.7 (left graph) **(c)** Figure 1.12

16. Determine whether the following graphs are bipartite. If so, give the partition into left and right vertices as in Figure 1.16*b*.





17. Suppose x and y are the only two vertices of odd degree in graph G, and x and y are not adjacent to each other. Show that G is connected if and only if the graph obtained from G by adding edge (x, y) is connected.

18. In the second part of the proof of Theorem 2, one can encounter the situation in which there exist paths P and P' between a and x, P of odd length and P' of even length, and these two paths have one or more vertices in common. One must show that a subset of the edges on these two paths forms an odd-length circuit. Let q be the first vertex on P, starting from a, that also lies on P'. Show that either the circuit from a along P to q and then back on P' to a, or the edge sequence from q along P to x and then back on P' to q, has odd length. In the latter case, if the edge sequence is not a circuit, then it has a vertex q' on both P and P'. Repeat the same reasoning considering the circuit on P from q to q' and then back on P' to q or the edge sequence from q' along P to x and back along P' to q'.

1.4 PLANAR GRAPHS

The most natural examples of graphs are street maps and telephone networks. The graphs that arise from such physical networks usually have the property that they can be depicted on a piece of paper without edges crossing (different edges meet only at vertices). We say that a graph is **planar** if it can be drawn on a plane without edges crossing. We use the term **plane graph** to refer to a planar depiction of a planar graph. The two graphs in Figure 1.17 are both planar. The graph in Figure 1.17*b* is a plane graph. The graph in Figure 1.17*a* is planar, since it can be redrawn in the form of the graph in Figure 1.17*b*.

Our principal focus in this section is determining whether a graph is planar. We take two approaches, both based on the AC Principle. The first approach involves a systematic method for trying to draw a graph edge-by-edge with no crossing edges, in the same spirit as when we tried to determine if two graphs are isomorphic. The second approach develops some theory with a goal of finding useful properties of planar graphs. If a graph does not satisfy one or more of these properties, then we know that it cannot be planar.

Remember that if a graph G has been drawn with edges crossing, this does not mean the graph is nonplanar. There may be another way to draw the graph without edges crossing, as illustrated by the graph in Figure 1.17*a*, which can be redrawn to be the plane graph in Figure 1.17*b*.

Probably the most important need today for testing whether a graph is planar arises in designing electronic circuits. Complex integrated circuits are nonplanar and require several layers of (planar) circuit connections in their wiring. But the number of layers is limited and so a major problem in integrated circuit design is decomposing a large circuit into a minimal number of subcircuits that are known to be planar.

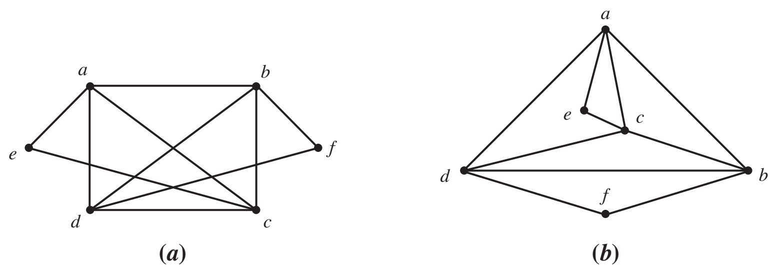


Figure 1.17

A more mundane, but still important, use of planarity testing arises in checking data-entry errors in planar networks. When a large planar graph such as a city's street network is entered on data terminals for computerized analysis, it is a common error-checking technique to test first whether the graph as typed in is indeed planar (most data-entry errors would make the graph nonplanar).

Planar graphs were first studied extensively by mathematicians over 100 years ago in connection with a map-coloring problem.

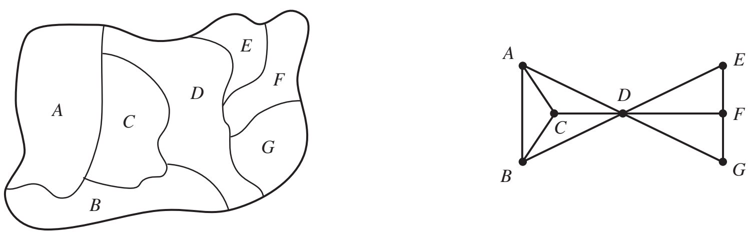
Example 1: Map Coloring

One of the most famous problems in mathematics concerns map coloring. The question is how many colors are needed to color countries on some map so that any pair of countries with a common border are given different colors. A map of countries is a planar graph with edges as borders and vertices where borders meet. See Figure 1.18*a*. However, a closely related planar graph called **a dual graph** of the map graph is more useful. The dual graph is obtained by making a vertex for each country and an edge between vertices corresponding to two countries with a common border. See Figure 1.18*b*. Normally, a vertex is also included for the unbounded region surrounding the map.

The question in the dual graph now is how many colors are needed to "color" the vertices such that adjacent vertices have different colors. In Figure 1.18*b*, vertices A, B, C, D form a complete subgraph on four vertices and so each requires a different color, four colors in all. With four colors, we can also properly color the remaining vertices.

One of the most famous unsolved problems in all of mathematics during the last century was the conjecture that *all* planar graphs can be properly colored with only four colors. In trying to resolve this conjecture, mathematicians developed a large theory about planar graphs. In 1976 the four-color conjecture was proven true by Appel and Haken using an immense computer-generated, case-by-case, exhaustive analysis (there were 1955 classes of graphical configurations to be considered, each involving numerous subcases). We will take a closer look at graph coloring in the next chapter. ■

We now use the AC Principle to try to find a planar drawing of a graph. We assume it to be planar. As with isomorphism between two graphs, we want to be



(*a*) (*b*)

Figure 1.18

able to conclude that a graph is not planar if our construction fails. We shall call our approach the **circle–chord method.** It starts by finding a circuit that contains all the vertices of our graph (though such circuits do not exist for all graphs, they are common in the types of graphs we will be considering in this section). We draw this circuit as a large circle. The remaining noncircuit edges, which we will call *chords,* must be drawn either inside the circle or outside the circle in a planar drawing.

We choose a first chord and draw it, say, outside the circle. If properly chosen, this chord will force certain other chords to be drawn inside the circle (if also placed outside the circle, they would have to cross the first chord). These inside chords will force still other chords to be drawn outside, and so on. After the first chord is drawn, the choice of placing subsequent chords inside or outside is forced. Thus, if we reach a point where a new chord will have to cross some previous chord, whether the new chord is drawn inside or outside, we can claim that the graph must be nonplanar. That is, our construction based on the assumption that the graph had a planar depiction led to a contradiction. If all the chords can be added without crossing other chords, then the graph is planar.

A critical decision is whether the first chord drawn should go inside or outside the circle. We claim that it makes no difference, because of the following inside–outside symmetry of a circle. Consider two maps of the earth, the first with the North Pole at the center of the map, the second with the South Pole at the center. Suppose each map has a path drawn in the Northern Hemisphere linking two cities on the equator. In the first map (with the Northern Hemisphere inside the equator) the path is inside the equator's circle, whereas in the second map the path is outside the equator. Think of the circle formed by the circuit in the circle–chord method as the equator and the first chord as the path between two cities. Whether the chord is inside or outside the circle is just a matter of which "map" of the earth one uses.

We note that very efficient algorithms exist to test whether a graph is planar, whereas there is no efficient algorithm known to test whether two graphs are isomorphic. The planarity testing algorithms are fairly complicated and beyond the scope of this text.

Example 2: Circle–Chord Method

Use the circle–chord method to determine whether the graph in Figure 1.19*a* is planar. Let us look for a circuit with all eight vertices. One possibility is *a-f-c-h-d-g-b-e-a*.

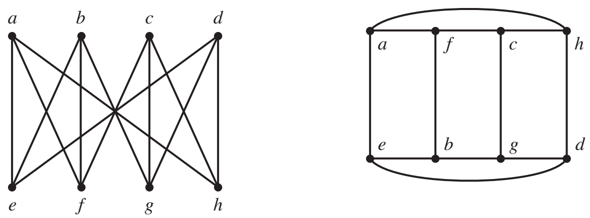
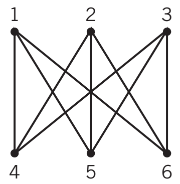


Figure 1.19 **(*a*)** **(*b*)**



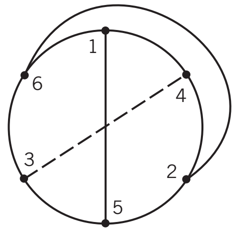


Figure 1.20 **(*a*)** **(*b*)**

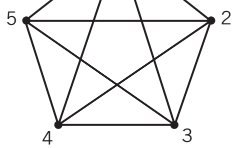
Now try to add the other four edges (*a, h*), (*b, f*), (*c, g*), (*d, e*). By inside–outside symmetry, we can start by drawing (*a, h*) outside. See Figure 1.19*b*. Then (*b, f*) and (*c, g*) must go inside. Then (*d, e*) must go outside. So the graph is planar, as shown in Figure 1.19*b*. ■

Example 3: Showing $K_{3,3}$ Is Nonplanar

Show that $K_{3,3}$, the graph in Figure 1.20*a*, is nonplanar. The notation $K_{3,3}$ indicates that this graph is a complete bipartite graph consisting of two sets of three vertices with each vertex in one set adjacent to all vertices in the other set. Applying the circle–chord method, we form a circuit containing all six vertices in $K_{3,3}$, and then try to add the remaining edges (not in the circuit) as inside and outside chords.

There are several choices for a 6-vertex circuit. Suppose we use the circuit *1–4–2–5–3–6–1* and draw it in a circle as shown in Figure 1.20*b*. Next the edges (*1*, *5*), (*2*, *6*), and (*3*, *4*) must be added. First draw chord (*1*, *5*). By the inside–outside symmetry of a circle discussed above, we can assume that (*1*, *5*) is drawn inside the circuit, as in Figure 1.20*b*. Then (*2*, *6*) must be drawn outside the circuit to avoid crossing chord (*1*, *5*). Finally, we must draw (*3*, *4*): if drawn outside the circuit, (*3*, *4*) would have to cross chord (*2*, *6*); if drawn inside the circuit, (*3*, *4*) would have to cross chord (*1*, *5*). Thus $K_{3,3}$ cannot be drawn in a planar depiction. Hence $K_{3,3}$ is nonplanar. ■

Using a mixture of theory and careful, case-by-case analysis, it is possible to prove that any nonplanar graph always contains a $K_{3,3}$ or a K_5 (the complete graph on five vertices shown in Figure 1.21*a*) as a subgraph or a slight modification of these two graphs. It is left as an exercise to show that the circle–chord method (used in



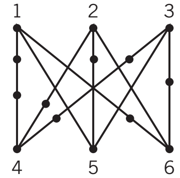


Figure 1.21 **(*a*)** **(*b*)**

Example 3) can be used to show that K_5 is nonplanar. Thus, these two graphs are the "reason" that a graph cannot be drawn in a planar fashion.

We have to allow a slight variation in $K_{3,3}$ and K_5 in nonplanarity analysis. Figure 1.21*b* shows a $K_{3,3}$, graph that has been **subdivided** by adding vertices in the middle of some of its edges. The resulting graph is no longer a $K_{3,3}$ and does not contain $K_{3,3}$ as a subgraph, yet it is still nonplanar (repeatedly adding a vertex in the middle of an edge cannot make a nonplanar graph planar). We say that a subgraph is a $\boldsymbol{K_{3,3}}$ **configuration** if it can be obtained from a $K_{3,3}$ by adding vertices in the middle of some edges. A $\boldsymbol{K_5}$ **configuration** is defined similarly. The following planar graph characterization theorem was first proved by the Polish mathematician Kuratowski.

Theorem 1 **(Kuratowski, 1930)**

A graph is planar if and only if it does not contain a subgraph that is a K_5 or $K_{3,3}$ configuration.

If the circle–chord method shows that a graph is nonplanar, then by Theorem 1 this graph has a subgraph that is a K_5 or $K_{3,3}$ configuration. Finding such a configuration can sometimes be tricky. However, the following observation is helpful: *Most small nonplanar graphs contain a* $K_{3,3}$ *configuration.* All but one of the nonplanar graphs in the exercises have $K_{3,3}$ configurations. Note also that the depiction of a $K_{3,3}$ in Figure 1.20*b* as a 6-vertex circle with three chords joining pairs of opposite vertices is the way that a $K_{3,3}$ configuration normally arises in a nonplanar graph.

Example 4: Finding a $\boldsymbol{K_{3,3}}$

Use the circle–chord method to determine whether the graph in Figure 1.22*a* is planar. If it is nonplanar, find a subgraph that is a $K_{3,3}$ configuration.

First we seek a circuit that visits all vertices. Many such circuits exist. Choose the circuit *a-b-c-d-e-f-g-h-a*, shown in Figure 1.22*b*. Say we pick (a, d) as our first chord to draw. By inside–outside symmetry, it makes no difference whether we draw (a, d) inside or outside the circuit. Put it inside. Then (b, e) must go outside to avoid intersecting (a, d). Next look for another chord reaching across the circle. Observe

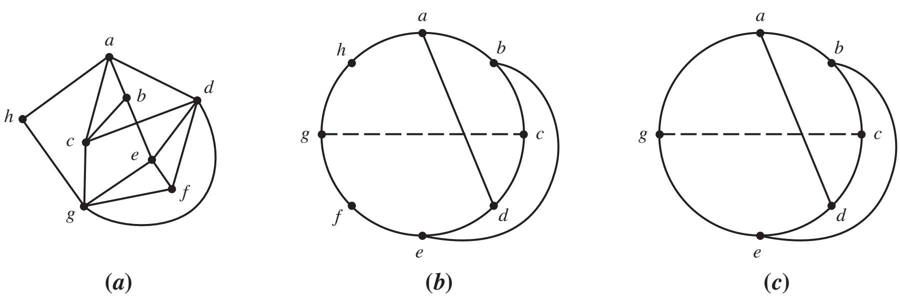


Figure 1.22

that chord (c, g) cannot be drawn inside without crossing (a, d), nor drawn outside without crossing (b, e). So the graph is nonplanar.

Next we look for a $K_{3,3}$ configuration. We can simplify the problem by restricting our attention to the nonplanar subgraph in Figure 1.22*b*, whose crossing edges proved that the full graph had to be nonplanar. A $K_{3,3}$ configuration has six vertices of degree 3 (corresponding to the vertices of a $K_{3,3}$) plus some number of vertices of degree 2 that subdivide the edges of a $K_{3,3}$. The way to find a $K_{3,3}$ configuration in a subgraph is to eliminate edge subdivisions in the graph (remove each vertex of degree 2 and combine the two edges at each such vertex into a single edge). Figure 1.22*c* shows the subgraph in Figure 1.22*b* with subdivisions removed. The graph in Figure 1.22*c* looks just like the depiction of a $K_{3,3}$ in Figure 1.20*b*. Thus, the subgraph in Figure 1.22*b* was a $K_{3,3}$ configuration. ■

Finding a $K_{3,3}$ configuration in the graph in Figure 1.22*a* (without using the subgraph in Figure 1.22*b*) would be difficult. The challenging problem in finding a $K_{3,3}$ configuration in a general nonplanar graph is the following. Let z be some vertex of degree 3 in the original graph and suppose that just two of the z's edges, say (z, r) and (z, q), are part of a $K_{3,3}$ configuration. Then z corresponds to a subdivision vertex in this $K_{3,3}$ configuration, and these two edges of z need to be fused into a single edge (r, q) to find the underlying $K_{3,3}$. That is, z disappears and a new edge (r, q) is created. Using the subgraph produced by the circle–chord method makes it much easier to identify vertices of degree 2 in a $K_{3,3}$ configuration whose two edges should be fused together.

There are many different plane graph depictions that can be drawn for a planar graph. For example, we can redraw the plane graph in Figure 1.23*a* by making the region bounded by the triangle (d, e, f) very large and bringing vertex a to the right side, as in Figure 1.23*b*. Now flip the part of the graph above and to the right of the triangle inside the triangle, obtaining the plane graph in Figure 1.23*c*. The triangle (d, e, f) has become the outside boundary of the whole graph. The boundary of any region can be converted to the outside boundary of the whole graph by a similar process.

Despite this variability in plane graph depictions of a planar graph, one important property of the plane depictions does not change. The number of regions is always

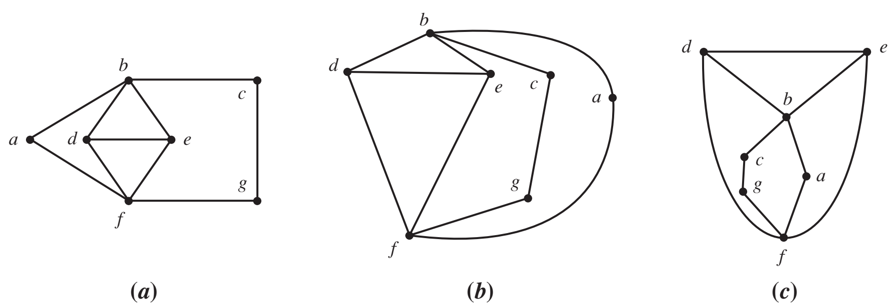


Figure 1.23

the same. For simplicity, assume that G is a connected planar graph. (Recall that *connected* means having paths between every pair of vertices.) If $\mathbf{v}$ and $\mathbf{e}$ denote the number of vertices and edges, respectively, in G, then a plane graph depiction of G will always have a number of regions $\mathbf{r}$ given by the formula, $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$. Remember that the unbounded region outside the graph is counted as a region. This remarkable formula for $\mathbf{r}$ was discovered by Euler 360 years ago.

Theorem 2 **Euler's Formula (1752)**

If G is a connected planar graph, then any plane graph depiction of G has $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$ regions.

Proof

Let us draw a plane graph depiction of G edge by edge. We choose successive edges so that at every stage we have a connected subgraph. Let G_n denote the connected plane graph obtained after n edges have been added, and let $\mathbf{v}_n$, $\mathbf{e}_n$, and $\mathbf{r}_n$ denote the number of vertices, edges, and regions in G_n, respectively. Initially we have G_1, which consists of one edge, its two end vertices, and the one (unbounded) region. Then $\mathbf{e}_1 = 1$, $\mathbf{v}_1 = 2$, $\mathbf{r}_1 = 1$, and so Euler's formula is valid for G_1, since $\mathbf{r}_1 = \mathbf{e}_1 - \mathbf{v}_1 + 2$: $1 = 1 - 2 + 2$. We obtain G_2 from G_1 by adding an edge at one of the vertices in G_1. In general, G_n is obtained from G_{n-1} by adding an nth edge at one of the vertices of G_{n-1}. The new edge might link two vertices already in G_{n-1}. If it does not, the other end vertex of the nth edge is a new vertex that must be added to G_n.

We will now use the method of induction (see Appendix A.2) to complete the proof. We have shown that the theorem is true for G_1. Next we assume that it is true for G_{n-1} for any $n > 1$, and prove that it is true for G_n. Let (x, y) be the nth edge that is added to G_{n-1} to get G_n. There are two cases to consider.

In the first case, x and y are both in G_{n-1}. Then they are on the boundary of a common region K of G_{n-1}, possibly the unbounded region [if x and y were not on a common region, edge (x, y) could not be drawn in a planar fashion, as required]. See Figure 1.24*a*. Edge (x, y) splits K into two regions. Then $\mathbf{r}_n = \mathbf{r}_{n-1} + 1$, $\mathbf{e}_n = \mathbf{e}_{n-1} + 1$, $\mathbf{v}_n = \mathbf{v}_{n-1}$. So each side of Euler's formula grows by 1. Hence, if the formula was true for G_{n-1}, it will also be true for G_n.

In the second case, one of the vertices x, y is not in G_{n-1}—say it is x. See Figure 1.24*b*. Then adding (x, y) implies that x is also added, but no new regions are formed (i.e., no existing regions are split). Thus $\mathbf{r}_n = \mathbf{r}_{n-1}$, $\mathbf{e}_n = \mathbf{e}_{n-1} + 1$, $\mathbf{v}_n = \mathbf{v}_{n-1} + 1$, and

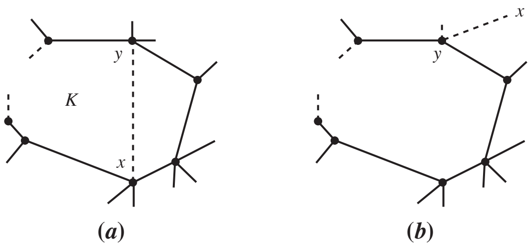


Figure 1.24

the value on each side of Euler's formula is unchanged. The validity of Euler's formula for G_{n-1} implies its validity for G_n.

So each increase in **r** is balanced in Euler's formula by an increase in **e** or **v**. By induction, the formula is true for all G_n's and hence true for the full graph G. ◆

Example 5: Using Euler's Formula

How many regions would there be in a plane graph with 10 vertices each of degree 3?

By Theorem 1 in Section 1.3, the sum of the degrees, 10×3, equals $2\mathbf{e}$, and so $\mathbf{e} = 15$. By Euler's formula, the number of regions **r** is

$$\mathbf{r} = \mathbf{e} - \mathbf{v} + 2 = 15 - 10 + 2 = 7 \quad ■$$

Theorem 2 has the following corollary that can often be used to show quickly that a graph is nonplanar.

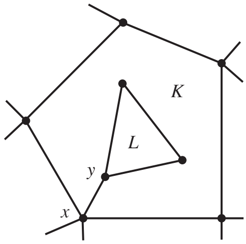
Corollary

If G is a connected planar graph with $\mathbf{e} > 1$, then $\mathbf{e} \leq 3\mathbf{v} - 6$.

Proof

Let us define the *degree of a region* analogously to the degree of a vertex to be the number of edges incident to a region—that is, the number of edges on its boundary. If an edge occurs twice along a boundary, as does (x, y) in region K in Figure 1.25*a*, the edge is counted twice in region K's degree; for example, region K has degree 10 and region L has degree 3 in Figure 1.25*a*. Observe that each region in a plane graph must have degree ≥ 3, for a region of degree 2 would be bounded by two edges joining the same pair of vertices and a region of degree 1 would be bounded by a loop edge (see Figure 1.25*b*), but parallel edges and loops are not allowed in graphs.

Since each region in a plane graph has degree ≥ 3, the sum of the degrees of all regions will be at least $3\mathbf{r}$. But this sum of degrees of all regions must equal $2\mathbf{e}$, since this sum counts each edge twice, that is, each of an edge's two sides is part of some boundary (this is the same type of argument as used to show that the sum of the



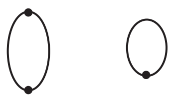


Figure 1.25 **(*a*)** **(*b*)**

vertices' degrees equals $2\mathbf{e}$). Thus, $2\mathbf{e} =$ (sum of regions' degrees) $\geq 3\mathbf{r}$, or $\frac{2}{3}\mathbf{e} \geq \mathbf{r}$. Combining this inequality with Euler's formula (Theorem 2), we have

$$\tfrac{2}{3}\mathbf{e} \geq \mathbf{r} = \mathbf{e} - \mathbf{v} + 2 \qquad \text{or} \qquad 0 \geq \tfrac{1}{3}\mathbf{e} - \mathbf{v} + 2$$

Solving for $\mathbf{e}$, we obtain $\mathbf{e} \leq 3\,\mathbf{v} - 6$. ◆

Example 6: Showing That K_5 Is Nonplanar

Use the Corollary to prove that K_5, the complete graph on 5 vertices, is nonplanar.

The graph K_5 has $\mathbf{v} = 5$ and $\mathbf{e} = 10$ (see Example 2 in Section 1.3 for how to find the number of edges in a complete graph). Then $3\mathbf{v} - 6 = 3 \times 5 - 6 = 9$. But the Corollary says that $\mathbf{e} \leq 3\mathbf{v} - 6$ must be true in a connected planar graph, and so assuming K_5 is a planar graph leads to a contradiction—namely, that the Corollary is not true. So K_5 cannot be planar. ■

The Corollary *should not be misinterpreted to mean* that if $\mathbf{e} \leq 3\mathbf{v} - 6$, then a connected graph is planar. Many nonplanar graphs also satisfy this inequality. For example, $K_{3,3}$ with $\mathbf{v} = 6$ and $\mathbf{e} = 9$ satisfies it.

Our two theorems and corollary have laid the foundation for a mathematical theory of planar graphs. In the process, we have acquired a practical aid for showing that a graph is nonplanar. One way to extend this theory is to make the inequality in the Corollary "stronger"—that is, to get a smaller upper bound on $\mathbf{e}$. Recall that the key step in proving the Corollary was the observation that every region has degree at least 3. This led to the inequality $2\mathbf{e} \geq 3\mathbf{r}$. Suppose that a certain connected graph G (with at least two edges) is known to be bipartite. By Theorem 2 in Section 1.3, all the circuits in a bipartite graph have even length. Then no region in this graph can have degree 3 (since this would imply a boundary circuit of length 3). Then every region in a bipartite planar graph must have degree ≥ 4. Summing the degrees of all regions, we now obtain the inequality $2\mathbf{e} =$ (sum of degrees of regions) $\geq 4\mathbf{r}$. Reworking the Corollary with the inequality $2\mathbf{e} \geq 4\mathbf{r}$, we have $\frac{2}{4}\mathbf{e} \geq \mathbf{r} = \mathbf{e} - \mathbf{v} + 2$, and hence

$$\mathbf{e} \leq 2\mathbf{v} - 4$$

Every connected planar graph that is bipartite must satisfy this inequality. Consider our "favorite" bipartite graph $K_{3,3}$. $K_{3,3}$ has $\mathbf{v} = 6$ and $\mathbf{e} = 9$ and, as noted above, satisfies the corollary inequality $\mathbf{e} \leq 3\mathbf{v} - 6$. But since it is bipartite, $K_{3,3}$ would also have to satisfy the new inequality $\mathbf{e} \leq 2\mathbf{v} - 4$ if it were planar. It does not: $9 \not\leq 2 \times 6 - 4$.

1.4 EXERCISES

Summary of Exercises The first six exercises involve determining whether various graphs are planar and drawing planar graphs in different ways. Exercise 10 involves duality. Exercises 15–26 build on Euler's formula and the corollary $\mathbf{e} \leq 3\mathbf{v} - 6$. The other exercises introduce new concepts.

1. Draw a dual graph of the planar graph (include a vertex for the unbounded region outside the graph) in

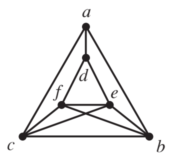
(a) Figure 1.17*b*

(b) Figure 1.18*b*

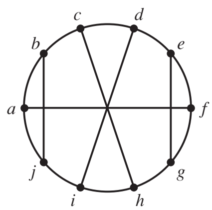
2. Show that K_5 is nonplanar by the method in Example 2.

3. Which of the following graphs are planar? Find $K_{3,3}$ or K_5 configurations in the nonplanar graphs (almost all are $K_{3,3}$).

(a)



(b)



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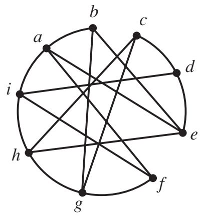

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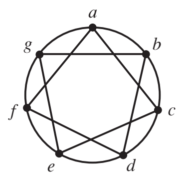
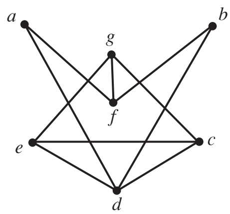


(e)



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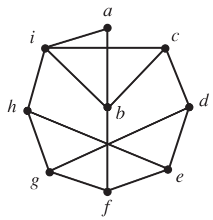
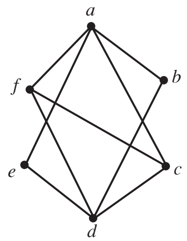
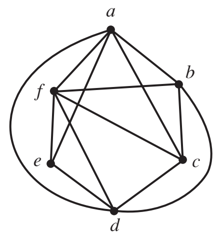


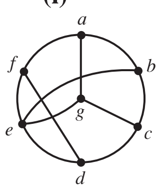
(g)



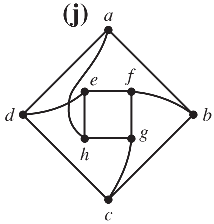
(h)



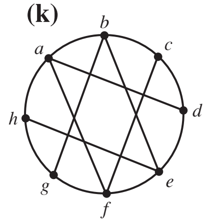
(i)



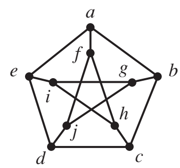
(j)



(k)



(l)



4. Redraw the graph in Figure 1.19*b* so that the infinite region is bounded by the circuit *a-f-c-h-a*.

5. **(a)** For what values of n is K_n planar?

(b) For what values of r and s is the complete bipartite graph $K_{r,s}$ planar? ($K_{r,s}$ is a bipartite graph with r vertices on the left side and s vertices on the right side and edges between all pairs of left and right vertices.)

6. A complete tripartite $K_{r,s,t}$ is a generalization of a complete bipartite graph (see part (b) of the previous exercise). There are three subsets of vertices: r in the first subset, s in the second subset, and t in the third subset. Every vertex in one particular subset is adjacent to every vertex in the other two subsets; that is, a vertex is adjacent to all vertices except those in its own subset. Determine all the triples r, s, t for which $K_{r,s,t}$ is planar.

7. In each case, give the values of **r, e,** or **v** (whichever is not given) assuming that the graph is planar. Then either draw a connected, planar graph with the property, if possible, or explain why no such planar graph can exist.

(a) Six vertices and seven regions

(b) Eight vertices and 13 edges

(c) Six vertices and 14 edges

(d) 14 edges and nine regions

(e) Six vertices all of degree 4

(f) Five regions and 10 edges

(g) Six regions all with four boundary edges

(h) Seven vertices all of degree 3

(i) 12 vertices and every region has four boundary edges

(j) 17 regions and every vertex has degree 5

8. If a connected planar graph with n vertices all of degree 4 has 10 regions, determine n.

9. A *line graph* $L(G)$ of a graph G has a vertex of $L(G)$ for each edge of G and an edge of $L(G)$ joining each pair of vertices corresponding to two edges in G with a common end vertex.

(a) Show that $L(K_5)$ is nonplanar.

(b) Find a planar graph whose line graph is nonplanar.

10. The construction of a dual $D(G)$ can be applied to any plane graph G: draw a vertex of $D(G)$ in the middle of each region of G and draw an edge e^* of $D(G)$ perpendicular to each edge e of G; e^* connects the vertices of $D(G)$ representing the regions on either side of e.

(a) A dual need not be a graph. It might have two edges between the same pair of vertices or a self-loop edge (from a vertex to itself). Find two planar graphs with duals that are not graphs because they contain these two forbidden situations.

(b) Show that the duals of the two different plane depictions of the graph in Figures 1.23*a* and 1.23*c* are isomorphic.

(c) Show that the degree of a vertex in the dual graph $D(G)$ equals the number of boundary edges of the corresponding region in the planar graph G.

(d) Find a planar graph that is isomorphic to its own dual.

(e) Show for any plane depiction of a graph G that the vertices of G correspond to regions in $D(G)$.

11. (a) Show that if a circuit in a planar graph encloses exactly two regions, each of which has an even number of boundary edges, then the circuit has even length.

(b) Show that if a circuit in a planar graph encloses a collection of regions, each of which has an even number of boundary edges, then the circuit has even length.

12. The *crossing number* $c(G)$ of a graph G is the minimum number of pairs of crossing edges in a depiction of G. For example, if G is planar, then $c(G) = 0$. Determine $c(G)$ for the following graphs:

(a) $K_{3,3}$ **(b)** K_5 **(c)** Figure 1.22*a*

13. A graph G is *critical nonplanar* if G is nonplanar but any subgraph obtained by removing a vertex is planar.

(a) Which of the following graphs are critical nonplanar?

(i) $K_{3,3}$ **(ii)** K_5 **(iii)** Figure 1.22*a*

(b) Show that critical nonplanar graphs must be connected and cannot have a vertex whose removal disconnects the graph.

14. A graph G is called *maximal planar* if G is planar but the addition of another edge between two nonadjacent vertices will make the graph nonplanar.

(a) Show that every region of a connected, maximal planar graph will be triangular.

(b) If a connected, maximal planar graph has n vertices, how many regions and edges does it have?

15. Suppose G is a planar graph that is not necessarily connected, as required in Euler's formula (Theorem 2). Recall that a *component* H is a connected subgraph of G with the property that there is no path between any vertex in H and any vertex of G not in H.

(a) Find the appropriate modification of Euler's formula for a planar graph with $\mathbf{c}$ components.

(b) Show that the corollary is valid for unconnected planar graphs.

16. Prove that if a graph G has 11 vertices, then either G or its complement $\overline{G}$ must be nonplanar. (*Hint*: Determine the total number N_{11} of edges in a complete graph on 11 vertices; if the result were false and G and its complement were each planar, how many of the N_{11} edges could be in each of these two graphs?)

17. Mimic the argument in the corollary to prove that $\mathbf{e} \leq 3\mathbf{r} - 6$ in a planar graph with each vertex of degree ≥ 3.

18. **(a)** Prove that every connected planar graph has a vertex of degree at most 5. [*Hint:* Assume that every vertex has degree at least 6 and obtain a contradiction as follows. Get a lower bound, involving $\mathbf{v}$, on the sum of the degrees of vertices (similar to the lower bound on the sum of the degrees of regions obtained in the proof of the corollary). Since this sum of degrees equals $2\mathbf{e}$, the corollary's bound on $\mathbf{e}$ can be used to get an upper bound, also involving $\mathbf{v}$, on this sum of degrees. Combining these two bounds yields the desired contradiction.]

(b) Show that part (a) immediately generalizes to any (unconnected) planar graph.

19. Prove that every connected planar graph with less than 12 vertices has a vertex of degree at most 4. [*Hint:* Assume that every vertex has degree at least 5 to obtain a lower bound on $\mathbf{e}$ (together with the upper bound on $\mathbf{e}$ in the corollary) that implies $\mathbf{v} \geq 12$.]

20. If G is a connected planar graph with all circuits of length at least k, show that the inequality $\mathbf{e} \leq 3\mathbf{v} - 6$ can be strengthened to $\mathbf{e} \leq \frac{k}{k-2}(\mathbf{v} - 2)$. (*Hint:* The degree of a region will be at least k.)

21. **(a)** Show that every circuit in the graph in Exercise 3(l) has at least five edges.

(b) Use part (a) and the result of Exercise 20 to show that this graph is nonplanar.

22. **(a)** Give an example of a graph with regions consisting solely of squares (regions bounded by four edges) and hexagons, and with vertices of degree at least 3.

(b) Write an expression for the sum of the degrees of the vertices (=2$\mathbf{e}$) in such a graph in terms of $\mathbf{v}$ and $\mathbf{s}$, the number of squares. Then use Exercise 17 to get an upper bound on 2$\mathbf{e}$. Deduce that any graph of the sort defined in part (a) has at least six squares.

(c) If each vertex has degree 3, show that any graph of the sort defined in part (a) has exactly six squares.

23. If G is a connected planar graph where $\mathbf{e} = 3\mathbf{v} - 6$, show that every region is triangular (has three boundary edges).

24. A *Platonic graph* is a planar graph in which all vertices have the same degree d_1 and all regions have the same number of bounding edges d_2, where $d_1 \geq 3$ and $d_2 \geq 3$. A Platonic graph is the "skeleton" of a Platonic solid, for example, an octahedron.

(a) If G is a Platonic graph with vertex and face degrees d_1 and d_2, respectively, then show that $\mathbf{e} = \frac{1}{2}d_1\mathbf{v}$ and $\mathbf{r} = (d_1/d_2)\mathbf{v}$.

(b) Using part (a) and Euler's formula, show that $\mathbf{v}(2d_1 + 2d_2 - d_1d_2) = 4d_2$.

(c) Since $\mathbf{v}$ and $4d_2$ are positive integers, we conclude from part (b) that $2d_1 + 2d_2 - d_1d_2 > 0$. Use this inequality to prove that $(d_1 - 2)(d_2 - 2) < 4$.

(d) From part (c), find the five possible pairs of positive (integral) values of d_1, d_2.

25. Suppose that l lines are drawn through a circle and these lines form p points of intersection (involving exactly two lines at each intersection). How many regions r are formed inside the circle by these lines? Assume that the lines end at the edge of the circle at $2l$ distinct points.

26. Consider an overlapping set of four circles A, B, C, D. One would like to position the circles so that every possible subset of the circles forms a region, e.g., four regions each contained in just one (different) circle, six regions formed by the intersection of two circles (AB, AC, AD, BC, BD, CD), four regions formed by the intersection of three of the four circles, and one region formed by the intersection of all four circles. Prove that it is not possible to have such a set of 15 bounded regions.

27. Show that the following graphs can be drawn on the surface of a doughnut (torus) without crossing edges:

(a) $K_{3,3}$ **(b)** K_5 **(c)** K_6

1.5 SUMMARY AND REFERENCES

This chapter introduced graphs, their applications, and some of their basic structures. This text takes a user-oriented approach to graph theory. Readers interested in a more formal graph theory text presenting the subject as an interesting area of pure mathematics should see the books by Bondy and Murty [1], West [3], or Wilson [4]. Section 1.1 introduced a set of illustrative graph models. The basic structure of graphs was explored in Section 1.2 under the guise of determining what makes two graphs different. Section 1.3 presented some useful edge-counting results. The final section introduced the important class of planar graphs. It surveyed ad hoc and theoretical approaches for determining whether a graph is planar.

The history of graph theory begins with the work of L. Euler in 1736 on Euler cycles (discussed in Section 2.1). Euler's formula for planar graphs, originally stated in terms of polyhedra, was proved in 1752. Bits of graph theory appeared in papers about topology and geometric games, but it was not until around 1850 that formal studies of graphs began to appear. One was A. Cayley's 1857 paper counting the number of trees (discussed in Chapter 3). Another was G. Kirchhoff's 1847 paper presenting an algebra of circuits and introducing graphs in the study of electrical circuits. This same paper contains Kirchhoff's famous current and voltage laws (Kirchhoff was 21 when he wrote this historic paper). The term graph was first used by J. Sylvester in 1877. The first book on graph theory, by D. Konig, did not appear until 1936. An excellent sourcebook on the history of graph theory is *Graph Theory 1736–1936* by Biggs, Lloyd, and Wilson [2].

See the General References (at the end of the book) for a list of other introductory texts on graph theory.

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* American Elsevier, New York, 1976.

2. N. Biggs, E. Lloyd, and R. Wilson, *Graph Theory 1736–1936,* Cambridge University Press, Cambridge, 1999.

3. D. West, *Introduction to Graph Theory,* 2nd ed., Prentice-Hall, Saddle River, N. J., 2001

4. R. Wilson, *Introduction to Graph Theory,* 4th ed., John Wiley and Sons, New York, 1997.

SUPPLEMENTARY EXERCISES

Summary of Exercises Graph theory is a field famous for its interesting problems. Several exercises introduce new graph concepts, such as strong connectedness (Exercise 14) and cut-set (Exercise 25). Exercise 30 is a very famous problem in Ramsey theory (see Appendix A.4). For more problems in graph theory, see any of the graph theory texts listed in the References at the end of this text.

1. Suppose that there are seven committees with each pair of committees having a common member and each person being on two committees. How many people are there?

2. Show that the complement of K_n, a complete graph on n vertices, is a set of n isolated vertices.

3. A graph is *regular* if all vertices have the same degree. If a graph with n vertices is regular of degree 3 and has 18 edges, determine n.

4. If a graph has 50 edges, what is the least number of vertices it can have?

5. Show that at least two vertices have the same degree in any graph with at least two vertices. (*Hint:* Be careful about vertices of degree 0.)

6. Show that an undirected graph with all vertices of degree ≥ 2 must contain a circuit (edges cannot be repeated in a circuit).

7. If every vertex in the graph G has degree 2, does every vertex lie on a circuit? Prove, or give a counterexample.

8. If every vertex in a graph G has degree $\geq d$, then show that G must contain a circuit of length at least $d + 1$.

9. If every vertex in a directed graph G has positive out-degree (at least one outwardly directed edge),

(a) Must G contain a directed circuit?

(b) Must every vertex of G be on a directed circuit?

10. If G is a connected graph that is not a complete graph, show that some vertex, call it x, has two neighbors, call them y, z, that are not adjacent to each other [that is, there are edges (x, y) and (x, z) but not edge (y, z)].

11. Show that if a graph is not connected, then its complement must be connected.

12. Show that removal of some vertex x disconnects the connected undirected graph G if and only if there are two vertices a and b in G such that all paths in G from a to b pass through x.

13. Let G be a connected graph such that G-x is not connected for all but two vertices x of G. Show that G is a path.

14. A directed graph G is called *strongly connected* if there is a directed path from x to y for any two vertices x, y in G. Direct the edges in the following graphs to make the graphs strongly connected. If not possible, explain why. (An application of this problem is making streets in a city one-way.)

(a) Figure 1.4 **(b)** Figure 1.2

15. Prove that G is strongly connected (see Exercise 14) if and only if G's vertices cannot be partitioned into two sets V_1, V_2 such that there are no edges from a vertex in V_1 to a vertex in V_2.

16. A *bridge* is an edge in a connected graph whose removal disconnects the graph. Show that if a graph G contains a bridge, then it cannot have a circuit that contains all vertices of G.

17. Show that if every edge in a connected graph G lies on a circuit, then G cannot have a bridge (a bridge is defined in Exercise 16).

18. Prove that the edges of a connected undirected graph G can be directed to create a strongly connected graph (see Exercise 14) if and only if there is no bridge in G (see Exercise 16).

19. Draw planar graphs with the following types of vertices, if possible:

(a) Six vertices of degree 3

(b) i vertices of degree i, $i = 1, 2, 3, 4, 5$

(c) Two vertices of degree 3 and four vertices of degree 5

20. Let I be a set of independent vertices in a graph G and let C be a set of vertices that forms a complete subgraph. Show that I and C have at most one vertex in common.

21. Is it possible to have a connected simple graph containing six vertices with degrees: 5, 2, 2, 2, 2, 1? If so, draw such a graph.

22. Mr. Megabucks invites three married couples to his penthouse for dinner. Upon arrival, Mr. Megabucks and the six guests shake the hands of some of the other people (none of the guests shakes hands with his or her own spouse). Suppose each of the six guests shakes a different number of hands (possibly one person shakes no hands). By building a graph model of this situation (with seven vertices for the six guests and Mr. Megabucks), determine exactly how many hands Mr. Megabucks must have shaken. (*Hint:* First determine the six different numbers of hands shaken by the different guests.)

23. A round-robin tournament can be represented by a complete directed graph with vertices for competitors and an edge $(\vec{a, b})$ if a beats b. Each competitor plays every other competitor once. A vertex's (competitor's) score will be its out-degree (number of victories). Show that if vertex x has a maximum score among vertices in a round-robin tournament, then for any other vertex y, either there is an edge $(\vec{x, y})$ or for some w there are edges $(\vec{x, w})$ and $(\vec{w, y})$.

24. Suppose the round-robin tournament graph (see Exercise 23) has no directed circuits. We define a ranking of vertices (competitors) as follows. A vertex with no outward edge has a rank of 0. In general, a vertex has rank k if it has an edge directed to a rank $k - 1$ vertex and all other edges directed to lower ranks

($\leq k - 1$). Show that a directed complete graph with no directed circuits always has such a ranking, and that each vertex will have a different rank.

25. A *trail* is a sequence of vertices with consecutive vertices joined by a distinct edge (no edge can be repeated). Unlike a path, a vertex can be visited any number of times in a trail. A *cycle* is a trail that starts and ends at the same vertex. A cycle that repeats no vertices is a circuit.

(a) Show that a subset of the vertices on a trail from x to y can be used to make a path from x to y.

(b) Prove, or give a counterexample: If x and y lie on a cycle, then they must lie on a circuit.

(c) Show that the edges in a cycle can be partitioned into a collection of circuits.

(d) Show that if C is an odd-length cycle, then a subset of C's edges forms an odd-length circuit.

26. Consider a collection of circles (of varying sizes) in the plane. Make a *circle graph* with a vertex for each circle and an edge between two vertices when they correspond to two circles that cross (if one circle properly contains another, there would be no edge).

(a) Draw a family of circles whose circle graph is isomorphic to K_4.

(b) Draw a family of circles whose circle graph is the graph in Figure 1.3 (ignoring edge directions).

(c) Draw a family of circles whose circle graph is isomorphic to $K_{3,3}$.

27. A *cut-set S* is a set of edges in a connected undirected graph G whose removal disconnects G, but such that no proper subset of S can disconnect G.

(a) Find a cut-set of minimal size in Figure 1.22*a*.

(b) Show that every cut-set has an even number of edges in common with any circuit (remember that 0 is an even number).

28. A graph with n vertices and $n + 2$ edges must contain two edge-disjoint circuits. Prove or give a counterexample.

29. Show that if an n-vertex graph has more than $\frac{1}{2}(n-1)(n-2)$ edges, then it must be connected. (*Hint:* The most edges possible in a disconnected graph will occur when there are two components, each complete subgraphs.)

30. Show that an n-vertex graph cannot be a bipartite graph if it has more than $\frac{1}{4}n^2$ edges.

31. Suppose that G is a connected graph containing no triangles and that every pair of two non-adjacent vertices in G has exactly two neighbors in common. Show that every vertex of G must have the same degree. (*Hint:* Show that any pair of adjacent vertices must have the same degree.)

32. (*Famous Ramsey theory problem*) Let each edge of a complete graph on six vertices be painted red or white. Show that there must always be either a red triangle of three edges or a white triangle of three edges.

33. **(a)** Find a graph that is isomorphic to its own complement.

(b) Show that any self-complementary graph [as in part (a)] must have either $4k$ or $4k + 1$ vertices, for some integer k. (*Hint:* Use the fact that G and $\overline{G}$ both must have the same number of edges.)

34. Suppose that each path in a certain 7-vertex planar graph contains an even number of edges (zero edge or two edges or four edges, etc.). Draw the graph. (*Hint:* This is a "trick" problem.)

35. Show that a directed graph has no directed circuits if and only if its vertices can be indexed $x_1, x_2, \ldots, x_n$, so that all edges are of the form $(\overrightarrow{x_i, x_j})$, $i < j$.

36. Let $K_{m,n}$ be a complete bipartite graph, with $m > n$. What is the size of the smallest edge cover of $K_{m,n}$? What is the size of the largest independent set?

37. A *line graph* $L(G)$ of a graph G has a vertex of $L(G)$ for each edge in G and an edge between two vertices in $L(G)$ corresponding to two edges of G with a common end vertex.

(a) Draw a line graph of the left graph in Figure 1.6.

(b) Show that each vertex in $L(K_n)$ has degree $2(n - 2)$.

(c) Find all graphs that are isomorphic to their own line graph.

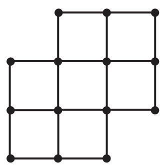
38. Show that if a graph H is the line graph (see Exercise 37) of some graph, then the edges of H can be partitioned into a collection of complete subgraphs such that each vertex of H is in exactly two such complete subgraphs.

39. An *automorphism* of a graph is an isomorphism $(1 - 1$ mapping preserving adjacency) of the vertices of a graph with themselves. Find an automorphism of the graph in

(a) Figure 1.1*a* **(b)** Figure 1.4 **(c)** Figure 1.13

40. **(a)** Show that there is no way to pair off the 14 vertices in the graph below with seven edges.

(b) Generalize part (a) to the problem of trying to use 31 dominoes to cover the 62 squares of an 8×8 chessboard with its two opposite corner squares removed.



41. Suppose circuits C_1 and C_2 have common edges (but $C_1 \neq C_2$). Show that the edges in $(C_1, \cup C_2) - (C_1, \cap C_2)$ form a circuit (or collection of circuits).

42. If the graph G has $2n$ vertices and no triangles, then show that G cannot have more than n^2 edges.