Chapter 1

Background Mathematics

1.1 ARITHMETIC, NOTATION, AND FORMULAS

Almost all of the mathematics used in this book involve only the four basic operations of addition, subtraction, multiplication, and division. If you can comfortably read about and then actually perform calculations using these four operations, you have all the math background you need. If you have a pocket calculator or a computer with a spreadsheet program, then you have the "machine power" to do whatever you need to do without resorting to pencil and paper.

Mathematical *notation*, the way we express what we want to calculate, can sometimes be confusing. Mathematical notation is the vocabulary of the language of mathematical concepts. Often a student will think he or she doesn't understand a concept when he or she simply is not familiar with the notation. It's like being given driving directions in a foreign language when you don't know the words for "turn left" or "turn right." To further complicate things, there is almost always more than one way to write a particular mathematical expression. Often the choice is a matter of style and/or convenience. In this section, I'll go through the various ways of writing different expressions involving only the four basic operations and explain why I will choose what I choose when I choose it.

I'll begin with the definition of a *variable*. A variable, simply speaking, is a letter or a name that represents a number. If I want to say, for example, that an item in a catalog costs the price listed in the catalog plus a $10 shipping and handling fee, I can write, "If the cost of an item is X dollars, then you must pay $X + 10$ dollars if you want to order any item from this catalog." It's a way of generalizing a relationship rather than having to recreate the relationship for each example.

I can get fancier and say that if Y represents the amount you must pay, then

$$Y = X + 10.$$

Some people like to use letters from the end of the alphabet; some like to use letters from the beginning of the alphabet; some like to use Greek letters. It doesn't matter which letters are used, as long as you're clearly told what number each of

Understanding the Mathematics of Personal Finance: An Introduction to Financial Literacy, by Lawrence N. Dworsky

these letters represent. Some authors of books and computer programs use variables that are case sensitive. That is, X and x represent different numbers. I don't do this either in this book or in my spreadsheets.

In some situations, it's convenient to use a whole word as the variable. In the above example, instead of letting Y be the cost, I could have written

$$\text{Cost} = X + 10.$$

The expression

$$Y = X + 10$$

is called a *formula*. A formula is a mathematician's version of a recipe. You put in X (in this case the catalog price for the item) and you get out Y (in this case the amount you must pay to have the item appear on your doorstep). It's conventional, but not necessary, for the variable that you're calculating to appear on the left-hand side of the equal sign and the variable(s) that you're supplying to appear on the right-hand side of the equal sign.

Typically, numbers that don't change are shown as numbers, such as the 10 in the above formula, and numbers that depend on your particular situation, such as X and Y in the above formula, are represented by letters. This isn't a law; it's just a common practice.

If, for example, the shipping cost depends on the item's weight, I could say that

$$Y = X + Z,$$

where Z is the shipping cost. If I do this, then I must refer you to a table or to another formula that explains how to calculate or look up the shipping cost before you can use this formula.

I'll start my discussion of the four basic operations with addition, spelling out some things that are probably obvious. I'm doing this in order to be able to draw a contrast with the other basic functions and also to start explaining the use of parentheses.

When adding numbers, it doesn't matter what order you do things in:

$$3 + 7 + 5 = 3 + 5 + 7 = 7 + 5 + 3 = \ldots$$

Note also the use of the expression "…". This means "and so on" and hopefully will be obvious in its intent when I use it.

I could also write the above addition example using parentheses to group some of the operations:

$$(3 + 7) + 5 = .$$

Writing it this way means "add 3 to 7 first, then add the result to 5." In this example, the parentheses don't contribute any value since the order of the additions doesn't matter. On the other hand, they don't introduce any error. In short, in this example, while the parentheses are harmless, they're also pointless.

Now, let's look at subtraction:

$$7 - 3 = 4.$$

This is pretty clear so far.

However, while 2 + 6 is the same as 6 + 2,

$$6 - 2 = 4 \text{ and } 2 - 6 = -4$$

are clearly not the same.

Getting a little more complicated,

$$(7 - 3) + 2 = 6.$$

The parentheses here mean first evaluate 7 − 3 (= 4) and then add the result to 2, yielding 6.

This is not the same as

$$7 - (3 + 2) = 2.$$

In this case, the instructions are to first add 3 + 2 (= 5) and then subtract the result from 7, yielding 2.

Subtraction differs from addition in the importance of notation because the order in which things are calculated matters.

If I were to just write

$$7 - 3 + 2 = ??,$$

I wouldn't know how to evaluate this because without the instructions added by the parentheses, I just don't know what to do first.[1]

Moving on to multiplication, the simplest notation (and one that's hardly ever used) is to use an "×" to indicate multiplication. Using this notation, it's clear to see that, as in the case of addition, order doesn't matter:

$$3 \times 2 \times 6 = 3 \times 6 \times 2 = 6 \times 2 \times 3 = \ldots = 36$$

One good reason why the "×" is hardly used to signify multiplication is that, once you're expecting formulas, you don't know whether this × means multiplication or is itself a variable representing another number.

For better or worse, there are many notations for multiplication. The important consideration is that the chosen notation must be clear and unambiguous.

When it is clear what I mean, I will just write the two numbers (and/or variables) that I want to multiply next to each other: $3x$, or xy. Obviously this won't work for multiplying 3 by 2, because 32 (or 23) would be interpreted as a two-digit number, not instructions to multiply the two single-digit numbers together.

When multiplying a number by a variable, it's common to put the number first: $3x$ means the same as $x3$ but is almost always written as $3x$.

This example also shows why multiplication is almost never written using an "x" to signify multiplication—the "x" is probably the most common choice of a letter for a variable, and writing $3 \times x$ to mean "multiply 3 by the variable x" is just a confusing mess.

[1] Computer programming languages usually resolve this kind of ambiguity by having a default procedure such as "when there are no explicit instructions (parentheses), work from left to right." I won't assume any such default procedures in this book.

The expression

$$3x(y+7)$$

can be interpreted two ways. The two ways are equivalent and both are valid.

The first interpretation is that you should do what's inside the parentheses first. That is, if y represents some cost or payment or whatever, let's say y = \$15.50, then add y to 7, giving 15.50 + 7 = 22.50. Then we have

$$3x(22.50).$$

This is simply three numbers multiplied together. The parentheses now are used just to keep the 3 from being tangled up with the 22.50. Since numbers multiplied together can be multiplied in any order, we have

$$3x(22.50) = 3(22.50)x = 67.50x$$

At this point we need a value for x or we just have to stop.

In order to do what I just did, I needed a value for the variable y. If I don't have a value for y, I can either leave things as they are for the time being, or I can "expand" the expression. This is the second interpretation: What's outside the parentheses multiplies everything that's inside the parentheses. Therefore,

$$3x(y+7) = 3x(y) + 3x(7) = 3xy + 21x$$

Whether the latter way of writing things is any clearer, or more useful, than the original expression is in the eye of the beholder.

Taking this one step further, what if I have

$$(12+4)(3+6).$$

The same rules apply; you just have to do a little more work:

$$(12+4)(3+6) = (16)(9) = 144.$$

This type of expression, when there are algebraic variables involved, often trips up students. Another correct way of evaluating this expression is to use the second interpretation above: What is outside the parentheses multiplies everything that is inside the parentheses. This time we have to remember that there are two sets of parentheses, so we have

$$\begin{aligned}(12+4)(3+6) &= 12(3+6) + 4(3+6) = 12(3) + 12(6) + 4(3) + 4(6)\\ &= 36 + 72 + 12 + 24 = 144.\end{aligned}$$

This is sometimes called "expanding" the expression.

An example of the same expression with algebraic variables is

$$(a+b)(c+d) = ac + ad + bc + bd.$$

In this book, I'll often be presenting formulas for use in calculating a number, typically a dollar value. This isn't an algebra book. You will have the working knowledge that you need as long as you understand the first interpretation, that is, put in the values for the variables then evaluate what's inside the parentheses (or sets of parentheses).

The last of the four basic operations is division. First, notation: The elementary school notation $6 \div 3$ is pretty much never used. Instead, recognizing that a division expression is the same as a fraction expression, 6 divided by 3 will be written as one of the following:

$$6/3 = \frac{6}{3} = 2.$$

If I want to use the first of these forms for multiple operations, then I have to get involved with parentheses, because order counts. That is,

$$18/6/3 =$$

is ambiguous because I don't know what to do first. My choices are

$$(18/6)/3 = 3/3 = 1$$

or

$$18/(6/3) = 18/2 = 9,$$

and I have no way of knowing which interpretation was intended.

Using the fraction notation, I can finesse the parentheses issue by working with different size fraction lines. That is,

$$\frac{\frac{18}{6}}{3} = \frac{3}{3} = 1$$

while

$$\frac{18}{\frac{6}{3}} = \frac{18}{2} = 9.$$

I could keep going with compound expressions—say, fractions involving sums or differences of numbers and variables inside the parentheses, and so on. But again, this is not an algebra book and my aim is not to trip you up but to give you clear rules for evaluating (finding the numeric value of) a formula when you're presented with it.

1.2 MINUS (NEGATIVE) SIGNS

The seemingly benign set of rules for manipulating a minus sign nevertheless manages to cause an endless set of headaches. Let's see if I can summarize these rules quickly and clearly:

1. (Not so much a rule as a reminder) When a sign is not shown, a positive sign is implied:

$$34 = +34,$$
$$(35) = (+35) = +(35).$$

2. Subtracting B from A is the same as adding –B to A:

$$7-5=7+(-5)=2.$$

3. As implied above, multiplying a positive number by a negative number yields a negative number:

$$(-3)(6)=-(3)(6)=-18,$$
$$(-6)=-(6)=(-1)(6).$$

4. Multiplying two negative numbers yields a positive number:

$$(-5)(-7)=+35.$$

5. Division rules are the same as multiplication rules. Dividing a positive number by a negative number or dividing a negative number by a positive number yields a negative number. Dividing a negative number by a negative number yields a positive number:

$$\frac{4}{-3}=\frac{-4}{3}=-\left(\frac{4}{3}\right)=-\frac{4}{3}=-1.33,$$
$$\frac{-4}{-3}=\frac{4}{3}=1.33.$$

1.3 LISTS AND SUBSCRIPTED VARIABLES

Throughout this book, I make frequent use of tables. Tables are lists of numbers that relate variables in different situations. This isn't as bad as it first sounds. I'm sure you've all seen this many times—everything from income tax tables that the Internal Revenue Service provides to automobile value depreciation tables.

Table 1.1 is a hypothetical automobile value depreciation table. Don't worry about what kind of car it is—I just made up the numbers for the sake of this example.

Looking from left to right, you see two columns: the age of the car and the car's wholesale price. Looking from top to bottom you see six rows. The top row contains the headings, or descriptions, of what the numbers beneath mean. Then there are

Table 1.1 Hypothetical Automobile Value Depreciation Table

Age of car (years)	Wholesale price ($)
0	32,000
1	26,500
2	21,300
3	18,000
4	15,500
5	13,250

Table 1.2 Hypothetical Automobile Depreciation Table with Air-Conditioning Option

Age of car (years)	Wholesale price ($)	Extra for air-conditioning
0	32,000	1,200
1	26,500	1,050
2	21,300	850
3	18,000	650
4	15,500	550
5	13,250	450

five rows of numbers. The numbers on each row "belong together." For example, when the car is 2 years old, the wholesale price is $21,300.

An important point about the headings is that whenever appropriate, the *units* should be listed. In Table 1.1, the age of the car is expressed in years. If I didn't say so, how would you know I didn't mean months, or decades? The value of the car is expressed in dollars. To be very precise, maybe I should have said U.S. dollars (if that's what I meant). Someone in Great Britain could easily assume that the prices are in pounds if I didn't clearly state otherwise.

Very often a table will have many columns. Table 1.2 is a repeat of Table 1.1, but with a third column added: How much more the car is worth if it has air-conditioning. Notice that I was a little sloppy here. I didn't say that the extra amount was in dollars. In this case, however, a little sloppiness is harmless. Once you know that we're dealing in dollars, you can be pretty sure that things will be consistent.

Again, the numbers in a given row belong together: A 3-year-old car is worth $18,000, and it is worth $650 more if it has air-conditioning.

Tables 1.1 and 1.2 tell you some dollar amounts based on the age of the car. It's therefore typical for the age of the car to appear in the leftmost column. I could have put the car's age in the middle column (of Table 1.2) or in the right column. Even though doing this wouldn't introduce any real errors, it makes things harder to read.

Whenever convenient, columns are organized from left to right in order of decreasing importance. That is, I could have made the air-conditioning increment the second column and the car value the third column (always count columns from the left), but again it's clearer if I put the more important number to the left of the less important number.

Some tables have many, many rows. The Life Tables presented in Chapter 10, the chapter about life insurance, have 102 rows—representing ages from 0 to 100, plus the heading row. The second column in the Life Tables is a number represented by the variable q, the third by the variable l, and so on. Don't worry about what these letters mean now; this is a topic in Chapter 10.

In Table 1.3, I've extracted a piece of the Life Table shown in Table 10.1. As you can see, for every age there are six associated pieces of information. Suppose I wanted to compare the values of q for two different ages, or to make some

Table 1.3 Part of the 2004 U.S. Life Table for All Men

Age	q	l	d	L	T	e
0	0.007475	100,000	747	99,344	7,517,501	75.2
1	0.000508	99,253	50	99,227	7,418,157	74.7
2	0.000326	99,202	32	99,186	7,318,929	73.8
3	0.000250	99,170	25	99,157	7,219,744	72.8
4	0.000208	99,145	21	99,135	7,120,586	71.8
5	0.000191	99,124	19	99,115	7,021,451	70.8
6	0.000182	99,105	18	99,096	6,922,336	69.8
7	0.000171	99,087	17	99,079	6,823,240	68.9
8	0.000152	99,070	15	99,063	6,724,161	67.9
9	0.000125	99,055	12	99,049	6,625,098	66.9
10	0.000105	99,043	10	99,038	6,526,049	65.9
11	0.000111	99,033	11	99,027	6,427,011	64.9
12	0.000162	99,022	16	99,014	6,327,984	63.9
13	0.000274	99,006	27	98,992	6,228,970	62.9
14	0.000431	98,978	43	98,957	6,129,978	61.9
15	0.000608	98,936	60	98,906	6,031,021	61.0
16	0.000777	98,876	77	98,837	5,932,116	60.0
17	0.000935	98,799	92	98,753	5,833,278	59.0
18	0.001064	98,706	105	98,654	5,734,526	58.1
19	0.001166	98,601	115	98,544	5,635,872	57.2
20	0.001266	98,486	125	98,424	5,537,328	56.2
21	0.001360	98,362	134	98,295	5,438,904	55.3
22	0.001419	98,228	139	98,158	5,340,609	54.4

generalizations of some sort. As I go through my discussion, I find that it's very cumbersome repeating terms like "the value of q for age 10" over and over again.

I can develop a much more concise, easy to read, notation by taking advantage of the fact that the left-hand column is a list of nonrepeating numbers that increase monotonically. By this I mean that 1 is below 0, 2 is below 1, 3 is below 2, and so on, so that it's easy to understand what row I'm looking at just by referring to the age (the left-hand column). Then I use a subscript (a little number placed low down on the right) tied to any variable that I want to discuss to tell you what I'm looking at. This is hard to describe but easy to show with examples:

q_3 refers to the value of q for age 3: $q_3 = 0.000250$.

q_{12} refers to the value of q for age 12: $q_{12} = 0.000162$.

d_{15} refers to the value of d for age 15: $d_{15} = 60$.

Now I can easily discuss the table using this subscript notation. In Table 1.3, q_{10} is the smallest of all the values of q, l_{22} is about 2% smaller than l_0, and so on. Asking why I'd want to be saying these things depends on the topic and the table under discussion. It's like asking why I'd ever want to multiply two numbers together.

1.4 CHANGES

When a number that you're interested in (the cost of a pound of coffee or the cost of a new home) changes, it's often more relevant to look at the percent change than it is to look at the absolute numbers.

For example, if you've been paying \$3.00 a pound for coffee and the price changes by \$2.00 up to \$5.00 a pound, this is a relatively big change. On the other hand, if you've been considering purchasing a new car for \$25,000 and the price changes by \$2.00 to \$25,002, relatively speaking, this is not a big difference.[2]

The standard way of calculating percent change is by subtracting the new value from the old value, and then by dividing this difference by the old value:

$$\%\ \text{Change} = 100\frac{\text{New value} - \text{Old value}}{\text{Old value}}.$$

The 100 multiplier is just to change the fractional quantity into a percentage.

Example: The pound of coffee mentioned above. The price was (the old value) \$3.00, and the price now is (the new value) \$5.00 so that

$$\%\ \text{Change} = 100\frac{\$5.00 - \$3.00}{\$3.00} = 100\left(\frac{2}{3}\right) = 66.7\%.$$

Remember that the units of the new and old values must be the same. Don't have the new value in pennies, or francs, or any other units, if the old value is in dollars (and vice versa). Also, because we're dividing dollars by dollars, or francs by francs, the percent change is called *dimensionless*. It has no units. This is reassuring, because if we were to first convert our dollar amounts into, say euros, and then calculate the percent change, we should certainly expect to get the same answer.

If the new value is smaller than the old value, then the percent change will be a negative number.

Example: The price of a new car dropped from \$30,000 to \$27,000. The percent change is

$$\%\ \text{Change} = 100\frac{\$27{,}000 - \$30{,}000}{\$30{,}000} = 100\frac{-3}{10} = -30\%$$

Looking at changes as percent changes or fractional changes (just don't multiply by 100) helps us to put things in perspective. We are comparing how much a number changes to how much the number used to be.

Percent change is not symmetrical in that changing a number and then changing it back doesn't give you the same results. For example, if I have something that used

[2] You might ask why not be concerned about the car price change—it's the same \$2. As I see it, the difference is that you'll have your new car about 5 or 8 years, but you buy a pound of coffee every few weeks.

to cost \$100 and now it costs \$150, the percent change was 50%. However, if the price then returns from \$150 back to \$100, we've reversed the titles of old and new values. In this latter case, the percent change is –33.3%.

In some situations we don't have an old value or a new value. Consider the statement "My bank account seems to swing back and forth between \$700 and \$800." How do you calculate the percent change?

In this case, where the numbers seem to be taking turns being the old and new numbers, it makes sense to calculate the average of the two numbers,

$$\text{Average} = \frac{\$700 + \$900}{2} = \$800,$$

and then to talk about a percent variation or sometimes a "percent swing" by subtracting the smaller number from the larger number and then dividing the result by this average:

$$\%\ \text{Variation} = 100\frac{\$900 - \$700}{\text{Average}} = 25\%.$$

In this case, the two percent change numbers about the average are

$$100\frac{\$900 - \$800}{\$800} = 12.5\%$$

and

$$100\frac{\$700 - \$800}{\$800} = -12.5\%.$$

This is often written as ±12.5%, which is read as "plus or minus 12.5%."

One last little item that makes writing and reading about small changes more convenient is that a small change in a number (not a percent change) is often denoted by the Greek uppercase letter delta (Δ). If, for example, we have a cost of something that changes from an old value of \$125.00 to a new value of \$127.00, we would write

$$\Delta\text{Cost} = \$127.00 - \$125.00 = \$2.00.$$

As above, the convention is to subtract the old value from the new value. If the new value is smaller than the old value, the result is negative. Note that ΔCost has the same units as the two numbers used to calculate it (in this example, dollars), and consequently, the two numbers must have the same units (both be dollars, or both be cents, or francs, or euros, and so on).

Sometimes, it's necessary to work these problems backward. Suppose I tell you that a store is having a "30% off sale" on all items. What would the sale price be on an \$80 purse? What we're looking at here is

$$100\frac{\text{New price} - \$80}{\$80} = -30\%.$$

I'm sure I could awe you with my prowess at algebraic manipulation, but it's really not called for. This can easily be solved in either of two ways:

1. If we're reducing the price by 30%, then we're keeping 70% of the original price: 0.7($80) = $56.
2. Thirty percent of $80 is $24. Reducing an $80 price by $24 leaves $80 − $24 = $56.

These simple calculations work correctly because we're looking for the new price. If we have to go the other way, the problem is a bit trickier. For example, "A purse that was reduced by 30% is now selling for $56. What was the original price?"

Without trying to awe you (or more likely to bore you) with the derivation, the formula you need here is

$$\text{Old price} = \frac{\text{New price}}{\dfrac{\%\ \text{Change}}{100} + 1}.$$

Putting in the appropriate numbers,

$$\text{Old price} = \frac{\$56}{\dfrac{-30}{100} + 1} = \frac{\$56}{-0.3 + 1} = \frac{\$56}{0.7} = \$80.$$

1.5 EXPONENTS

An exponent is another neat notation. Suppose I want to multiply an expression by itself (called "squaring" the expression):

$$(j+7)(j+7) = (j+7)^2.$$

The little "2" placed high up in the upper right means "square the expression" or more directly, "write the expression down twice, making it clear that you mean multiplication." This is also sometimes called "raising the expression to the power of 2."

Similarly, I can "cube" the expression

$$(j+7)(j+7)(j+7) = (j+7)^3$$

and so on.

In general, $(\text{anything})^n$ is called raising the expression "anything" to the nth power.

The following discussion of exponents is not needed in order to understand the book; I just thought that some readers might be curious as to why raising an expression to the power of 2 is called "squaring the expression" and raising it to the power of 3 is called "cubing the expression."

The area inside a rectangle is calculated by multiplying the rectangle's length by its width. A square is a rectangle whose length is equal to its width. In other words, all four sides of a square are the same length. The area of a square is therefore calculated by taking the square's length (or width) and multiplying it by itself. Consequently multiplying a number by itself is called "squaring" and raising a number to the second power is just multiplying the number by itself. Incidentally, if you start with the area of a square and want to find the length of its sides, the procedure is called "finding the square root."

Similarly, all the edges of a cube are the same, and consequently, you find the volume of the cube by "cubing," that is, raising the length of any edge to the third power.

It's also possible to raise expressions to "non-integer" powers, for example,

$$(j+7)^{2.5}.$$

This cannot be explained without the use of logarithms, which is a topic that is beyond what you need to understand this book. I will use this notation when I have to calculate, for example, 2.5 years' worth of interest on a loan. Your calculator or spreadsheet will handle this correctly; you don't need to worry about it.

1.6 SUMMATIONS

This topic is *not* necessary for understanding the rest of the book. It's an introduction, however, to a powerful notation that allows us to work with numbers from arbitrarily large lists in a very concise manner.

Looking at Table 1.3 again, suppose that I want to add up the values of e from e_5 to e_8. That is, add up $e_5 + e_6 + e_7 + e_8$. Just for the record, I don't know why I'd ever want to do this with a Life Table. There are cases, however, such as adding up the interest payments on a loan, where I often want to do this.

The shorthand notation involves the use of the uppercase Greek letter sigma:

$$\sum_{i=5}^{8} e_i.$$

How to read this: Beneath the sigma you see "$i = 5$." Above the sigma you see "8." This means that we want to add up all the terms to the right of the sigma for $i = 5$, 6, 7, and 8:

$$\sum_{i=5}^{8} e_i = e_5 + e_6 + e_7 + e_8.$$

To finish the job, we have to look at the appropriate e values in the table:

$$\sum_{i=5}^{8} e_i = e_5 + e_6 + e_7 + e_8 = 70.8 + 69.8 + 68.9 + 67.9.$$

I won't bother actually adding these numbers up because I don't really care about the answer; I'm just showing how the notation works.

But why bother with such an esoteric notation just to show that I want to add up four numbers? The convenience is that I can represent huge sums of numbers concisely.

For example, Table 10.1 has 101 rows of numbers representing some information from age = 0 to age = 100. To show that I want to add up all the e values in this table, I write

$$\sum_{i=0}^{100} e_i.$$

This can be extended to more complicated expressions, such as

$$\sum_{i=5}^{8} (e_i + 7)^2 = (e_5 + 7)^2 + (e_6 + 7)^2 + \ldots$$

The variable i is sometimes called the "index." Note that it can also be used directly in the expression to be evaluated:

$$\sum_{i=5}^{8} ie_i = 5e_5 + 6e_6 + 7e_7 + 8e_8.$$

1.7 GRAPHS AND CHARTS

When looking at relationships between variables, the formula tells it all. Very often, however, a picture is indeed worth a thousand words in "giving us a feeling" for what the formula is telling us.

We will often be presented with a graph that we'll study to gain some insight into the information the graph is presenting. Conversely, we will often need to be able to create a graph to show a formula that we are interested in. I'll take this latter approach first.

Let's start with a simple formula:

$$y = 27{,}000 - 2{,}000x.$$

This formula gives us a value for the variable y when we give it a value for the variable x. These variables might stand for the depreciation of a car's value, the interest on a loan, the number of years that you will hold a loan, and so on.

Before I draw a graph, I have to know what values I will have to consider. Suppose that this formula tells us the value of a car over time: x will be the age of the car in years. The lowest value that x can possibly have is 0—that's the year when the car is brand new. There's no mathematical reason why x can't be less than 0 (negative numbers), but it doesn't make sense when I want x to stand for the age of the car. Let's say I only want to look at the first 10 years of the car's life. The largest value x can have is 10.

Next, I'll create a table showing y for various values of x. If possible, I recommend that you create this or a similar table on a computer spreadsheet. Since I don't know what spreadsheet program you're using, I can't give detailed directions for creating a graph from this table—but every spreadsheet program I've seen for the past 25 years has had graphing capability, so consult your manuals or do some searching online.

Table 1.4 Data for the First Graph Example

X	Y ($)
0	27,000
2	23,000
4	19,000
6	15,000
8	11,000
10	7,000

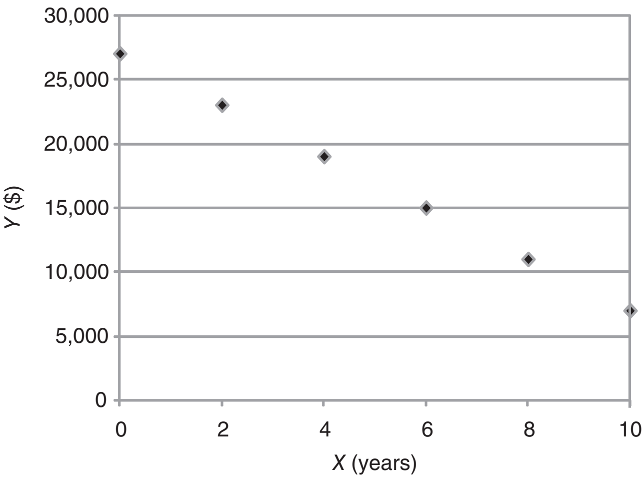


Figure 1.1 Graph of the data in Table 1.4.

As is seen in Table 1.4 and Figure 1.1, it's conventional to show the units for a variable in parenthesis after the variable, for example, X (years) in Figure 1.1.

Table 1.4 shows the data generated using the above formula, for even number values of x between 0 and 10. Why even number values? Here I have a choice. I can get very detailed ($x = 0, 0.2, 0.4, 0.6, \ldots, 9.8, 10$) or I can get very sparse ($x = 0$, 5, 10). There is no magic answer for what is the best thing to do. This is, unfortunately, as much an art as a science. You want enough detail so that the graph conveys all the details of the data, but you don't want to bury yourself (or your computer) in reams of numbers.

Figure 1.1 shows the graph for the data in Table 1.4. The horizontal line along the bottom (the horizontal axis) has the label "X" beneath it and shows numbered points between 0 and 10. The vertical line along the right (the vertical axis) has the label "Y" to the left of it and shows numbered points between 0 and 30,000.

Each diamond corresponds to a row in the table. If you draw a vertical line up from the number 4 on the horizontal axis and draw a horizontal line to the right from the number 18,000 on the vertical axis, there is a small diamond at the point where these lines cross.

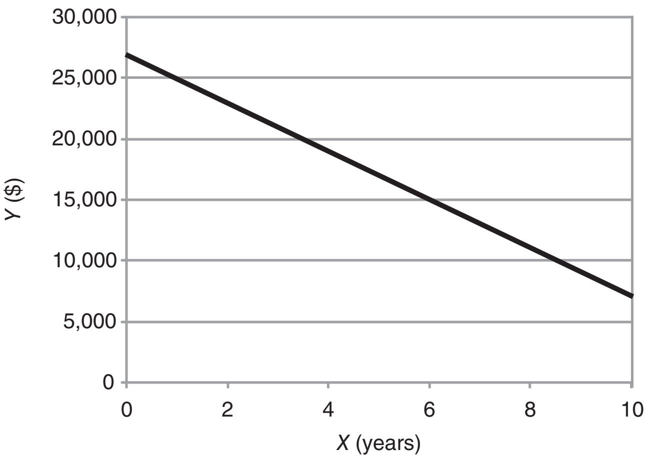


Figure 1.2 Continuous interpolation graph of data in Table 1.4.

Looking at the graph can quickly give you an idea of what's happening: the car value is dropping rapidly with passing years.

What about, say, $X = 5$? There is no diamond on the graph. I could have included $X = 5$ and the corresponding Y value in the table. Alternatively, once I'm sure that there are no "surprises" in the curve, I can draw a continuous curve rather than a discrete point, as is shown in Figure 1.2.

In Figure 1.2, I've taken the same data that I used for Figure 1.1 but I connected all the diamonds (and I'm not showing the diamonds). This is a very convenient way to show things—when you're very sure you know what's happening. Some formulas do funny things. Fortunately, we'll only be looking at fairly "well-behaved" formulas in the upcoming chapters, so there's no reason to dwell on mathematical curiosities. Connecting the dots between data points on a graph is called interpolation. Connecting these data points can sometimes lead to erroneous conclusions. Suppose the horizontal axis represents a number of apples and the vertical axis represents the cost of this number of apples in a store. The data might not be obvious because the store owner is giving discounts on large purchases. Connecting the points with a continuous line might give the impression that because you can estimate the cost of 1.5 apples, you can go into the store and buy 1.5 apples. In other words, connecting the points might give an incorrect impression that any value on the horizontal axis is possible.

When a graph describes how some variable such as the car value in the example above changes due to the change of another variable (the year in the example above), the graph is often entitled either "car values *as a function of* time," "car values versus time," or "car values with respect to time." This is just a shorthand jargon to tell you to expect a graph with car values on the vertical axis and time on the horizontal axis.

Graphs are especially useful for comparing two or more sets of data, that is, data that came from tables with three or more columns. The first column almost invariably becomes the horizontal axis on a graph.

Table 1.5 Data for the Second Graph Example

X	*Y*	*Z*
0	27,000	35,000
2	23,000	29,000
4	19,000	23,000
6	15,000	17,000
8	11,000	11,000
10	7,000	5,000

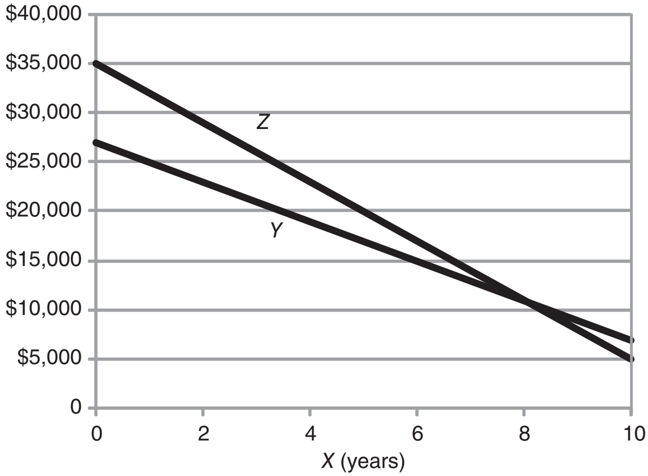


Figure 1.3 Graph of the data in Table 1.5.

Consider Table 1.5 and Figure 1.3. The table has three columns, labeled *X*, *Y*, and *Z*. Again considering car depreciation, *Y* and *Z* could represent the values of two different brands of cars, while *X* still represents the age of the car in years. In the figure, I've put the *Y* and *Z* labels inside the graph itself, each label nearest to the appropriate curve. The left (vertical) axis relates to both *Y* and *Z*.

Looking at Figure 1.3, it's pretty clear that car brand *Z*, while costing more when new, depreciates faster than car brand *Y*, and after about 8 years, brand *Z* is actually worth less than brand *Y*.

Another popular style of graph is the "histogram." The word histogram is derived from Greek roots and has to do with a drawing with information set upright. For practical purposes, it's another way to show some graphical information.

Figure 1.4 shows the same data as Figure 1.1—that is, the data of Table 1.4. As you can see, there is very little difference between these two figures. In the histogram version (Fig. 1.4), there is a solid bar reaching up from the horizontal axis to the place where Figure 1.4 had the diamond. Both graphs are read exactly the same way.

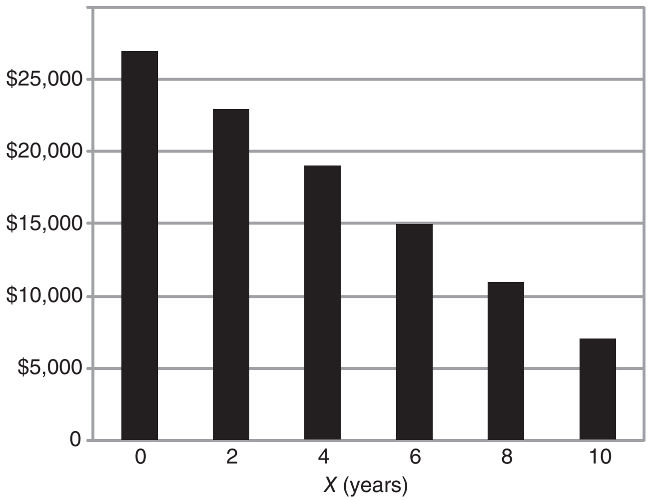


Figure 1.4 Histogram of the data in Table 1.1.

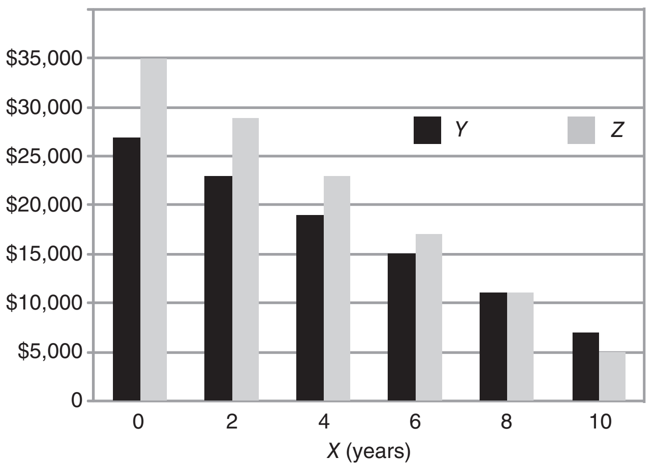
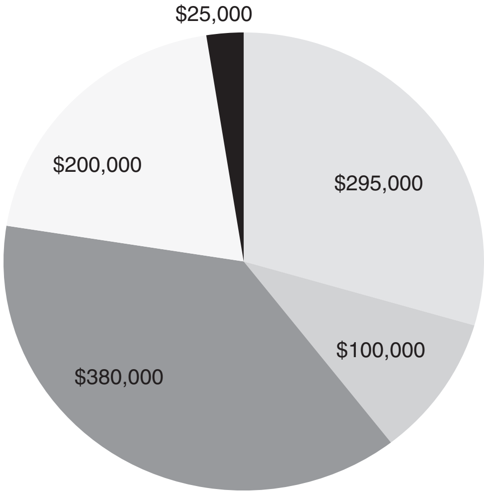


Figure 1.5 Histogram of two car depreciations.

Figure 1.5 shows the histogram version of Table 1.5, where I am comparing two sets of data. Note that in this case, I am explaining which set of bars belongs to which variable by means of little squares, appropriately labeled. This type of histogram can be extended to three or more variables, but things start getting very crowded and hard to grasp.

The histograms I've just shown aren't the only kinds of histograms; there are many variations on the theme. For the purposes of this book, however, these are enough.



Category	*Amount*
Road maintenance	$295,000
Capital	$100,000
Schools	$380,000
Salaries	$200,000
Electricity	$25,000

Figure 1.6 Pie chart example.

The last type of graph that I want to show is the pie chart. Pie charts are typically used when you want to see how a total amount of some quantity is divided into pieces, or "slices of the pie." A typical use is to show the major sections of an organization's budget.

An example of a pie chart is shown in Figure 1.6. This is a hypothetical breakdown of a community's total budget of $1,000,000. Each of the sections of the pie is labeled with the amount of money it represents and the legend on the right shows what this money is being spent on. The area of each pie slice is proportional to the fraction of the total that the slice represents as a fraction of the whole pie. In Figure 1.6, the salaries slice, representing $200,000, is twice as big as the capital slice, representing $100,000. As with the histogram, there are many variations of the pie chart, but Figure 1.6 is fairly representative. A pie chart can comfortably hold 8 or 10 categories. If you try to squeeze in many more than that, the chart starts getting too crowded to read.

1.8 APPROXIMATIONS

We frequently don't need to know an answer to many decimal places. When we give someone directions to drive to our house, we usually say something like, "Get off the highway at exit 14, go right and follow the road for about 12 mi until you see an old church on the right."

We could have said "Follow the road for 11.87 mi" but "about 12" gives enough information to tell someone when to start looking for the church. I don't need to delve into the theory of approximations. Instead, I'll use some commonsense rules, such as "about 14 mi" means that the number is closer to 14 than it is to 13 or 15.

The mathematical expression

$$x \approx 14$$

means that "x is approximately equal to 14." Other ways of writing this are $x \sim 14$ and $x \cong 14$.

The number 2,123,774 has seven *significant figures*. It is approximately 2 million. I have to be careful, however, not to say that the number is approximately 2,000,000 because when I do this I am implying that I precisely know all the digits I put down: that is, that the last digit is indeed 0 and not 4.

All of the above issues are resolved by using a notation sometimes called "scientific" notation, which specifies exactly how many digits of a number are known and how many are just place keepers (zeros to the left of the decimal place). For the purposes of this book, I'll be careful just to use commonsense approximations and not worry about the mathematical implications.

Another name for approximating is *rounding*. When I know a currency amount to the nearest penny and I *round* it to the nearest dollar, I'm approximating the amount to two fewer significant figures than I started with. The rules are simple. If the amount after the decimal place is 50 or less, just drop this amount. If the amount after the decimal place is 51 or more, add 1 to the number of dollars and drop the amount after the decimal place. A few examples:

$151.23 rounded to the nearest dollar is $151.

$26.76 rounded to the nearest dollar is $27.

$315.00 rounded to the nearest dollar is $315.

You can also round to the nearest ten dollars, to the nearest million dollars, and so on. For example,

$514,676.26 rounded to the nearest hundred thousand dollars is $515 hundred thousand dollars—not $515,000.00 or $515,000.

Rounding to the nearest billion dollars is usually restricted to members of Congress.

1.9 RATES—AVERAGE AND INSTANTANEOUS

This section is useful for understanding the mathematics of the average and incremental IRS income tax rates discussed in Chapter 9. It is not necessary for understanding Chapter 9 or anything else in this book.

Suppose I were to take a walk for 25 minutes. I'm walking along a marked track, so I know exactly how far I've walked at all times. Every few seconds, I write down how long I've been walking and how far I've walked. After I've finished my walk, I produce the graph shown in Figure 1.7, interpolating my data as described above.

At the end of 25 minutes, I've walked about 2,200 feet. As the graph shows, I started off walking at a good pace and then I slowed down. From about 8 minutes to about 13 minutes, I hardly moved at all. Then I started walking faster and faster, until the end of my walk. The total distance traveled (2,200 feet) divided by the total time spent (25 minutes) is called the average rate of distance traveled. On this graph,

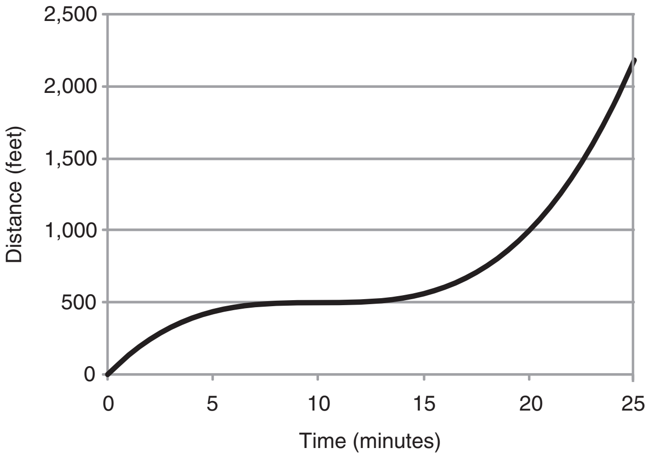


Figure 1.7 Graph of my walk—distance versus time.

it will be measured in feet per minute. "Rate" usually refers to something changing over time, so when I say "rate of distance traveled," the phrase "with respect to time" is implied. When the horizontal axis of a graph is something other than time, the definition of rate must be spelled out carefully.

The rate of distance with respect to time is such a common number that it has been given its own name, *speed*. If the distance is measured in feet and the time is measured in minutes (these are called the *units* of measurement), then the speed is measured in feet per minute. Speed can of course also be measured in miles per hour, inches per second, and so on. Since my speed was varying over the course of my walk, dividing the total distance traveled by the total time spent gives me the *average speed*:

$$\text{Average speed} = \frac{2{,}200 \text{ feet}}{25 \text{ minutes}} \approx 88 \frac{\text{feet}}{\text{minute}}.$$

If I draw a straight line connecting the start of my walk (time = 0, distance = 0) to the end of my walk (time = 25, distance ~ 2,200), this line represents how my walk would have been graphed had I walked at a constant speed, identically the average speed (Figure 1.8).

Mathematically, the property of a line describing the change in the vertical axis over the length of the line divided by the change in the horizontal axis over the length of the line is called the *slope* of the line. When a graph is showing distance on the vertical axis and time on the horizontal axis, the slope is the speed.

If I were to draw a line from my position at, say, time = 10 to time = 20, then the slope of this line would be my average walking speed between times 10 and 20.

Finally, I would like to be able to mathematically describe my speed at different times during the walk. Graphically, this means that I want to know the slopes of lines "tangent to" my graph at arbitrary points (just touching the graph at these

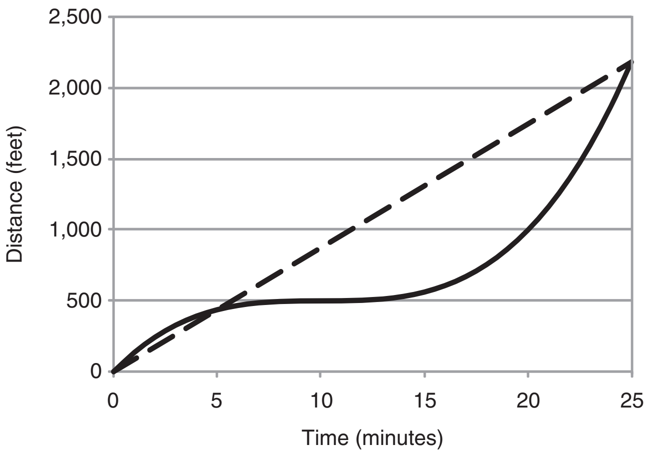


Figure 1.8 Graph of my walk showing average speed.

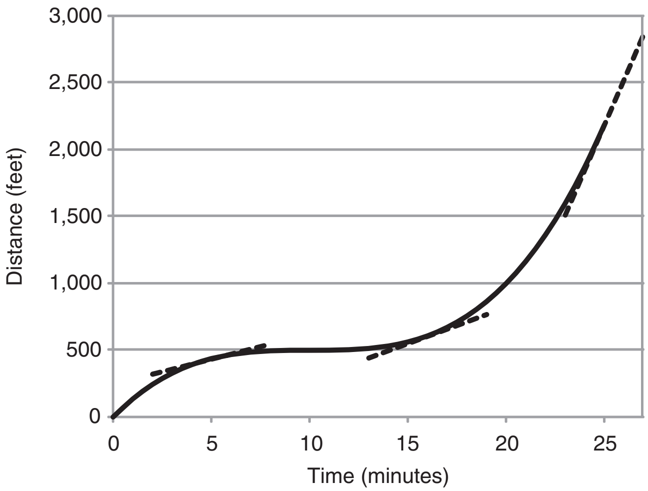


Figure 1.9 Graph of my walk showing several instantaneous speeds.

points). A few examples of this are shown in Figure 1.9. Mathematically, this is a topic in differential calculus, which we certainly aren't going to get into.

Fortunately, what I'm describing is very easy to picture intuitively. If I ask, for example, "How fast was I walking at time = 16?" then mathematically I am asking, "What is the slope of the line tangent to the curve at time = 16?"

In Figure 1.9, you see that I am asking about the line that's "bumping up" to and just touching the curve at time = 16. The answer is, at time = 16, I was walking 54 feet per minute. You can also see that I was walking a little slower than this at time = 5 and a good bit faster than this at time = 25.

Also, if you can picture drawing a tangent line to the curve at about time = 10, you will see that the line would be horizontal, that is, have zero slope, which means zero speed, which also means standing still. At time = 10, I may have paused to tie a shoelace.

When dealing with graphs where the horizontal axis is something other than time, some of the terminology changes a bit. You'll see this when I present an IRS tax curve in Chapter 9. In that case, the horizontal axis will be taxable income and the vertical axis will be tax owed to the IRS. I will be interested in tax rate. Since both of the axes have units of dollars, the tax rate has the units of dollars per dollar. Since these units can be converted to pennies per penny or Swiss francs per Swiss francs, without the resultant number changing, a tax rate is a *dimensionless* quantity.

The average tax rate will look, on the graph, just the same as the average speed on the graphs in this chapter. However, since there is no time axis in the tax curve, the term "instantaneous tax rate" would be inappropriate. Instead, we will use the term "incremental tax rate."

If Figures 1.7–1.9 represented an IRS tax curve instead of a description of my walk, then the straight line of Figure 1.8 would be my average tax rate, and the last dotted line of Figure 1.9 would be my incremental tax rate. Don't worry if these terms seem strange to you; I'll start from the beginning and go through them all in Chapter 9.

1.10 INEQUALITIES AND RANGES OF NUMBERS

The symbol $<$ means "is less than," as in $3 < 4$. If x represents the numbers of the months of the year, for example, $x < 4$ means x could be 1, 2, or 3. The symbol $\leq$ means "is less than or equal to," so that $x \leq 4$ in the above example means x could be 1, 2, 3, or 4.

Similarly, the symbols $>$ and $\geq$ mean "is greater than" and "is greater than or equal to," respectively. For some reason, these latter symbols are rarely used. Instead, the more common approach is to say that $x < 3$ means that x is less than 3, and $3 < x$ means that 3 is less than x, or equivalently, that x is equal to or greater than 3.

These symbols let us describe a range of numbers conveniently. For example,

$$3 < x < 7$$

means that x is somewhere between 3 and 7, but is not equal to either 3 or 7, while

$$3 < x \leq 7$$

means that x is somewhere between 3 and 7, possibly 7 itself, but not 3.

This notation will prove to be very useful in describing income tax brackets. For example, if y is your income, then the simple table

Income	Taxes
$y < \$10{,}000$	\$25
$\$10{,}000 \le y < 22{,}500$	\$235
$\$22{,}500 \le y$	\$1,000

means that if your income is less than \$10,000, your tax is \$25; if your income is \$10,000 or more but less than \$22,500, your tax is \$235; and if your income is \$22,500 or more, your tax is \$1,000. (This is not an actual tax table; I'm just interested in describing ranges of numbers using inequality signs here.)

PROBLEMS

1. Evaluate the following arithmetic expressions:

(a) $7 - (12 - 5)$

(b) $12(14 - 6)$

(c) $\dfrac{16-(3+7)}{3(7-5)}$

(d) $(12 - 2)(7 + 3)$

(e) $12 - 2(7 + 3)$

(f) $(12 - 2)7 + 3$

(g) $6.2 + 1/3$

2. Given that $x = 6$, $y = 2$, and $z = 3$, evaluate the following arithmetic expressions:

(a) $x + y + z$

(b) $z(x - 3)(y + 2)$

(c) $\dfrac{\dfrac{x+2}{y-3}}{z-4} + 2.25$

(d) $x(x - 1)(x + 2)$

3. Refer to the table for the problem set 3:

T (hours)	P (\$)	N (number of wallets sold)
0	25.00	6
2	24.00	2
4	23.00	2
6	21.00	4
8	18.00	8
10	14.00	16

This table is for the price (P) of a wallet at a luggage counter over the course of a day set by a store owner who would be getting a shipment of new wallets the following day and wanted to make sure that he had cleared the old wallets out of his inventory before the new ones came in. T is the time in hours, starting the count from when the store opened. N is the number of wallets sold at the corresponding price (same row in the table).

(a) If we use subscripted notation to refer to the table entries for T and P, what is T_1, P_3, N_4, and T_5?

(b) How many wallets were sold at more than \$20 per wallet? How many wallets were sold all together?

(c) If the storekeeper paid \$12 each for these wallets, not counting overhead, did he make or lose money that day? If overhead (rent, electricity, etc.) costs \$100 a day, did the storekeeper make or lose money that day?

4. Let's extend problem 3 to a 2-day rather than a 1-day wallet sale. On the second day, the storekeeper decides to repeat his change-the-price-every-two-hours strategy, but he drops each price by 10% from its amount on the first day. His sales in each 2-hour period on the second day are 50% higher than they were on the first day.

(a) Create a table just like the table in problem 3, except use numbers for the second day.

(b) Using the same assumption about overhead as in problem 3, how much money did the storekeeper make or lose on the second day?

5. Sketch a histogram based on the table of problem 3 using T as the horizontal axis and N as the vertical axis.

6. The table below shows a business plan to put items on sale. Shown are the original price and the sale price of a list of items. Calculate the percent change of all the items to the nearest percent and put these numbers in the empty percent change locations in the table. Calculate the percent sale numbers (% Sale = 100 – % Change). Then plot a graph of these data, placing the original price on the horizontal axis and the percent sale on the vertical axis.

Original price (\$)	Sale price (\$)	% Change	% Sale
289.99	217.49		
249.99	199.99		
127.50	102.00		
99.99	84.99		
59.79	53.99		
37.50	33.75		

7. For $x = 0.5$ and then again for $x = 1.2$, fill out the table below:

n	$(0.25 + x)^n$ $x = 0.5$	$(0.25 + x)^n$ $x = 1.2$
0		
1		
2		
3		
4		

(Use a calculator or a spreadsheet for this problem; doing it by hand is a very tedious job.) Can you draw any conclusions about numbers raised to the 0th power and numbers raised to the 1st power?

8. Approximate (round) the following numbers to the nearest dollar: \$12.87, \$22.22, \$53.50, and \$1,719.88.

9. The following table describes a walk I recently took. I started at 1:30 and walked until 3:30:

Time walked (miles)	Distance
1:30	0
1:45	0.25
2:00	0.5
2:15	0.75
2:30	1.00
2:45	1.50
3:00	2.00
3:15	2.50
3:30	3.00

Draw a graph using time walking (in hours) as the horizontal axis and distance walked (in miles) as the vertical axis. What was my average speed for the entire walk? (Don't forget to specify your units.) Can you estimate the instantaneous speeds near the beginning of the walk and near the end of the walk?