PART I

DETERMINISTIC THEORY

CHAPTER 1

Introduction

The cavities discussed in Part I consist of a region of finite extent bounded by conducting walls and filled with a uniform dielectric (usually free space). After a brief discussion of fundamentals of electromagnetic theory, the general properties of cavity modes and their excitation will be given in this chapter. The remaining three chapters of Part I give detailed expressions for the modal resonant frequencies and field structures, quality (Q) factor [1], and Dyadic Green's Functions [2] for commonly used cavities of separable geometries (rectangular cavity in Chapter 2, circular cylindrical cavity in Chapter 3, and spherical cavity in Chapter 4). The International System of Units (SI) is used throughout.

1.1 MAXWELL'S EQUATIONS

Since this book deals almost exclusively with time-harmonic fields, the field and source quantities have a time variation of $\exp(-i\omega t)$, where the angular frequency ω is given by $\omega = 2\pi f$. The time dependence is suppressed throughout. The differential forms of Maxwell's equations are most useful in modal analysis of cavity fields. If we follow Tai [2], the three independent Maxwell equations are:

$$\nabla \times \vec{E} = i\omega\vec{B}, \tag{1.1}$$

$$\nabla \times \vec{H} = \vec{J} - i\omega\vec{D}, \tag{1.2}$$

$$\nabla \bullet \vec{J} = i\omega\rho, \tag{1.3}$$

where $\vec{E}$ is the electric field strength (volts/meter), $\vec{B}$ is the magnetic flux density (teslas), $\vec{H}$ is the magnetic field strength (amperes/meter), $\vec{D}$ is the electric flux density (coulombs/meter^2), $\vec{J}$ is the electric current density (amperes/meter^2), and ρ is the electric charge density (coulombs/meter^3). Equation (1.1) is the differential form of Faraday's law, (1.2) is the differential form of the Ampere-Maxwell law, and (1.3) is the equation of continuity.

Electromagnetic Fields in Cavities: Deterministic and Statistical Theories, by David A. Hill

Two dependent Maxwell equations can be obtained from (1.1)–(1.3). Taking the divergence of (1.1) yields:

$$\nabla \bullet \vec{B} = 0 \tag{1.4}$$

Taking the divergence of (1.2) and substituting (1.3) into that result yields

$$\nabla \bullet \vec{D} = \rho \tag{1.5}$$

Equation (1.4) is the differential form of Gauss's magnetic law, and (1.5) is the differential form of Gauss's electric law. An alternative point of view is to consider (1.1), (1.2), and (1.5) as independent and (1.3) and (1.4) as dependent, but this does not change any of the equations. Sometimes a magnetic current is added to the right side of (1.1) and a magnetic charge is added to the right side of (1.4) in order to introduce duality [3] into Maxwell's equations. However, we choose not to do so.

The integral or time dependent forms of (1.1)–(1.5) can be found in numerous textbooks, such as [4]. The vector phasors, for example $\vec{E}$, in (1.1)–(1.5) are complex quantities that are functions of position $\vec{r}$ and angular frequency ω, but this dependence will be omitted except where required for clarity. The time and space dependence of the real field quantities, for example electric field $\vec{\mathcal{E}}$, can be obtained from the vector phasor quantity by the following operation:

$$\vec{\mathcal{E}}(\vec{r}, t) = \sqrt{2}\,\mathrm{Re}[\vec{E}(\vec{r}, \omega)\exp(-i\omega t)], \tag{1.6}$$

where Re represents the real part. The introduction of the $\sqrt{2}$ factor in (1.6) follows Harrington's notation [3] and eliminates a 1/2 factor in quadratic quantities, such as power density and energy density. It also means that the vector phasor quantities represent root-mean-square (RMS) values rather than peak values.

In order to solve Maxwell's equations, we need more information in the form of the constitutive relations. For isotropic media, the constitutive relations are written:

$$\vec{D} = \varepsilon \vec{E}, \tag{1.7}$$

$$\vec{B} = \mu \vec{H}, \tag{1.8}$$

$$\vec{J} = \sigma \vec{E}, \tag{1.9}$$

where ε is the permittivity (farads per meter), μ is the permeability (henrys/meter), and σ is the conductivity (siemens/meter). In general, ε, μ, and σ are frequency dependent and complex. Actually, there are more general constitutive relations [5] than those shown in (1.7)–(1.9), but we will not require them.

In many problems, $\vec{J}$ is treated as a source current density rather than an induced current density, and the problem is to determine $\vec{E}$ and $\vec{H}$ subject to specified boundary conditions. In this case (1.1) and (1.2) can be written:

$$\nabla \times \vec{E} = i\omega\mu\vec{H}, \tag{1.10}$$

$$\nabla \times \vec{H} = \vec{J} - i\omega\varepsilon\vec{E} \tag{1.11}$$

Equations (1.10) and (1.11) are two vector equations in two vector unknowns ($\vec{E}$ and $\vec{H}$) or equivalently six scalar equations in six scalar unknowns. By eliminating either $\vec{H}$ in (1.10) or $\vec{E}$ in (1.11), we can obtain inhomogeneous vector wave equations:

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = i\omega\mu\vec{J}, \tag{1.12}$$

$$\nabla \times \nabla \times \vec{H} - k^2 \vec{H} = \nabla \times \vec{J}, \tag{1.13}$$

where $k = \omega\sqrt{\mu\varepsilon}$. Chapters 2 through 4 will contain sections where dyadic Green's functions provide compact solutions to (1.12) and (1.13) and satisfy the boundary conditions at the cavity walls.

1.2 EMPTY CAVITY MODES

Consider a simply connected cavity of arbitrary shape with perfectly conducting electric walls as shown in Figure 1.1. The interior of the cavity is filled with a homogeneous dielectric of permittivity ε and permeability μ. The cavity has volume V and surface area S. Because the walls have perfect electric conductivity, the tangential electric field at the wall surface is zero:

$$\hat{n} \times \vec{E} = 0, \tag{1.14}$$

where $\hat{n}$ is the unit normal directed outward from the cavity. Because the cavity is source free and the permittivity is independent of position, the divergence of the electric field is zero:

$$\nabla \bullet \vec{E} = 0 \tag{1.15}$$

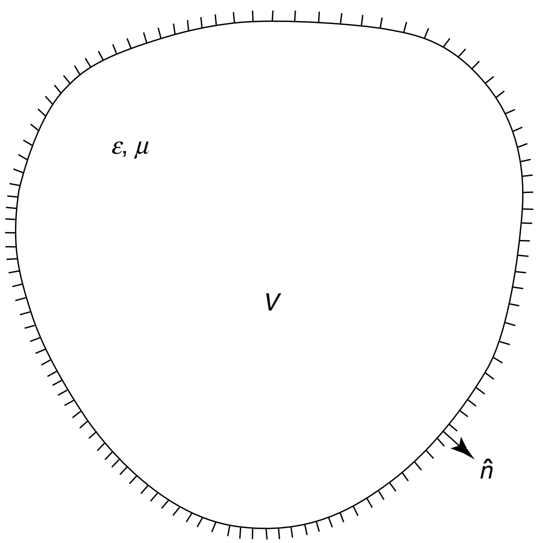

FIGURE 1.1 Empty cavity of volume V with perfectly conducting walls.

If we set the current $\vec{J}$ equal to zero in (1.12), we obtain the homogeneous vector wave equation:

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = 0 \tag{1.16}$$

We can work directly with (1.16) in determining the cavity modes, but it is simpler and more common [6, 7] to replace the double curl operation by use of the following vector identity (see Appendix A):

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \bullet \vec{E}) - \nabla^2 \vec{E} \tag{1.17}$$

Since the divergence of $\vec{E}$ is zero, (1.17) can be used to reduce (1.16) to the vector Helmholtz equation:

$$(\nabla^2 + k^2)\vec{E} = 0. \tag{1.18}$$

The simplest form of the Laplacian operator ∇^2occurs in rectangular coordinates, where $\nabla^2 \vec{E}$ reduces to:

$$\nabla^2 \vec{E} = \hat{x}\nabla^2 E_x + \hat{y}\nabla^2 E_y + \hat{z}\nabla^2 E_z, \tag{1.19}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors.

We assume that the permittivity ε and the permeability μ of the cavity are real. Then nontrivial (nonzero) solutions of (1.14), (1.15), and (1.18) occur when k is equal to one of an infinite number of discrete, real eigenvalues k_p (where $p = 1, 2, 3, \ldots$). For each eigenvalue k_p, there exists an electric field eigenvector $\vec{E}_p$. (There can be degenerate cases where two or more eigenvectors have the same eigenvalue.) The pth eigenvector satisfies:

$$(-\nabla \times \nabla \times + k_p^2)\vec{E}_p = (\nabla^2 + k_p^2)\vec{E}_p = 0 \quad (\text{in } V), \tag{1.20}$$

$$\nabla \bullet \vec{E}_p = 0 \quad (\text{in } V), \tag{1.21}$$

$$\hat{n} \times \vec{E}_p = 0 \quad (\text{on } S). \tag{1.22}$$

For convenience (and without loss of generality), each electric field eigenvector can be chosen to be real ($\vec{E}_p = \vec{E}_p^*$, where * indicates complex conjugate).

The corresponding magnetic field eigenvector $\vec{H}_p$ can be determined from (1.1) and (1.8):

$$\vec{H}_p = \frac{1}{i\omega_p \mu} \nabla \times \vec{E}_p, \tag{1.23}$$

where the angular frequency ω_p is given by:

$$\omega_p = \frac{k_p}{\sqrt{\mu\varepsilon}} \tag{1.24}$$

Hence, the pth normal mode of the resonant cavity has electric and magnetic fields, $\vec{E}_p$ and $\vec{H}_p$, and a resonant frequency f_p ($= \omega_p/2\pi$). The magnetic field is then pure imaginary ($\vec{H}_p = -\vec{H}_p^*$) and has the same phase throughout the cavity (as does $\vec{E}_p$).

For the pth mode, the time-averaged values of the electric stored energy $\overline{W}_{ep}$ and the magnetic stored energy $\overline{W}_{mp}$ are given by the following integrals over the cavity volume [3]:

$$\overline{W}_{ep} = \frac{\varepsilon}{2} \iiint_V \vec{E}_p \bullet \vec{E}_p^* \mathrm{d}V, \tag{1.25}$$

$$\overline{W}_{mp} = \frac{\mu}{2} \iiint_V \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}V \tag{1.26}$$

(The complex conjugate in (1.25) is not actually necessary when $\vec{E}_p$ is real, but it increases the generality to cases where $\vec{E}_p$ is not chosen to be real.) In general, the complex Poynting vector $\vec{S}$ is given by [3]:

$$\vec{S} = \vec{E} \times \vec{H}^* \tag{1.27}$$

If we apply Poynting's theorem to the pth mode, we obtain [6]:

$$\oint\!\!\!\oint_S (\vec{E}_p \times \vec{H}_p^*) \bullet \hat{n} \mathrm{d}S = 2i\omega_p(\overline{W}_{ep} - \overline{W}_{mp}) \tag{1.28}$$

Since $\hat{n} \times \vec{E}_p = 0$ on S, the left side of (1.28) equals zero, and for each mode we have:

$$\overline{W}_{ep} = \overline{W}_{mp} = \overline{W}_p/2 \tag{1.29}$$

Thus, the time-averaged electric and magnetic stored energies are equal to each other and are equal to one half the total time-averaged stored energy $\overline{W}_p$ at resonance. However, since (1.23) shows that the electric and magnetic fields are 90 degrees out of phase, the total energy in the cavity oscillates between electric and magnetic energy.

Up to now we have discussed only the properties of the fields and the energy of an individual cavity mode. It is also important to know what the distribution of the resonant frequencies is. In general, this depends on cavity shape, but the problem has been examined from an asymptotic point of view for electrically large cavities. Weyl [8] has studied this problem for general cavities, and Liu et al. [9] have studied the problem in great detail for rectangular cavities. For a given value of wavenumber k, the asymptotic expression (for large $kV^{1/3}$) for the number of modes N_s with eigenvalues less than or equal to k is [8, 9]:

$$N_s(k) \cong \frac{k^3 V}{3\pi^2} \tag{1.30}$$

The subscript s on N indicates that (1.30) is a smoothed approximation, whereas N determined by mode counting has step discontinuities at each mode. It is usually more useful to know the number of modes as a function of frequency. In that case, (1.30) can be written:

$$N_s(f) \cong \frac{8\pi f^3 V}{3c^3} \tag{1.31}$$

where c ($= 1/\sqrt{\mu\varepsilon}$) is the speed of light in the medium (usually free space). The f^3 dependence in (1.31) indicates that the number of modes increases rapidly at high frequencies.

The mode density D_s is also an important quantity because it is an indicator of the separation between the modes. By differentiating (1.30), we obtain:

$$D_s(k) = \frac{\mathrm{d}N_s(k)}{\mathrm{d}k} \cong \frac{k^2 V}{\pi^2} \tag{1.32}$$

The mode density as a function of frequency is obtained by differentiating (1.31):

$$D_s(f) = \frac{\mathrm{d}N_s(f)}{\mathrm{d}f} \cong \frac{8\pi f^2 V}{c^3} \tag{1.33}$$

The f^2 dependence in (1.33) indicates that the mode density also increases rapidly for high frequencies. The approximate frequency separation (in Hertz) between modes is given by the reciprocal of (1.33).

1.3 WALL LOSSES

For cavities with real metal walls, the wall conductivity σ_w is large, but finite. In this case, the eigenvalues and resonant frequencies become complex. An exact calculation of the cavity eigenvalues and eigenvectors is very difficult, but an adequate approximate treatment is possible for highly conducting walls. This allows us to obtain an approximate expression for the cavity quality factor Q_p [1].

The exact expression for the time-average power $\bar{P}_p$ dissipated in the walls can be obtained by integrating the normal component of the real part of the Poynting vector (defined in 1.27) over the cavity walls:

$$\bar{P}_p = \oiint_S \mathrm{Re}(\vec{E}_p \times \vec{H}_p^*) \bullet \hat{n}\mathrm{d}S \tag{1.34}$$

For simplicity and to compare with earlier work [6], we assume that the cavity medium and the cavity walls have free-space permeability μ_0, as shown in Figure 1.2. Using a vector identity, we can rewrite (1.34) as:

$$\bar{P}_p = \oiint_S \mathrm{Re}[(\hat{n} \times \vec{E}_p) \bullet \vec{H}_p^*]\mathrm{d}S \tag{1.35}$$

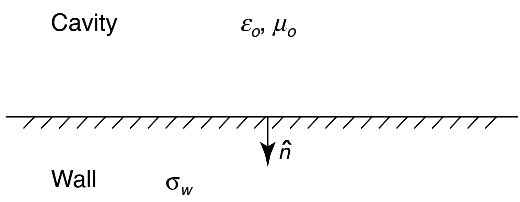

FIGURE 1.2 Cavity wall with conductivity σ_W.

In (1.35), we can approximate $\vec{H}_p$ by its value for the case of the lossless cavity. For $\hat{n} \times \vec{E}_p$, we can use the surface impedance boundary condition [10]:

$$\hat{n} \times \vec{E}_p \cong \eta \vec{H}_p \quad \text{on } S \tag{1.36}$$

where:

$$\eta \cong \sqrt{\frac{\omega_p \mu_0}{i\sigma_w}} \tag{1.37}$$

By substituting (1.36) and (1.37) into (1.35), we obtain:

$$\bar{P}_p \cong R_s \oiint_S \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}S \tag{1.38}$$

where the surface resistance R_s is the real part of η:

$$R_s \cong \mathrm{Re}(\eta) \cong \sqrt{\frac{\omega_p \mu_0}{2\sigma_w}} \tag{1.39}$$

The quality factor Q_p for the pth mode is given by [1, 6]:

$$Q_p = \omega_p \frac{\overline{W}_p}{\bar{P}_p} \tag{1.40}$$

where $\overline{W}_p$ $(= 2\overline{W}_{mp} = 2\overline{W}_{ep})$ is the time-averaged total stored energy. Substituting (1.26) and (1.38) into (1.40), we obtain:

$$Q_p \cong \omega_p \frac{\mu_0 \iiint_V \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}V}{R_s \oiint_S \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}S} \tag{1.41}$$

where $\vec{H}_p$ is the magnetic field of the pth cavity mode without losses. An alternative to (1.41) can be obtained by introducing the skin depth δ [3]:

$$Q_p \cong \frac{2 \iiint_V \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}V}{\delta \oiint_S \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}S} \tag{1.42}$$

where $\delta = \sqrt{2/(\omega_p \mu_0 \sigma_w)}$. In order to accurately evaluate (1.41) or (1.42), we need to know the magnetic field distribution of the pth mode, and in general this depends on the cavity shape and resonant frequency ω_p. This will be pursued in the next three chapters.

A rough approximation for (1.42) has been obtained by Borgnis and Papas [6]:

$$Q_p \cong \frac{2 \iiint\limits_V \mathrm{d}V}{\delta \oiint\limits_S \mathrm{d}S} = \frac{2V}{\delta S} \tag{1.43}$$

For highly conducting metals, such as copper, δ is very small compared to the cavity dimensions. Hence, the quality factor Q_p is very large. This is why metal cavities make very effective resonators. Even though (1.43) is a very crude approximation to (1.42)—it essentially assumes that $\vec{H}_p$ is independent of position—it is actually close to another approximation that has been obtained by two unrelated methods. Either by taking a modal average about the resonant frequency for rectangular cavities [9] or by using a plane-wave integral representation for stochastic fields in a multimode cavity of arbitrary shape (see either Section 8.1 or [11]), the following expression for Q has been obtained:

$$Q \cong \frac{3V}{2\delta S} \tag{1.44}$$

Hence, (1.43) exceeds (1.44) by a factor of only $\frac{4}{3}$. It is actually possible to improve the approximation in (1.43) and bring it into agreement with (1.44) by imposing the boundary conditions for $\vec{H}_p$ on S. If we take the z axis normal to S at a given point, then the normal component H_{pz} is zero on S. However, the x component is at a maximum because it is a tangential component:

$$H_{px} = H_{pm} \quad \text{on } S \tag{1.45}$$

We can make a similar argument for H_{py}. Hence, we can approximate the surface integral in (1.42) as:

$$\oiint\limits_S \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}S \cong 2|H_{pm}|^2 S \tag{1.46}$$

For the volume integral, we can assume that all three components of $\vec{H}_p$ contribute equally if the cavity is electrically large. However, since each rectangular component is a standing wave with approximately a sine or cosine spatial dependence, then a factor of $\frac{1}{2}$ occurs from integrating a sine-squared or cosine-squared dependence over an integer number of half cycles in V. Hence, the volume integral in (1.42) can be written:

$$\iiint\limits_V \vec{H}_p \bullet \vec{H}_p^* \mathrm{d}V \cong \frac{3}{2}|H_{pm}|^2 V \tag{1.47}$$

If we substitute (1.46) and (1.47) into (1.42), then we obtain:

$$Q_p \cong \frac{2}{\delta}\frac{(3/2)|H_{pm}|^2 V}{2|H_{pm}|^2 S} = \frac{3V}{2\delta S} \tag{1.48}$$

which is in agreement with (1.44). Hence, the single-mode approximation, the modal average for rectangular cavities [9], and the plane-wave integral representation for stochastic fields in a multimode cavity [11] all yield the same approximate value for Q.

When cavities have no loss, the fields of a resonant mode oscillate forever in time with no attenuation. However, with wall loss present, the fields and stored energy decay with time after any excitation ceases. For example, the incremental change in the time-averaged total stored energy in a time increment dt can be written:

$$\mathrm{d}\overline{W}_p = -\bar{P}_p \mathrm{d}t \tag{1.49}$$

By substituting (1.40) into (1.49), we can derive the following first-order differential equation:

$$\frac{\mathrm{d}\overline{W}_p}{\mathrm{d}t} = -\frac{\omega_p}{Q_p}\overline{W}_p \tag{1.50}$$

For the initial condition, $\overline{W}_p|_{t=0} = \overline{W}_{p0}$, the solution to (1.50) is:

$$\overline{W}_p = \overline{W}_{p0}\exp(-t/\tau_p), \quad \text{for } t \geq 0 \tag{1.51}$$

where $\tau_p = Q_p/\omega_p$. Hence, the energy decay time τ_p of the pth mode is the time required for the time-average energy to decay to 1/e of its initial value. Equations (1.49)–(1.51) assume that the decay time τ_p is large compared to the averaging period $1/f_p$. This is assured if Q_p is large.

By a similar analysis when the energy is switched off at $t = 0$, we find that the fields of the pth mode, $\vec{E}_p$ and $\vec{H}_p$, also have an exponential decay, but that the decay time is $2\tau_p$. This is equivalent to replacing the resonant frequency ω_p for a lossless cavity by the complex frequency $\omega_p\left(1-\frac{i}{2Q_p}\right)$ corresponding to a lossy cavity [6]. We can use this result to determine the bandwidth of the pth mode [6]. If E_{pm} is any scalar component of the electric field of the pth mode, then its time dependence $\widetilde{E}_{pm}(t)$ when the mode is suddenly excited at $t = 0$ can be written:

$$\widetilde{E}_{pm}(t) = E_{pm0}\exp\left(-i\omega_p t - \frac{\omega_p t}{2Q_p}\right)\mathrm{U}(t), \tag{1.52}$$

where U is the unit step function and E_{pm0} is independent of t. The Fourier transform of (1.52) is:

$$E_{pm}(\omega) = \frac{E_{pm0}}{2\pi}\int_0^\infty \exp\left[-i\omega_p t - \frac{\omega_p t}{2Q_p} + i\omega t\right]\mathrm{d}t, \tag{1.53}$$

which can be evaluated to yield:

$$E_{pm0}(\omega) = \frac{E_{pm0}}{2\pi} \frac{1}{i(\omega_p - \omega) + \dfrac{\omega_p}{2Q_p}} \tag{1.54}$$

The absolute value of (1.54) is:

$$|E_{pm}(\omega)| = \frac{|E_{pm0}|Q_p}{\pi\omega_p} \frac{1}{\sqrt{1 + \left[\dfrac{2Q_p(\omega - \omega_p)}{\omega_p}\right]^2}} \tag{1.55}$$

The maximum of (1.55) occurs at $\omega = \omega_p$:

$$|E_{pm}(\omega_p)| = \frac{|E_{pm0}|Q_p}{\pi\omega_p} \tag{1.56}$$

This maximum value is seen to be proportional to Q_p. The frequencies at which (1.55) drops to $\frac{1}{\sqrt{2}}$ times its maximum value are called the half-power frequencies, and their separation $\Delta\omega$ (or Δf in Hertz) is related to Q_p by:

$$\frac{\Delta\omega}{\omega_p} = \frac{\Delta f}{f_p} = \frac{1}{Q_p} \tag{1.57}$$

Hence Q_p is a very important property of a cavity mode because it controls both the maximum field amplitude and the mode bandwidth.

1.4 CAVITY EXCITATION

Cavities are typically excited by short monopoles, small loops, or apertures. Complete theories for the excitation of modes in a cavity have been given by Kurokawa [12] and Collin [13]. According to Helmholtz's theorem, the electric field in the interior of a volume V bounded by a closed surface S can be written as the sum of a gradient and a curl as follows [13]:

$$\begin{aligned} \vec{E}(\vec{r}) = -\nabla &\left[\iiint_V \frac{\nabla_0 \bullet \vec{E}(\vec{r}_0)}{4\pi R} \mathrm{d}V_0 - \oiint_S \frac{\hat{n} \bullet \vec{E}(\vec{r}_0)}{4\pi R} \mathrm{d}S_0 \right] \\ &+ \nabla \times \left[\iiint_V \frac{\nabla_0 \times \vec{E}(\vec{r}_0)}{4\pi R} \mathrm{d}V_0 - \oiint_S \frac{\hat{n} \times \vec{E}(\vec{r}_0)}{4\pi R} \mathrm{d}S_0 \right], \end{aligned} \tag{1.58}$$

where $R = |\vec{r} - \vec{r}_0|$ and $\hat{n}$ is the outward unit normal to the surface S. Equation (1.58) gives the conditions for which the electric field $\vec{E}(\vec{r})$ can be either a purely solenoidal or a purely irrotational field. A purely solenoidal (zero divergence) field must satisfy the conditions $\nabla \bullet \vec{E} = 0$ in V and $\hat{n} \bullet \vec{E} = 0$ on S. In this case, there is no volume or surface charge associated with the field. In the following chapters, we will see that some modes are purely solenoidal in the volume V, but are not purely solenoidal because the mode has surface charge ($\hat{n} \bullet \vec{E} \neq 0$ on S). A purely irrotational or lamellar field (zero curl) must satisfy the conditions $\nabla \times E = 0$ in V and $\hat{n} \times E = 0$ on S. For a cavity with perfectly conducting walls, $\hat{n} \times E = 0$ on S. However, for a time varying field, $\nabla \times E \neq 0$ in V. Hence, in general the electric field is not purely solenoidal or irrotational.

For the modal expansion of the electric field, we follow Collin [13]. The solenoidal modes $\vec{E}_p$ satisfy (1.20)–(1.22). The irrotational modes $\vec{F}_p$ are solutions of:

$$(\nabla^2 + l_p^2)\vec{F}_p = 0 \quad (\text{in } V), \tag{1.59}$$

$$\nabla \times \vec{F}_p = 0 \quad (\text{in } V), \tag{1.60}$$

$$\hat{n} \times \vec{F}_p = 0 \quad (\text{on } S) \tag{1.61}$$

These irrotational modes are generated from scalar functions Φ_p that are solutions of:

$$(\nabla^2 + l_p^2)\Phi_p = 0 \quad (\text{in } V), \tag{1.62}$$

$$\Phi_p = 0 \quad (\text{on } S), \tag{1.63}$$

$$l_p \vec{F}_p = \nabla \Phi_p \tag{1.64}$$

The factor l_p in (1.64) yields the desired normalization for $\vec{F}_p$ when Φ_p is normalized.

The $\vec{E}_p$ modes are normalized so that:

$$\iiint_V \vec{E}_p \bullet \vec{E}_p \mathrm{d}V = 1 \tag{1.65}$$

(The normalization in (1.65) can be made consistent with the energy relationship in (1.25) if we set $\overline{W} = \varepsilon$.) The scalar functions Φ_p are similarly normalized:

$$\iiint_V \Phi_p^2 \mathrm{d}V = 1 \tag{1.66}$$

From (1.64), the normalization for the $\vec{F}_p$ modes can be written:

$$\iiint_V \vec{F}_p \bullet \vec{F}_p \mathrm{d}V = \iiint_V l_p^{-2} \nabla\Phi_p \bullet \nabla\Phi_p \mathrm{d}V \tag{1.67}$$

To evaluate the right side of (1.67), we use the vector identity for the divergence of a scalar times a vector:

$$\nabla \bullet (\Phi_p \nabla \Phi_p) = \Phi_p \nabla^2 \Phi_p + \nabla\Phi_p \bullet \nabla\Phi_p \tag{1.68}$$

From (1.62), (1.63), (1.68), and the divergence theorem, we can evaluate the right side of (1.67):

$$\iiint_V l_p^{-2}\nabla\Phi_p \bullet \nabla\Phi_p \mathrm{d}V = \iiint_V \Phi_p^2 \mathrm{d}V + l_p^{-2} \oiint_S \Phi_p \frac{\partial\Phi_p}{\partial n}\mathrm{d}S = 1, \tag{1.69}$$

since the second integral on the right side is zero. Thus the $\vec{F}_p$ modes are also normalized:

$$\iiint_V \vec{F}_p \bullet \vec{F}_p \mathrm{d}V = 1 \tag{1.70}$$

We now turn to mode orthogonality. To show that the $\vec{E}_p$ and $\vec{F}_p$ modes are orthogonal, we begin with the following vector identity:

$$\nabla \bullet (\vec{F}_q \times \nabla \times \vec{E}_p) = \nabla \times \vec{F}_q \bullet \nabla \times \vec{E}_p - \vec{F}_q \bullet \nabla \times \nabla \times \vec{E}_p \tag{1.71}$$

Substituting (1.20) and (1.60) into the right side of (1.71), we obtain:

$$\nabla \bullet (\vec{F}_q \times \nabla \times \vec{E}_p) = -k_p^2 \vec{F}_q \bullet \vec{E}_p \tag{1.72}$$

Using the divergence theorem and the vector identity, $\vec{A} \bullet \vec{B} \times \vec{C} = \vec{C} \bullet \vec{A} \times \vec{B}$, in (1.72), we can obtain:

$$k_p^2 \iiint_V \vec{F}_q \bullet \vec{E}_p \mathrm{d}V = -\oiint_S \hat{n} \times \vec{F}_q \bullet \nabla \times \vec{E}_p \mathrm{d}S \tag{1.73}$$

Substituting (1.61) into (1.73), we obtain the desired orthogonality result:

$$k_p^2 \iiint_V \vec{F}_q \bullet \vec{E}_p \mathrm{d}V = 0 \tag{1.74}$$

The modes $\vec{E}_p$ are also mutually orthogonal. By dotting $\vec{E}_q$ into (1.20), reversing the subscripts, subtracting the results, and integrating over V, we obtain:

$$(k_q^2 - k_p^2) \iiint_V \vec{E}_p \bullet \vec{E}_q = \iiint_V (\vec{E}_p \bullet \nabla \times \nabla \times \vec{E}_q - \vec{E}_q \bullet \nabla \times \nabla \times \vec{E}_p)\mathrm{d}V \tag{1.75}$$

By using the vector identity, $\nabla \bullet \vec{A} \times \vec{B} = \vec{B} \bullet \nabla \times \vec{A} - \vec{A} \bullet \nabla \times \vec{B}$, the right side of (1.75) can be rewritten:

$$(k_q^2 - k_p^2) \iiint_V \vec{E}_p \bullet \vec{E}_q = \iiint_V \nabla \bullet (\vec{E}_q \times \nabla \times \vec{E}_p - \vec{E}_p \times \nabla \times \vec{E}_q)\mathrm{d}V \tag{1.76}$$

By using the divergence theorem and (1.22), we obtain the desired result:

$$(k_q^2 - k_p^2) \iiint_V \vec{E}_p \bullet \vec{E}_q = -\oiint_S (\hat{n} \times \vec{E}_p \bullet \nabla \times \vec{E}_q - \hat{n} \times \vec{E}_q \bullet \nabla \times \vec{E}_p) \mathrm{d}S = 0 \quad (1.77)$$

When $k_q^2 \neq k_p^2$, the modes $\vec{E}_p$ and $\vec{E}_q$ are orthogonal. For degenerate modes that have the same eigenvalue ($k_p = k_q$), we can use the Gram-Schmidt orthogonalization procedure to construct a new subset of orthogonal modes [13].

We now consider cavity excitation by an electric current $\vec{J}$. The electric field $\vec{E}$ satisfies (1.12). We can expand the electric field in terms of the $\vec{E}_p$ and $\vec{F}_p$ modes:

$$\vec{E} = \sum_p (A_p \vec{E}_p + B_p \vec{F}_p), \quad (1.78)$$

where A_p and B_p are constants to be determined. Substitution of (1.78) into (1.12) yields

$$\sum_p [(k_p^2 - k^2) A_p \vec{E}_p - k^2 B_p \vec{F}_p] = i\omega\mu \vec{J} \quad (1.79)$$

If we scalar multiply (1.79) by $\vec{E}_p$ and $\vec{F}_p$ and integrate over the volume V, we obtain:

$$(k_p^2 - k^2) A_p = i\omega\mu \iiint_V \vec{E}_p(\vec{r}\,') \bullet \vec{J}(\vec{r}\,') \mathrm{d}V', \quad (1.80)$$

$$-k^2 B_p = i\omega\mu \iiint_V \vec{F}_p(\vec{r}\,') \bullet \vec{J}(\vec{r}\,') \mathrm{d}V' \quad (1.81)$$

Substitution of (1.80) and (1.81) into (1.78) gives the solution for $\vec{E}$:

$$\vec{E}(\vec{r}) = i\omega\mu \iiint_V \sum_p \left[\frac{\vec{E}_p(\vec{r})\vec{E}_p(\vec{r}\,')}{k_p^2 - k^2} - \frac{\vec{F}_p(\vec{r})\vec{F}_p(\vec{r}\,')}{k^2} \right] \bullet \vec{J}(\vec{r}\,') \mathrm{d}V' \quad (1.82)$$

The summation quantity is the dyadic Green's function $\overleftrightarrow{G}_e$ for the electric field in the cavity [2, 13]:

$$\overleftrightarrow{G}_e(\vec{r}, \vec{r}\,') = \sum_p \left[\frac{\vec{E}_p(\vec{r})\vec{E}_p(\vec{r}\,')}{k_p^2 - k^2} - \frac{\vec{F}_p(\vec{r})\vec{F}_p(\vec{r}\,')}{k^2} \right] \quad (1.83)$$

The summation over integer p actually represents a triple sum over a triple set of integers. The specific details will be given in the next three chapters.

Equations (1.82) and (1.83) have singularities at $k^2 = k_p^2$. However, if we include wall loss as in Section 1.3, then we can replace k_p by $k_p(1 - \frac{i}{2Q_p})$. Then there are no singularities for real k (except at the source point, $r = r'$, which will be discussed later).

1.5 PERTURBATION THEORIES

When a cavity shape is deformed or the dielectric is inhomogeneous, the analysis is generally difficult, and numerical methods are required. However, if the shape deformation or the dielectric inhomogeneity is small, then perturbation techniques [14] are applicable.

1.5.1 Small-Sample Perturbation of a Cavity

If a small sample of dielectric or magnetic material of volume V_s is introduced into a cavity (as in Figure 1.3), the resonant frequency ω_p of the cavity is changed by a small amount $\delta\omega$. If the sample has loss, then $\delta\omega$ becomes complex and a damping factor occurs (the cavity Q is changed). If the sample is properly positioned, the measurement of the complex frequency change $\delta\omega$ can be used to infer the complex permittivity or permeablility of the sample [15].

If $\vec{E}_p$ and $\vec{H}_p$ are the unperturbed fields of the pth cavity mode and $\vec{E}_1$ and $\vec{H}_1$ are the perturbation fields due to the introduced sample, then the total perturbed fields $\vec{E}'$ and $\vec{H}'$ are:

$$\vec{E}' = \vec{E}_p + \vec{E}_1, \tag{1.84}$$

$$\vec{H}' = \vec{H}_p + \vec{H}_1 \tag{1.85}$$

The (complex) frequency of oscillation is $\omega_p + \delta\omega$. Outside the sample, the magnetic and electric flux densities, $\vec{B}'$ and $\vec{D}'$, are given by:

$$\vec{B}' = \vec{B}_p + \vec{B}_1 = \mu(\vec{H}_p + \vec{H}_1), \tag{1.86}$$

$$\vec{D}' = \vec{D}_p + \vec{D}_1 = \varepsilon(\vec{E}_p + \vec{E}_1) \tag{1.87}$$

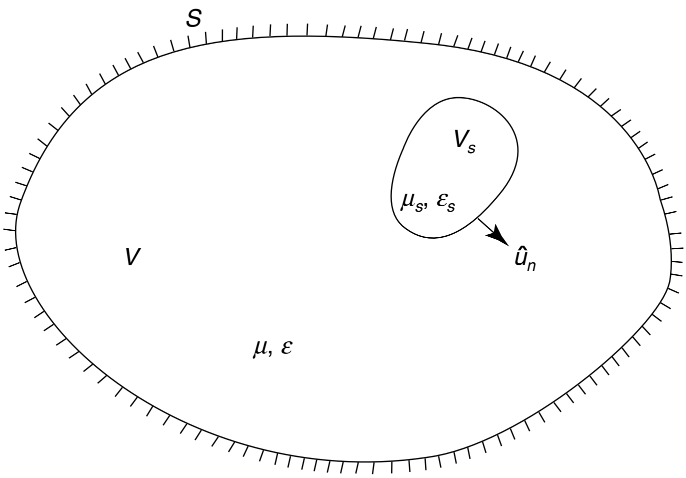

FIGURE 1.3 Cavity with a small sample of material.

Inside the sample, we have:

$$\vec{B}' = \mu_s \vec{H}' = \vec{B}_p + \vec{B}_1 = \mu \vec{H}_p + \mu[\kappa_{sm}(\vec{H}_p + \vec{H}_1) - \vec{H}_p], \tag{1.88}$$

$$\vec{D}' = \varepsilon_s \vec{E}' = \vec{D}_p + \vec{D}_1 = \varepsilon \vec{E}_p + \varepsilon[\kappa_{se}(\vec{E}_p + \vec{E}_1) - \vec{E}_p], \tag{1.89}$$

where μ_s and ε_s are the permeability and permittivity of the sample and κ_{sm} and κ_{se} are the relative permeability and permittivity of the sample. Here we assume that the sample is isotropic, but for anisotropic materials these quantities become tensors.

Throughout the cavity, the total fields satisfy Maxwell's curl equations:

$$\nabla \times (\vec{E}_p + \vec{E}_1) = i(\omega_p + \delta\omega)(\vec{B}_p + \vec{B}_1), \tag{1.90}$$

$$\nabla \times (\vec{H}_p + \vec{H}_1) = -i(\omega_p + \delta\omega)(\vec{D}_p + \vec{D}_1) \tag{1.91}$$

The unperturbed fields satisfy:

$$\nabla \times \vec{E}_p = i\omega_p \vec{B}_p, \tag{1.92}$$

$$\nabla \times \vec{H}_p = -i\omega_p \vec{D}_p \tag{1.93}$$

Subtracting (1.92) from (1.90) and (1.93) from (1.91), we obtain:

$$\nabla \times \vec{E}_1 = i[\omega_p + \delta\omega(\vec{B}_p + \vec{B}_1)], \tag{1.94}$$

$$\nabla \times \vec{H}_1 = -i[\omega_p \vec{D}_1 + \delta\omega(\vec{D}_p + \vec{D}_1)] \tag{1.95}$$

If we scalar multiply (1.94) by $\vec{H}_p$ and (1.95) by $\vec{E}_p$ and add the results, we obtain:

$$\begin{aligned} &\vec{H}_p \bullet \nabla \times \vec{E}_1 + \vec{E}_p \bullet \nabla \times \vec{H}_1 \\ &= -i\omega_p(\vec{E}_p \bullet \vec{D}_1 - \vec{B}_1 \bullet \vec{H}_p) - i\delta\omega(\vec{E}_p \bullet \vec{D}_p + \vec{E}_p \bullet \vec{D}_1 - \vec{H}_p \bullet \vec{B}_p - \vec{H}_p \bullet \vec{B}_1) \end{aligned} \tag{1.96}$$

Using (1.92)–(1.95) and vector identities, we can write the right side of (1.96) in the two following forms:

$$\begin{aligned} &\vec{H}_p \bullet \nabla \times \vec{E}_1 + \vec{E}_p \bullet \nabla \times \vec{H}_1 \\ &= \vec{E}_1 \bullet \nabla \times \vec{H}_p + \vec{H}_1 \bullet \nabla \times \vec{E}_p - \nabla \bullet (\vec{H}_p \times \vec{E}_1 + \vec{E}_p \times \vec{H}_1) \\ &= -i\omega_p(\vec{D}_p \bullet \vec{E}_1 - \vec{B}_p \bullet \vec{H}_1) - \nabla \bullet (\vec{H}_p \times \vec{E}_1 + \vec{E}_p \times \vec{H}_1) \end{aligned} \tag{1.97}$$

If we substitute (1.94) and (1.95) into (1.97) and evaluate the result outside the sample, we obtain:

$$i\delta\omega(\varepsilon \vec{E}_p \bullet \vec{E}_p + \varepsilon \vec{E}_p \bullet \vec{E}_1 - \mu \vec{H}_p \bullet \vec{H}_p - \mu \vec{H}_p \bullet \vec{H}_1) = \nabla \bullet (\vec{H}_p \times \vec{E}_1 + \vec{E}_0 \times \vec{H}_1) \tag{1.98}$$

The perturbation fields $\vec{E}_1$ and $\vec{H}_1$ are not necessarily small everywhere in the cavity. However, if (1.98) is integrated over the volume $V - V_s$, it is possible to neglect contributions of terms involving $\vec{E}_1$ and $\vec{H}_1$ when the sample volume V_s is small. Taking into account that $\vec{E}_p$ and $\vec{E}_1$ are normal to S, and using the divergence theorem and vector identities, we obtain:

$$-i\delta\omega \int\limits_{V-V_s} (\vec{B}_p \bullet \vec{H}_p - \vec{D}_p \bullet \vec{E}_p)\mathrm{d}V = \int\limits_{\Sigma} [(\hat{u}_n \times \vec{E}_1) \bullet \vec{H}_p + (\hat{u}_n \times \vec{H}_1) \bullet \vec{E}_p]\mathrm{d}\Sigma, \quad (1.99)$$

where $\hat{u}_n$ is the outward unit normal from the sample and Σ is the surface of the sample.

Comparing the right sides of (1.96) and (1.97), we obtain:

$$\begin{aligned} &i\omega_p(\vec{E}_1 \bullet \vec{D}_p - \vec{B}_p \bullet \vec{H}_1) + i(\omega_p + \delta\omega)(\vec{B}_1 \bullet \vec{H}_p - \vec{E}_p \bullet \vec{D}_1) \\ &+ i\delta\omega(\vec{H}_p \bullet \vec{B}_p - \vec{E}_p \bullet \vec{D}_p) = \nabla \bullet (\vec{E}_1 \times \vec{H}_p + \vec{H}_1 \times \vec{E}_p) \end{aligned} \quad (1.100)$$

If we neglect $\delta\omega$ in the factor $(\omega_p + \delta\omega)$, integration of (1.100) over the sample volume yields:

$$\begin{aligned} &i\delta\omega \int\limits_{V_s} (\vec{B}_p \bullet \vec{H}_p - \vec{D}_p \bullet \vec{E}_p)\mathrm{d}V_s + i\omega_p \int\limits_{V_s} (\vec{E}_1 \bullet \vec{D}_p - \vec{E}_p \bullet \vec{D}_1 - \vec{B}_p \bullet \vec{H}_1 + \vec{B}_1 \bullet \vec{H}_p)\mathrm{d}V_s \\ &= \int\limits_{\Sigma} [(\hat{u}_n \times \vec{E}_1) \bullet \vec{H}_p + (\hat{u}_n \times \vec{H}_1) \bullet E_p]\mathrm{d}\Sigma \end{aligned} \quad (1.101)$$

The surface integrals in (1.99) and (1.101) are equal. Thus we can equate the left sides of (1.99) and (1.101) to obtain:

$$\frac{\delta\omega}{\omega_p} = \frac{\int\limits_{V_s} [(\vec{E}_1 \bullet \vec{D}_p - \vec{E}_p \bullet \vec{D}_1) - (\vec{H}_1 \bullet \vec{B}_p - \vec{H}_p \bullet \vec{B}_1)]\mathrm{d}V_s}{\int\limits_{V} (\vec{E}_p \bullet \vec{D}_p - \vec{H}_p \bullet \vec{B}_p)\mathrm{d}V} \quad (1.102)$$

Inside the sample, we can write the constitutive relations, (1.7) and (1.8), in more convenient forms:

$$\vec{D}_1 = \varepsilon_0\vec{E} + \vec{P} \quad \text{and} \quad \vec{B}_1 = \mu_0\vec{H}_1 + \mu_0\vec{M}, \quad (1.103)$$

where ε_0 and μ_0 are the permittivity and permeability of free space, $\vec{P}$ is the electric polarization, and $\vec{M}$ is the magnetic polarization. For convenience, we will assume in the rest of this section that the cavity permittivity $\varepsilon = \varepsilon_0$ and the cavity permeability $\mu = \mu_0$. If we substitute (1.103) into (1.102), we obtain:

$$\frac{\delta\omega}{\omega_p} = \frac{\mu_0 \int\limits_{V_s} \vec{H}_p \bullet \vec{M}\mathrm{d}V_s - \int\limits_{V_s} \vec{E}_p \bullet \vec{P}\mathrm{d}V_s}{\int\limits_{V} (\vec{E}_p \bullet \vec{D}_p - \vec{H}_p \bullet \vec{B}_p)\mathrm{d}V} \quad (1.104)$$

If the sample volume V_s is very small, $\vec{E}_p$ and $\vec{H}_p$ are nearly constant throughout the sample volume, and (1.104) can be approximated as:

$$\frac{\delta\omega}{\omega_p} = \frac{\mu_0\vec{H}_p \bullet \vec{P}_m - \vec{E}_p \bullet \vec{P}_e}{\int\limits_{V} (\vec{E}_p \bullet \vec{D}_p - \vec{H}_p \bullet \vec{B}_p)\mathrm{d}V}, \quad (1.105)$$

where $\vec{P}_e$ and $\vec{P}_m$ are the quasi-static electric and magnetic dipole moments induced in the sample by the cavity modal fields $(\vec{E}_p, \vec{H}_p)$.

For a spherical sample of radius a, the induced dipole moments are [15, 16]:

$$\vec{P}_e = 4\pi a^3 \varepsilon_0 \frac{\kappa_{se}-1}{\kappa_{se}+2} \vec{E}_p(P), \tag{1.106}$$

$$\vec{P}_m = 4\pi a^3 \frac{\kappa_{sm}-1}{\kappa_{sm}+2} \vec{H}_p(P), \tag{1.107}$$

where P is the location of the center of the sphere. If we substitute (1.25), (1.26), (1.29), (1.106), (1.107) into (1.05), we obtain the following resonant frequency shift:

$$\frac{\delta\omega}{\omega_p} = -\frac{2\pi a^3}{\overline{W}} \left[\mu_0 \frac{\kappa_{sm}-1}{\kappa_{sm}+2} |\vec{H}_p(P)|^2 + \varepsilon_0 \frac{\kappa_{se}-1}{\kappa_{se}+2} |\vec{E}_p(P)|^2 \right] \tag{1.108}$$

Equation (1.108) is the desired mathematical result, which can be applied to a number of measurements. Consider first the case where the spherical sample is located at a point where the electric field $\vec{E}_p(P)$ is zero. If the relative permeability κ_{sm} of the sample is known, then (1.108) can be used to determine the square of the magnetic field at P:

$$|\vec{H}_p(P)|^2 = -\frac{\delta\omega}{\omega_p} \frac{\overline{W}}{2\pi a^3 \mu_0} \frac{\kappa_{sm}+2}{\kappa_{sm}-1} \tag{1.109}$$

If the magnitude of the square of the magnetic field at P is known (measured), then (1.108) can be used to determine κ_{sm}:

$$\kappa_{sm} = 2 \frac{\dfrac{\pi a^3}{\overline{W}} \mu_0 |\vec{H}_p(P)|^2 - \dfrac{\delta\omega}{\omega_p}}{\dfrac{2\pi a^3}{\overline{W}} \mu_0 |\vec{H}_p(P)|^2 + \dfrac{\delta\omega}{\omega_p}} \tag{1.110}$$

If $\delta\omega$ is real, then κ_{sm} is real and the sample has no magnetic loss. However, if $\delta\omega$ is complex, then κ_{sm} is complex and the sample does have magnetic loss. The imaginary part of the resonant frequency is related to the cavity Q from the expression for a complex resonant frequency $\omega_p(1-\frac{i}{Q})$. Hence the change in the imaginary part of the resonant frequency is determined from the change in Q. This is typically determined by measuring the half-power bandwidth, which is given by (1.57).

In the analogous case, the spherical sample is located at a point where the magnetic field $\vec{H}_p(P)$ is zero. If the relative permittivity κ_{se} of the sample is known, then (1.108) can be used to determine the square of the electric field at P:

$$|\vec{E}_p(P)|^2 = -\frac{\delta\omega}{\omega_p} \frac{\overline{W}}{2\pi a^3 \varepsilon_0} \frac{\kappa_{se}+2}{\kappa_{se}-1} \tag{1.111}$$

This method has been used to map the electric field along the axis of a linear accelerator [15]. If the magnitude of the square of the electric field at P is known (measured), then (1.108) can be used to determine κ_{se}:

$$\kappa_{se} = 2\frac{\dfrac{\pi a^3}{\overline{W}}\varepsilon_0|\vec{E}_p(P)|^2 - \dfrac{\delta\omega}{\omega_p}}{\dfrac{2\pi a^3}{\overline{W}}\varepsilon_0|\vec{E}_p(P)|^2 + \dfrac{\delta\omega}{\omega_p}} \tag{1.112}$$

Similar to (1.110), $\delta\omega$ can be either real (lossless dielectric sample) or complex (lossy dielectric sample).

1.5.2 Small Deformation of Cavity Wall

Here we consider the change in the resonant frequency of a cavity mode due to a small deformation in the cavity wall. This case is useful in determining the effects of small accidental deformations or intentional displacements of pistons or membranes on the resonant frequencies.

Our derivation is similar to that of Argence and Kahan [7], but with somewhat different notation. We first write Maxwell's equation for the curl of $\vec{E}_p$ and the complex conjugate for Maxwell's equation for the curl of $\vec{H}_p$ for the pth mode of the unperturbed cavity:

$$\nabla \times \vec{E}_p = i\omega_p\mu\vec{H}_p, \tag{1.113}$$

$$\nabla \times \vec{H}_p^* = -i\omega_p\varepsilon\vec{E}_p^*, \tag{1.114}$$

where the electric current term is omitted in (1.114) for this source-free case. If we scalar multiply (1.113) by $\vec{H}_p^*$ and (1.114) by $\vec{E}_p$ and take the difference, we obtain:

$$\vec{H}_p^* \bullet \nabla \times \vec{E}_p - \vec{E}_p \bullet \nabla \times \vec{H}_p^* = -i\omega_p(\mu\vec{H}_p \bullet \vec{H}_p^* - \varepsilon\vec{E}_p \bullet \vec{E}_p^*) \tag{1.115}$$

If we integrate (1.115) over the volume V, the two terms on the right side can be written in terms of the time-averaged magnetic and electric energies from (1.25) and (1.26). The left side of (1.115) can be converted to a divergence via a vector identity and converted to a surface integral over S by use of the divergence theorem. The result is:

$$-\oint\!\!\oint_S (\vec{E}_p \times \vec{H}_p^*) \bullet \hat{n}\,\mathrm{d}S = 2i\omega(\overline{W}_{mp} - \overline{W}_{ep}) \tag{1.116}$$

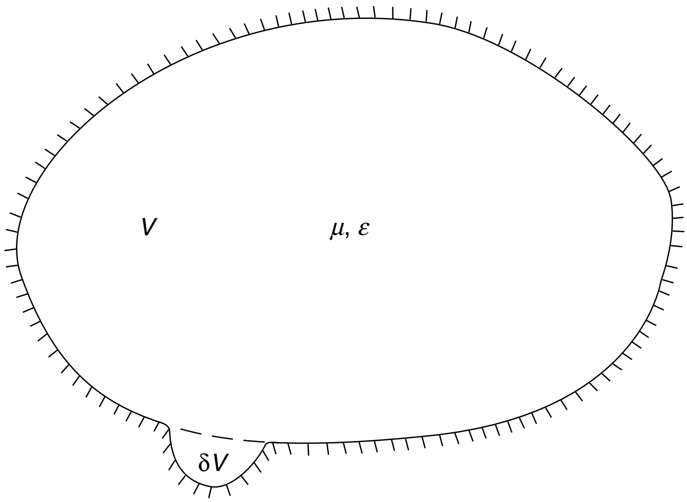

FIGURE 1.4 Cavity with a small deformation δV in the cavity wall.

Equation (1.116) can be written in the form:

$$\Phi_p = -\oiint_S (\vec{E}_p \times \vec{H}_p^*) \bullet \hat{n} \mathrm{d}S = 2i\omega \iiint_V \tau_p \mathrm{d}V, \tag{1.117}$$

where:

$$\tau_p = \frac{\mu}{2} \vec{H}_p \bullet \vec{H}_p^* - \frac{\varepsilon}{2} \vec{E}_p \bullet \vec{E}_p^*, \tag{1.118}$$

which is the difference between the time-average magnetic and electric energy densities.

We consider now a small deformation in the cavity wall, as shown in Figure 1.4. We write the perturbed electric field $\vec{E}'$ and magnetic field $\vec{H}'$as in (1.84) and (1.85). The resonant frequency of the deformed cavity is $\omega_p + \delta\omega$. The analogy to (1.117) for the perturbed cavity is:

$$\Phi' = \Phi_p + \delta\Phi = 2i(\omega_p + \delta\omega) \iiint_{V+\delta V} (\tau_p + \delta\tau) \mathrm{d}V \tag{1.119}$$

Subtracting (1.117) from (1.119) and neglecting second-order terms, we obtain:

$$\delta\Phi = 2i\omega_p \iiint_V \delta\tau \mathrm{d}V + 2i\delta\omega \iiint_V \tau \mathrm{d}V + 2i\omega_p \iiint_{\delta V} \tau \mathrm{d}V \tag{1.120}$$

The perturbed fields satisfy the following Maxwell curl equations, which are equivalent to (1.90) and (1.91):

$$\nabla \times (\vec{H}_p + \vec{H}_1) = i\varepsilon(\omega_p + \delta\omega)(\vec{E}_p + \vec{E}_1), \tag{1.121}$$

$$\nabla \times (\vec{E}_p + \vec{E}_1) = -i\mu(\omega_p + \delta\omega)(\vec{H}_p + \vec{H}_1) \tag{1.122}$$

By subtracting the complex conjugate of (1.114) from (1.121) and (1.113) from (1.122), we obtain:

$$\nabla \times \vec{H}_1 = i\varepsilon(\omega_p \vec{E}_1 + \vec{E}_p \delta\omega), \tag{1.123}$$

$$\nabla \times \vec{E}_1 = -i\mu(\omega_p \vec{H}_1 + \vec{H}_p \delta\omega) \tag{1.124}$$

We can write τ' in a manner analogous to (1.118):

$$\tau' = \frac{\mu}{2}(\vec{H}_p + \vec{H}_1) \bullet (\vec{H}_p^* + \vec{H}_1^*) - \frac{\mu}{2}(\vec{E}_p + \vec{E}_1) \bullet (\vec{E}_p^* + \vec{E}_1^*) \tag{1.125}$$

If we subtract (1.118) from (1.125) and ignore second order terms (such as $\vec{H}_1 \bullet \vec{H}_1^*$), we obtain:

$$\delta\tau = \tau' - \tau_p = \frac{\mu}{2}(\vec{H}_p \bullet \vec{H}_1^* + \vec{H}_p^* \bullet \vec{H}_1) - \frac{\varepsilon}{2}(\vec{E}_p \bullet \vec{E}_1^* + \vec{E}_p^* \bullet \vec{E}_1) \tag{1.126}$$

By substituting the curl equations from this section into (1.126) and using a vector identity, we can multiply the result by $2i\omega_p$to obtain:

$$2i\omega_p \delta\tau = i\nabla \bullet \mathrm{Im}(\vec{E}_p \times \vec{H}_1) + i\varepsilon\delta\omega \vec{E}_p \bullet \vec{E}_p^* \tag{1.127}$$

If we substitute (1.127) into (1.120), we obtain:

$$\begin{aligned} \delta\Phi = 2i \oint\!\!\!\oint_S [\mathrm{Im}(\vec{E}_p^* \times \delta\vec{H}_1)] \bullet \hat{n} \mathrm{d}S + i\delta\omega \iiint_V (\mu \vec{H}_p \bullet \vec{H}_p^* + \varepsilon \vec{E}_p \bullet \vec{E}_p^*) \mathrm{d}V \\ + i\omega_p \iiint_{\delta V} (\mu \vec{H}_p \bullet \vec{H}_p^* - \varepsilon \vec{E}_p \bullet \vec{E}_p^*) \mathrm{d}V \end{aligned} \tag{1.128}$$

Because the cavity walls are assumed to be perfectly conducting, the tangential component of the electric field is zero and $\delta\Phi = 0$. Similarly:

$$\oint\!\!\!\oint_S [\mathrm{Im}(\vec{E}_p^* \times \vec{H}_1)] \bullet \hat{n} \mathrm{d}V = 0 \tag{1.129}$$

By using $\delta\Phi = 0$ and (1.129) in (1.128), we obtain the desired result for the relative shift in the resonant frequency of the deformed cavity:

$$\frac{\delta\omega}{\omega_p} = -\frac{\iiint\limits_{\delta V}(\mu\vec{H}_p \bullet \vec{H}_p^* - \varepsilon\vec{E}_p \bullet \vec{E}_p^*)\mathrm{d}V}{\iiint\limits_{V}(\mu\vec{H}_p \bullet \vec{H}_p^* + \varepsilon\vec{E}_p \bullet \vec{E}_p^*)\mathrm{d}V} \tag{1.130}$$

Equation (1.130) can be written in a simpler form if we define time-average electric and magnetic energy densities for the pth mode:

$$\bar{w}_{pe} = \frac{\varepsilon}{2}\vec{E}_p \bullet \vec{E}_p^* \quad \text{and} \quad \bar{w}_{pm} = \frac{\mu}{2}\vec{H}_p \bullet \vec{H}_p^* \tag{1.131}$$

If we substitute (1.131) into (1.130), we can simplify the result to:

$$\begin{aligned}\frac{\delta\omega}{\omega_p} &= \frac{-1}{\overline{W}_p}\iiint\limits_{\delta V}(\bar{w}_{pm} - \bar{w}_{pe})\mathrm{d}V \\ &\approx \frac{(\bar{w}_{pe} - \bar{w}_{pm})\delta V}{\overline{W}_p}\end{aligned} \tag{1.132}$$

In the second result in (1.132), $\bar{w}_{pe}$ and $\bar{w}_{pm}$ are the time-averaged electric and magnetic energies at the volume deformation. Equation (1.132) shows that if the cavity is compressed ($\delta V < 0$) in a region where $\bar{w}_{pm} > \bar{w}_{pe}$, then $\delta\omega > 0$ and the resonant frequency is increased. However, if $\delta V < 0$ and $\bar{w}_{pm} < \bar{w}_{pe}$, then $\delta\omega < 0$ and the resonant frequency is decreased. For δV positive, the results are reversed. The result in (1.132) is identical to that given by Borgnis and Papas [6].

PROBLEMS

1-1 Derive (1.3) from (1.2) and (1.5). This shows that the continuity equation can be derived from two of Maxwell's equations.

1-2 Show that (1.17) is satisfied in rectangular coordinates where $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$. Combine that result with (1.15) and (1.16) to derive the vector Helmholtz equation in (1.18).

1-3 Apply the boundary condition, $\hat{n} \times \vec{E}_p = 0$ on S, to (1.28) to show that $\overline{W}_{ep} = \overline{W}_{mp}$ as in (1.29). Hint: use the vector identity (A19). Is the boundary condition, $\hat{n} \bullet \vec{H} = 0$ on S, sufficient to derive the same result?

1-4 Using the smoothed approximations in (1.31) and (1.33), determine the mode number and mode density for an empty cavity of volume 1 m^3 at a frequency of 1 GHz. What is the mode separation?

1-5 Show that the $1/e$ decay time of the fields of the pth mode is $2Q_p/\omega_p$.

1-6 In (1.82), show that the coupling of the current source $\vec{J}$ to $\vec{F}_p$ is zero if $\nabla \bullet \vec{J} = 0$ and the normal component of $\vec{J}$ is zero at the boundary of the source region. Hint: use the divergence theorem.

1-7 Does a small loop current, $\vec{J} = \hat{\phi}\frac{I_0}{\rho_0}\delta(\rho-\rho_0)$, satisfy the current conditions for problem 1-6?

1-8 Does a short dipole current, $\vec{J} = I_0\delta(x)\delta(y)U\left(\frac{l}{2}-|z|\right)$, satisfy the current conditions for problem 1-6?

1-9 Consider a small lossless dielectric sphere, $\text{Re}(\kappa_{se}) > 1, \text{Im}(\kappa_{se}) = 0$, and $\kappa_{sm} = 0$, inserted in a lossless cavity. From (108), what is the sign of the resonant frequency shift $\delta\omega$? What is the physical explanation for this sign?

1-10 Consider a small lossy dielectric sphere, $\text{Re}(\kappa_{se}) > 1, \text{Im}(\kappa_{se}) > 0$, and $\kappa_{sm} = 0$, inserted in a lossless cavity. From (108), what is the sign of the imaginary part of the frequency shift $\text{Im}(\delta\omega)$? What is the physical explanation for this sign?