

## CHAPTER 1

# Introduction

### **1.1 THE NEED FOR BETTER FINANCIAL MODELING OF ASSET PRICES**

Major debacles in financial markets since the mid-1990s such as the Asian financial crisis in 1997, the bursting of the dot-com bubble in 2000, the subprime mortgage crisis that began in the summer of 2007, and the days surrounding the bankruptcy of Lehman Brothers in September 2008 are constant reminders to risk managers, portfolio managers, and regulators of how often extreme events occur. These major disruptions in the financial markets have led researchers to increase their efforts to improve the flexibility and statistical reliability of existing models that seek to capture the dynamics of economic and financial variables. Even if a catastrophe cannot be predicted, the objective of risk managers, portfolio managers, and regulators is to limit the potential damages.

The failure of financial models has been identified by some market observers as a major contributor—indeed some have argued that it is the single most important contributor—for the latest global financial crisis. The allegation is that financial models used by risk managers, portfolio managers, and even regulators simply did not reflect the realities of real-world financial markets. More specifically, the underlying assumption regarding asset returns and prices failed to reflect real-world movements of these quantities. Pinpointing the criticism more precisely, it is argued that the underlying assumption made in most financial models is that distributions of prices and returns are normally distributed, popularly referred to as the “normal model.” This probability distribution—also referred to as the *Gaussian distribution* and in lay terms the “bell curve”—is the one that dominates the teaching curriculum in probability and statistics courses in all business schools. Despite its popularity, the normal model flies in the face of what has been well documented regarding asset prices and returns. The preponderance of the empirical evidence has led to the following three stylized facts

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regarding financial time series for asset returns: (1) they have *fat tails* (*heavy tails*), (2) they may be *skewed*, and (3) they exhibit *volatility clustering*.

The “tails” of the distribution are where the extreme values occur. Empirical distributions for stock prices and returns have found that the extreme values are more likely than would be predicted by the normal distribution. This means that between periods where the market exhibits relatively modest changes in prices and returns, there will be periods where there are changes that are much higher (i.e., crashes and booms) than predicted by the normal distribution. This is not only of concern to financial theorists, but also to practitioners who are, in view of the frequency of sharp market down turns in the equity markets noted earlier, troubled by, in the words of Hoppe (1999), the “. . . compelling evidence that something is rotten in the foundation of the statistical edifice . . . used, for example, to produce probability estimates for financial risk assessment.” Fat tails can help explain larger price fluctuations for stocks over short time periods than can be explained by changes in fundamental economic variables as observed by Shiller (1981).

The normal distribution is a *symmetric distribution*. That is, it is a distribution where the shape of the left side of the probability distribution is the mirror image of the right side of the probability distribution. For a skewed distribution, also referred to as a *nonsymmetric distribution*, there is no such mirror imaging of the two sides of the probability distribution. Instead, typically in a skewed distribution one tail of the distribution is much longer (i.e., has greater probability of *extreme values* occurring) than the other tail of the probability distribution, which, of course, is what we referred to as fat tails. Volatility clustering behavior refers to the tendency of large changes in asset prices (either positive or negative) to be followed by large changes, and small changes to be followed by small changes.

The attack on the normal model is by no means recent. The first fundamental attack on the assumption that price or return distribution are not normally distributed was in the 1960s by Mandelbrot (1963). He strongly rejected normality as a distributional model for asset returns based on his study of commodity returns and interest rates. Mandelbrot conjectured that financial returns are more appropriately described by a non-normal stable distribution. Since a normal distribution is a special case of the stable distribution, to distinguish between Gaussian and non-Gaussian stable distributions, the latter are often referred to as *stable Paretian* distributions or *Lévy stable* distributions.<sup>1</sup> We will describe these distributions later in this book.

<sup>1</sup>The stable Paretian distribution is so-named because the tails of the non-Gaussian stable distribution have Pareto power-type decay. The Lévy stable distribution is

Mandelbrot's early investigations on returns were carried further by Fama (1963a, 1963b), among others, and led to a consolidation of the hypothesis that asset returns can be better described as a stable Paretian distribution. However, there was obviously considerable concern in the finance profession by the findings of Mandelbrot and Fama. In fact, shortly after the publication of the Mandelbrot paper, Cootner (1964) expressed his concern regarding the implications of those findings for the statistical tests that had been published in prominent scholarly journals in economics and finance. He warned that (Cootner, 1964, p. 337):

*Almost without exception, past econometric work is meaningless. Surely, before consigning centuries of work to the ash pile, we should like to have some assurance that all our work is truly useless. If we have permitted ourselves to be fooled for as long as this into believing that the Gaussian assumption is a workable one, is it not possible that the Paretian revolution is similarly illusory?*

Although further evidence supporting Mandelbrot's empirical work was published, the "normality" assumption remains the cornerstone of many central theories in finance. The most relevant example for this book is the pricing of options or, more generally, the pricing of contingent claims. In 1900, the father of modern option pricing theory, Louis Bachelier, proposed using *Brownian motion* for modeling stock market prices.<sup>2</sup> Inspired by his work, Samuelson (1965) formulated the log-normal model for stock prices that formed the basis for the well-known Black-Scholes option pricing

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named in honor of Paul Lévy for his seminal work introducing and characterizing the class of non-Gaussian stable distributions.

<sup>2</sup>There are several reasons why Brownian motion is a popular process. First, Brownian motion is the milestone of the theory of stochastic processes. However, more realistic general processes that are better suited for financial modeling such as Lévy, additive or self-similar processes (all of which we discuss in this book) have been developed only since the mid-1990s (see Samorodnitsky and Taqqu, 1994, Sato, 1999, and Embrechts and Maejima, 2002). Most of the practical problems of mathematical finance can be solved by taking into consideration these new processes. For example, the concept of stochastic integral with respect to Brownian motion was introduced in 1933 and only in the 1990s has the general theory of stochastic integration with respect to semimartingale appeared. From a practical point of view, the second reason for the popularity of Brownian motion is that the normal distribution allows one to solve real-world pricing problems such as option prices as estimations and simulations in a few seconds, and most of the problems have a closed-form solution that can be easily used.

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model. Black and Scholes (1973) and Merton (1974) introduced pricing and hedging theory for the options market employing a stock price model based on the *exponential Brownian motion*. The model greatly influences the way market participants price and hedge options; in 1997, Merton and Scholes were awarded the Nobel Prize in Economic Science.

Despite the importance of option theory as formulated by Black, Scholes, and Merton, it is widely recognized that on Black Monday, October 19, 1987, the Black-Scholes formula failed. The reason for the failure of the model particularly during volatile periods is its underlying assumptions necessary to generate a closed-form solution to price options. More specifically, it is assumed that returns are normally distributed and that return volatility is constant over the option's life. The latter assumption means that regardless of an option's strike price, the implied volatility (i.e., the volatility implied by the Black-Scholes model based on observed prices in the options market) should be the same. Yet, it is now an accepted fact that in the options market, implied volatility varies depending on the strike price. In some options markets, for example, the market for individual equities, it is observed that, for options, implied volatility decreases with an option's strike price. This relationship is referred to as *volatility skew*. In other markets, such as index options and currency options, it is observed that at-the-money options tend to have an implied volatility that is lower than for both out-of-the-money and in-the-money options. Since graphically this relationship would show that implied volatility decreases as options move from out-of-the-money options to at-the-money options and then increase from at-the-money options to in-the-money options, this relationship between strike price and implied volatility is called *volatility smile*. Obviously, both volatility skew and volatility smile are inconsistent with the assumption of a constant volatility.

Consequently, since the mid-1990s there has been growing interest in non-normal models not only in academia but also among financial practitioners seeking to try to explain extreme events that occur in financial markets. Furthermore, the search for proper models to price complex financial instruments and to calibrate the observed prices of those instruments quoted in the market has motivated studies of more complex models. There is still a good deal of work to be done on financial modeling using alternative non-normal distributions that have recently been proposed in the finance literature. In this book, we explain these univariate and multivariate models (both discrete and continuous) and then show their applications to explaining stock price behavior and pricing options.

In the balance of this chapter we describe some background information that is used in the chapters ahead. At the end of the chapter we provide an overview of the book.

## 1.2 THE FAMILY OF STABLE DISTRIBUTION AND ITS PROPERTIES

As noted earlier, Mandelbrot and Fama observed fat tails for many asset price and return data. For assets whose returns or prices exhibit fat-tail attributes, non-normal distribution models are required to accurately model the tail behavior and compute probabilities of extreme returns. The candidates for non-normal distributions that have been proposed for modeling extreme events in addition to the  $\alpha$ -stable Paretian distribution include mixtures of two or more normal distributions, Student  $t$ -distributions, hyperbolic distributions, and other scale mixtures of normal distributions, gamma distributions, extreme value distributions. The class of stable Paretian distributions (which includes  $\alpha$ -stable Paretian distribution as a special case) are simply referred to as *stable distributions*.

Although we cover the stable distribution in considerable detail in Chapter 3, here we only briefly highlight the key features of this distribution.

### 1.2.1 Parameterization of the Stable Distribution

In only three cases does the density function of a stable distribution have a closed-form expression. In the general case, stable distributions are described by their characteristic function that we describe in Chapter 3. A characteristic function provides a third possibility (besides the cumulative distribution function and the probability density function) to uniquely define a probability distribution. At this point, we just state the fact that knowing the characteristic function is mathematically equivalent to knowing the probability density function or the cumulative distribution function. What is important to understand is that the characteristic function (and thus the density function) of a stable distribution is described by four parameters:  $\mu$ ,  $\sigma$ ,  $\alpha$ , and  $\beta$ .<sup>3</sup>

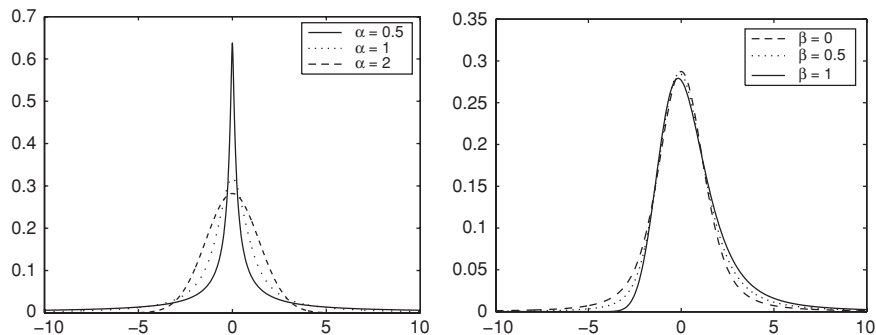
The  $\mu$  and  $\sigma$  parameters are measures of central location and scale, respectively. The parameter  $\alpha$  determines the tail weight or the distribution's kurtosis with  $0 < \alpha \leq 2$ . The  $\beta$  determines the distribution's skewness. When the  $\beta$  of a stable distribution is zero, the distribution is symmetric around  $\mu$ . Stable distributions allow for skewed distributions when  $\beta \neq 0$  and fat tails; this means a high probability for extreme events relative to the normal distribution when  $\alpha < 0$ . The value of  $\beta$  can range from

<sup>3</sup>There are many different possible parameterizations of stable distributions. For an overview the reader is referred to Zolotarev (1986). The parameterization used here is the one introduced by Samorodnitsky and Taquq (1994).

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$-1$  to  $+1$ . When  $\beta$  is positive, a stable distribution is skewed to the right; when  $\beta$  is negative, a stable distribution is skewed to the left. Figure 1.1 shows the effect on tail thickness of the density as well as peakedness at the origin relative to the normal distribution (collectively the “kurtosis” of the density) for the case of where  $\mu = 0$ ,  $\sigma = 1$ , and  $\beta = 0$ . As the values of  $\alpha$  decrease, the distribution exhibits fatter tails and more peakedness at the origin. Figure 1.1 illustrates the influence of  $\beta$  on the skewness of the density function for the case where  $\alpha = 1.5$ ,  $\mu = 0$ , and  $\sigma = 1$ . Increasing (decreasing) values of  $\beta$  result in skewness to the right (left).

There are only four stable distributions that possess a closed-form expression for their density function. The case where  $\alpha = 2$  (and  $\beta = 0$ , which plays no role in this case) and with the re-parameterization in the scale parameter  $\sigma$ , yields the normal distribution. Thus, the normal distribution is one of the four special cases of the stable distribution, one that possesses a closed-form expression. The second occurs when  $\alpha = 1$  and  $\beta = 0$ . In this case we have the *Cauchy distribution*, which, although symmetric, is characterized by much fatter tails than the normal distribution. When we have  $\alpha = 0.5$  and  $\beta = 1$ , the resulting density function is the *Lévy distribution*.<sup>4</sup>



**FIGURE 1.1** Stable Density:  $\mu = 0$ ,  $\sigma = 1$ ,  $\beta = 0$ , and varying  $\alpha$  (left);  $\alpha = 1.5$ ,  $\mu = 0$ ,  $\sigma = 1$ , and varying  $\beta$  (right)

<sup>4</sup>The probability mass of the Lévy distribution is concentrated on the interval  $(\mu, +\infty)$ . The phenomenon that the domain of a stable distribution differs from the whole real line can only occur for values of  $\alpha$  strictly less than one and in the case of maximal skewness, that is, for  $\beta = +1$  or  $\beta = -1$ . In the former case, the support of the distribution equals the interval  $(\mu, +\infty)$  whereas in the latter case it equals  $(-\infty, \mu)$ .

And finally, the fourth case with a closed-form density is the *reflected Lévy distribution* with parameters  $\alpha = 0.5$  and  $\beta = -1$ , so-called because this distribution can be obtained from the Lévy distribution by reflecting the graph of the density at the vertical axis.

### **1.2.2 Desirable Properties of the Stable Distributions**

An attractive feature of stable distributions, not shared by other probability distribution models, is that they allow generalization of financial theories based on normal distributions and, thus, allow construction of a coherent and general framework for financial modeling. These generalizations are possible only because of two specific probabilistic properties that are unique to stable distributions (both normal and non-normal): (1) the stability property and (2) the Central Limit Theorem.

The stability property was briefly mentioned before and denotes the fact that the sum of two independent  $\alpha$ -stable random variables follows—up to some correction of scale and location—again the same stable distribution. This property, which is well known for the special case of the normal distribution, becomes important in financial applications such as portfolio choice theory or when measuring returns on different time-scales. The second property, also well known for the normal distribution, generalizes to the stable case. Specifically, by the Central Limit Theorem, appropriately normalized sums of independent and identically distributed (i.i.d.) random variables with finite variance converge weakly<sup>5</sup> to a normal random variable, and with infinite variance, the sums converge weakly to a stable random variable. This gives a theoretical basis for the use of stable distributions when heavy tails are present and stable distributions are the only distributional family that has its own domain of attraction—that is, a large sum of appropriately standardized i.i.d. random variables will have a distribution that converges to a stable one. This is a unique feature and its fundamental implications for financial modeling are the following: If changes in a stock price, interest rate, or any other financial variable are driven by many independently occurring small shocks, then the only appropriate distributional model for these changes is a stable model (normal or non-normal stable).

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<sup>5</sup>Weak convergence of a sequence of random variables to a distribution function  $F$  means that the distribution functions  $F_1, F_2, \dots$  of  $X_1, X_2, \dots$  converge pointwise in every point of continuity of  $F$  to the distribution function  $F$ .

### 1.2.3 Considerations in the Use of the Stable Distribution

Despite the empirical evidence rejecting the normal distribution and in support of the stable distribution, there have been several barriers to the application of stable distribution models, both conceptual and technical. The major problem is that the variance of the stable non-normal distributions equals infinity. This fact can be explained by the tail behavior of stable distributions. One can show that the density function of a stable distribution with index of stability  $\alpha$  “behaves like”  $|x|^{-\alpha-1}$  and consequently all moments  $E|X|^p$  with  $p \geq \alpha$  do not exist. In particular, the mean only exists for  $\alpha > 1$ .

A second criticism of the stable distribution concerns the fact that without a general expression for stable probability densities—except the four cases identified above—one cannot directly implement estimation methodologies for fitting these densities. Today, because of advances in computational finance, there are methodologies for fitting densities for stable distributions that we describe in later chapters. Nevertheless, there remains the problem of using the  $\alpha$ -stable distribution in option pricing models because of its infinite moments of order higher than  $\alpha$ .

Finally, the empirical evidence of observed market returns, although inconsistent with the normal distribution and better explained by the  $\alpha$ -stable distribution, still is not a good fit to that distribution. More specifically, the tails of the distribution for asset returns are heavier than the normal distribution but thinner than the  $\alpha$ -stable distribution.<sup>6</sup>

To overcome the drawbacks of the  $\alpha$ -stable distribution, the tails of an  $\alpha$ -stable random variable can be appropriately tempered or truncated in order to obtain a proper distribution that can be utilized to price derivatives. Several alternatives to the  $\alpha$ -stable distribution have been proposed in the literature. One alternative is the *classical tempered stable* (CTS) distribution—introduced under the names truncated Lévy flight, KoBoL, and CGMY<sup>7</sup>—and its extension, the KR distribution. The *modified tempered stable* (MTS) distribution is another alternative.<sup>8</sup> These distributions, sometimes called the *tempered stable distributions*,<sup>9</sup> have not only heavier tails than the

<sup>6</sup>See Grabchak and Samorodnitsky (2010).

<sup>7</sup>The truncated Lévyflight, KoBoL, and CGMY were introduced by Koponen (1995), Boyarchenko and Levendorskiĭ (2000), and Carr et al. (2002), respectively.

<sup>8</sup>The KR and the MTS distribution are analyzed in Kim et al. (2008) and Kim et al. (2009), respectively.

<sup>9</sup>Rosiński (2007) extended CTS distribution under the name of the tempered stable distribution, and KR distribution is included in this extension, but MTS distribution is not (see Bianchi et al., 2010).



normal distribution and thinner than the  $\alpha$ -stable distribution, but also have finite moments for all orders and exponential moments of some order. Thus an exponential Lévy model can be constructed. Recently, Menn and Rachev (2009) introduced the so-called *smoothly truncated stable* (STS) random variable in order to provide a practical framework to extend option pricing theory to the  $\alpha$ -stable model.

### **1.3 OPTION PRICING WITH VOLATILITY CLUSTERING**

The arbitrage pricing of options is based on the martingale approach described in Harrison and Kreps (1979) and subsequently by Harrison and Pliska (1981). According to this approach, option prices can be obtained by taking the expectation for the payoff function of the given underlying asset under a so-called *risk-neutral measure* (or *equivalent martingale measure*), which generally differs from the market measure estimated from historical data. The option price is effected by the risk-neutral measure. In the Black-Scholes model, for example, the return distribution is assumed to be a normal distribution and the price of a European call and put option given by a simple explicit form that depends on two main parameters: the risk-free rate and the variance.

Practitioners prefer to use the word *volatility* (the square root of the variance). Two types of volatilities can be observed in the market: (1) the volatility that can be inferred from stock prices, and (2) the so-called *implied volatility* (which we mentioned earlier in this chapter) that is embedded in option prices. The former is the volatility defined under the market measure, and the latter is usually viewed as a predictor of the future stock market volatility, and it can be considered as the risk-neutral volatility. Furthermore, one can assume a constant volatility or a time-varying one, depending on the statistical model one wants to employ. In particular, as observed by Corcuera et al. (2009), the implied volatility, calculated by inverting the formula of a given pricing model, strictly depends on the model selected.

Exponential Lévy models have been proposed to overcome the problems arising from the Black-Scholes model. Unfortunately, in spite of the skewness and the heavy-tail properties of the price-driving process, the exponential Lévy model has been rejected based on empirical evidence because it cannot explain the volatility clustering effect of a time series of observed returns. As noted in section 1.1, volatility clustering behavior refers to the tendency of large changes in asset prices (either positive or negative) to be followed by large changes and small changes to be followed by small changes. Furthermore, Lévy models provide a suitable fit to observed option prices

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for a single maturity, but not over all the maturities simultaneously.<sup>10</sup> That is, the volatility surface cannot be exactly fit with these kinds of models. In order to overcome this deficiency of Lévy-based models, one can utilize both stochastic volatility models and discrete-time generalized autoregressive conditional heteroscedastic (GARCH) models to price derivatives under the assumption of unknown volatility.

Understanding the behavior of return volatility is important for forecasting as well as pricing option-type derivative instruments since volatility is a proxy for risk. There are two important directions in the literature for modeling for non constant volatility: (1) continuous-time stochastic volatility processes<sup>11</sup> represented in general by a bivariate diffusion process and (2) the discrete-time autoregressive conditionally heteroscedastic (ARCH) model of Engle (1982) or its generalization (GARCH) as first defined by Bollerslev (1986).

There are different ways to construct continuous-time stochastic volatility models. The first way changes the volatility parameter of the Black-Scholes model to a stochastic one and considers a bivariate diffusion process. Hull and White (1987) and Heston (1993) used an Itô process as the volatility process. Recently, Barndorff-Nielsen and Shephard (2001) defined the squared volatility process as an Ornstein-Uhlenbeck process driven by a Lévy subordinator. The second way to build models with dependence in increments is to time change a Lévy process by a positive increasing process with dependent increments. This second way to construct stochastic volatility model goes back to Mandelbrot and Taylor (1967) and Clark (1973) who modeled the asset models price as a geometric Brownian motion subordinated by an independent Lévy subordinator. Mandelbrot and Taylor assumed an  $\alpha$ -stable distributed subordinator and Clark a log-normal one.<sup>12</sup>

Based on the previous construction, a stochastic time driven by a positive increasing Lévy process with dependent increments can be taken into consideration.<sup>13</sup> They take homogeneous Lévy processes and generate the desired volatility properties by subordinating them to the time integral of a Cox-Ingersoll-Ross (CIR) process.<sup>14</sup> The randomness of the CIR process induces stochastic volatility, while mean reversion in this process induces volatility clustering.

<sup>10</sup>See Corcuera et al. (2009).

<sup>11</sup>Schoutens (2003) examined the performance of various stochastic volatility models. See also Cont and Tankov (2004).

<sup>12</sup>See De Giovanni et al. (2008).

<sup>13</sup>This stochastic volatility model has been proposed in Carr et al. (2002).

<sup>14</sup>This model is introduced in Cox et al. (1985).

The main advantage of the continuous-time models is that a closed-form solution for European option prices is available; in contrast, in general, this is not a property of discrete models. However, in GARCH models, volatility is observable at each time point, thereby making the estimation procedure a much easier task than the one in continuous-time models. Duan (1995) investigated the pricing problem in the presence of lognormal stock returns and a GARCH volatility dynamic. Duan's result relies on the existence of a representative agent with constant relative risk aversion or constant absolute risk aversion. Heston and Nandi (2000), derived a semi-analytical pricing formula for European options for a normal GARCH model. The advantage of their closed-form solution is that the calibration technique is much easier to implement, even if the explanatory power of the model is poor.

Even if GARCH models are a bit mechanical, the methodology is useful since their diffusion limits contain many well-known stochastic volatility models. From an estimation perspective, GARCH models may have distinct advantages over stochastic volatility models. Continuous-time stochastic volatility models are difficult to implement because, with discrete observations on the underlying asset price process, the volatility is not readily identifiable. Furthermore, time-continuity models impose the possibility of continuous trading in order to construct the hedge portfolio this is not feasible in reality. To overcome this problem, implied volatilities are extracted from current option prices. In contrast, GARCH models have the advantage that volatility is observable from the history of asset prices. Consequently, it is possible to price options solely on the basis of observable history of the underlying asset process without requiring information on derivative prices.

In this book, we test performance of option pricing models using the S&P 500 index (SPX) option and the S&P 100 index (OEX) option. The former is a European style option while the latter is American style. Both options are traded on the Chicago Board Options Exchange. All market data are obtained from Option Metrics's Ivy DB in the Wharton Research Data Services.

### **1.3.1 Non-Gaussian GARCH Models**

When fitting GARCH models to return series, it is often found that the residuals still tend to be heavy tailed. One reason is that the normally distributed innovation is insufficient to describe the residual of return distributions. In general, the skewness and leptokurtosis observed for financial data cannot be captured by a GARCH model with innovations that are normally distributed. To allow for particularly heavy-tailed conditional (and unconditional) return distributions, GARCH processes with non-normal distribution have been considered (see Mittnik et al., 1998).

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Although asset return distributions are known to be conditionally leptokurtic, only a few studies have investigated the option pricing problem with GARCH dynamics and non-Gaussian innovations. Menn and Rachev (2009) considered *smoothly truncated stable* innovations and Christoffersen et al. (2010) investigated GARCH option pricing with inverse Gaussian and skewed variance-gamma innovations.<sup>15</sup> Kim et al. (2010) studied parametric models based on tempered stable distributions.

Another important direction in the financial literature is to estimate the risk-neutral return distribution and risk-neutral return volatility dependence using nonparametric techniques. Barone-Adesi et al. (2008) proposed the so-called filtered historical simulation method in which a random choice among the observed historical innovation sample is used to simulate the future innovation behavior.<sup>16</sup>

### 1.4 MODEL DEPENDENCIES

An important topic in quantitative finance is obtaining a reliable estimate of dependencies among financial instruments. This is fundamental in solving portfolio allocation problems or finding a fair price for derivatives whose underlying is a basket of instruments. Multivariate normal distributions are usually considered to model these dependencies, and the correlation matrix becomes the most important parameter to look at. However, correlation cannot explain joint extreme events since it can deal only with linear dependencies. More sophisticated techniques are needed to model the dependency structures observed in financial markets, particularly after the recent financial crisis that highlighted the failure of the Gaussian one-factor copula model in pricing collateralized debt obligations (CDOs).<sup>17</sup>

The more intuitive approach considers non-normal multivariate distributions by looking at a more flexible structure.<sup>18</sup> A second approach considers the copula framework,<sup>19</sup> which involves modeling the joint multivariate distribution in two steps: first by selecting a function to model the dependency structure, and second by selecting a proper model for the marginals. The first approach has its foundation in distribution theory and, in a certain sense, it is more elegant; the second approach offers a framework that can easily be understood by practitioners and offers sufficient flexibility that

<sup>15</sup>See also Christoffersen et al. (2006).

<sup>16</sup>See also Ait-Sahalia and Lo (2000), and Badescu and Kulperger (in press).

<sup>17</sup>See Brigo et al. (2010).

<sup>18</sup>See McNeil et al. (2005).

<sup>19</sup>See Embrechts et al. (2003) and references therein.

allows its adaption to stylized empirical facts observed in financial markets. The benefits of the copula framework are that it is quite simple to estimate and simulate, and for this reason in recent years it has become popular among financial practitioners.

A model must have three fundamental characteristics: (1) It has to be sophisticated enough to try to explain the major phenomena observed in financial markets; (2) it has to be simple enough to be calibrated; and (3) it has to be easily understood by practitioners. For these reasons, the normal distribution is a cornerstone in quantitative finance. Even if good a number of researchers found good results in applications to finance, only a few of these models have become market standards for the financial industry. Furthermore, it is not always true that the best model is the most popular. In this book, we will introduce two examples of nonstandard multivariate models for stock returns, with an application to portfolio selection. In particular, in Chapters 9 and 10 we analyze a multi-tail  $t$ -distribution and propose an algorithm to calibrate and simulate it, and then employ a skewed- $t$  copula together with a one-dimensional time-series process allowing for volatility clustering in order to take into account the stylized facts of the time series of log-returns.

## **1.5 MONTE CARLO**

Even if we consider the return process of assets modeled by a Brownian motion (that is, the return distribution is assumed to be normal), we do not have a closed-form solution to price complex path-dependent options. In GARCH models, explicit-form solutions are not given for options possessing a complex payoff function and even for European call/put. If we do not have an efficient analytical solution for pricing options, a classical way to price them is to employ the Monte Carlo method.<sup>20</sup>

Monte Carlo integration methods are based on the generation of a large number of simulations. These methods are based on the idea of evaluating an expectation (that is, an integral) by sampling from a set of possible scenarios. For this reason, the generation of random numbers is the fundamental tool used in the Monte Carlo integration method. Algorithms for generating normal and Poisson distributed random numbers are well known and easy to find in the literature. However, more sophisticated methods are required for more complex distributions, such as  $\alpha$ -stable and tempered stable distributions. Approximation by a compounded Poisson distribution or a series

<sup>20</sup>A detailed introduction is provided in Glasserman (2004).

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representation are two possible methods to simulate Lévy processes.<sup>21</sup> In order to be able to price financial instruments in a non-normal setting, in this book we provide an overview of the simulation algorithm that can be utilized to generate random samples starting from simple uniform random variable and ending with more complex infinitely divisible distributions.

### 1.6 ORGANIZATION OF THE BOOK

In this book we mainly focused on the application of non-normal distributions for modeling the behavior of stock price returns (more specifically, log returns). Both univariate and multivariate models are analyzed from a practical point of view, explaining the necessary theory to understand these models.

This book includes a brief introduction to fundamental probability distributions that will be used in later chapters. In particular, the  $\alpha$ -stable and tempered stable distributions are described in detail from both a theoretical and empirical perspective.

Starting from the notion of a distribution, we describe some fundamental stochastic processes such as Brownian motion and Poisson process. Then, we introduce Lévy processes, giving examples of pure jump processes and time-changed Brownian motion. For these stochastic processes, the change of measure problem is discussed in order to provide a tool to find the link between the market measure and a risk-neutral measure, and thereby for pricing, to price financial derivatives within a Lévy framework.

In Chapters 6 and 7, we go into depth regarding recent results for continuous-time modeling of stock prices with Lévy processes. Commencing with the Black-Scholes model, we investigate time changed, exponential tempered stable, and stochastic volatility models.

Chapter 8 provides a wide spectrum of methods for the simulation of infinitely divisible distributions and Lévy processes with a view toward option pricing.

In Chapters 9 and 10, we investigate two approaches to deal with non-normal multivariate distributions. Both chapters provide insight into portfolio allocation assuming a multi-tail  $t$ -distribution and a non-Gaussian multivariate model. The use of a copula function together with time-series analysis needed for modeling joint extreme events and volatility clustering is the subject of Chapter 10.

<sup>21</sup>See Asmussen and Glynn (2007) for a complete overview.

The last part is the core of the book: discrete option pricing models with volatility clustering. Non-Gaussian GARCH models for option pricing are investigated in detail. In particular, we critically assess different approaches to price options by using the information content of historical time series for the underlying. In the book's final chapter, Chapter 15, we provide an algorithm to price American-style options under non-normal discrete-time models with volatility clustering.

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