CHAPTER 1

INTRODUCTION

1.1 MICROELECTROMECHANICAL SYSTEMS

MEMS, *microelectromechanical systems*, are systems that consist of small-scale electrical and mechanical components for specific purposes. MEMS were translated into systems with electrical and mechanical components but have extended their boundaries to include optical, radio-frequency, and nano devices. As a result, depending on the components included and applications desired, MEMS have different names: for example, MOEMS (micro-optoelectromechanical systems) for optical applications, RF MEMS (radio-frequency MEMS) to refer to radio-frequency components and applications, and NEMS (nanoelectromechanical systems) if the systems include at least one component whose dimension is less than 1 μm. When MEMS use bio-related material (e.g., strands of DNA) to detect desired targets or to manipulate cells, the corresponding MEM system is currently called bioMEMS. Different names may refer to MEMS: *microsystems technology* (MST) in Europe and *micromachines* in Japan. Throughout this book, MEMS will be referred to as systems that include at least one set of electrical and mechanical components for a specific purpose. Depending on the specific purpose, more components, such as a reflective surface for a micromirror, can be added to a MEMS device. A typical dimension of a component of MEMS varies from 1 μm to a few hundred micrometers, and the overall size is approximately less than 1 mm. In this book we describe MEMS principles via a unified approach

Principles of Microelectromechanical Systems, By Ki Bang Lee

and newly developed closed-form solutions. Readers are assumed to be familiar with mathematical background at the third-year college and university level.

1.2 COUPLED SYSTEMS

MEMS are coupled systems since they consist of electrical and mechanical components; the mechanical behavior of MEMS are in general coupled with the electrical behavior. For example, let us consider the first electrostatic MEMS device (Fig. 1.1), presented by Nathanson et al. in the 1960s to filter or amplify electrical signals using the resonance of an electroplated cantilever. When an input signal (electrical signal) is applied across the end of the cantilever and the actuation electrode on a substrate, the electrical attractive force, given by Coulomb's law, actuates the cantilever, and a detection circuit formed under the cantilever detects the filtered or amplified electrical signal that is generated by the mechanical vibration of the cantilever.

Since the development of the first MEMS device, many other MEMS have been developed. For example, as one of the important components of MEMS, the parallel plate shown in Fig. 1.2 (similar to the cantilever of Fig. 1.1) is widely used in many microdevices that employ electrostatic forces for actuation of a microstructure or detection of a physical quantity. The typical parallel plate shown in Fig. 1.2 illustrates the basic knowledge that is required to understand MEMS behavior. The parallel plate consists of a movable plate suspended by flexures, a stationary plate, and a voltage source to supply voltage or electrical charge to the movable and stationary plates. The flexures are used to support the movable plate and act as a spring. The gap between plates can be adjusted when a force (e.g., electrostatic force or inertial force) acts on the plate.

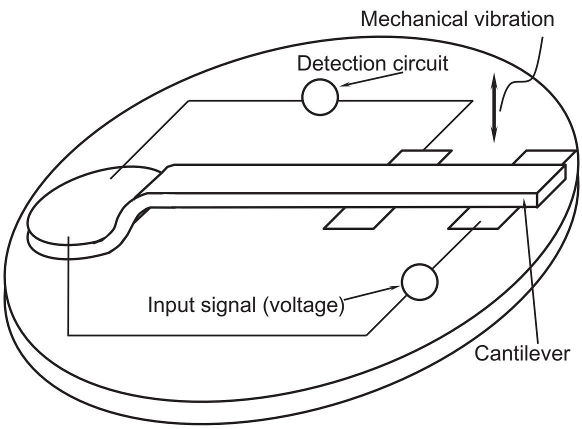


Figure 1.1 Resonant gate transistor.

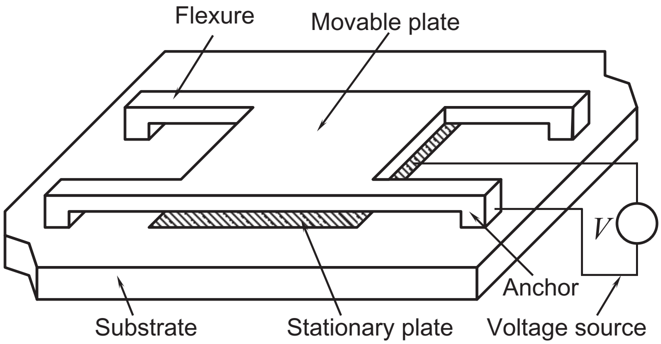


Figure 1.2 Parallel plate.

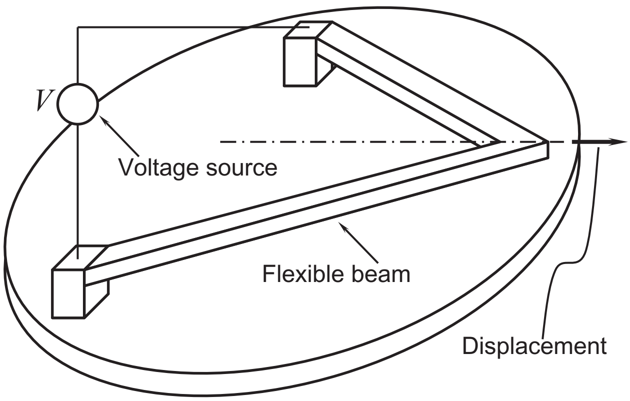


Figure 1.3 Electrothermal actuator.

Let us suppose that we apply a voltage across the movable and stationary plates. Upon applying the voltage, positive charges (or negative charges, depending on the electrical connection) are accumulated on the movable plate while opposite charges are accumulated on the stationary plate. As a result, the positive and negative charges on the plates generate an attractive force, the electrostatic force, which can push down the movable plate. The movable plate is displaced until the spring force (restoring force) due to the flexures balances the electrostatic force; that is, the displaced movable plate is in equilibrium while the voltage is applied. However, when the voltage is greater than a critical voltage called the *pull-in voltage*, the movable plate collapses into the lower plate.

A thermal actuator (Fig. 1.3) utilizes the thermal expansion due to Joule heating. As a voltage source supplies electrical current through the flexible beam that acts as a heater, heat is generated in the heater. The thermal

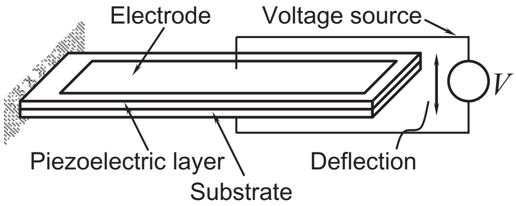


Figure 1.4 Piezoelectric actuator.

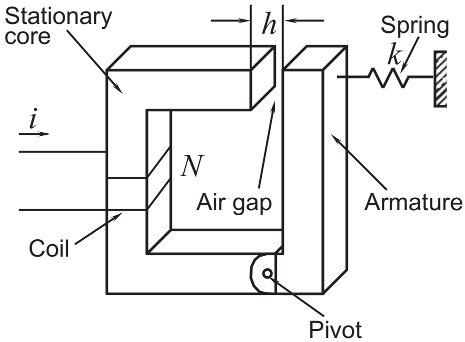


Figure 1.5 Electromagnetic relay.

expansion of the beam provides the displacement shown in the figure. The displacement depends on the voltage applied, the resistance of the beam, and the stiffness. Therefore, the mechanical behavior (e.g., displacement) of thermal actuators is coupled with the electrical and thermal behavior.

A piezoelectric actuator (Fig. 1.4) utilizes a piezoelectric material whose shape is deformed when exposed to an electric field. In Fig. 1.4 a piezoelectric layer is glued or deposited on a substrate. A thin conductive electrode is placed or deposited on the piezoelectric layer so that the layer is exposed to an electric field when a voltage source applies a voltage across the layer. In this situation, the layer expands or contracts, depending on the polarity of the voltage. For example, if the piezoelectric layer expands in the longitudinal direction, the right end of the actuator moves downward. The end of the actuator moves upward when the polarity of the voltage is reversed. The mechanical behavior of the piezoelectric actuator is then coupled with the piezoelectric constants that relate the voltage to the deformation of the piezoelectric layer, the mechanical properties (e.g., Young's modulus), and the layer geometry.

Electromagnetic force is also used to actuate microstructures. Figure 1.5 shows a model of an electromagnetic relay, one type of electromagnetic actuator. The relay consists of a movable bar (called an *armature*), a stationary core connected to the movable bar, a coil to generate magnetic field in the movable bar and stationary core, and a spring to provide the movable bar with a restoring force. When an electric current is applied to the coil, the relay is magnetized to generate an attractive force between the movable bar and the stationary core, and the movable bar is then attached to the stationary core. If the current is removed, the movable bar returns to its initial position under the restoring force of the spring. Thus, the mechanical behavior of electromagnetic actuators depends on the applied current, the magnetic and mechanical properties of the material used, the geometry of the actuator, and the stiffness of the spring.

As briefly discussed above, actuators use electricity to generate mechanical motion such as displacement, and the resulting mechanical behaviors are then coupled with electrical behavior, material properties, geometry, and so on. As a result of the coupling, the mechanical behavior is, in general, related nonlinearly to electric input (e.g., applied voltage) except in a few cases, or are expressed as complicated functions of electric input. To understand these nonlinear actuators and sensors, numerical analyses have been widely used. For example, to obtain the sensitivity to voltage of the capacitance of a parallel plate (Fig. 1.2), numerical analyses have been used to solve the equilibrium equation that governs the equilibrium position of the movable plate. Therefore, researchers, designers, and students have required commercial software to solve a problem or the skill to develop codes or programs that obtain the solution numerically. This book is designed to provide analytical closed-form solutions of both linear and nonlinear actuators in which mechanical behavior and electrical behavior are coupled. Since most MEMS-based sensors use actuators to measure physical quantities, this book can be used to design and analyze sensors.

1.3 KNOWLEDGE REQUIRED

As discussed in the foregoing section, MEMS are systems that consist of mechanical and electrical components and that may also involve other components, such as a reflective layer for a micromirror, depending on the purpose. Since the mechanical behavior of MEMS are coupled with other behavior, we should study interdisciplinary subjects in the fields of science and engineering to understand the coupled behaviors. Figure 1.6 shows an overview of the knowledge required for the research and development of MEMS and their derivatives. Because the most convenient and controllable energy is the electrical energy, electrical and electronic engineering covering electromagnetics, circuit theory, or signal processing is required to control phenomena associated with electric charge (i.e., electron, current). For example, from the point of view of electrical engineering, the parallel plate of Fig. 1.2 may be considered to be a capacitor consisting of movable and stationary plates, so knowledge of electrical engineering is necessary to calculate the capacitance of the

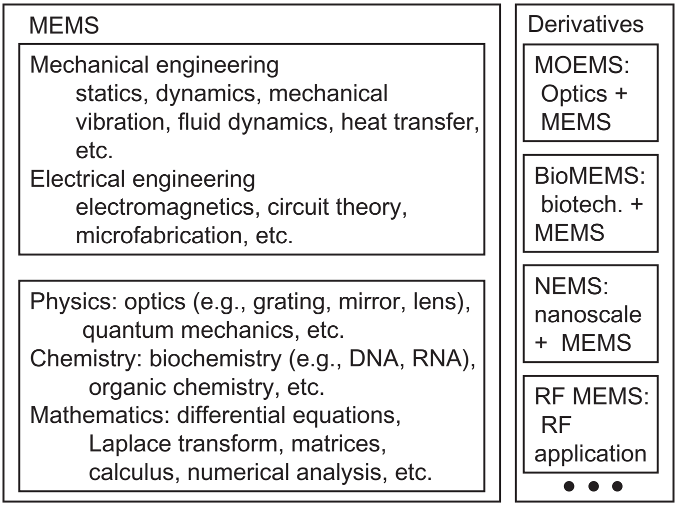


Figure 1.6 Knowledge required to understand MEMS.

parallel plate and to obtain the electrostatic force acting on the movable plate as a function of the interplate gap and the applied voltage. Similarly, since the magnetic relay of Fig. 1.5 is an electromagnet with a variable air gap, we use the magnetic energy that is stored in the electromagnet and calculate the magnetic force pulling the movable bar into the stationary core.

Physically, MEMS are mechanical structures that are designed for specific purposes. For desired functions, components of MEMS must be mechanically stable, vibrate if the mechanical resonance is utilized, and be deformed if deformation or displacement is needed. For the design and analysis of mechanical components, we need statics for the mechanical structure design, dynamics and vibration for resonance and mechanical vibration, heat transfer for thermal actuation, and fluid dynamics for the evaluation of damping due to the movement of microstructures. Let us consider the parallel plate in Fig. 1.2 as an example of a mechanical structure. Since the four flexures support the movable plate under the electrostatic force, we need statics to determine the flexure dimensions: the length, width, and thickness of the flexures. If the movable plate operates at resonance for a mechanical filter, we should use our knowledge of dynamics or mechanical vibration to design the resonant frequency desired. If we wish to set up a mechanical quality factor that affects the bandwidth of a mechanical filter, we should evaluate the damping force or damping coefficient that can be provided by fluid dynamics. If we wish to design an accelerometer or acceleration switch using the parallel plate shown in Fig. 1.2, we need to know the dynamics and mechanical vibration.

All the above-mentioned knowledge is coupled, so the design of a parallel plate for a specific application is very complicated even though the parallel plate in Fig. 1.2 looks simple. In addition to these complexities, we may need physics, chemistry, mathematics, and other subject areas, and MEMS may have different names, as described in Fig. 1.6: MOEMS if a MEMS device involves at least one optical component; RF MEMS if a MEMS device is designed for a radio-frequency application such as an RF filter; bioMEMS if a MEMS device is used for biological applications such as the detection of DNA strands; NEMS if at least one dimension of the mechanical structure is less than 1 μm; and perhaps other names in future applications if mechanical structures with electrical components are used for a specific purpose.

1.4 DIMENSIONAL ANALYSIS

Dimensional analysis and dimensionless numbers allow us to investigate complicated or coupled systems such as the MEMS described in Section 1.3. Using dimensional analysis and experimental results (or numerical simulation), we can find relationships between variables that are involved in a problem or system. If we apply dimensional analysis to a governing equation that describes a physical phenomenon and cannot be solved due to its nonlinearity, we can obtain useful dimensionless numbers that play crucial roles in describing the phenomenon. We begin with easy dimensionless numbers with which we are familiar.

Let us begin by considering the ratio of the circumference of a circle to its diameter, the well-known constant. Figure 1.7a, b, and c show a circular column, a rectangular column, and an arbitrarily shaped body, respectively. As the radius and height of the circular column (Fig. 1.7a) are represented by r and t, respectively, the perimeter l of the top view, the top-view area A, and the volume V are given by

$$l = 2\pi r = \pi d$$

$$A = \pi r^2 = \frac{\pi}{4} d^2$$

$$V = \pi r^2 t = \frac{\pi}{4} d^2 t$$

where π denotes the ratio of the circumference of a circle to its diameter and d represents the diameter of the column. It is worth noting that the perimeter, area, and volume are proportional to the diameter, the square of the diameter, and the product of the area and thickness, respectively. It is also noted that if the circular column become n times larger than its original dimensions d and t, the corresponding length, area, and volume will be nl, n^2A, and n^3V, respectively. Manipulating the equations above gives dimensionless forms as follows:

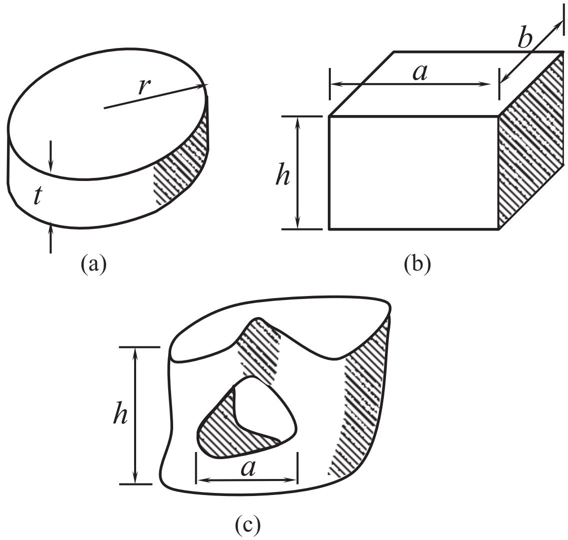


Figure 1.7 Various three-dimensional objects.

$$\frac{l}{d} = \pi$$

$$\frac{A}{r^2} = \pi \quad \text{or} \quad \frac{A}{d^2} = \frac{\pi}{4}$$

$$\frac{V}{d^2 t} = \frac{\pi}{4} \quad \text{or} \quad \frac{V}{d^3} = \frac{\pi}{4}\frac{t}{d}$$

In the preceding equations, the diameter d may be considered a characteristic length that represents the dimension of the circular column. The first two equations above give the constants (numbers) on their right-hand sides, and the third equation also yields a constant if the t/d remains unchanged. In this case, the dimensionless numbers l/d and A/d^2 remain unchanged even though the diameter becomes larger or smaller. However, the dimensionless number V/d^3 is proportional to the dimensionless number t/d. If the diameter and thickness become n times the original dimensions, the resulting length and area are, respectively, nl and n^2A, and the volume becomes n^3V since t/d does not change for a uniform transform (i.e., $nt/nd = t/d$). The preceding equations may be expressed in more general dimensionless forms as

$$f_1\left(\frac{l}{d}\right) = \frac{l}{d} - \pi = 0$$

$$f_2\left(\frac{A}{d^2}\right)=\frac{A}{d^2}-\frac{\pi}{4}=0$$

$$f_3=\left(\frac{V}{d^3},\frac{t}{d}\right)=\frac{V}{d^3}-\frac{\pi}{4}\frac{t}{d}=0$$

Similarly, for the rectangular column of Fig. 1.7b, the perimeter l of the top view, the top-view area A, and the volume V are given by

$$l=2(a+b)$$

$$A=ab$$

$$V=abh$$

We transform the equations above into dimensionless equations as follows:

$$f_4\left(\frac{l}{a},\frac{b}{a}\right)=\frac{l}{a}-2\left(1+\frac{b}{a}\right)=0$$

$$f_5\left(\frac{A}{a^2},\frac{b}{a}\right)=\frac{A}{a^2}-\frac{b}{a}=0$$

$$f_6\left(\frac{V}{a^3},\frac{b}{a},\frac{h}{a}\right)=\frac{V}{a^3}-\frac{b}{a}\frac{h}{a}=0$$

The dimensionless equations f_4, f_5, and f_6 represent functions for the perimeter and area of the top view of the rectangular column and the volume, respectively. It should be noted that the dimensionless length l/a, area A/a^2, and volume V/a^3 are expressed as functions of dimensionless variables b/a and h/a. This concept may be extended into more general cases.

Let us consider the complex three-dimensional structure shown in Fig. 1.7c. We wish to obtain the dimension a, the area, and the volume of the structure as functions of a characteristic length. The relations may be used to build a miniature or larger structure. Let l (not shown in Fig. 1.7c), h, A, and V represent a length, the height, the area, and the volume of a structure, respectively. The following equations can be written for a dimensional analysis:

$$f_7(a,l,h)=0$$

$$f_8(A,a,l,h)=0$$

$$f_9(V,a,l,h)=0$$

Let l be a characteristic length of a structure. Since a, l, and h have the dimensions of length and A and V have the dimensions of the square and cube of length, respectively, the dimensionless equations are given by

$$f_{10}\left(\frac{a}{l},\frac{h}{l}\right)=0$$

$$f_{11}\left(\frac{A}{l^2},\frac{a}{l},\frac{h}{l}\right)=0$$

$$f_{12}\left(\frac{V}{l^3},\frac{a}{l},\frac{h}{l}\right)=0$$

When the preceding equations are set up by dividing the arguments of the function by l, l_2, or l_3, the number of arguments is reduced by one in each equation. Rearranging the preceding equations yields the length a, the area A, and the volume V in dimensionless forms:

$$\frac{a}{l}=f_{13}\left(\frac{h}{l}\right)$$

$$\frac{A}{l^2}=f_{14}\left(\frac{a}{l},\frac{h}{l}\right)$$

$$\frac{V}{l^3}=f_{15}\left(\frac{a}{l},\frac{h}{l}\right)$$

For the enlargement or contraction of the structure, the ratio of linear dimensions, h/l, remains constant, and then the ratio a/l also becomes constant. The first equation above becomes $a/l = c_1$, where c_1 is a constant. Consequently, the first equation above gives a linear relation between a and l:

$$a=c_1 l \tag{1.1}$$

Similarly, the equations for the area A and the volume can be expressed as

$$A=c_2 l^2 \tag{1.2}$$

$$V=c_3 l^3 \tag{1.3}$$

where c_2 and c_3 denote constants. These three equations state that for any structures in three-dimensional space, if the shape of the structure remains unchanged for enlargement or contraction, the length from one point to another, the area of any portion of the structure, and the volume of the structure are proportional to a characteristic length and to the square and cube of the characteristic length, respectively. During derivation of equations (1.1) to (1.3), the characteristic length can be taken to be any dimension: for example, the width or the height. If the height h is selected as the characteristic length, the foregoing equations may be expressed as follows: $a = d_1h$, $A = d_2h^2$, *and* $V = d_3h^3$, where d_1, d_2, and d_3 represent constants.

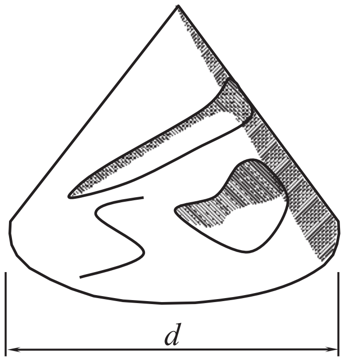


Figure 1.8 A piece of chocolate.

Example 1.1 A chocolate company decides to build an enlarged model of a piece of chocolate for an advertisement. The piece of chocolate shown in Fig. 1.8 will be enlarged n times. In other words, the dimension d will be nd in the model. In order to build the model and paint the outside, the company must calculate the length of the company logo S shown in Fig. 1.8 and the area and volume of the model. If the logo length is l_p, the outside area A_p, and the volume V_p, find the logo length of the model, the required volume of the piece of chocolate, and the outside area. If the company paints the outside to a thickness of t (wet paint), determine the volume of paint required.

As discussed above, the dimension, area, and volume of the model are proportional to a characteristic length and to the characteristic length squared and cubed, respectively. Let the logo length of model be l_m, the outside area A_m, and the volume V_m. For the length, area, and volume of the prototype (original) piece of chocolate, (1.1) to (1.3) give

$$l_p = c_1 d \tag{a}$$

$$A_p = c_2 d^2 \tag{b}$$

$$V_p = c_3 d^3 \tag{c}$$

where c_1, c_2, and c_3 denote constants for the relationships between the prototype and the model. The preceding equations also hold for the model as follows:

$$l_m = c_1 nd \tag{d}$$

$$A_m = c_2 (nd)^2 \tag{e}$$

$$V_m = c_3 (nd)^3 \tag{f}$$

From the preceding equations, we thus find relationships that will hold for both the prototype and the model:

$$l_m = nl_p \tag{g}$$

$$A_m = n^2 A_p \tag{h}$$

$$V_m = n^3 V_p \tag{i}$$

The volume of paint required for the model is given by $tA_m = n^2 tA_p$. According to (g) to (i), if the dimensions of a model are n times those of a prototype (i.e., geometrically similar), the linear dimension, area, and volume of the model are increased to n, n^2, and n^3 times those of the prototype, respectively. For example, if a structure is magnified by a factor of 10, any length, area, and volume of the magnified structure become, respectively, 10, 10^2, and 10^3 times those of the original structure.

In the foregoing discussion and Example 1.1, the dimensional analysis has been described for similar structures. These concepts may be extended to involve more general cases that are related to force, stress, energy, or any other physical quantities. As a physical quantity, force such as the weight of a structure is measured in newtons (N) if we use SI units (an abbreviated form of the French term corresponding to "international system of units"). Weight may be measured in other units: for example, lb_f (pound-force). To avoid any confusion associated with force units such as N and lb_f, F is used to represent the force dimension in dimensional analysis. Similarly, L and T represent the dimensions of length and time, respectively. In many cases, force (F), length (L), and time (T) are used as the fundamental units if physical quantities involved in a problem are expressed using force, length, and time. If force, length, and time are used as the fundamental units, the system of units is called the *F–L–T system*. Other dimensions that can be derived from the fundamental units are known as *derived units*. Derived units can be derived easily from basic equations. For example, the mass of a structure is a derived unit that is defined as $FL^{-1}T^2$, since the mass may be expressed as $m = F/a$, where F and a denote the force acting on the mass and the acceleration of the structure, respectively. If physical quantities under consideration cannot be derived from the preceding fundamental units (F–L–T units), the physical unit may be added to the list of fundamental units. For example, the temperature for thermal study and the electric charge for electric phenomena can be considered as additional units. As the fundamental units, the temperature and electric charge are represented by θ and Q, respectively. Fundamental and derived units that are widely used in scientific and engineering problems are shown in Table 1.1. In physics, researchers may use mass as a fundamental unit instead of force. In this case, the dimensions in Table 1.1 can be converted into an M–L–T system as MLT^{-2} is substituted for force F. For example, FL of the F–L–T system, representing energy, is converted to ML^2T^{-2} in the M–L–T system. The F–L–T and M–L–T systems yield the same results for the dimensional analysis of a physical problem.

TABLE 1.1 Dimensions of Physical Quantities

Physical Quantities	Dimensions	Physical Quantities	Dimensions
Force	F	Thermal conductivity	$FT^{-1}\theta^{-1}$
Length, displacement	L	Electric charge	Q
Time	T	Current	QT^{-1}
Mass	$FL^{-1}T^2$	Voltage	FLQ^{-1}
Density	$FL^{-4}T^2$	Resistance	$FLTQ^{-2}$
Velocity	LT^{-1}	Permittivity	$F^{-1}L^{-2}Q^2$
Acceleration	LT^{-2}	Capacitance	$F^{-1}L^{-1}Q^2$
Energy	FL	Magnetic field strength	$L^{-1}T^{-1}Q$
Stress, pressure	FL^{-2}	Magnetic flux density	$FL^{-1}T\,Q^{-1}$
Viscosity	$FL^{-2}T$	Permeability	FT^2Q^{-2}
Angle	dimensionless	Inductance	FLT^2Q^{-2}
Temperature	θ		

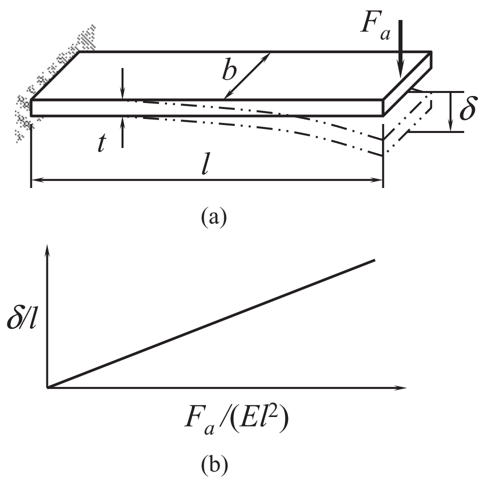


Figure 1.9 Cantilever subjected to a force.

Using the fundamental and derived units listed in Table 1.1, let us perform dimensional analysis that may be used to obtain a relation among many variables. Consider the cantilever beam of Young's modulus E (dimensions: N/m^2; a modulus relating stress to strain), length l, width b, and thickness t, as shown in Fig. 1.9a. For this simple problem we know from statics (that will be dealt with in Chapter 3) that a solution for the deflection δ at the end of a cantilever under an applied force F_a is given by

$$\delta = \frac{1}{3}\frac{F_a l^3}{EI} \tag{1.4}$$

where I denotes the moment of inertia of the cantilever, defined as $bt^3/12$. However, in order to know how to conduct the dimensional analysis of this problem, it is assumed that at this stage we don't know the solution. From Fig. 1.9a we know that F_a, E, δ, b, t, and l enter into the problem. The problem may be expressed as

$$f(F_a, E, \delta, b, t, l) = 0 \tag{1.5}$$

From Table 1.1 we know the dimensions of the variables as follows (E has the dimension of stress):

$$F_a = [F]$$
$$E = [F/L^2]$$
$$\delta = [L], \quad b = [L], \quad t = [L], \quad l = [L]$$

These variables involve the two fundamental dimensions F and L. If we divide the first equation by the second to eliminate the fundamental unit F, we have

$$\frac{F_a}{E} = \frac{[F]}{[F/L^2]} = [L^2]$$

which involves only the fundamental dimension L. We have δ, b, t, and l for the fundamental dimension of L, and any of them may be selected to represent L. Using l for a characteristic length representing the fundamental dimension L, the preceding equation can be converted into the following dimensionless number:

$$\frac{F_a}{El^2} = \frac{[L^2]}{[L^2]} = [0]$$

where [0] states that the number Fa/El^2 is dimensionless. Similarly, the other variables, δ, b, and t, can be converted into corresponding dimensionless numbers by dividing the variables by the characteristic length l, and the original problem can be expressed as

$$f\left(\frac{F_a}{El^2}, \frac{\delta}{l}, \frac{b}{l}, \frac{t}{l}\right) = 0$$

Since we wish to obtain an expression for the beam deflection δ, the foregoing equation may be expressed as follows:

$$\frac{\delta}{l} = f_1\left(\frac{F_a}{El^2}, \frac{b}{l}, \frac{t}{l}\right) \tag{1.6}$$

Equation (1.6) states that the deflection of the cantilever in Fig. 1.9a is expressed as the four dimensionless numbers Fa/El^2, δ/l, b/l, and t/l. Equation (1.6) is all that we can obtain from dimensional analysis. However, if we obtain more information from experiments or numerical simulations, the relationship (1.6) can be expressed in a more accurate expression. From an experiment or numerical analysis, we can find more relationships:

$$\frac{\delta}{l} = c_1 \frac{F_a}{El^2} \qquad \frac{\delta}{l} = c_2 \left(\frac{b}{l}\right)^{-1} \qquad \frac{\delta}{l} = c_3 \left(\frac{t}{l}\right)^{-3} \tag{1.7}$$

These equations are substituted into equation (1.6) and we then have

$$\frac{\delta}{l} = f_1\left(\frac{F_a}{El^2}, \frac{b}{l}, \frac{t}{l}\right) = c\frac{F_a}{El^2}\left(\frac{b}{l}\right)^{-1}\left(\frac{t}{l}\right)^{-3} = c\frac{F_a l^2}{Ebt^3}$$

or

$$\delta = c\frac{F_a l^3}{Ebt^3}$$

where c is a constant that represents the product of c_1, c_2, and c_3 and can be obtained from the experiment or numerical analysis (e.g., Fig. 1.9b) and obtained easily from a graph of $F_a l^2/(Ebt^3)$ against δ/l. We know that the constant will be 4 when the equation above is compared with the analytical solution, (1.4).

In the procedure used to obtain the dimensionless equation above, note that after selecting variables as characteristic variables (E and l), the other variables were divided by the variables selected. Note also that the number of resulting dimensionless variables is reduced by the number of fundamental units. In the cantilever problem in Fig. 1.9a, the number of dimensional variables of (1.5) was 6, but in the dimensionless form, the number of dimensionless variables was reduced to 4 (6 – 2, where 2 is the number of fundamental units, F and L in this case). This procedure is generalized by *Buckingham's π-theorem* (Buckingham, 1914), which may be stated as follows:

If n variables (υ_1, υ_2, … , υ_n), which can be expressed by N fundamental units, are involved in a problem, the dimensional equation for the problem may be expressed as

$$f(\upsilon_1, \upsilon_2, \upsilon_3, \ldots, \upsilon_n) = 0 \tag{1.8}$$

and the corresponding dimensionless equation can be a function of the $n - N$ dimensionless variables (π_1, π_2, π_3, … , π_{n-N}), as follows:

$$g(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-N}) = 0 \tag{1.9}$$

As discussed in the foregoing problem (Fig. 1.9a), equation (1.9) can give useful information and becomes more accurate if more information, such as data from an experiment or numerical analysis, is available.

The problem shown in Fig. 1.9a can also be solved using the π-theorem. In Fig. 1.9a we have six dimensional variables (F_a, E, δ, b, t, and l), and the problem of obtaining the deflection at the end of the cantilever can be expressed as

$$f(F_a, E, \delta, b, t, l) = 0$$

Since all the dimensional variables are expressed by the fundamental units of F and L, we expect that four dimensionless variables ($6 - 2 = 4$) will appear in a dimensionless equation:

$$g(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{1.10}$$

δ, b, t, and l have the dimension of L, and then we take the first three dimensionless variables as $\pi_1 = \delta/l$, $\pi_2 = b/l$, and $\pi_3 = t/l$. The fourth variable, π_4, may be expressed in the form

$$\pi_4 = F_a E^a l^b = [F][F/L^2]^a [L]^b = [F^{1+a} L^{-2a+b}]$$

for π_4 to be a dimensionless number (i.e., [0]), we have two equations,

$$1 + a = 0 \quad \text{and} \quad -2a + b = 0$$

From these equations, $a = -1$ and $b = 2a = -2$ are obtained. Substituting a and b into π_4 above gives

$$\pi_4 = F_a E^{-1} l^{-2} = \frac{F_a}{El^2}$$

The dimensionless variables (π_1, π_2, π_3 and π_4) are substituted into (1.10) to yield

$$g\left(\frac{\delta}{l}, \frac{b}{l}, \frac{t}{l}, \frac{F_a}{El^2}\right) = 0$$

The preceding equation is the same as (1.6). For more general applications of dimensional analysis, more examples are presented below.

Example 1.2 Parallel plates are used widely in MEMS to actuate microstructures and to sense physical quantities. As illustrated in Fig. 1.10, a parallel plate consists of an upper plate on a lower plate, and a voltage V may be applied across the plates, which are separated by a gap h. The length and width of the upper plate are l_1 and l_2 (Fig. 1.10) and the thickness and fringing field effect

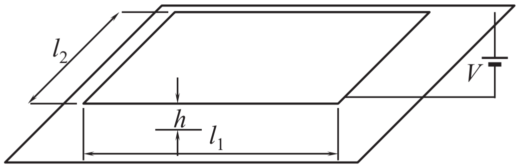


Figure 1.10 Dimensions of a parallel plate.

can be neglected if the thickness and the gap are much less than the length and width. The lower plate can be considered an infinite plate since its length and width are much larger than those of the upper plate. We wish to obtain a dimensionless equation that represents the electrostatic force F_e acting on the upper plate. Obtain the dimensionless equation in a simple form if from an experiment for $h << l_1$ and l_2, the electrostatic force F is proportional to the length and width of the upper plate, l_1 and l_2, and inversely proportional to the square of the gap h.

We derive the analytic solution of the electrostatic force for $h << l_1$ and l_2 in Chapter 7. At this stage it is assumed that we do not know the relation, in order to study the dimensional analysis for an electrostatic problem. Let ε represent the permittivity of the material (e.g., air) between the plates. The dimensional equation f for the problem may be expressed as

$$f(F_e, \varepsilon, V, l_1, l_2, h) = 0 \tag{a}$$

and the dimension of the variables in equation (a) is written, with reference to Table 1.1, as follows:

$$\begin{aligned} F_e &= [F] \\ \varepsilon &= [Q^2/FL^2] \\ V &= [FL/Q] \\ l_1 &= l_2 = h = [L] \end{aligned} \tag{b}$$

We have three fundamental variables (F, Q, and L) and the six dimensional variables above. We thus expect three dimensionless variables and may write the dimensionless equation g as

$$g(\pi_1, \pi_2, \pi_3) = 0 \tag{c}$$

where π_1, π_2 and π_3 are the dimensionless variables to be found. If l_1 is taken as the characteristic length that represents the length of the parallel plate, we have the following dimensionless variables:

$$\pi_1 = \frac{l_2}{l_1}$$
$$\pi_2 = \frac{h}{l_1} \tag{d}$$
$$\pi_3 = F_e \varepsilon^c V^d l_1^{\,e} = [F]\left[\frac{Q^2}{FL^2}\right]^c \left[\frac{FL}{Q}\right]^d [L]^e = \left[F^{1-c+d} L^{-2c+d+e} Q^{2c-d}\right]$$

For π_3 to be dimensionless (i.e., [0]) we have a set of equations to construct the dimensionless variable π_3:

$$\begin{aligned} 1-c+d&=0\\ -2c+d+e&=0\\ 2c-d&=0 \end{aligned} \tag{e}$$

which gives $c = -1$, $d = -2$, and $e = 0$, and we find the third dimensionless variable as

$$\pi_3 = \frac{F_e}{\varepsilon V^2} \tag{f}$$

Substituting equations (d) and (f) into (c), we have the dimensionless equation

$$g(\pi_1,\pi_2,\pi_3) = g\left(\frac{l_2}{l_1},\frac{h}{l_1},\frac{F_e}{\varepsilon V^2}\right) = 0$$

or

$$\frac{F_e}{\varepsilon V^2} = g_1\left(\frac{l_2}{l_1},\frac{h}{l_1}\right) \tag{g}$$

Equation (g) states that the ratio (i.e., π_3) of the electrostatic force acting on the upper plate to the product of the permittivity and the square of voltage is a function of the dimensionless ratios l_2/l_1 and h/l_1. From experiments of the parallel plate, it was found that the electrostatic force F is proportional to the length and width of the upper plate, l_1 and l_2, and inversely proportional to the square of the gap h. Using this information, equation (g) is simplified as follows:

$$\frac{F_e}{\varepsilon V^2} = c\frac{l_2}{l_1}\Big/\left(\frac{h}{l_1}\right)^2 = c\frac{A}{h^2} \tag{h}$$

where c represents a constant and A denotes the area of the upper plate, defined as $A = l_1 l_2$. The constant is also determined by the experimental data.

If we perform an experiment or numerical simulation and if we plot a graph of $F_e/\varepsilon V^2$ against A/h^2, the constant c is obtained from the slope of the graph. From the theoretical derivation described in Chapter 7, the solution for the electrostatic force acting on the parallel plate for $h \ll l_1$ and $h \ll l_2$ is given by

$$F_e = \frac{1}{2}\frac{\varepsilon A V^2}{h^2} \tag{i}$$

From equations (h) and (i) we know that the constant c must be 1/2.

The analysis to evaluate electrostatic force discussed in Example 1.2 may be extended to obtain the electrostatic force acting on more complicated structures across which a voltage is applied. Consider the structures in Fig. 1.11 under electrostatic force due to the applied voltage V. Let the dimensions of the structures be represented by a, b, c, d, ... and the interstructure gap be denoted by h. When we repeat the dimensional analysis discussed in Example 1.2, the following equation is obtained:

$$\frac{F_e}{\varepsilon V^2} = g\left(\frac{h}{a}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \cdots\right) \tag{1.11}$$

In equation (1.11), for convenience, dimension a was taken as the characteristic dimension. Any physical dimensions, such as b, c, ... , can be selected as the characteristic dimension. If b is taken as the characteristic length, the argument of equation (1.11) may be (h/b, a/b, c/b, d/b, ...). In many microstructures and nanostructures, only one dimension, such as h in Fig. 1.11, varies with time

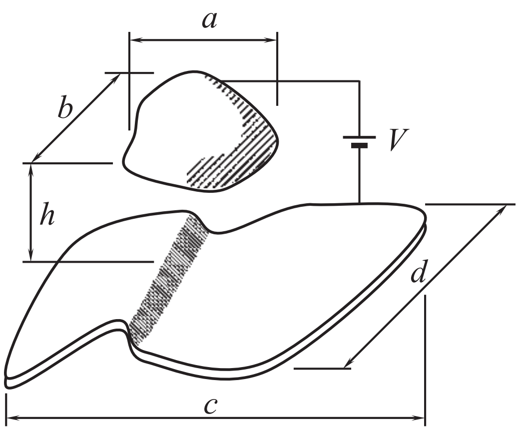


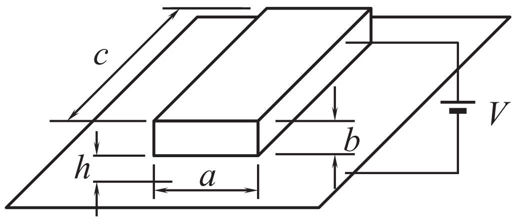
Figure 1.11 Arbitrarily shaped structures under electrostatic force.

or can be adjusted, while the other dimensions (b, c, d, ... in Fig. 1.11) remain unchanged since the dimensions are formed during fabrication of the structures. Then the dimension ratios (b/a, c/a, ...) in equation (1.11) also remain unchanged, and we rewrite (1.11) in the simpler form

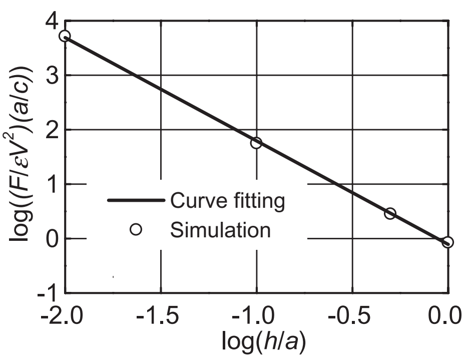
$$F_e = \varepsilon V^2 g_1\left(\frac{h}{a}\right) \tag{1.12}$$

Application of (1.11) is described in the following example.

Example 1.3 A conductive strip of width $a = 10\,\mu$m, thickness $b = 5\,\mu$m, and length c (much larger than a and b) is positioned over an infinite conductive plate as shown in Fig. 1.12a. The gap h of the strip can be adjusted from the plate and a voltage is applied across the strip and the plate. The length is much larger than the dimensions a, b, and h, so the problem in Fig. 1.12a can be considered to be a two-dimensional problem. Using commercial software for two-dimensional electrostatic analysis, we obtain the electrostatic forces (per unit meter) of 4.5685×10^{-3} N, 4.9595×10^{-5} N, 2.5008×10^{-6} N, and 7.4195×10^{-7} N at $h = 0.1 \times 10^{-6}$ m, 1×10^{-6} m, 5×10^{-6} m, and 10×10^{-6} m, respectively. It is desired to find a relation between the force F_e and the gap h in the form $F_e = ph^q$, where p and q are constants.



(a)



(b)

Figure 1.12 Electrostatic force acting on a strip configured over a plate.

In designing MEM devices, this type of problem will be encountered, which cannot be solved by an analytical method or whose analytical solution is in very a complicated form or expressed in an infinite series. In this example, since the thickness b and gap h are comparable to the width a, equation (i) of Example 1.2 for the parallel-plate force generates significant errors. Thus, we require an expression that can be used to evaluate the force. For this purpose, we can use (1.11), which was obtained from dimensional analysis.

Let us write (1.11) with the dimensions a, b, and c:

$$\frac{F_e}{\varepsilon V^2} = g\left(\frac{h}{a}, \frac{b}{a}, \frac{c}{a}\right) \tag{a}$$

Since c is much greater than a and b, the problem is considered a two-dimensional problem, and the electrostatic force F_e is then proportional to the length, c. Thus, equation (a) is written in the form

$$\frac{F_e}{\varepsilon V^2} = \frac{c}{a} g_1\left(\frac{h}{a}, \frac{b}{a}\right) \tag{b}$$

where g_1 is a new function of h/a and b/a. However, since b/a remains constant while h/a varies, equation (b) may be written as

$$\frac{F_e}{\varepsilon V^2} = \frac{c}{a} g_2\left(\frac{h}{a}\right)$$

or

$$\frac{F_e}{\varepsilon V^2}\frac{a}{c} = g_2\left(\frac{h}{a}\right) \tag{c}$$

where g_2 is a function of h/a when b/a remains unchanged. As shown in Fig. 1.12b, the dimensionless force defined on the left-hand side of equation (c) is plotted as circles against h/a on a log-log scale (base 10). The force–gap relation is linear in the log-log graph and can be expressed as a linear equation:

$$\log\left(\frac{F_e}{\varepsilon V^2}\frac{a}{c}\right) = c_1 \log\frac{h}{a} + c_2 \tag{d}$$

From a linear curve fitting (e.g., graphical method or least-squares method) using the force data given, we find that $c_1 = -1.8990$ and $c_2 = -0.1084$. Substituting the coefficients into equation (d) leads us to

$$\frac{F_e}{\varepsilon V^2}\frac{a}{c} = 10^{c_1 \log(h/a) + c_2} = 10^{c_2}\left(\frac{h}{a}\right)^{c_1} = 0.7791\left(\frac{h}{a}\right)^{-1.8990} \tag{e}$$

Rearranging equation (e) gives us the electrostatic force F_e in the form $F_e = ph^q$:

$$F_e = 0.7791\varepsilon V^2 \frac{c}{a}\left(\frac{a}{h}\right)^{1.8990} \tag{f}$$

It should be noted that equation (f) is valid for width $a = 10 \times 10^{-6}$ m, thickness $b = 5 \times 10^{-6}$ m, $a \ll c$, $b \ll c$, and $0.1 \times 10^{-6}\,\text{m} \le h \le 10 \times 10^{-6}\,\text{m}$ (the range of the input data). To validate equation (f) we can substitute $a = 10 \times 10^{-6}$ m, $b = 5 \times 10^{-6}$ m, $h = 5 \times 10^{-6}$ m, $c = 1$ m, $\varepsilon = 8.854 \times 10^{-12}$ F/m, and $V = 1$ V to give $F_e = 2.5727 \times 10^{-6}$ N, which is very close to 2.5008×10^{-6} N from numerical analysis. For $a = 10 \times 10^{-6}$ m, $b = 5 \times 10^{-6}$ m, $h = 5 \times 10^{-6}$ m, $c = 10a = 100 \times 10^{-6}$ m, $\varepsilon = 8.854 \times 10^{-12}$ F/m, and $V = 10$ V, equation (f) yields $F_e = 2.5727 \times 10^{-8}$ N.

It is worth noting that (1.11) holds for any structures that are geometrically similar; that is, (f) holds for any structures of $b/a = 0.5$, $c/a \gg 1$, and $0.1 \le h/a \le 1$. For example, (f) can be used to evaluate the electrostatic force of the structure, which is suspended over a large plate and has the following dimensions: $a = 20 \times 10^{-6}$ m, $b = 10 \times 10^{-6}$ m, $h = 10 \times 10^{-6}$ m, and $c = 10a = 200 \times 10^{-6}$ m. The structure is the doubled structure of that above and generates the same magnitude of electrostatic force (2.5727×10^{-8} N) if the same voltage (10 V) is applied.

If a microstructure immersed in a gas or liquid is moved, a drag force due to the viscosity is generated and acts as a damper that decays the energy stored in the microstructure. The drag force or damping force plays an important role since it is related to the damping coefficient or quality factor (this subject is dealt with in detail in Chapter 5). The drag force is also expressed in dimensionless form. The following example shows how to use dimensional analysis to obtain the damping force.

Example 1.4 A microplate of length l and width w (Fig. 1.13) moves at a velocity of u toward a lower plate that is much larger than the microplate. During the motion of the microplate, the upper plate, spaced by a gap h from the lower plate, squeezes air between the plates so that the air moves out to atmosphere and the pressure under the microplate is increased. If the microplate is moved up, the pressure is lowered and the air will move in from the atmosphere. The pressure acting on the microplate is a source of damping force. Experiments show that the damping force F_d depends on the velocity u, the viscosity μ, the length l, the width w, and the gap h. We wish to find a relation between the damping force and the variables. If the damping force F_d is inversely proportional to the cubic of the gap h, refine the relation.

This subject is covered in detail in Chapter 6. At this stage we use dimensional analysis to obtain the relation among the parameters. The dimensional equation of the problem may be expressed as

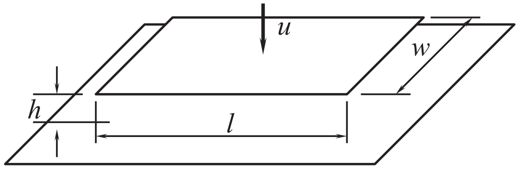


Figure 1.13 Parallel plate subjected to a damping force.

$$f(F_d, \mu, u, l, w, h) = 0 \tag{a}$$

and the dimension of the variables in equation (a) is written, with reference to Table 1.1, as follows:

$$\begin{aligned} F_d &= [F] \\ \mu &= [FT/L^2] \\ u &= [L/T] \\ l = w = h &= [L] \end{aligned} \tag{b}$$

We have three fundamental variables (F, L, and T) and the six dimensional variables above. Three dimensionless variables will then be involved in this problem and the corresponding dimensionless equation g may be written as

$$g(\pi_1, \pi_2, \pi_3) = 0 \tag{c}$$

where π_1, π_2, and π_3 are the dimensionless variables to be found. If l is taken as the characteristic length that represents the upper plate, we have the following dimensionless variables:

$$\begin{aligned} \pi_1 &= \frac{w}{l} \\ \pi_2 &= \frac{h}{l} \\ \pi_3 &= F_e \mu^c u^d l^e = [F]\left[\frac{FT}{L^2}\right]^c \left[\frac{L}{T}\right]^d [L]^e = [F^{1+c} L^{-2c+d+e} T^{c-d}] \end{aligned} \tag{d}$$

Setting $\pi_3 = [0]$ gives a set of equations to construct the dimensionless variable π_3:

$$
\begin{aligned}
1 + c &= 0 \\
-2c + d + e &= 0 \\
c - d &= 0
\end{aligned} \tag{e}
$$

which gives $c = -1$, $d = -1$, and $e = -1$, and we find the third dimensionless variable as

$$\pi_3 = \frac{F_d}{\mu u l} \tag{f}$$

Substituting the first two equations of (d) and (f) into (c), we have the dimensionless equation

$$g(\pi_1, \pi_2, \pi_3) = g\left(\frac{w}{l}, \frac{h}{l}, \frac{F_d}{\mu u l}\right) = 0$$

or

$$\frac{F_d}{\mu u l} = g_1\left(\frac{w}{l}, \frac{h}{l}\right) \tag{g}$$

For refinement of equation (g), we use the inverse proportionality of the damping force F_d to the cubic of the gap h. Using the proportionality, equation (g) is rewritten in the form

$$\frac{F_d}{\mu u l} = \left(\frac{l}{h}\right)^3 g_2\left(\frac{w}{l}\right) \tag{h}$$

Even though equation (h) is valid, a slightly different form may be obtained for our convenience. Since the product of dimensionless numbers generates a dimensionless number, (g) can be written

$$\frac{F_d}{\mu u l} = g_3\left(\frac{w}{l}, \frac{w}{l} \Big/ \frac{h}{l}\right) = g_3\left(\frac{w}{l}, \frac{w}{h}\right) \tag{i}$$

If proportionality is used, equation (i) is converted into

$$\frac{F_d}{\mu u l} = \left(\frac{w}{h}\right)^3 g_4\left(\frac{w}{l}\right) \tag{j}$$

After studying squeeze damping in Chapter 6, we obtain the following closed-form equation for the damping force:

$$F_d = \beta \frac{\mu u w^3 l}{h^3} \tag{k}$$

where β denotes a function of the ratio w/l. It is noted that if we define $\beta = g_4(w/l)$, (k) is the same as (j). It is noted that comparing (h) and (j) gives

$$g_4 = \left(\frac{l}{w}\right)^3 g_2\left(\frac{w}{l}\right)$$

The damping problem discussed in Example 1.4 may be extended to involve more dimensions that affect the damping force. If additional dimensions are represented by $a, b, c, \ldots$, we have the following dimensional equation:

$$f(F_d, \mu, u, l, w, h, a, b, c, \ldots) = 0 \tag{1.13}$$

When the procedure used in example 1.4 is repeated, the dimensionless equation for the general case is given by

$$\frac{F_d}{\mu u l} = \left(\frac{w}{h}\right)^3 g\left(\frac{w}{l}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right)$$

or

$$F_d = \left(\frac{w}{h}\right)^3 \mu u l \, g\left(\frac{w}{l}, \frac{a}{l}, \frac{b}{l}, \frac{c}{l}, \cdots\right) \tag{1.14}$$

Equation (1.14) can be interpreted as follows: The damping force due to squeezed gas is proportional to the viscosity of the surrounding gas or fluid and to the velocity perpendicular to the plate. Furthermore, the damping force is inversely proportional to the cubic of the interplate gap. Other dimensions, such as the size of the perforation and the plate shape (e.g., rectangle or circle), also affect the damping force, and their effect on the force may be involved in a correction factor such as the function g of equation (1.14).

So far, to conduct dimensional analysis for problems, we have assumed that we did not know exact principles or governing equations but knew parameters involved in the problems. If we know governing equations of problems that cannot be solved due to their nonlinearity or for which it is difficult to obtain analytical solutions, we can also employ dimensional analysis to anticipate which parameters affect the solutions, to design experiments or numerical analysis, or to obtain closed-form expressions from experiments or numerical analysis. We begin by considering the cantilever deflection problem, which has already been studied in association with Fig. 1.9.

The governing equation for deflection y of the cantilever at distance x is given by (its derivation is dealt with in Chapter 3)

$$EI\frac{d^2y}{dx^2} = M \tag{1.15}$$

where E, I, and M denote Young's modulus, the moment of inertia of the beam defined as $bt^3/12$, and the moment acting on the beam cross section, respectively. The deflection y is easily obtained since the equation is linear and the moment M is defined as $F_a(l - x)$. To obtain a dimensionless equation from equation (1.15), we introduce the following dimensionless parameters:

$$X = \frac{x}{l}$$

$$Y = \frac{y}{\delta}$$

where the dimensionless parameters X and Y represent x and y, respectively, and l and δ are taken as the characteristic dimensions for the dimensionless parameters. It is noted that X and Y are normalized by the maximum values (i.e., l, δ) of the corresponding dimensional variables x and y, and then the maximum X and Y are unity. Recalling the chain rule of differentiation, the first and second derivatives of y with respect to x are obtained as

$$\frac{dy}{dx} = \frac{dX}{dx}\frac{dy}{dX} = \frac{1}{l}\frac{d\delta Y}{dX} = \frac{\delta}{l}\frac{dY}{dX}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{1}{l}\frac{d}{dX}\left(\frac{\delta}{l}\frac{dY}{dX}\right) = \frac{\delta}{l^2}\frac{d^2Y}{dX^2}$$

Substituting the preceding equations into the governing equation with $M = F_a(l - x) = F_a l(1 - X)$, we have

$$EI\frac{\delta}{l^2}\frac{d^2Y}{dX^2} = F_a l(1 - X)$$

or in dimensionless form,

$$\frac{d^2Y}{dX^2} = G(1 - X) \tag{1.16}$$

where G is a dimensionless force defined as

$$G = \frac{F_a l^3}{EI\delta}$$

The foregoing governing equation in dimensionless form states that the dimensionless deflection Y is a function of the dimensionless force G and the dimensionless position X. This statement can be expressed as follows:

$$f(Y, G, X) = 0$$

or

$$Y = g(G, X)$$

For a small deflection (i.e., $\delta/l << 1$) we can find from an experiment or numerical simulation that there is a linear relation between Y and G and that the following equation holds:

$$Y = Gg_1(X)$$

Substituting Y, G, and X defined earlier into the foregoing equation leads us to

$$\frac{y}{\delta} = \frac{F_a l^3}{EI\delta} g_1\left(\frac{x}{l}\right)$$

or

$$y = \frac{F_a l^3}{EI} g_1\left(\frac{x}{l}\right) \tag{1.17}$$

The exact solution of this problem, which is dealt with in Chapter 3, is given by

$$y = \frac{F_a l^3}{6EI}\left[-\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^2\right]$$

Comparing the preceding equation with (1.17), we find that

$$g_1 = \frac{1}{6}\left[-\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^2\right]$$

If g_1 of (1.17) is not known, we may obtain the coefficient from experiment.

As shown, if we make a governing equation dimensionless, we can find the dimensionless parameters that are involved in the governing equation. Another example, involving manipulating governing equations to obtain dimensionless equations and variables, is presented below.

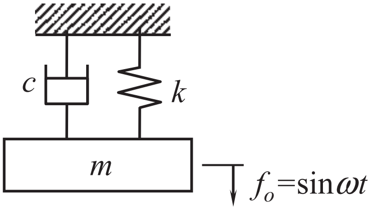


Figure 1.14 Vibrating system.

Example 1.5 In physics many vibrating systems consist of a mass m, a spring of stiffness k to generate a restoring force, and a damper with a damping coefficient c. The model for the systems is illustrated in Fig. 1.14, in which the mass is actuated by a time-varying force $f_o \sin \omega t$, where f_o, ω, and t denote the force amplitude, angular frequency, and time, respectively. The governing equation for the response of the mass to the force is given by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = f_0 \sin \omega t \tag{a}$$

where y is the response of the mass. We wish to obtain the dimensionless form of equation (a) and find dimensionless parameters that are involved in the problem.

To obtain the dimensionless equation, we first define dimensionless parameters for y and t as follows:

$$\begin{aligned} Y &= \frac{y}{y_c} \\ T &= \frac{t}{\tau} \end{aligned} \tag{b}$$

where Y and T represent the dimensionless displacement and time, and y_c and t denote the characteristic displacement and time, respectively, which will be defined later. Using the foregoing dimensionless parameters and recalling the chain rule of differentiation yields the time derivatives

$$\frac{d}{dt} = \frac{dT}{dt}\frac{d}{dT} = \frac{1}{\tau}\frac{d}{dT}$$

Similarly,

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{\tau^2}\frac{d^2}{dT^2}$$

Substituting the preceding four equations into the original governing equation, we have

$$m\frac{1}{\tau^2}\frac{d^2(y_cY)}{dT^2}+c\frac{1}{\tau}\frac{d(y_cY)}{dT}+ky_cY=f_0\sin\omega\tau T$$

Dividing the foregoing equation by ky_c leads to

$$\frac{m}{k\tau^2}\frac{d^2Y}{dT^2}+\frac{c}{k\tau}\frac{dY}{dT}+Y=\frac{f_0}{ky_c}\sin\omega\tau T \tag{c}$$

In equation (c) the characteristic displacement y_c and time τ are arbitrary and can be taken to reduce equation (c) to simplest form. For this purpose we may set both the first coefficient on the left-hand side of equation (c) and the coefficient on the right-hand side equal to unity as follows:

$$\begin{aligned}\frac{m}{k\tau^2}&=1\\ \frac{f_0}{ky_c}&=1\end{aligned} \tag{d}$$

The two equations (d) give the following characteristic time and displacement to provide the simplest dimensionless form:

$$\begin{aligned}\tau&=\sqrt{\frac{m}{k}}=\frac{1}{\omega_n}\\ y_c&=\frac{f_0}{k}\end{aligned} \tag{e}$$

where ω_n denotes the natural frequency of the vibrating system, defined as $\sqrt{k/m}$, and y_c represents the static displacement. Using the definition of τ and y_c above and introducing the quality factor $Q=\sqrt{mk}/c$, we obtain the dimensionless governing equation as follows:

$$\frac{d^2Y}{dT^2}+\frac{1}{Q}\frac{dY}{dT}+Y=\sin\Omega T \tag{f}$$

where Ω is the dimensionless frequency, defined as $\Omega = \omega/\omega_n$. Therefore, the dimensionless response may be expressed as

$$g(Y,Q,\Omega,T)=0$$

or

$$Y = g_1(Q, \Omega, T) \tag{g}$$

This equation shows clearly that the response Y is a function of the quality factor Q, which is defined by m, c, and k, the dimensionless angular frequency Ω, and the dimensionless time T. The corresponding analytic solution is dealt with in Chapter 5.

PROBLEMS

1.1 The size of the piece of chocolate in Fig. 1.8 is increased n times and its weight is proportional to the mass. If the density of the model is half that of the original piece, determine the weight of the model.

1.2 Suppose that an astronaut visits a planet on which the gravitational acceleration is one-tenth that on Earth. He finds a giant ant that consists of the same material as, but whose size is 100 times greater than, that of the ant on Earth. He returns to Earth with the giant and puts it in a cage. By dimensional analysis, describe what happens to the giant ant.

1.3 A fixed–fixed beam is subjected to a load q (N/m) that is distributed uniformly along the beam. The governing equation for deflection y at x from one end of the beam is given by

$$EI\frac{d^4y}{dx^4} - N_0\frac{d^2y}{dx^2} = q$$

where E, I, and N_o denote Young's modulus (N/m^2), the moment of inertia of the beam (m^4), and the tension (N) acting along the beam, respectively. Find the dimensionless equation and suggest an expression for the solution in dimensionless form.

1.4 A movable plate of length l and width w in a gas moves into an infinite stationary plate at a velocity of u (Fig. P1.4). The movable plate is perforated to reduce the damping force. The plate gap, the pitch, and the size of the perforation are denoted by h, p, and f, respectively. Find an expression for the damping force F_d in dimensionless form. If experiments show that the damping force is inversely proportional to the gap cubed, refine the expression.

1.5 The sound of a bell depends on the natural frequency of the bell structure. The bell is made of a material of Young's modulus E (N/m^2) and density ρ (kg/m^3). If the natural frequency of the original bell is f_p, find the natural frequency of an n-fold increased model that is geometrically similar to the original, but whose Young's modulus and density are changed from the original values.

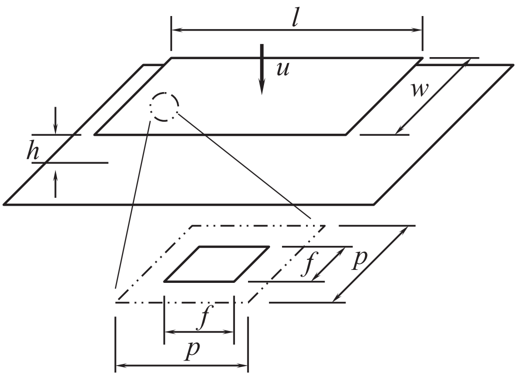


Figure P1.4 Perforated plate.