ONE: INTRODUCTION

Linear programming is concerned with the optimization (minimization or maximization) of a linear function while satisfying a set of linear equality and/or inequality constraints or restrictions. The linear programming problem was first conceived by George B. Dantzig around 1947 while he was working as a mathematical advisor to the United States Air Force Comptroller on developing a mechanized planning tool for a time–staged deployment, training, and logistical supply program. Although the Soviet mathematician and economist L. V. Kantorovich formulated and solved a problem of this type dealing with organization and planning in 1939, his work remained unknown until 1959. Hence, the conception of the general class of linear programming problems is usually credited to Dantzig. Because the Air Force refers to its various plans and schedules to be implemented as "programs," Dantzig's first published paper addressed this problem as "Programming in a Linear Structure." The term "linear programming" was actually coined by the economist and mathematician T. C. Koopmans in the summer of 1948 while he and Dantzig strolled near the Santa Monica beach in California.

In 1949 George B. Dantzig published the "simplex method" for solving linear programs. Since that time a number of individuals have contributed to the field of linear programming in many different ways, including theoretical developments, computational aspects, and exploration of new applications of the subject. The simplex method of linear programming enjoys wide acceptance because of (1) its ability to model important and complex management decision problems, and (2) its capability for producing solutions in a reasonable amount of time. In subsequent chapters of this text we shall consider the simplex method and its variants, with emphasis on the understanding of the methods.

In this chapter, we introduce the linear programming problem. The following topics are discussed: basic definitions in linear programming, assumptions leading to linear models, manipulation of the problem, examples of linear problems, and geometric solution in the feasible region space and the requirement space. This chapter is elementary and may be skipped if the reader has previous knowledge of linear programming.

1.1 THE LINEAR PROGRAMMING PROBLEM

We begin our discussion by formulating a particular type of linear programming problem. As will be seen subsequently, any general linear programming problem may be manipulated into this form.

Basic Definitions

Consider the following linear programming problem. Here, $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is the *objective function* (or *criterion function*) to be minimized and will be denoted by z. The coefficients $c_1, c_2, \ldots, c_n$ are the (known) *cost coefficients* and

$x_1, x_2, \ldots, x_n$ are the *decision variables* (variables, structural variables, or activity levels) to be determined.

$$\begin{array}{lccccccccc}
\text{Minimize} & c_1x_1 & + & c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & \geq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & \geq & b_2 \\
& \vdots & & \vdots & +\cdots+ & \vdots & & \vdots \\
& a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & \geq & b_m \\
& x_1, & & x_2, & \ldots, & x_n & \geq & 0.
\end{array}$$

The inequality $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ denotes the ith *constraint* (or restriction or functional, structural, or technological constraint). The coefficients a_{ij} for $i = 1, \ldots, m$, $j = 1, \ldots, n$ are called the *technological coefficients.* These technological coefficients form the *constraint matrix* **A**.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

The column vector whose ith component is b_i, which is referred to as the *right–hand side vector*, represents the minimal requirements to be satisfied. The constraints $x_1, x_2, \ldots, x_n \geq 0$ are the *nonnegativity constraints*. A set of values of the variables $x_1, \ldots, x_n$ satisfying all the constraints is called a *feasible point* or a *feasible solution.* The set of all such points constitutes the *feasible region* or the *feasible space.*

Using the foregoing terminology, the linear programming problem can be stated as follows: Among all feasible solutions, find one that minimizes (or maximizes) the objective function.

Example 1.1

Consider the following linear problem:

$$\begin{array}{lccccc}
\text{Minimize} & 2x_1 & + & 5x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \geq & 6 \\
& -x_1 & - & 2x_2 & \geq & -18 \\
& x_1, & & x_2 & \geq & 0.
\end{array}$$

In this case, we have two decision variables x_1 and x_2. The objective function to be minimized is $2x_1 + 5x_2$. The constraints and the feasible region are illustrated in Figure 1.1. The optimization problem is thus to find a point in the feasible region having the smallest possible objective value.

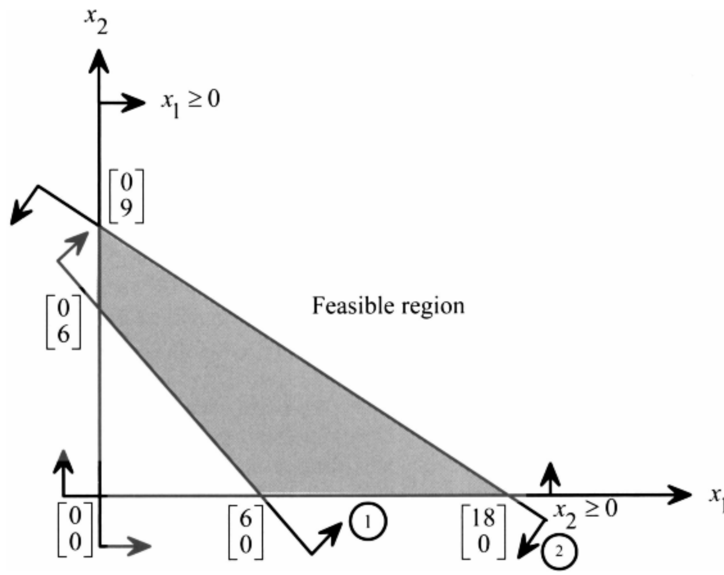

Figure 1.1. Illustration of the feasible region.

Assumptions of Linear Programming

To represent an optimization problem as a linear program, several assumptions that are implicit in the linear programming formulation discussed previously are needed. A brief discussion of these assumptions is given next.

1. *Proportionality*. Given a variable x_j, its contribution to cost is $c_j x_j$ and its contribution to the *i*th constraint is $a_{ij} x_j$. This means that if x_j is doubled, say, so is its contribution to cost and to each of the constraints. To illustrate, suppose that x_j is the amount of activity *j* used. For instance, if $x_j = 10$, then the cost of this activity is $10c_j$. If $x_j = 20$, then the cost is $20c_j$, and so on. This means that no savings (or extra costs) are realized by using more of activity *j*; that is, there are no economies or returns to scale or discounts. Also, no setup cost for starting the activity is realized.
2. *Additivity*. This assumption guarantees that the total cost is the sum of the individual costs, and that the total contribution to the *i*th restriction is the sum of the individual contributions of the individual activities. In other words, there are no substitution or interaction effects among the activities.
3. *Divisibility*. This assumption ensures that the decision variables can be divided into any fractional levels so that non-integral values for the decision variables are permitted.
4. *Deterministic*. The coefficients c_j, a_{ij}, and b_i are all known deterministically. Any probabilistic or stochastic elements inherent in

demands, costs, prices, resource availabilities, usages, and so on are all assumed to be approximated by these coefficients through some deterministic equivalent.

It is important to recognize that if a linear programming problem is being used to model a given situation, then the aforementioned assumptions are implied to hold, at least over some anticipated operating range for the activities. When Dantzig first presented his linear programming model to a meeting of the Econometric Society in Wisconsin, the famous economist H. Hotelling critically remarked that in reality, the world is indeed nonlinear. As Dantzig recounts, the well-known mathematician John von Neumann came to his rescue by countering that the talk was about "Linear" Programming and was based on a set of postulated axioms. Quite simply, a user may apply this technique if and only if the application fits the stated axioms.

Despite the seemingly restrictive assumptions, linear programs are among the most widely used models today. They represent several systems quite satisfactorily, and they are capable of providing a large amount of information besides simply a solution, as we shall see later, particularly in Chapter 6. Moreover, they are also often used to solve certain types of nonlinear optimization problems via (successive) linear approximations and constitute an important tool in solution methods for linear discrete optimization problems having integer-restricted variables.

Problem Manipulation

Recall that a linear program is a problem of minimizing or maximizing a linear function in the presence of linear inequality and/or equality constraints. By simple manipulations the problem can be transformed from one form to another equivalent form. These manipulations are most useful in linear programming, as will be seen throughout the text.

INEQUALITIES AND EQUATIONS

An inequality can be easily transformed into an equation. To illustrate, consider the constraint given by $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$. This constraint can be put in an equation form by subtracting the nonnegative *surplus* or *slack variable* x_{n-i} (sometimes denoted by s_i) leading to $\sum_{j=1}^{n} a_{ij}x_j - x_{n-i} = b_i$ and $x_{n+i} \geq 0$. Similarly, the constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ is equivalent to $\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i$ and $x_{n+i} \geq 0$. Also, an equation of the form $\sum_{j=1}^{n} a_{ij}x_j = b_i$ can be transformed into the two inequalities $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ and $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$, although this is not the practice.

NONNEGATIVITY OF THE VARIABLES

For most practical problems the variables represent physical quantities, and hence must be nonnegative. The simplex method is designed to solve linear programs where the variables are nonnegative. If a variable x_j is unrestricted in sign, then it can be replaced by $x'_j - x''_j$ where $x'_j \geq 0$ and $x''_j \geq 0$. If $x_1, \ldots, x_k$ are some k

variables that are all unrestricted in sign, then only one additional variable x'' is needed in the equivalent transformation: $x_j = x'_j - x''$ for $j = 1, ..., k$, where $x'_j \geq 0$ for $j = 1,\ldots,k$, and $x'' \geq 0$. (Here, $-x''$ plays the role of representing the most negative variable, while all the other variables x_j are x'_j above this value.) Alternatively, one could solve for each unrestricted variable in terms of the other variables using any equation in which it appears, eliminate this variable from the problem by substitution using this equation, and then discard this equation from the problem. However, this strategy is seldom used from a data management and numerical implementation viewpoint. Continuing, if $x_j \geq \ell_j$, then the new variable $x'_j = x_j - \ell_j$ is automatically nonnegative. Also, if a variable x_j is restricted such that $x_j \leq u_j$, where we might possibly have $u_j \leq 0$, then the substitution $x'_j = u_j - x_j$ produces a nonnegative variable x'_j.

MINIMIZATION AND MAXIMIZATION PROBLEMS

Another problem manipulation is to convert a maximization problem into a minimization problem and conversely. Note that over any region,

$$\text{maximum} \sum_{j=1}^{n} c_j x_j = -\text{minimum} \sum_{j=1}^{n} -c_j x_j.$$

Hence, a maximization (minimization) problem can be converted into a minimization (maximization) problem by multiplying the coefficients of the objective function by −1. After the optimization of the new problem is completed, the objective value of the old problem is −1 times the optimal objective value of the new problem.

Standard and Canonical Formats

From the foregoing discussion, we have seen that any given linear program can be put in different equivalent forms by suitable manipulations. In particular, two forms will be useful. These are the standard and the canonical forms. A linear program is said to be in *standard format* if all restrictions are equalities and all variables are nonnegative. The simplex method is designed to be applied only after the problem is put in standard form. The canonical form is also useful, especially in exploiting duality relationships. A minimization problem is in *canonical form* if all variables are nonnegative and all the constraints are of the $\geq$ type. A maximization problem is in canonical form if all the variables are nonnegative and all the constraints are of the $\leq$ type. The standard and canonical forms are summarized in Table 1.1.

Linear Programming in Matrix Notation

A linear programming problem can be stated in a more convenient form using matrix notation. To illustrate, consider the following problem:

Table 1.1. Standard and Canonical Forms

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
STANDARD FORM	Minimize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m$ $x_j \geq 0, \quad j = 1, \ldots, n.$	Maximize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m$ $x_j \geq 0, \quad j = 1, \ldots, n.$
CANONICAL FORM	Minimize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, \ldots, m$ $x_j \geq 0, \quad j = 1, \ldots, n.$	Maximize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m$ $x_j \geq 0, \quad j = 1, \ldots, n.$

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1,\ldots,m \\ & x_j \geq 0, \quad j = 1,\ldots,n. \end{aligned}$$

Denote the row vector $(c_1, c_2, \ldots, c_n)$ by **c**, and consider the following column vectors **x** and **b**, and the $m \times n$ matrix **A**.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

Then the problem can be written as follows:

$$\begin{aligned} \text{Minimize} \quad & \mathbf{cx} \\ \text{subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

The problem can also be conveniently represented via the columns of **A**. Denoting **A** by $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ where $\mathbf{a}_j$ is the *j*th column of **A**, the problem can be formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b} \\ & x_j \geq 0, \quad j = 1,\ldots,n. \end{aligned}$$

1.2 LINEAR PROGRAMMING MODELING AND EXAMPLES

The modeling and analysis of an operations research problem in general, and a linear programming problem in particular, evolves through several stages. The *problem formulation phase* involves a detailed study of the system, data collection, and the identification of the specific problem that needs to be analyzed (often the encapsulated problem may only be part of an overall system problem), along with the system constraints, restrictions, or limitations, and the objective function(s). Note that in real-world contexts, there frequently already exists an operating solution and it is usually advisable to preserve a degree of *persistency* with respect to this solution, i.e., to limit changes from it (e.g., to limit the number of price changes, or decision option modifications, or changes in percentage resource consumptions, or to limit changing some entity contingent on changing another related entity). Such issues, aside from technological or structural aspects of the problem, should also be modeled into the problem constraints.

The next stage involves the construction of an abstraction or an idealization of the problem through a *mathematical model*. Care must be taken to ensure that the model satisfactorily represents the system being analyzed, while keeping the model mathematically tractable. This compromise must be made judiciously, and the underlying assumptions inherent in the model must be properly considered. It must be borne in mind that from this point onward, the solutions obtained will be solutions to the model and not necessarily solutions to the actual system unless the model adequately represents the true situation.

The third step is to *derive a solution.* A proper technique that exploits any special structures (if present) must be chosen or designed. One or more optimal solutions may be sought, or only a heuristic or an approximate solution may be determined along with some assessment of its quality. In the case of multiple objective functions, one may seek *efficient* or *Pareto–optimal* solutions, that is, solutions that are such that a further improvement in any objective function value is necessarily accompanied by a detriment in some other objective function value.

The fourth stage is *model testing, analysis, and (possibly) restructuring.* One examines the model solution and its sensitivity to relevant system parameters, and studies its predictions to various what–if types of scenarios. This analysis provides insights into the system. One can also use this analysis to ascertain the reliability of the model by comparing the predicted outcomes with the expected outcomes, using either past experience or conducting this test retroactively using historical data. At this stage, one may wish to *enrich* the model further by incorporating other important features of the system that have not been modeled as yet, or, on the other hand, one may choose to *simplify* the model.

The final stage is *implementation.* The primary purpose of a model is to interactively aid in the decision–making process. The model should never *replace* the decision maker. Often a "frank–factor" based on judgment and experience needs to be applied to the model solution before making policy decisions. Also, a model should be treated as a "living" entity that needs to be nurtured over time, i.e., model parameters, assumptions, and restrictions should be periodically revisited in order to keep the model current, relevant, and valid.

We describe several problems that can be formulated as linear programs. The purpose is to exhibit the varieties of problems that can be recognized and expressed in precise mathematical terms as linear programs.

Feed Mix Problem

An agricultural mill manufactures feed for chickens. This is done by mixing several ingredients, such as corn, limestone, or alfalfa. The mixing is to be done in such a way that the feed meets certain levels for different types of nutrients, such as protein, calcium, carbohydrates, and vitamins. To be more specific, suppose that n ingredients $j = 1,..., n$ and m nutrients $i = 1,..., m$ are considered. Let the unit cost of ingredient j be c_j and let the amount of ingredient j to be used be x_j. The cost is therefore $\sum_{j=1}^{n} c_j x_j$. If the amount of the final product needed is b, then we must have $\sum_{j=1}^{n} x_j = b$. Further suppose that a_{ij} is the amount of nutrient i present in a unit of ingredient j, and that the acceptable lower and upper limits of nutrient i in a unit of the chicken feed are ℓ_i' and u_i', respectively. Therefore, we must have the constraints $\ell_i' b \le \sum_{j=1}^{n} a_{ij} x_j \le u_i' b$ for $i = 1, \ldots, m$. Finally, because of shortages, suppose that the mill cannot acquire more than u_j units of ingredient j. The problem of mixing the ingredients such that the cost is minimized and the restrictions are met can be formulated as follows:

$$
\begin{array}{lrcrcrcr}
\text{Minimize} & c_1x_1 & + & c_2x_2 & +\cdots+ & c_nx_n & & \\
\text{subject to} & x_1 & + & x_2 & +\cdots+ & x_n & = & b \\
& b\ell_1' \le a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & \le & bu_1' \\
& b\ell_2' \le a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & \le & bu_2' \\
& \vdots & & \vdots & & \vdots & & \vdots \\
& b\ell_m' \le a_{m1}x_1 & + & a_{m2}x_2 & -\cdots+ & a_{mn}x_n & \le & bu_m' \\
& & & & & 0 \le x_1 & \le & u_1 \\
& & & & & 0 \le x_2 & \le & u_2 \\
& & & & & \vdots & & \\
& & & & & 0 \le x_n & \le & u_n.
\end{array}
$$

Production Scheduling: An Optimal Control Problem

A company wishes to determine the production rate over the planning horizon of the next T weeks such that the known demand is satisfied and the total production and inventory cost is minimized. Let the known demand rate at time t be $g(t)$, and similarly, denote the production rate and inventory at time t by $x(t)$ and $y(t)$, respectively. Furthermore, suppose that the initial inventory at time 0 is y_0 and that the desired inventory at the end of the planning horizon is y_T. Suppose that the inventory cost is proportional to the units in storage, so that the inventory cost is given by $c_1 \int_0^T y(t)\,dt$ where $c_1 > 0$ is known. Also, suppose that the production cost is proportional to the rate of production, and is therefore given by $c_2 \int_0^T x(t)\,dt$. Then the total cost is $\int_0^T [c_1 y(t) - c_2 x(t)]\,dt$. Also note that the inventory at any time is given according to the relationship

$$y(t) = y_0 - \int_0^t [x(\tau) - g(\tau)]\,d\tau, \text{ for } t \in [0, T].$$

Suppose that no backlogs are allowed; that is, all demand must be satisfied. Furthermore, suppose that the present manufacturing capacity restricts the production rate so that it does not exceed b_1 at any time. Also, assume that the available storage restricts the maximum inventory to be less than or equal to b_2. Hence, the production scheduling problem can be stated as follows:

$$
\begin{array}{lll}
\text{Minimize} & \int_0^T [c_1 y(t) + c_2 x(t)]\,dt & \\
\text{subject to} & y(t) = y_0 + \int_0^t [x(\tau) - g(\tau)]\,d\tau, & \text{for} \quad t \in [0, T] \\
& y(T) = y_T & \\
& 0 \le x(t) \le b_1, & \text{for} \quad t \in [0, T] \\
& 0 \le y(t) \le b_2, & \text{for} \quad t \in [0, T].
\end{array}
$$

The foregoing model is a linear control problem, where the *control variable* is the production rate $x(t)$ and the *state variable* is the inventory level $y(t)$. The problem can be approximated by a linear program by discretizing the continuous variables x and y. First, the planning horizon $[0, T]$ is divided into n smaller periods $[0,\Delta],[\Delta,2\Delta],\ldots,[(n-1)\Delta,n\Delta]$, where $n\Delta = T$. The production rate, the inventory, and the demand rate are assumed constant over each period. In particular, let the production rate, the inventory, and the demand rate in period j be x_j, y_j, and g_j, respectively. Then, the production scheduling problem can be approximated by the following linear program (why?).

$$\begin{array}{lll} \text{Minimize} & \sum_{j=1}^{n} (c_1\Delta)y_j + \sum_{j=1}^{n} (c_2\Delta)x_j & \\ \text{subject to} & y_j = y_{j-1} + (x_j - g_j)\Delta, & j = 1,\ldots,n \\ & y_n = y_T & \\ & 0 \le x_j \le b_1, & j = 1,\ldots,n \\ & 0 \le y_j \le b_2, & j = 1,\ldots,n. \end{array}$$

Cutting Stock Problem

A manufacturer of metal sheets produces rolls of standard fixed width w and of standard length ℓ. A large order is placed by a customer who needs sheets of width w and varying lengths. In particular, b_i sheets with length ℓ_i and width w for $i = 1,\ldots, m$ are ordered. The manufacturer would like to cut the standard rolls in such a way as to satisfy the order and to minimize the waste. Because scrap pieces are useless to the manufacturer, the objective is to minimize the number of rolls needed to satisfy the order. Given a standard sheet of length ℓ, there are many ways of cutting it. Each such way is called a *cutting pattern*. The jth cutting pattern is characterized by the column vector $\mathbf{a}_j$ where the ith component of $\mathbf{a}_j$, namely a_{ij}, is a nonnegative integer denoting the number of sheets of length ℓ_i in the jth pattern. For instance, suppose that the standard sheets have length $\ell = 10$ meters and that sheets of lengths 1.5, 2.5, 3.0, and 4.0 meters are needed. The following are typical cutting patterns:

$$\mathbf{a}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \ldots$$

Note that the vector $\mathbf{a}_j$ represents a cutting pattern if and only if $\sum_{i=1}^{m} a_{ij}\ell_i \le \ell$ and each a_{ij} is a nonnegative integer. The number of cutting patterns n is finite. If we let x_j be the number of standard rolls cut according to the jth pattern, the problem can be formulated as follows:

$$\begin{array}{lll} \text{Minimize} & \sum_{j=1}^{n} x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij}x_j \geq b_i, & i = 1,\ldots,m \\ & x_j \geq 0, & j = 1,\ldots,n \\ & x_j \text{ integer}, & j = 1,\ldots,n. \end{array}$$

If the integrality requirement on the x_j variables is dropped, the problem is a linear program. Of course, the difficulty with this problem is that the number of possible cutting patterns n is very large, and also, it is not computationally feasible to enumerate each cutting pattern and its column $\mathbf{a}_j$ beforehand. The decomposition algorithm of Chapter 7 is particularly suited to solve this problem, where a new cutting pattern is generated at each iteration (see also Exercise 7.28). In Section 6.7 we suggest a method for handling the integrality requirements.

The Transportation Problem

The Brazilian coffee company processes coffee beans into coffee at m plants. The coffee is then shipped every week to n warehouses in major cities for retail, distribution, and exporting. Suppose that the unit shipping cost from plant i to warehouse j is c_{ij}. Furthermore, suppose that the production capacity at plant i is a_i and that the demand at warehouse j is b_j. It is desired to find the production–shipping pattern x_{ij} from plant i to warehouse j, $i = 1,\ldots, m$, $j = 1,\ldots, n$, which minimizes the overall shipping cost. This is the well–known *transportation problem*. The essential elements of the problem are shown in the network of Figure 1.2. The transportation problem can be formulated as the following linear program:

$$\begin{array}{lll} \text{Minimize} & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} & \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} \leq a_i, & i = 1,\ldots,m \\ & \sum_{i=1}^{m} x_{ij} = b_j, & j = 1,\ldots,n \\ & x_{ij} \geq 0, & i = 1,\ldots,m, \quad j = 1,\ldots,n. \end{array}$$

Capital Budgeting Problem

A municipal construction project has funding requirements over the next four years of \$2 million, \$4 million, \$8 million, and \$5 million, respectively. Assume that all of the money for a given year is required at the beginning of the year.

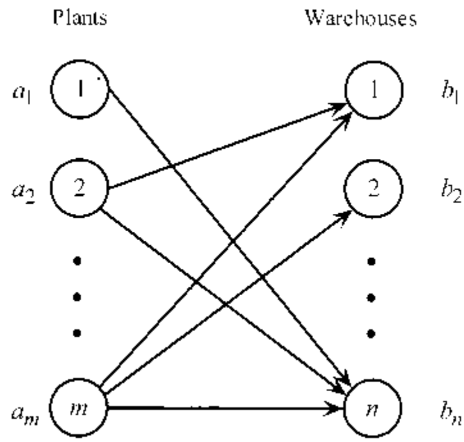

Figure 1.2. The transportation problem.

The city intends to sell exactly enough long–term bonds to cover the project funding requirements, and all of these bonds, regardless of when they are sold, will be paid off (*mature*) on the same date in a distant future year. The long–term bond market interest rates (that is, the costs of selling bonds) for the next four years are projected to be 7 percent, 6 percent, 6.5 percent, and 7.5 percent, respectively. Bond interest paid will commence one year after the project is complete and will continue for 20 years, after which the bonds will be paid off. During the same period, the short–term interest rates on time deposits (that is, what the city can earn on deposits) are projected to be 6 percent, 5.5 percent, and 4.5 percent, respectively (the city will clearly not invest money in short-term deposits during the fourth year). What is the city's optimal strategy for selling bonds and depositing funds in time accounts in order to complete the construction project?

To formulate this problem as a linear program, let x_j, $j = 1,...,4$, be the amount of bonds sold at the beginning of each year j. When bonds are sold, some of the money will immediately be used for construction and some money will be placed in short–term deposits to be used in later years. Let y_j, $j = 1,...,3$, be the money placed in time deposits at the beginning of year j. Consider the beginning of the first year. The amount of bonds sold minus the amount of time deposits made will be used for the funding requirement at that year. Thus, we may write

$$x_1 - y_1 = 2.$$

We could have expressed this constraint as $\geq$. However, it is clear in this case that any excess funds will be deposited so that equality is also acceptable.

Consider the beginning of the second year. In addition to bonds sold and time deposits made, we also have time deposits plus interest becoming available from the previous year. Thus, we have

$$1.06y_1 - x_2 - y_2 = 4.$$

The third and fourth constraints are constructed in a similar manner.

Ignoring the fact that the amounts occur in different years (that is, the time value of money), the unit cost of selling bonds is 20 times the interest rate. Thus, for bonds sold at the beginning of the first year we have $c_1 = 20(0.07)$. The other cost coefficients are computed similarly.

Accordingly, the linear programming model is given as follows:

$$\begin{aligned}
\text{Minimize} \quad & 20(0.07)\,x_1 + 20(0.06)\,x_2 + 20(0.065)\,x_3 + 20(0.075)\,x_4 \\
\text{subject to} \quad & x_1 - y_1 = 2 \\
& 1.06\,y_1 - x_2 - y_2 = 4 \\
& 1.055\,y_2 - x_3 - y_3 = 8 \\
& 1.045\,y_3 - x_4 = 5 \\
& x_1,\ x_2,\ x_3,\ x_4,\ y_1,\ y_2,\ y_3 \geq 0.
\end{aligned}$$

Tanker Scheduling Problem

A shipline company requires a fleet of ships to service requirements for carrying cargo between six cities. There are four specific routes that must be served daily. These routes and the number of ships required for each route are as follows:

ROUTE #	ORIGIN	DESTINATION	NUMBER OF SHIPS PER DAY NEEDED
1	Dhahran	New York	3
2	Marseilles	Istanbul	2
3	Naples	Mumbai	1
4	New York	Marseilles	1

All cargo is compatible, and therefore only one type of ship is needed. The travel time matrix between the various cities is shown.

	NAPLES	MARSEILLES	ISTANBUL	NEW YORK	DHAHRAN	MUMBAI	
Naples	0	1	2	14	7	7	
Marseilles	1	0	3	13	8	8	
Istanbul	2	3	0	15	5	5	t_{ij} matrix (days)
New York	14	13	15	0	17	20	
Dhahran	7	8	5	17	0	3	
Mumbai	7	8	5	20	3	0	

It takes one day to off–load and one day to on–load each ship. How many ships must the shipline company purchase?

In addition to nonnegativity restrictions, there are two types of constraints that must be maintained in this problem. First, we must ensure that ships coming off of some route get assigned to some (other) route. Second, we must ensure that each route gets its required number of ships per day. Let x_{ij} be the number of ships per day coming off of route i and assigned to route j. Let b_i represent the number of ships per day required on route i.

To ensure that ships from a given route get assigned to other routes, we write the constraint

$$\sum_{j=1}^{4} x_{ij} = b_i, \qquad i = 1,\ldots,4.$$

To ensure that a given route gets its required number of ships, we write the constraint

$$\sum_{k=1}^{4} x_{ki} = b_i, \qquad i = 1,\ldots,4.$$

Computing the cost coefficients is a bit more involved. Since the objective is to minimize the total number of ships, let c_{ij} be the number of ships required to ensure a continuous daily flow of one ship coming off of route i and assigned to route j. To illustrate the computation of these c_{ij}-coefficients, consider c_{23}. It takes one day to load a ship at Marseilles, three days to travel from Marseilles to Istanbul, one day to unload cargo at Istanbul, and two days to head from Istanbul to Naples—a total of seven days. This implies that seven ships are needed to ensure that one ship will be assigned daily from route 2 to route 3 (why?). In particular, one ship will be on- loading at Marseilles, three ships en route from Marseilles to Istanbul, one ship off–loading at Istanbul, and two ships en route from Istanbul to Naples.

In general, c_{ij} is given as follows:

c_{ij} = one day for on–loading + number of days for transit on route i
+ one day for off–loading
+ number of days for travel from the destination of route i to the origin of route j.

Therefore, the tanker scheduling problem can be formulated as follows:

$$\begin{aligned}
\text{Minimize}\quad & 36x_{11} + 32x_{12} + 33x_{13} + 19x_{14} + 10x_{21} - 8x_{22} + 7x_{23} \\
& +20x_{24} + 12x_{31} + 17x_{32} + 16x_{33} + 29x_{34} + 23x_{41} \\
& +15x_{42} + 16x_{43} + 28x_{44} \\
\text{subject to}\quad & \sum_{j=1}^{4} x_{ij} = b_i, \qquad i = 1, 2, 3, 4 \\
& \sum_{k=1}^{4} x_{ki} = b_i, \qquad i = 1, 2, 3, 4 \\
& x_{ij} \geq 0, \qquad i, j = 1, 2, 3, 4,
\end{aligned}$$

where $b_1 = 3$, $b_2 = 2$, $b_3 = 1$, and $b_4 = 1$.

It can be easily seen that this is another application of the transportation problem (it is instructive for the reader to form the origins and destinations of the corresponding transportation problem).

Multiperiod Coal Blending and Distribution Problem

A southwest Virginia coal company owns several mines that produce coal at different given rates, and having known quality (ash and sulfur content) specifications that vary over mines as well as over time periods at each mine. This coal needs to be shipped to silo facilities where it can be possibly subjected to a beneficiation (cleaning) process, in order to partially reduce its ash and sulfur content to a desired degree. The different grades of coal then need to be blended at individual silo facilities before being shipped to customers in order to satisfy demands for various quantities having stipulated quality specifications. The aim is to determine optimal schedules over a multiperiod time horizon for shipping coal from mines to silos, cleaning and blending the coal at the silos, and distributing the coal to the customers, subject to production capacity, storage, material flow balance, shipment, and quality requirement restrictions, so as to satisfy the demand at a minimum total cost, including revenues due to rebates for possibly shipping coal to customers that is of a better quality than the minimum acceptable specified level.

Suppose that this problem involves $i = 1,\ldots, m$ mines, $j = 1,\ldots, J$ silos, $k = 1,\ldots, K$ customers, and that we are considering $t = 1,\ldots, T$ (≥ 3) time periods. Let p_{it} be the production (in tons of coal) at mine i during period t, and let a_{it} and s_{it} respectively denote the ash and sulfur percentage content in the coal produced at mine i during period t. Any excess coal not shipped must be stored at the site of the mine at a per–period storage cost of c_i^M per ton at mine i, where the capacity of the storage facility at mine i is given by M_i.

Let A_1 denote the permissible flow transfer arcs (i, j) from mine i to silo j, and let $F_i^1 = \{j : (i, j) \in A_1\}$ and $R_j^1 = \{i : (i, j) \in A_1\}$. The transportation cost per ton from mine i to silo j is denoted by c_{ij}^1, for each $(i, j) \in A_1$. Each silo j has a storage capacity of S_j, and a per–ton storage cost of c_j^S, per period. Assume that at the beginning of the time horizon, there exists an initial amount of q_j^0 tons of coal stored at silo j, having an ash and sulfur percentage content of a_j^0 and s_j^0, respectively. Some of the silos are equipped with beneficiation or cleaning facilities, where any coal coming from mine i to such a silo j is cleaned at a cost of c_{ij}^B per ton, resulting in the ash and sulfur content being respectively attenuated by a factor of $\beta_{ij} \in (0,1]$ and $\gamma_{ij} \in (0,1]$, and the total weight being thereby attenuated by a factor of $\alpha_{ij} \in (0,1]$ (hence, for one ton input, the output is α_{ij} tons, which is then stored for shipment). Note that for silos that do not have any cleaning facilities, we assume that $c_{ij}^B = 0$, and $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 1$.

Let A_2 denote the feasible flow transfer arcs (j, k) from silo j to customer k, and let $F_j^2 = \{k : (j,k) \in A_2\}$, and $R_k^2 = \{j : (j,k) \in A_2\}$. The transportation cost per ton from silo j to customer k is denoted by c_{jk}^2, for each $(j,k) \in A_2$. Additionally, if t_1 is the time period for a certain mine to silo shipment (assumed to occur at the beginning of the period), and t_2 is the time period for a continuing silo to customer shipment (assumed to occur at the end of the period), then the *shipment lag* between the two coal flows is given by $t_2 - t_1$. A maximum of a three–period shipment lag is permitted between the coal production at any mine and its ultimate shipment to customers through any silo, based on an estimate of the maximum clearance time at the silos. (Actual shipment times from mines to silos are assumed to be negligible.) The demand placed (in tons of coal) by customer k during period t is given by d_{kt}, with ash and sulfur percentage contents being respectively required to lie in the intervals defined by the lower and upper limits $[\ell_{kt}^a, u_{kt}^a]$ and $[\ell_{kt}^s, u_{kt}^s]$. There is also a revenue earned of r_{kt} per–ton per–percentage point that falls below the maximum specified percentage u_{kt}^a of ash content in the coal delivered to customer k during period t.

To model this problem, we first define a set of *principal decision variables* as $y_{ijt}^{k\tau}$ = amount (tons) of coal shipped from mine i to silo j in period t, with continued shipment to customer k in period τ (where $\tau = t, t+1, t+2$, based on the three–period shipment lag restriction), and $y_{jk\tau}^0$ = amount (tons) of coal that is in initial storage at silo j, which is shipped to customer k in period τ (where $\tau = 1,2,3$, based on a three period dissipation limit). Controlled by these principal decisions, there are four other *auxiliary decision variables* defined as follows: $x_{i\delta}^M$ = slack variable that represents the amount (tons) of coal remaining in storage at mine i during period δ; $x_{j\delta}^S$ = accumulated storage amount (tons) of coal in silo j during period δ; $z_{k\tau}^a$ = percentage ash content in the blended coal that is ultimately delivered to customer k in period τ; and $z_{k\tau}^s$ = percentage sulfur content in the blended coal that is ultimately delivered to customer k in period τ. The linear programming model is then given as follows, where the objective function records the transportation, cleaning, and storage costs, along with the revenue term over the horizon $1,\ldots,T$ of interest. The respective sets of constraints represent the flow balance at the mines, storage capacity restrictions at the mines, flow balance at the silos, storage capacity restrictions at the silos, the dissipation of the initial storage at the silos, the demand satisfaction constraints, the ash content identities, the quality bound specifications with respect to the ash content, the sulfur content identities, the quality bound specifications with respect to the sulfur content, and the remaining logical nonnegativity restrictions. (All undefined variables and summation terms are assumed to be zero. Also, see Exercises 1.19-1.21.)

$$\text{Minimize} \quad \sum_{j=1}^{J} \sum_{i \in R_j^1} \sum_{k \in F_j^2} \sum_{t=1}^{T} \sum_{\tau=t}^{\min\{t+2,T\}} [c_{ij}^1 + c_{ij}^B + c_{jk}^2 + (\tau - t + 1)c_j^S] y_{ijt}^{k\tau} - \sum_{i=1}^{m} \sum_{\delta=1}^{T} c_i^M x_{i\delta}^M$$

$$+ \sum_{j=1}^{J} \sum_{k \in F_j^2} \sum_{\tau=1}^{3} c_{jk}^2 y_{jk\tau}^0 + \sum_{j=1}^{J} \sum_{t=1}^{3} c_j^S [q_j^0 - \sum_{\tau<t} \sum_{k \in F_j^2} y_{jk\tau}^0]$$

$$- \sum_{k=1}^{K} \sum_{\tau=1}^{T} (u_{k\tau}^a - z_{k\tau}^a) d_{k\tau} r_{k\tau}$$

$$\text{subject to} \quad \sum_{t=1}^{\delta} p_{it} - \sum_{j \in F_i^1} \sum_{t=1}^{\delta} \sum_{k \in F_j^2} \sum_{\tau=t}^{t+2} y_{ijt}^{k\tau} = x_{i\delta}^M, \quad i = 1,\ldots,m,\ \delta = 1,\ldots,T$$

$$0 \le x_{i\delta}^M \le M_i, \quad i = 1,\ldots,m,\ \delta = 1,\ldots,T$$

$$\sum_{i \in R_j^1} \sum_{k \in F_j^2} \sum_{t=\max\{1,\delta-2\}}^{\delta} \sum_{\tau=\delta}^{t+2} \alpha_{ij} y_{ijt}^{k\tau} + [q_j^0 - \sum_{k \in F_j^2} \sum_{\tau=1}^{\min\{\delta-1,3\}} y_{jk\tau}^0] = x_{j\delta}^S, \quad j = 1,\ldots,J,\ \delta = 1,\ldots,T$$

$$0 \le x_{j\delta}^S \le S_j, \quad j = 1,\ldots,J,\ \delta = 1,\ldots,T$$

$$\sum_{k \in F_j^2} \sum_{\tau=1}^{3} y_{jk\tau}^0 = q_j^0, \quad j = 1,\ldots,J$$

$$\sum_{j \in R_k^2} \sum_{i \in R_j^1} \sum_{t=\max\{1,\tau-2\}}^{\tau} \alpha_{ij} y_{ijt}^{k\tau} + \sum_{j \in R_k^2} y_{jk\tau}^0 = d_{k\tau}, \quad k = 1,\ldots,K,\ \tau = 1,\ldots,T$$

$$z_{k\tau}^a d_{k\tau} = \sum_{j \in R_k^2} \sum_{i \in R_j^1} \sum_{t=\max\{1,\tau-2\}}^{\tau} a_{it} \beta_{ij} y_{ijt}^{k\tau} + \sum_{j \in R_k^2} a_j^0 y_{jk\tau}^0, \quad k = 1,\ldots,K,\ \tau = 1,\ldots,T$$

$$\ell_{k\tau}^a \le z_{k\tau}^a \le u_{k\tau}^a, \quad k = 1,\ldots,K,\ \tau = 1,\ldots,T$$

$$z_{k\tau}^{s} d_{k\tau} = \sum_{j \in R_k^2} \sum_{i \in R_j^1} \sum_{t=\max\{1,\tau-2\}}^{\tau} s_{it} \gamma_{ij} y_{ijt}^{k\tau} + \sum_{j \in R_k^2} s_j^0 y_{jk\tau}^0,$$

$$k = 1,\ldots,K, \tau = 1,\ldots,T$$

$$\ell_{k\tau}^{s} \leq z_{k\tau}^{s} \leq u_{k\tau}^{s}, \ k = 1,\ldots,K, \tau = 1,\ldots,T$$

$$y_{ijt}^{k\tau} \geq 0, (i,j) \in A_1, t = 1,\ldots,T, k = 1,\ldots,K, \ \ \tau = t,\ldots,t+2,$$

$$y_{jk\tau}^{0} \geq 0, (j,k) \in A_2, \tau = 1,2,3.$$

1.3 GEOMETRIC SOLUTION

We describe here a geometric procedure for solving a linear programming problem. Even though this method is only suitable for very small problems, it provides a great deal of insight into the linear programming problem. To be more specific, consider the following problem:

$$\begin{array}{lrcl} \text{Minimize} & \mathbf{cx} & & \\ \text{subject to} & \mathbf{Ax} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}. \end{array}$$

Note that the feasible region consists of all vectors $\mathbf{x}$ satisfying $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Among all such points, we wish to find a point having a minimal value of $\mathbf{cx}$. Note that points having the same objective value z satisfy the equation $\mathbf{cx} = z$, that is, $\sum_{j=1}^{n} c_j x_j = z$. Since z is to be minimized, then the plane (line in a two–dimensional space) $\sum_{j=1}^{n} c_j x_j = z$ must be moved parallel to itself in the direction that minimizes the objective the most. This direction is $-\mathbf{c}$, and hence the plane is moved in the direction of $-\mathbf{c}$ as much as possible, while maintaining contact with the feasible region. This process is illustrated in Figure 1.3. Note that as the optimal point $\mathbf{x}^*$ is reached, the line $c_1x_1 + c_2x_2 = z^*$, where $z^* = c_1x_1^* + c_2x_2^*$, cannot be moved farther in the direction $-\mathbf{c} = (-c_1, -c_2)$, because this will only lead to points outside the feasible region. In other words, one cannot move from $\mathbf{x}^*$ in a direction that makes an acute angle with $-\mathbf{c}$, i.e., a direction that reduces the objective function value, while remaining feasible. We therefore conclude that $\mathbf{x}^*$ is indeed an optimal solution. Needless to say, for a maximization problem, the plane $\mathbf{cx} = z$ must be moved as much as possible in the direction $\mathbf{c}$, while maintaining contact with the feasible region.

The foregoing process is convenient for problems having two variables and is obviously impractical for problems with more than three variables. It is worth noting that the optimal point $\mathbf{x}^*$ in Figure 1.3 is one of the five corner points that are called *extreme points*. We shall show in Section 3.1 that if a linear program in standard or canonical form has a finite optimal solution, then it has an optimal corner (or extreme) point solution.

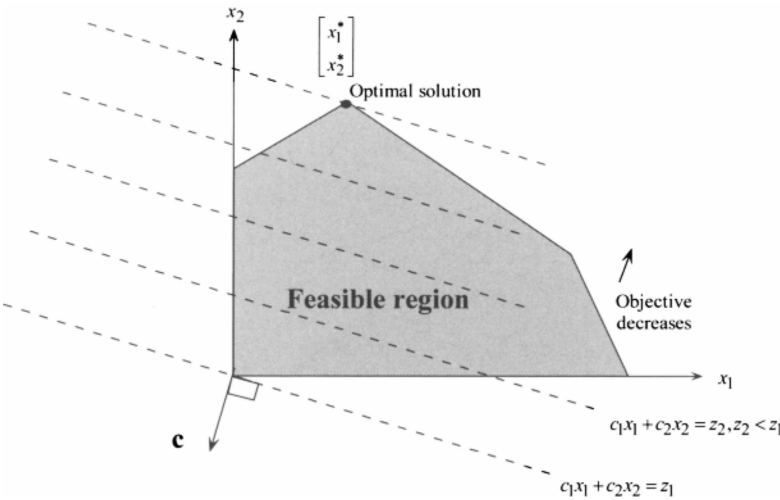

Figure 1.3. Geometric solution.

Example 1.2

$$\begin{array}{lrcrcl} \text{Minimize} & -x_1 & - & 3x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & 2x_2 & \leq & 8 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

The feasible region is illustrated in Figure 1.4. For example, consider the second constraint. The equation associated with this constraint is $-x_1 + 2x_2 = 8$. The gradient or the partial derivative vector of the linear function $-x_1 + 2x_2$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Hence, $-x_1 + 2x_2$ increases in any direction making an acute angle with $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and decreases in any direction making an acute angle $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Consequently, the region feasible to $-x_1 - 2x_2 \leq 8$ relative to the equation $-x_1 + 2x_2 - 8$ is as shown in Figure 1.4 and encompasses points having decreasing values of $-x_1 - 2x_2$ from the value 8. (Alternatively, this region may be determined relative to the equation $-x_1 + 2x_2 = 8$ by testing the feasibility of a point, for example, the origin.) Similarly, the region feasible to the first constraint is as shown. (Try adding the constraint $-2x_1 + 3x_2 \geq 0$ to this figure.) The nonnegativity constraints restrict the points to be in the first quadrant. The equations $-x_1 - 3x_2 = z$, for different values of z, are called the *objective contours* and are represented by dotted lines in Figure 1.4. In particular

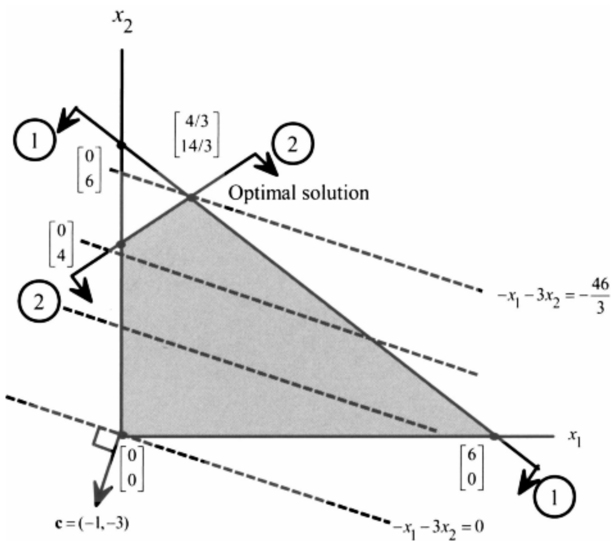

Figure 1.4. Numerical example.

the contour $-x_1 - 3x_2 = z = 0$ passes through the origin. We move onto lower valued contours in the direction $-\mathbf{c} = (1, 3)$ as much as possible until the optimal point (4/3, 14/3) is reached.

In this example we had a unique optimal solution. Other cases may occur depending on the problem structure. All possible cases that may arise are summarized below (for a minimization problem).

1. *Unique Optimal Solution.* If the optimal solution is unique, then it occurs at an extreme point. Figures 1.5a and b show a unique optimal solution. In Figure 1.5a the feasible region is *bounded*; that is, there is a ball of finite radius centered at, say, the origin that contains the feasible region. In Figure 1.5b the feasible region is not bounded. In each case, however, a finite unique optimal solution is obtained.
2. *Alternative Optimal Solutions*. This case is illustrated in Figure 1.6. Note that in Figure 1.6a the feasible region is bounded. The two corner points $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are optimal, as well as any point on the line segment joining them. In Figure 1.6b the feasible region is unbounded but the optimal objective is finite. Any point on the "ray" with vertex $\mathbf{x}^*$ in Figure 1.6b is optimal. Hence, the *optimal solution set* is unbounded.

In both cases (1) and (2), it is instructive to make the following observation. Pick an optimal solution $\mathbf{x}^*$ in Figure 1.5 or 1.6, corner point or not. Draw the normal vectors to the constraints passing through $\mathbf{x}^*$ pointing in the outward direction with respect to the feasible region. Also, construct the vector $-\mathbf{c}$ at $\mathbf{x}^*$. Note that the "cone" spanned by the normals to the constraints passing through $\mathbf{x}^*$ contains the vector $-\mathbf{c}$. This is in fact the necessary and sufficient condition for $\mathbf{x}^*$ to be optimal, and will be formally established later. Intuitively, when this condition occurs, we can see that there is no direction along which a motion is possible that would improve the objective function while remaining feasible. Such a direction would have to make an acute angle with $-\mathbf{c}$ to improve the objective value and simultaneously make an angle $\geq 90°$ with respect to each of the normals to the constraints passing through $\mathbf{x}^*$ in order to maintain feasibility for some step length along this direction. This is impossible at any optimal solution, although it is possible at any nonoptimal solution.

3. *Unbounded Optimal Objective Value.* This case is illustrated in Figure 1.7 where both the feasible region and the optimal objective value are unbounded. For a minimization problem, the plane $\mathbf{cx} = z$ can be moved in the direction $-\mathbf{c}$ indefinitely while always inter– secting with the feasible region. In this case, the optimal objective value is unbounded (with value $-\infty$) and *no optimal solution exists.*

Examining Figure 1.8, it is clear that there exists no point (x_1, x_2) satisfying these inequalities. The problem is said to be *infeasible, inconsistent,* or with an *empty feasible region.* Again, we say that *no optimal solution exists* in this case.

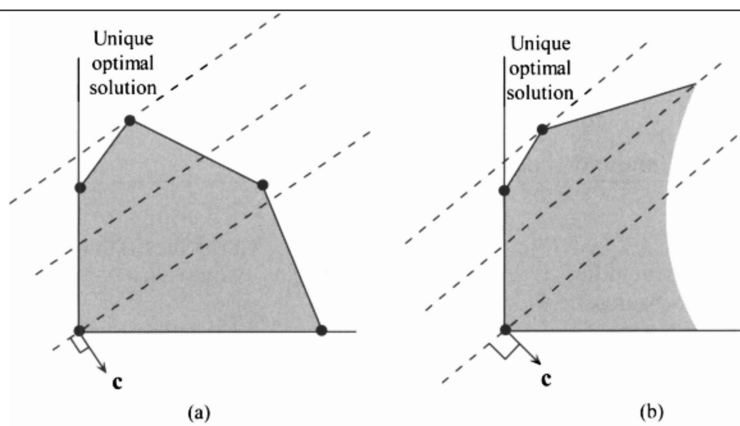

Figure 1.5. Unique optimal solution: (a) Bounded region. (b) Unbounded region.

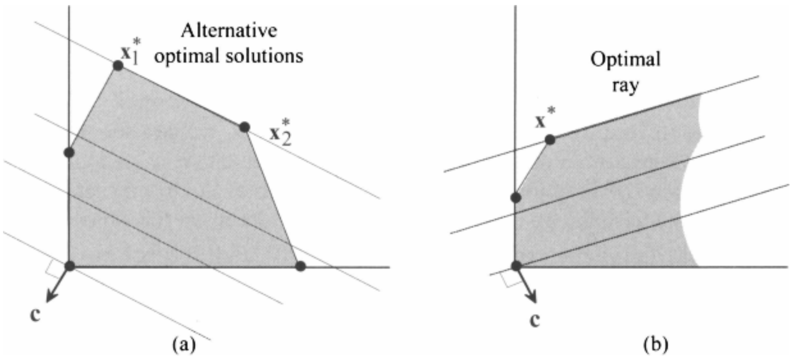

Figure 1.6. Alternative optima: (a) Bounded region. (b) Unbounded region.

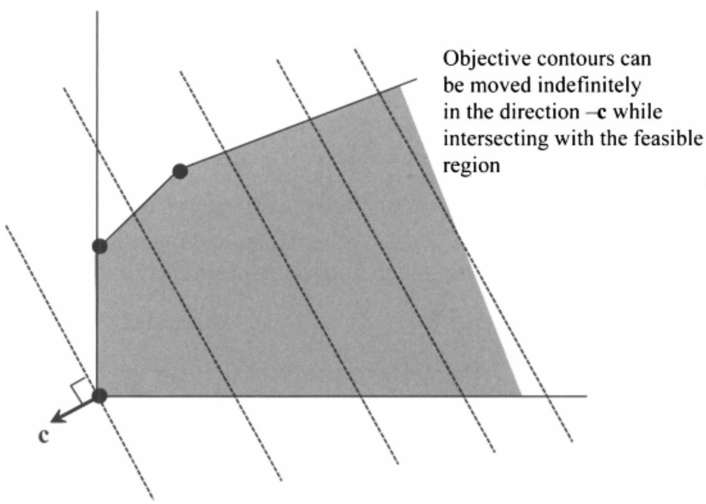

Figure 1.7. Unbounded optimal objective value.

4. *Empty Feasible Region.* In this case, the system of equations and/or inequalities defining the feasible region is *inconsistent.* To illustrate, consider the following problem:

$$\begin{array}{lrcrcl} \text{Minimize} & -2x_1 & - & 3x_2 & & \\ \text{subject to} & -x_1 & + & 2x_2 & \leq & 2 \\ & 2x_1 & - & x_2 & \leq & 3 \\ & & & x_2 & \geq & 4 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

1.4 THE REQUIREMENT SPACE

The linear programming problem can be interpreted and solved geometrically in another space, referred to as the requirement space.

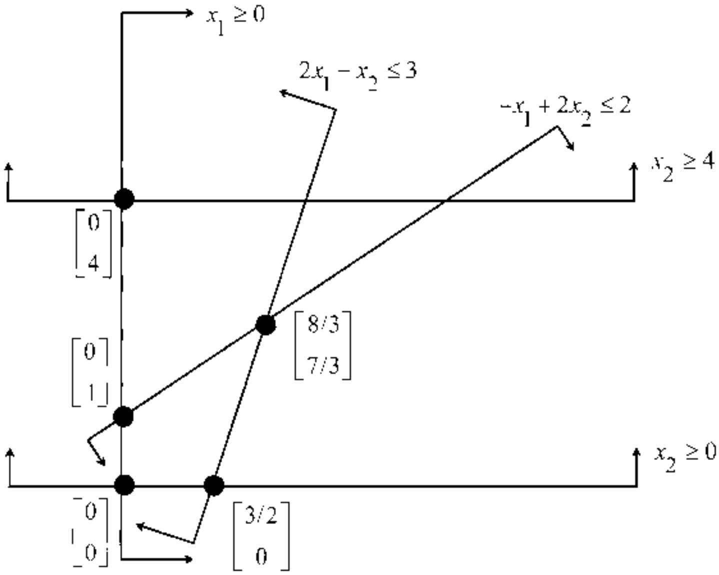

Figure 1.8. An example of an empty feasible region.

Interpretation of Feasibility

Consider the following linear programming problem in standard form:

$$\begin{array}{lrcl} \text{Minimize} & \mathbf{cx} & & \\ \text{subject to} & \mathbf{Ax} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}, \end{array}$$

where $\mathbf{A}$ is an $m \times n$ matrix whose *j*th column is denoted by $\mathbf{a}_j$. The problem can be rewritten as follows:

$$\text{Minimize} \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b}$$

$$x_j \geq 0, \qquad j = 1,\ldots,n.$$

Given the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, we wish to find nonnegative scalars $x_1, x_2, \ldots, x_n$ such that $\sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b}$ and such that $\sum_{j=1}^{n} c_j x_j$ is minimized. Note, however, that the collection of vectors of the form $\sum_{j=1}^{n} \mathbf{a}_j x_j$, where $x_1, x_2, \ldots, x_n \geq 0$, is the cone generated by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ (see Figure 1.9). Thus, the problem has a feasible solution if and only if the vector $\mathbf{b}$ belongs to this cone. Since the vector $\mathbf{b}$ usually reflects requirements to be satisfied, Figure 1.9 is referred to as illustrating the *requirement space*.

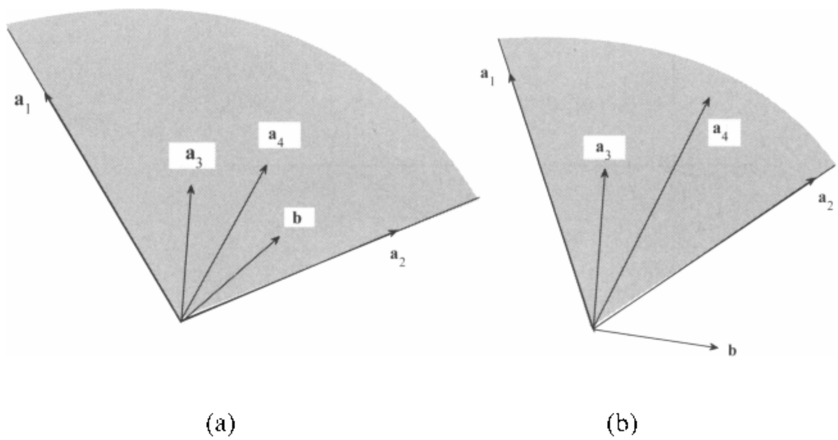

Figure 1.9. Interpretation of feasibility in the requirement space: (a) Feasible region is not empty. (b) Feasible region is empty.

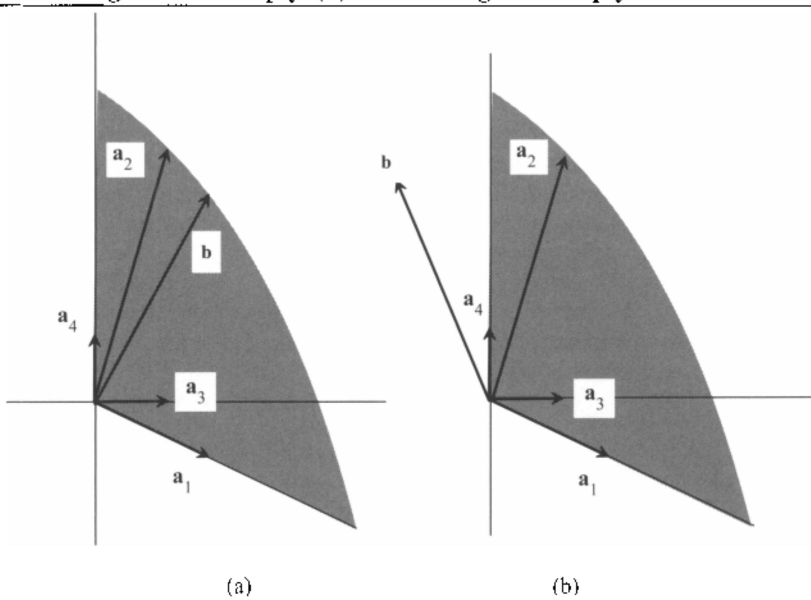

Figure 1.10. Illustration of the requirement space: (a) System 1 is feasible. (b) System 2 is inconsistent.

Example 1.3

Consider the following two systems:

System 1:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & & & = & 2 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 3 \\ x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

System 2:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & & & = & -1 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 2 \\ x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

Figure 1.10 shows the requirement space of both systems. For System 1, the vector **b** belongs to the cone generated by the vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and hence admits feasible solutions. For the second system, **b** does not belong to the corresponding cone and the system is then inconsistent.

The Requirement Space and Inequality Constraints

We now illustrate the interpretation of feasibility for the inequality case. Consider the following inequality system:

$$\sum_{j=1}^{n} \mathbf{a}_j x_j \leq \mathbf{b}$$

$$x_j \geq 0, \quad j = 1,\ldots, n.$$

Note that the collection of vectors $\sum_{j=1}^{n} \mathbf{a}_j x_j$, where $x_j \geq 0$ for $j = 1,\ldots,n$, is the cone generated by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. If a feasible solution exists, then this cone must intersect the collection of vectors that are less than or equal to the requirement vector **b**. Figure 1.11 shows both a feasible system and an infeasible system.

Optimality

We have seen that the system $\sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b}$ and $x_j \geq 0$ for $j = 1,\ldots, n$ is feasible if and only if **b** belongs to the cone generated by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. The variables $x_1, x_2, \ldots, x_n$ must be chosen so that feasibility is satisfied and $\sum_{j=1}^{n} c_j x_j$ is minimized. Therefore, the linear programming problem can be stated as follows. Find nonnegative $x_1, x_2, \ldots, x_n$ such that

$$\begin{bmatrix} c_1 \\ \mathbf{a}_1 \end{bmatrix} x_1 + \begin{bmatrix} c_2 \\ \mathbf{a}_2 \end{bmatrix} x_2 + \cdots + \begin{bmatrix} c_n \\ \mathbf{a}_n \end{bmatrix} x_n = \begin{bmatrix} z \\ \mathbf{b} \end{bmatrix}.$$

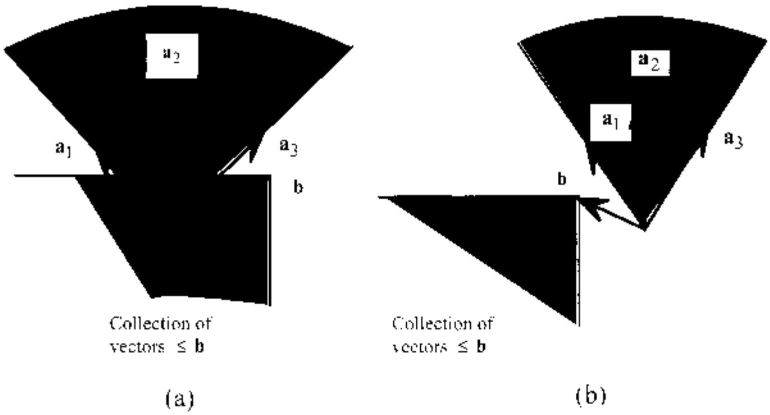

Figure 1.11. Requirement space and inequality constraints: (a) System is feasible. (b) System is infeasible.

where the objective z is to be minimized. In other words we seek to represent the vector $\begin{bmatrix} z \\ \mathbf{b} \end{bmatrix}$, for the smallest possible z, in the cone spanned by the vectors $\begin{bmatrix} c_1 \\ \mathbf{a}_1 \end{bmatrix}, \begin{bmatrix} c_2 \\ \mathbf{a}_2 \end{bmatrix}, \ldots, \begin{bmatrix} c_n \\ \mathbf{a}_n \end{bmatrix}$. The reader should note that the price we must pay for including the objective function explicitly in the requirement space is to increase the dimensionality from m to $m + 1$.

Example 1.4

$$\begin{array}{lrcrcl} \text{Minimize} & -2x_1 & - & 3x_2 & & \\ \text{subject to} & x_1 & + & 2x_2 & \leq & 2 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

Add the slack variable $x_3 \geq 0$. The problem is then to choose $x_1,\ x_2,\ x_3 \geq 0$ such that

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} z \\ 2 \end{bmatrix},$$

where z is to be minimized. The cone generated by the vectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is shown in Figure 1.12. We want to choose a vector $\begin{bmatrix} z \\ 2 \end{bmatrix}$ in this cone having a minimal value for z. This gives the optimal solution $z^* = -4$ with $x_1^* = 2$ (the multiplier associated with $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$) and $x_2^* = x_3^* = 0$.

Example 1.5

$$\begin{array}{lrcrcl} \text{Minimize} & -2x_1 & - & 3x_2 & & \\ \text{subject to} & x_1 & + & 2x_2 & \geq & 2 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

Obviously the optimal objective value is unbounded. We illustrate this fact in the requirement space. Subtracting the slack (or surplus) variable $x_3 \geq 0$, the problem can be restated as follows: Find $x_1,\ x_2,\ x_3 \geq 0$ such that

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} z \\ 2 \end{bmatrix},$$

and such that z is minimized. The cone generated by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is shown in Figure 1.13. We want to choose $\begin{bmatrix} z \\ 2 \end{bmatrix}$ in this cone having the smallest possible value for z. Note that we can find points of the form $\begin{bmatrix} z \\ 2 \end{bmatrix}$ in the cone having an arbitrarily small value for z. Therefore, the objective value z can be driven to $-\infty$, or it is unbounded.

1.5 NOTATION

Throughout the text, we shall utilize notation that is, insofar as possible, consistent with generally accepted standards for the fields of mathematics and operations research. In this section, we indicate some of the notation that may require special attention, either because of its infrequency of use in the linear programming literature, or else because of the possibility of confusion with other terms.

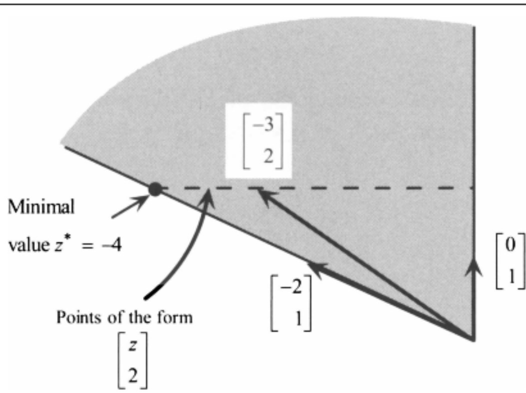

Figure 1.12. Optimal objective value in the requirement space.

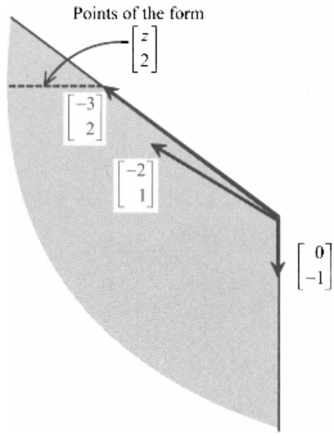

Figure 1.13. Unbounded optimal objective value in the requirement space.

In Chapter 2, we shall review material on vectors and matrices. We indicate vectors by lowercase, boldface Greek or Roman letters or numerals, such as **a**, **b**, **x**, **1**, **λ**; matrices by uppercase, boldface Greek or Roman letters, such as **A**, **B**, **N**, **Φ**; and all scalars by Greek or Roman letters or numerals that are not boldface, such as a, b, 1, ε. Column vectors are generally denoted by subscripts, such as $\mathbf{a}_j$, unless clear in the context. When special emphasis is required, row vectors are indicated by superscripts, such as $\mathbf{a}^i$. A superscript t will denote the transpose operation.

In calculus, the partial derivative, indicated by $\partial z / \partial x$, represents the rate of change in the variable z with respect to (a unit increase in) the variable x. We shall also utilize the symbol $\partial z / \partial \mathbf{x}$ to indicate the vector of partial derivatives of z with respect to each element of the vector **x**. That is, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then

$$\frac{\partial z}{\partial \mathbf{x}} = \left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots, \frac{\partial z}{\partial x_n} \right).$$

Also, we shall sometimes consider the partial derivative of one vector with respect to another vector, such as $\partial \mathbf{y} / \partial \mathbf{x}$. If $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

Note that if z is a function of the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$, then $\partial z / \partial \mathbf{x}$ is called the *gradient* of z.

We shall, when necessary, use (a, b) to refer to the *open interval* $a < x < b$, and $[a, b]$ to refer to the *closed interval* $a \le x \le b$. Finally we shall utilize the standard set operators $\cup$, $\cap$, $\subset$, and $\in$ to refer to union, intersection, set inclusion, and set membership, respectively.

EXERCISES

[1.1] Fred has $5000 to invest over the next five years. At the beginning of each year he can invest money in one– or two–year time deposits. The bank pays 4 percent interest on one–year time deposits and 9 percent (total) on two–year time deposits. In addition, West World Limited will offer three–year certificates starting at the beginning of the second year. These certificates will return 15 percent (total). If Fred reinvests his money that is available every year, formulate a linear program to show him how to maximize his total cash on hand at the end of the fifth year.

[1.2] A manufacturer of plastics is planning to blend a new product from four chemical compounds. These compounds are mainly composed of three elements: A, B, and C. The composition and unit cost of these chemicals are shown in the following table:

CHEMICAL COMPOUND	1	2	3	4
Percentage A	35	15	35	25
Percentage B	20	65	35	40
Percentage C	40	15	25	30
Cost/kilogram	20	30	20	15

The new product consists of 25 percent element A, at least 35 percent element B, and at least 20 percent element C. Owing to side effects of compounds 1 and 2, they must not exceed 25 percent and 30 percent, respectively, of the content of the new product. Formulate the problem of finding the least costly way of blending as a linear program.

[1.3] An agricultural mill manufactures feed for cattle, sheep, and chickens. This is done by mixing the following main ingredients: corn, limestone, soybeans, and fish meal. These ingredients contain the following nutrients: vitamins, protein, calcium, and crude fat. The contents of the nutrients in standard units for each kilogram of the ingredients are summarized in the following table:

	NUTRIENT UNITS			
INGREDIENT	VITAMINS	PROTEIN	CALCIUM	CRUDE FAT
Corn	8	10	6	8
Limestone	6	5	10	6
Soybeans	10	12	6	6
Fish meal	4	8	6	9

The mill is contracted to produce 12, 8, and 9 (metric) tons of cattle feed, sheep feed, and chicken feed. Because of shortages, a limited amount of the ingredients is available—namely, 9 tons of corn, 12 tons of limestone, 5 tons of soybeans, and 6 tons of fish meal. The price per kilogram of these ingredients is, respectively, \$0.20, \$0.12, \$0.24, and \$0.12. The minimal and maximal units of the various nutrients that are permitted is summarized below for a kilogram of the cattle feed, the sheep feed, and the chicken feed.

	NUTRIENT UNITS							
	VITAMINS		PROTEIN		CALCIUM		CRUDE FAT	
PRODUCT	MIN	MAX	MIN	MAX	MIN	MAX	MIN	MAX
Cattle feed	6	∞	6	∞	7	∞	4	8
Sheep feed	6	∞	6	∞	6	∞	4	6
Chicken feed	4	6	6	∞	6	∞	4	5

Formulate this feed-mix problem so that the total cost is minimized.

[1.4] Consider the problem of locating a new machine to an existing layout consisting of four machines. These machines are located at the following coordinates in two-dimensional space: $\begin{pmatrix}3\\1\end{pmatrix}$, $\begin{pmatrix}0\\-3\end{pmatrix}$, $\begin{pmatrix}-2\\2\end{pmatrix}$, and $\begin{pmatrix}1\\4\end{pmatrix}$. Let the coordinates of the new machine be $\begin{pmatrix}x_1\\x_2\end{pmatrix}$. Formulate the problem of finding an optimal location as a linear program for each of the following cases:

a. The sum of the distances from the new machine to the four machines is minimized. Use the *street distance* (also known as *Manhattan distance* or *rectilinear* distance); for example, the distance from $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ to the first machine located at $\begin{pmatrix}3\\1\end{pmatrix}$ is $|x_1 - 3| + |x_2 - 1|$.
b. Because of various amounts of flow between the new machine and the existing machines, reformulate the problem where the sum of the weighted distances is minimized, where the weights corresponding to the four machines are 6, 4, 7, and 2, respectively.
c. In order to avoid congestion, suppose that the new machine must be located in the square $\{(x_1, x_2) : -1 \le x_1 \le 2,\ 0 \le x_2 \le 1\}$. Formulate Parts (a) and (b) with this added restriction.
d. Suppose that the new machine must be located so that its distance from the first machine does not exceed 2. Formulate the problem with this added restriction.

[1.5] The technical staff of a hospital wishes to develop a computerized menu-planning system. To start with, a lunch menu is sought. The menu is divided into three major categories: vegetables, meat, and dessert. At least one equivalent serving of each category is desired. The cost per serving of some suggested items as well as their content of carbohydrates, vitamins, protein, and fats is summarized below:

	CARBO–HYDRATES	VITAMINS	PROTEIN	FATS	COST IN $/SERVING
Vegetables					
Peas	1	3	1	0	0.10
Green beans	1	5	2	0	0.12
Okra	1	5	1	0	0.13
Corn	2	6	1	2	0.09
Macaroni	4	2	1	1	0.10
Rice	5	1	1	1	0.07
Meat					
Chicken	2	1	3	1	0.70
Beef	3	8	5	2	1.20
Fish	3	6	6	1	0.63
Dessert					
Orange	1	3	1	0	0.28
Apple	1	2	0	0	0.42
Pudding	1	0	0	0	0.15
Jello	1	0	0	0	0.12

Suppose that the minimal requirements of carbohydrates, vitamins, protein, and fats per meal are respectively 5, 10, 10, and 2.

a. Formulate the menu–planning problem as a linear program.

b. Many practical aspects have been left out in the foregoing model. These include planning the breakfast, lunch, and supper menus together, weekly planning so that different varieties of food are used, and special menus for patients on particular diets. Discuss in detail how these aspects can be incorporated in a comprehensive menu–planning system.

[1.6] A cheese firm produces two types of cheese: swiss cheese and sharp cheese. The firm has 60 experienced workers and would like to increase its working force to 90 workers during the next eight weeks. Each experienced worker can train three new employees in a period of two weeks during which the workers involved virtually produce nothing. It takes one man–hour to produce 10 pounds of Swiss cheese and one man–hour to produce 6 pounds of sharp cheese. A work week is 40 hours. The weekly demands (in 1000 pounds) are summarized below:

	WEEK							
CHEESE TYPE	1	2	3	4	5	6	7	8
Swiss cheese	11	12	13	18	14	18	20	20
Sharp cheese	8	8	10	8	12	13	12	12

Suppose that a trainee receives the same full salary as an experienced worker. Further suppose that overaging destroys the flavor of the cheese, so that inventory is limited to one week. How should the company hire and train its new employees so that the labor cost is minimized over this 8–week period? Formulate the problem as a linear program.

[1.7] A company wishes to plan its production of two items with seasonal demands over a 12–month period. The monthly demand of item 1 is 100,000 units during the months of October, November, and December; 10,000 units during the months of January, February, March, and April; and 30,000 units during the remaining months. The demand of item 2 is 50,000 during the months of October through February and 15,000 during the remaining months. Suppose that the unit product cost of items 1 and 2 is $5.00 and $8.50, respectively, provided that these were manufactured prior to June. After June, the unit costs are reduced to $4.50 and $7.00 because of the installation of an improved manufacturing system. The total units of items 1 and 2 that can be manufactured during any particular month cannot exceed 120,000 for Jan- Sept, and 150,000 for Oct–Dec. Furthermore, each unit of item 1 occupies 2 cubic feet and each unit of item 2 occupies 4 cubic feet of inventory space. Suppose that the maximum inventory space allocated to these items is 150,000 cubic feet and that the holding cost per cubic foot during any month is $0.20. Formulate the production scheduling problem so that the total cost of production and inventory is minimized.

[1.8] A textile mill produces five types of fabrics. The demand (in thousand yards) over a quarter–year time horizon for these fabrics is 16, 48, 37, 21, and 82, respectively. These five fabrics are woven, finished, and sold in the market at prices 0.9, 0.8, 0.8, 1.2, and 0.6 $/per yard, respectively. Besides weaving and finishing the fabrics at the mill itself, the fabrics are also purchased woven from outside sources and are then finished at the mill before being sold. If the unfinished fabrics are purchased outside, the costs in $/per yard for the five fabrics are 0.8, 0.7, 0.75, 0.9, and 0.7, respectively. If produced at the mill itself, the respective costs are 0.6, 0.5, 0.6, 0.7, and 0.3 $/per yard. There are two types of looms that can produce the fabrics at the mill, that is, there are 10 Dobbie looms and 80 regular looms. The production rate of each Dobbie loom is 4.6, 4.6, 5.2, 3.8, and 4.2 yards per hour for the five fabrics. The regular looms have the same production rates as the Dobbie looms, but they can only produce fabric types 3, 4, and 5. Assuming that the mill operates seven days a week and 24 hours a day, formulate the problem of optimally planning to meet the demand over a quarter–year horizon as a linear program. Is your formulation a transportation problem? If not, reformulate the problem as a transportation problem.

[1.9] A steel manufacturer produces four sizes of I beams: small, medium, large, and extra large. These beams can be produced on any one of three machine types: A, B, and C. The lengths in feet of the I beams that can be produced on the machines per hour are summarized below:

	MACHINE		
BEAM	A	B	C
Small	350	650	850
Medium	250	400	700
Large	200	350	600
Extra large	125	200	325

Assume that each machine can be used up to 50 hours per week and that the hourly operating costs of these machines are respectively $30.00, $50.00, and

\$80.00. Further suppose that 12,000, 6000, 5000, and 7000 feet of the different size I beams are required weekly. Formulate the machine scheduling problem as a linear program.

[1.10] An oil refinery can buy two types of oil: light crude oil and heavy crude oil. The cost per barrel of these types is respectively \$20 and \$15. The following quantities of gasoline, kerosene, and jet fuel are produced per barrel of each type of oil.

	GASOLINE	KEROSENE	JET FUEL
Light crude oil	0.4	0.2	0.35
Heavy crude oil	0.32	0.4	0.2

Note that 5 percent and 8 percent, respectively, of the light and heavy crude oil are lost during the refining process. The refinery has contracted to deliver 1 million barrels of gasoline, 500,000 barrels of kerosene, and 300,000 barrels of jet fuel. Formulate the problem of finding the number of barrels of each crude oil that satisfies the demand and minimizes the total cost as a linear program.

[1.11] A lathe is used to reduce the diameter of a steel shaft whose length is 36 in. from 14 in. to 12 in. The speed x_1 (in revolutions per minute), the depth feed x_2 (in inches per minute), and the length feed x_3 (in inches per minute) must be determined. The duration of the cut is given by $36/x_2x_3$. The compression and side stresses exerted on the cutting tool are given by $30x_1 + 4500x_2$ and $40x_1 + 5000x_2 + 5000x_3$ pounds per square inch, respectively. The temperature (in degrees Fahrenheit) at the tip of the cutting tool is $200 + 0.5x_1 + 150(x_2 + x_3)$. The maximum compression stress, side stress, and temperature allowed are 150,000 psi, 100,000 psi, and 800°F, respectively. It is desired to determine the speed (which must be in the range from 600 rpm to 800 rpm), the depth feed, and the length feed such that the duration of the cut is minimized. In order to use a linear model, the following approximation is made. Since $36/x_2x_3$ is minimized if and only if x_2x_3 is maximized, it was decided to replace the objective by the maximization of the minimum of x_2 and x_3. Formulate the problem as a linear model and comment on the validity of the approximation used in the objective function.

[1.12] A television set manufacturing firm has to decide on the mix of color and black–and–white TVs to be produced. A market research indicates that, at most, 2000 units and 4000 units of color and black–and–white TVs can be sold per month. The maximum number of man–hours available is 60,000 per month. A color TV requires 20 man–hours and a black–and–white TV requires 15 man–hours to manufacture. The unit profits of the color and black–and–white TVs are \$60 and \$30, respectively. It is desired to find the number of units of each TV type that the firm must produce in order to maximize its profit. Formulate the problem as a linear program.

[1.13] A production manager is planning the scheduling of three products on four machines. Each product can be manufactured on each of the machines. The unit production costs (in \$) are summarized below.

PRODUCT	MACHINE 1	2	3	4
1	4	4	5	7
2	6	7	5	6
3	12	10	8	11

The time (in hours) required to produce a unit of each product on each of the machines is summarized below.

PRODUCT	MACHINE 1	2	3	4
1	0.3	0.25	0.2	0.2
2	0.2	0.3	0.2	0.25
3	0.8	0.6	0.6	0.5

Suppose that 3000, 6000, and 4000 units of the products are required, and that the available machine–hours are 1500, 1200, 1500, and 2000, respectively. Formulate the scheduling problem as a linear program.

[1.14] A furniture manufacturer has three plants that need 500, 700, and 600 tons of lumber weekly. The manufacturer may purchase the lumber from three lumber companies. The first two lumber manufacturers virtually have an unlimited supply and, because of other commitments, the third manufacturer cannot ship more than 500 tons weekly. The first lumber manufacturer uses rail for transportation and there is no limit on the tonnage that can be shipped to the furniture facilities. On the other hand, the last two lumber companies use trucks that limit the maximum tonnage that can be shipped to any of the furniture companies to 200 tons. The following table gives the transportation cost from the lumber companies to the furniture manufacturers ($ per ton).

LUMBER COMPANY	FURNITURE FACILITY 1	2	3
1	1	3	5
2	3.5	4	4.8
3	3.5	3.6	3.2

Formulate the problem as a linear program.

[1.15] A company manufactures an assembly consisting of a frame, a shaft, and a ball bearing. The company manufactures the shafts and frames but purchases the ball bearings from a ball bearing manufacturer. Each shaft must be processed on a forging machine, a lathe, and a grinder. These operations require 0.6 hour, 0.3 hour, and 0.4 hour per shaft, respectively. Each frame requires 0.8 hour on a forging machine, 0.2 hour on a drilling machine, 0.3 hour on a milling machine, and 0.6 hour on a grinder. The company has 5 lathes, 10 grinders, 20 forging machines, 3 drillers, and 6 millers. Assume that each machine operates a maximum of 4500 hours per year. Formulate the problem of finding the maximum number of assembled components that can be produced as a linear program.

[1.16] A corporation has $30 million available for the coming year to allocate to its three subsidiaries. Because of commitments to stability of personnel employment and for other reasons, the corporation has established a minimal level of funding for each subsidiary. These funding levels are \$3 million, \$5 million, and \$8 million, respectively. Owing to the nature of its operation, subsidiary 2 cannot utilize more than \$17 million without major new capital expansion. The corporation is unwilling to undertake such an expansion at this time. Each subsidiary has the opportunity to conduct various projects with the funds it receives. A rate of return (as a percent of investment) has been established for each project. In addition, certain projects permit only limited investment. The data of each project are given below:

SUBSIDIARY	PROJECT	RATE OF RETURN	UPPER LIMIT OF INVESTMENT
	1	7%	\$6 million
1	2	5%	\$5 million
	3	8%	\$9 million
	4	5%	\$7 million
2	5	7%	\$10 million
	6	9%	\$4 million
3	7	10%	\$6 million
	8	8%	\$3 million

Formulate this problem as a linear program.

[1.17] A 10–acre slum in New York City is to be cleared. The officials of the city must decide on the redevelopment plan. Two housing plans are to be considered: low–income housing and middle–income housing. These types of housing can be developed at 20 and 15 units per acre, respectively. The unit costs of the low– and middle–income housing are \$17,000 and \$25,000. The lower and upper limits set by the officials on the number of low–income housing units are 80 and 120. Similarly, the number of middle–income housing units must lie between 40 and 90. The combined maximum housing market potential is estimated to be 190 (which is less than the sum of the individual market limits due to the overlap between the two markets). The total mortgage committed to the renewal plan is not to exceed \$2.5 million. Finally, it was suggested by the architectural adviser that the number of low–income housing units be at least 50 units greater than one–half the number of the middle income housing units.

a. Formulate the minimum cost renewal planning problem as a linear program and solve it graphically.
b. Resolve the problem if the objective is to maximize the number of houses to be constructed.

[1.18] Consider the following problem of launching a rocket to a fixed altitude b in a given time T while expending a minimum amount of fuel. Let $u(t)$ be the acceleration force exerted at time t and let $y(t)$ be the rocket altitude at time t. The problem can be formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & \int_0^T |u(t)|\, dt \\ \text{subject to} \quad & \ddot{y}(t) = u(t) - g \end{aligned}$$

$$y(T) = b$$
$$y(t) \geq 0, \quad t \in [0,T],$$

where g is the gravitational force and $\ddot{y}$ is the second derivative of the altitude y. Discretize the problem and reformulate it as a linear programming problem. In particular, formulate the problem for $T = 15$, $b = 20$, and $g = 32$. (*Hint*: Replace the integration by a proper summation and the differentiation by difference equations. Make the change of variables $|u_j| = x_j$, $\forall j$, based on the discretization, and note that $x_j \geq u_j$ and $x_j \geq -u_j$.)

[1.19] Consider the Multiperiod Coal Blending and Distribution Problem presented in Section 1.2. For the developed model, we have ignored the effect of any production and distribution decisions made prior to period $t = 1$ that might affect the present horizon problem, and we have also neglected to consider how production decisions during the present horizon might impact demand beyond period T (especially considering the shipment lag phenomenon). Provide a detailed discussion on the impact of such considerations, and suggest appropriate modifications to the resulting model.

[1.20] Consider the Multiperiod Coal Blending and Distribution Problem presented in Section 1.2, and assume that the *shipment lag* is zero (but that the initial storage at the silos is still assumed to be dissipated in three periods). Defining the *principal decision variables* as y_{ijt}^k = amount (tons) of coal shipped from mine i through silo j to customer k during a particular time period t, and $y_{jk\tau}^0$ = amount (tons) of coal that is in initial storage at silo j, which is shipped to customer k in period τ (where τ = 1, 2, 3), and defining all other auxiliary variables as before, reformulate this problem as a linear program. Discuss its size, structure, and assumptions relative to the model formulated in Section 1.2.

[1.21] Consider the Multiperiod Coal Blending and Distribution Problem presented in Section 1.2. Suppose that we define the *principal decision variables* as y_{ijt}^1 = amount (tons) of coal shipped from mine i to silo j in period t; y_{jkt}^2 = amount (tons) of coal shipped from silo j to customer k in period t; and y_{jkt}^0 = amount (tons) of coal that is in initial storage at silo j, which is shipped to customer k in period t (where t = 1, 2, 3), with all other auxiliary decision variables being defined as before. Assuming that coal shipped from any mine i to a silo j will essentially affect the ash and sulfur content at silo j for only three periods as before (defined as the *shipment lag*), derive expressions for the percentage ash and percentage sulfur contents of the coal that would be available for shipment at silo j during each period t, in terms of the foregoing decision variables. Discuss how you might derive some fixed constant estimates for these values. Using these estimated constant values, formulate the coal distribution and blending problem as a linear program in terms of the just–mentioned decision variables. Discuss the size, structure, and assumptions of this model relative to the model formulated in Section 1.2. Also, comment on the nature of the resulting model if we used the actual expressions for the ash and sulfur content of coal at each silo j during each period t, in lieu of the constant estimates.

[1.22] A region is divided into m residential and central business districts. Each district is represented by a node, and the nodes are interconnected by links representing major routes. People living in the various districts go to their business in the same and/or at other districts so that each node attracts and/or generates a number of trips. In particular, let a_{ij} be the number of trips generated at node i with final destination at node j and let b_{ij} be the time to travel from node i to node j. It is desired to determine the routes to be taken by the people living in the region.

a. Illustrate the problem by a suitable network.
b. Develop some measures of effectiveness for this *traffic assignment problem*, and for each measure, devise a suitable model.

[1.23] Consider the problem of scheduling court hearings over a planning horizon consisting of n periods. Let b_j be the available judge–hours in period j, h_{ij} be the number of hearings of class i arriving in period j, and a_i be the number of judge–hours required to process a hearing of class i. It is desired to determine the number of hearings x_{ij} of class i processed in period j.

a. Formulate the problem as a linear program.
b. Modify the model in Part (a) so that hearings would not be delayed for too long.

[1.24] Consider the following multiperiod, multiproduct *Production- Inventory Problem*. Suppose that we are examining T periods $t = 1,..., T$, and some n products $i = 1,..., n$. There is an initial inventory of y_{i0} that is available at hand for each product i, $i = 1,..., n$. We need to principally determine the level of production for each product $i = 1,..., n$ during each of the periods $t = 1,..., T$, so as to meet the forecasted demand d_{it} for each product i during each period t when it occurs, at a minimal total cost. To take advantage of varying demands and costs, we are permitted to produce excess quantities to be stored for later use. However, each unit of product i consumes a storage space s_i, and incurs a storage cost of c_{it} during period t, for $i = 1,..., n$, $t = 1,..., T$. The total storage space anticipated to be available for these n products during period t is S_t, $t = 1$,..., T. Furthermore, each unit of product i requires h_i hours of labor to produce, where the labor cost per hour when producing a unit of product i during period t is given by ℓ_{it}, for $i = 1,..., n$, $t = 1,..., T$. The total number of labor hours available to manufacture these n products during period t is given by H_t. Formulate a linear program to solve this production–inventory control system problem, prescribing production as well as inventory levels for each product over each time period.

[1.25] Suppose that there are m sources that generate waste and n disposal sites. The amount of waste generated at source i is a_i and the capacity of site j is b_j. It is desired to select appropriate transfer facilities from among K candidate facilities. Potential transfer facility k has a fixed cost f_k, capacity q_k, and unit processing cost α_k per ton of waste. Let c_{ik} and $\overline{c}_{kj}$ be the unit shipping costs from source i to transfer station k and from transfer station k to disposal site j, respectively. The problem is to choose the transfer facilities and the shipping pattern that minimize the total capital and operating costs of the transfer stations plus the transportation costs. Formulate this *distribution problem*. (*Hint*: Let y_k be 1 if transfer station k is selected and 0 otherwise.)

[1.26] A governmental planning agency wishes to determine the purchasing sources for fuel for use by n depots from among m bidders. Suppose that the maximum quantity offered by bidder i is a_i gallons and that the demand of depot j is b_j gallons. Let c_{ij} be the unit delivery cost for bidder i to the jth depot.

a. Formulate the problem of minimizing the total purchasing cost as a linear program.
b. Suppose that a discount in the unit delivery cost is offered by bidder i if the ordered quantity exceeds the level α_i. How would you incorporate this modification in the model developed in Part (a)?

[1.27] The quality of air in an industrial region largely depends on the effluent emission from n plants. Each plant can use m different types of fuel. Suppose that the total energy needed at plant j is b_j British Thermal Units per day and that c_{ij} is the effluent emission per ton of fuel type i at plant j. Further suppose that fuel type i costs c_i dollars per ton and that each ton of this fuel type generates α_{ij} British Thermal Units at plant j. The level of air pollution in the region is not to exceed b micrograms per cubic meter. Finally, let γ_j be a meteorological parameter relating emissions at plant j to air quality in the region.

a. Formulate the problem of determining the mix of fuels to be used at each plant.
b. How would you incorporate technological constraints that prohibit the use of certain mixes of fuel at certain plants?
c. How could you ensure equity among the plants?

[1.28] For some industry, let $p(Q)$, $Q \geq 0$, be an inverse demand curve, that is, $p(Q)$ is the price at which a quantity Q will be demanded. Let $f(Q)$, $Q \geq 0$, be a supply or marginal cost curve, that is, $f(Q)$ is the price at which the industry is willing to supply Q units of the product. Consider the problem of determining a (perfect competition) equilibrium price and production quantity p^* and Q^* for $p(\cdot)$ and $f(\cdot)$ of the type shown in Figure 1.14 via the intersection of these supply and demand curves. Note that Q^* is obtainable by finding that Q, which maximizes the area under the demand curve minus the area under the supply curve. Now, suppose that $f(\cdot)$ is not available directly, but being a marginal cost curve, is given implicitly by

$$\int_0^Q f(x)\,dx = \min\{\mathbf{cy}: \mathbf{Ay} = \mathbf{b},\ \boldsymbol{\alpha}\,\mathbf{y} = Q,\ \mathbf{y} \geq 0\},$$

where $\mathbf{A}$ is $m \times n$, $\boldsymbol{\alpha}$ and $\mathbf{c}$ are $1 \times n$, and $\mathbf{y}$ is an n–vector representing a set of production activities. Further, suppose that the demand curve $p(\cdot)$ is approximated using the steps shown in the figure. (Let there be s steps used, each of height H_i and width W_i for $i = 1,\ldots, s$.) With this approximation, formulate the problem of determining the *price–quantity equilibrium* as a linear program.

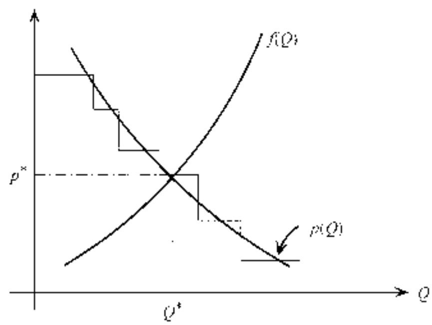

Figure 1. 14. Supply demand equilibrium for Exercise 1.28

[1.29] Consider the following *two–stage stochastic program with recourse.* Suppose that in a time–stage process, we need to make some immediate decision ("here and now") that is represented by the decision variable vector $\mathbf{x} \geq \mathbf{0}$. (For example, $\mathbf{x}$ might represent some investment or production decisions.) This decision must satisfy the constraints $\mathbf{Ax} = \mathbf{b}$. Subsequent to this, a random demand vector $\mathbf{d}$, say, will be realized according to some probability distribution. Based on the initial decision $\mathbf{x}$ and the realization $\mathbf{d}$, we will need to make a recourse decision $\mathbf{y} \geq \mathbf{0}$ (e.g., some subsequent additional production decisions), such that the constraint $\mathbf{Dx} + \mathbf{Ey} = \mathbf{d}$ is satisfied. The total objective cost for this pair of decisions is given by $\mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y}$. Assume that the random demand vector $\mathbf{d}$ can take on a discrete set of possible values $\mathbf{d}_1$, $\mathbf{d}_2$, ..., $\mathbf{d}_q$ with respective known probabilities $p_1, p_2,\ldots,p_q$, where $\sum_{r=1}^{q} p_r = 1$, $p_r > 0$, $\forall r = 1,\ldots,q$. (Accordingly, note that each realization $\mathbf{d}_r$ will prompt a corresponding recourse decision $\mathbf{y}_r$.) Formulate the problem of minimizing the total expected cost subject to the initial and recourse decision constraints as a linear programming problem. Do you observe any particular structure in the overall constraint coefficient matrix with respect to the possible nonzero elements? (Chapter 7 explores ways for solving such problems.)

[1.30] Consider the following linear programming problem.

$$\begin{array}{lrcrcrcr}
\text{Minimize} & x_1 & - & 2x_2 & - & 3x_3 & & \\
\text{subject to} & -x_1 & + & 3x_2 & + & x_3 & \leq & 13 \\
& x_1 & + & 2x_2 & + & 3x_3 & \geq & 12 \\
& 2x_1 & - & x_2 & + & x_3 & = & 4 \\
& x_1, & & x_2 & & \text{unrestricted} & & \\
& & & & & x_3 & \leq & -3.
\end{array}$$

a. Reformulate the problem so that it is in standard format.
b. Reformulate the problem so that it is in canonical format.
c. Convert the problem into a maximization problem.

[1.31] Consider the following problem:

$$\begin{array}{lrcrcr}
\text{Maximize} & x_1 & - & x_2 & & \\
\text{subject to} & -x_1 & + & 3x_2 & \leq & 0 \\
& -3x_1 & + & 2x_2 & \geq & -3 \\
& x_1, & & x_2 & \geq & 0.
\end{array}$$

a. Sketch the feasible region in the (x_1, x_2) space.
b. Identify the regions in the (x_1, x_2) space where the slack variables x_3 and x_4, say, are equal to zero.
c. Solve the problem geometrically.
d. Draw the requirement space and interpret feasibility.

[1.32] Consider the feasible region sketched in Figure 1.5*b*. Geometrically, identify conditions on the objective gradient vector **c** for which the different points in the feasible region will be optimal and identify the vectors **c** for which no optimum exists. (Assume a minimization problem.)

[1.33] Sketch the feasible region of the set $\{\mathbf{x}: \mathbf{Ax} \leq \mathbf{b}\}$ where **A** and **b** are as given below. In each case, state whether the feasible region is empty or not and whether it is bounded or not.

a. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$

b. $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 5 \end{bmatrix}$

c. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -12 \\ 0 \end{bmatrix}$

[1.34] Consider the following problem:

$$\begin{array}{lrcrcl} \text{Maximize} & 3x_1 & + & x_2 & & \\ \text{subject to} & -x_1 & + & 2x_2 & \leq & 0 \\ & & & x_2 & \leq & 4. \end{array}$$

a. Sketch the feasible region.
b. Verify that the problem has an unbounded optimal solution value.

[1.35] Consider the following problem:

$$\begin{array}{lrcrcl} \text{Maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 2 \\ & 4x_1 & + & 6x_2 & \leq & 9 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

a. Sketch the feasible region.
b. Find two alternative optimal extreme (corner) points.
c. Find an infinite class of optimal solutions.

[1.36] Consider the following problem:

$$\begin{array}{lrcrcrcrcl} \text{Maximize} & -x_1 & - & x_2 & + & 2x_3 & + & x_4 & & \\ \text{subject to} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & \geq & 6 \\ & x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & \leq & 4 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

a. Introduce slack variables and draw the requirement space.
b. Interpret feasibility in the requirement space.
c. You are told that an optimal solution can be obtained by having at most two positive variables while all other variables are set at zero. Utilize this statement and the requirement space to find an optimal solution.

[1.37] Consider the following problem:

$$\begin{array}{lrcrcl} \text{Maximize} & 3x_1 & + & 6x_2 & & \\ \text{subject to} & -x_1 & + & 2x_2 & \leq & 2 \\ & -2x_1 & + & x_2 & \leq & 0 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

a. Graphically identify the set of all alternative optimal solutions to this problem.
b. Suppose that a secondary priority objective function seeks to maximize $-3x_1 + x_2$ over the set of alternative optimal solutions identified in Part (a). What is the resulting solution obtained?
c. What is the solution obtained if the priorities of the foregoing two objective functions is reversed?

[1.38] Consider the problem: Minimize $\mathbf{cx}$ subject to $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Suppose that one component of the vector $\mathbf{b}$, say b_i, is increased by one unit to $b_i + 1$.

a. What happens to the feasible region?
b. What happens to the optimal objective value?

[1.39] From the results of the previous problem, assuming $\partial z^* / \partial b_i$ exists, is it ≤ 0, $= 0$, or ≥ 0?

[1.40] Solve Exercises 1.38 and 1.39 if the restrictions $\mathbf{Ax} \geq \mathbf{b}$ are replaced by $\mathbf{Ax} \leq \mathbf{b}$.

[1.41] Consider the problem: Minimize $\mathbf{cx}$ subject to $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Suppose that a new constraint is added to the problem.

a. What happens to the feasible region?
b. What happens to the optimal objective value z^*?

[1.42] Consider the problem: Minimize $\mathbf{cx}$ subject to $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Suppose that a new variable is added to the problem.

a. What happens to the feasible region?
b. What happens to the optimal objective value z^*?

[1.43] Consider the problem: Minimize $\mathbf{cx}$ subject to $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Suppose that a constraint, say constraint i, is deleted from the problem.

a. What happens to the feasible region?
b. What happens to the optimal objective value z^*?

[1.44] Consider the problem: Minimize $\mathbf{cx}$ subject to $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Suppose that a variable, say, x_k, is deleted from the problem.

a. What happens to the feasible region?
b. What happens to the optimal objective value z^*?

NOTES AND REFERENCES

1. Linear programming and the simplex method were developed by Dantzig in 1947 in connection with military planning. A great deal of work has influenced the development of linear programming, including World War II operations and the need for scheduling supply and maintenance operations as well as training of Air Force personnel: Leontief's input - output model [1951], von Neumann's Equilibrium Model [1937], Koopmans' Model of Transportation [1949], the Hitchcock transportation problem [1941], the work of Kantorovich [1958], von Neumann-Morgenstern game theory [1944], and the rapid progress in electronic computing machines have also impacted the field of linear programming. The papers by Dantzig [1982] and by Albers and Reid [1986] provide good historical developments.
2. Linear programming has found numerous applications in the military, the government, industry, and urban engineering. See Swanson [1980], for example.
3. Linear programming is also frequently used as a part of general computational schemes for solving nonlinear programming problems, discrete programs, combinatorial problems, problems of optimal control, and programming under uncertainty.
4. Exercise 1.8 is derived from Camm et al. [1987], who discuss some modeling issues. For more detailed discussions, see Woolsey and Swanson [1975], Woolsey [2003], and Swanson [1980]. The Multiperiod Coal Blending and Distribution Problem discussed in Section 1.2, and Exercises 1.19–1.21, are adapted from Sherali and Puri [1993] (see also Sherali and Saifee [1993]). For a discussion on structured modeling and on computer–assisted/artificial intelligence approaches to linear

programming modeling, see Geoffrion [1987], Greenberg [1983], and Murphy and Stohr [1986], for example.

5. For discussions on diagnosing infeasibilities and related *Irreducible Infeasible System* (IIS) issues, see Amaldi et al. [2003], Greenberg [1996], Greenberg and Murphy [1991], and Parker and Ryan [1996].
6. For an excellent discussion on modeling issues and insights related to formulating and solving real-world problems, the reader is referred to Brown and Rosenthal [2008] (also, see the other references cited therein).