I. Euclidean geometry

1. A brief history of early geometry

⧫ 1.1 Prehellenistic mathematics

This brief chapter is not meant as a complete history of geometry or even of Greek geometry prior to Euclid, but contains only enough of the history to put Euclid's *The Elements* in context. As such, we discuss only those civilizations that had a direct impact on the development of Greek mathematics. Thus, we make no mention of the extensive knowledge held by the Chinese, Hindus, and Aztecs, for example.

Pictures from tombs in Egypt show that a rope was the main instrument used by surveyors and engineers. Indeed, the name for this class of professionals translates as "rope pullers" or "rope stretchers." The following group of activities was suggested in Dr. Stephen Luecking's paper, "Introducing Geometry with a Neolithic Tool Kit." You will need a length of string, preferably hemp or cotton since nylon cord tends to stretch; some pushpins; and a nice rock. In doing these activities, pretend you are an Egyptian surveyor, laying out a pyramid. The typical classroom contains entirely too many flat and straight surfaces. A sandbox or beach would be a more realistic setting.

▷ **Activity 1.1.** Take a bit of string about as long as your arm and find its midpoint. Explain your procedure. Tie a knot to mark this midpoint.

▷ **Activity 1.2.** Draw an angle (make this fairly large) and figure out how to bisect it, using only your string with its midpoint knot and some pushpins. Do not use your string as a compass to perform the traditional ruler and compass construction, but come up with another approach. Explain your procedure.

▷ **Activity 1.3.** Draw a straight line (you may use a ruler for this). Using only your string with its midpoint knot and some pushpins, figure out how to erect a perpendicular from a given point on the line. (Do not use your string as a compass to perform the traditional ruler and compass construction, but come up with another approach.) Explain your procedure.

▷ **Activity 1.4.** Draw a straight line (using a ruler). Using only your string with its midpoint knot and some pushpins, figure out how to drop a perpendicular from a given point not on the line. (Do not use your string as a compass to perform the traditional ruler and compass construction, but come up with another approach.) Explain your procedure.

▷ **Activity 1.5.** Now use a marker to make thirteen evenly spaced marks on your string, with the first and last near the ends. Use this to form a 3-4-5 right triangle. Check your angle with a protractor. If you are too far off, then your marks are not evenly spaced, so start over.

▷ **Activity 1.6.** Using your marked string, find at least two ways to construct a line parallel to a given line. Do not use your string as a compass to perform the traditional ruler and compass construction, but come up with another approach. Explain your procedure.

▷ **Activity 1.7.** Now find a rock and tie it to a bit of string. Use this to construct a horizontal line.

With only such simple tools, the Egyptian surveyors were able to carry out the annual survey of the Nile River plots of farmland and build immense structures like the pyramids. From surviving papyri, it was clear that there were precise rules and rigorous checks for the profession. For example, the base of the Great Pyramid at Giza is not quite square, but rather a trapezoid with one side 8 inches shorter than the other. However, the two diagonals are very nearly equal in length, so the requisite check did not find the error.

The Greeks credited the invention of geometry to the Egyptians. Early Egyptian and Babylonian mathematics is fairly well documented as records were kept on papyrus or clay tablets, both of which were sometimes preserved due to the arid climate. These show some evidence of influence from India. The Egyptians and Babylonians were well advanced in arithmetic and simple algebra, including compound interest problems. The annual flooding of the Nile and consequent obliteration of boundary markers and erosion of the river banks show that the Egyptians were accurate and efficient surveyors. Both the Babylonians and the Egyptians were clearly excellent engineers, building the pyramids, large multistory buildings, terraced farms, and extensive irrigation systems. Archaeologists have found tables of multiplication, squares and square roots, cubes and cube roots, and compound interest. Both the Babylonians and the Egyptians knew how to solve the quadratic equation (though only for the positive rational solutions) by completing the square. This and other procedures were evidently taught by rote: a list of steps that lead to the solution, with no explanation.

In geometry, the Egyptians and Babylonians knew of the 3-4-5 right triangle and so at least one special case of the Pythagorean Theorem. The formula $\frac{(AB+CD)(BC+AD)}{4}$ is cited for the area of a quadrilateral. Note that this formula only gives the correct answer for a rectangle.

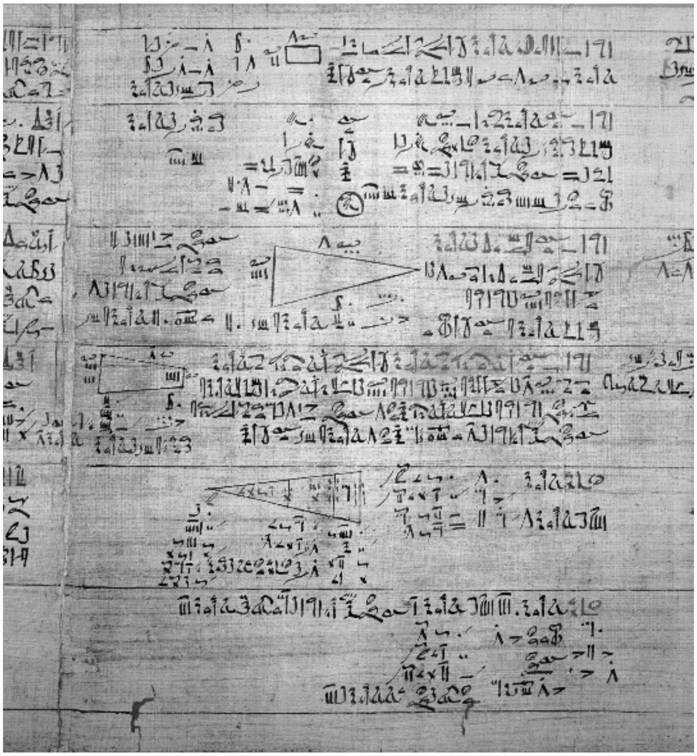
Here is a problem taken from the Rhind papyrus, a 1550 BC copy of a document written in 1850 BC.[1] This gives a flavor of what their mathematics was like. First, a text description of the problem and its solution are given, then a list of steps.

Find the volume of a cylindrical granary of diameter 9 and height 10: Take away $\frac{1}{9}$ of 9, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. Multiply 64 times 10; it makes 640 cubed cubits.

1	8
2	16
4	32
\8	64
1	64
\10	640

From this we can see how the mathematics is treated. The procedure is given as a list of steps. We know that the formula for the volume of a cylinder is $V = \pi r^2 h$, or $V = \frac{\pi}{4} d^2 h$, where r is the radius, d the diameter, and h the height. What these Egyptians seem to be doing is starting with the diameter d, then dividing that by 9 and subtracting this from the diameter. Thus the result of the first line is $d - \frac{1}{9}d = \frac{8}{9}d$. They next square this quantity (by doubling it successively to get 8, 16, 32, and finally 64) and then multiply this by $h = 10$. The first four lines of the list of steps show how they approached the problem of squaring a number: begin with eight, two times eight is 16, then two times sixteen (or four times eight) is 32, and finally two times 32 (or eight times eight) is 64. The sixth line indicates the product of the previous computation and the height 10 (the \ marks seems to indicate the result of a computation). Thus the formula implicit in this computation is $V = (\frac{8}{9}d)^2 h = \frac{64}{81} d^2 h$. They are using the approximation $\frac{\pi}{4} \approx \frac{64}{81}$ or $\pi \approx \frac{256}{81} = 3.1604938$, a fairly accurate approximation.

[1] From Fauvel and Gray, *The History of Mathematics: A Reader*.



(*Source*: Rhind Mathematical Papyrus, treatise claiming to be a copy of a 12th dynasty work, Thebes, Hyksos period, 15th dynasty, c.1550 BC (papyrus) (see also 116092) by Second Intermediate Period Egyptian (c.1750–c.1650 BC) British Museum, London, UK/The Bridgeman Art Library).

Perhaps one reason for the nature of the mathematics in documents from civilizations such as in Egypt and Babylon is that they seem to have been reasoning by analogy and that their mathematical procedures were based on experimentation and observation. Thus, it appears that they treated mathematical formulae as facts, which would not be subject to discussion or debate.

♦ 1.2 Greek mathematics before Euclid

While Egypt and Babylon had an extensive knowledge of mathematical facts (some correct and some not), it was in Greece that the notion first appeared that formulae and geometric facts should be derived by deductive reasoning rather than by analogy or experiment. It is not clear why this happened, but this was a major change in how people thought. The Greeks have a history of liking to argue and pursuing philosophical inquiries. Public debates were a popular amusement as described, for example, in the plays of Euripides and Aristophanes. For the purposes of these debates and for legal arguments, strict rules of logic were developed by, among others, Socrates, Plato, and Aristotle.

A highly skilled slave class and freemen ran most businesses, farms, and households and provided technical and unskilled labor, while the aristocracy provided the capital. This freed the aristocracy from day-to-day responsibilities. Perhaps this stratification of society led to a corresponding stratification of pursuits: theory versus practice, abstraction and deduction versus experimentation and practical application.

Of early Greek mathematics, very little is known. Few contemporary writings exist, since the writing materials (parchment, usually) and the climate were not as favorable as the desert climates of the Egyptians and Babylonians. Also, Euclid's *The Elements* was so successful that few earlier texts were ever cited afterwards. Of the mathematicians before Euclid, we know a few by name. The first and most influential of these was Thales [ca. 624–546 BC]. He is known to have studied in Egypt. He is credited with being the first to apply the deductive rules of logic to geometry, giving the first mathematical "proofs." He invented words for several major new concepts: cosmos, geometry, and mathematics. Among the theorems which Thales is said to have proved are the following:

> The base angles of an isosceles triangle are equal.
> If two lines intersect, the vertical angles formed are equal.
> Two triangles satisfying the Angle-Side-Angle criterion are congruent.
> An angle inscribed in a semicircle is a right angle.

Most of these facts were already known, some by the Babylonians, but the first evidence of their proofs is attributed to Thales. These proofs were more rudimentary and less sophisticated than those of Euclid but seminal in introducing the idea that geometrical truths must be justified by proof rather than accepted as doctrine or observed by direct experience.

Pythagoras of Samos [ca. 570–500 BC] may have studied under Thales. He traveled to Egypt and may have visited India. He founded a secret society in the Greek colonies in southern Italy which lasted 200 years. This society, really more of a commune, was devoted to mathematics, philosophy, and natural science. All discoveries of the society were attributed to the founder, so it is difficult to figure out who discovered what or even when. The Pythagoreans invested a great deal of mystical significance to numbers and Pythagoras is said to have claimed that "All Things are Number." This gave rise to the study of numerology but also led to solid achievements in mathematics, particularly in the field of number theory. Among their geometric discoveries were the properties of parallel lines, the sum of the angles in a triangle, and, of course, the Pythagorean Theorem.

The most startling discovery of the Pythagoreans was the existence of irrational numbers. Recall that a number is irrational if it cannot be expressed as the quotient of two integers. The proof given below is essentially the same as the one given by Aristotle.[2]

Theorem 1.1. $\sqrt{2}$ is irrational.

Proof: Assume that $\sqrt{2}$ is rational. Then it can be expressed as

$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers, and we additionally assume that the p and q have no common factors, so that the fraction is expressed in lowest terms.

$$\begin{aligned} 2 &= \frac{p^2}{q^2}, \\ 2q^2 &= p^2. \end{aligned}$$

[2]Aristotle, *Prior Analectics*, I. 23.

From this, it follows that p^2 must be an even number. By the proof given in Example 2 of Appendix A.3, it follows that p itself must be even. Thus, p can be written in the form $p = 2n$ for some integer n, and so

$$\begin{aligned} 2q^2 = p^2 &= (2n)^2 = 4n^2, \\ q^2 &= 2n^2. \end{aligned}$$

Therefore q^2 is also even, and so q must be even. We have thus arrived at the conclusion that both p and q are even, contradicting our assumption that p and q have no common factors. We conclude that $\sqrt{2}$ cannot be rational. □

That $\sqrt{2}$ is irrational was profoundly disturbing to the Pythagoreans, indicating that the precious integers were not adequate to describe all the numbers they could construct. Prior to this discovery, it was thought that a finite line segment is made up of finitely many points. The discovery of the irrational numbers indicated that this could not be true. Two quantities are called *commensurable* if they have a common unit of measure. For example, $\frac{4}{3}$ and $\frac{3}{4}$ can both be expressed in terms of twelfths: $\frac{4}{3} = \frac{16}{12}$ and $\frac{3}{4} = \frac{9}{12}$, so that these numbers are commensurable. Similarly, $\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$ are commensurable. The discovery that 1 and $\sqrt{2}$ are incommensurable led to the finding of other irrational numbers.

A vow of secrecy covered this disturbing flaw in the perfection of the logical structure of the universe. It is said that a member of the Pythagorean society divulged the fact to an outsider while traveling by sea, and so a storm blew up out of a clear day and he drowned—clearly an act of divine retribution. (A variant of this legend claims that his colleagues threw him overboard.) The discovery of such nonintuitive facts may have led to the desire to put geometry on a more rational basis. A member of the Pythagorean society during its last years was Hippocrates of Chios [ca. 470–410 BC]. He seems to have been the first to develop a presentation of geometry as a logical chain of propositions based on a few initial definitions.

Plato [428–347 BC] is known to have traveled to Egypt and visited the Pythagoreans in southern Italy before returning to Athens to found his Academy. He was not himself a mathematician but appreciated that geometry was an excellent playground for logical thinking. He is said to have had "Let no one ignorant of geometry enter here" carved over the doorway to the Academy. He wrote that geometry was the finest training for the mind and as such essential for philosophers and statesmen. In describing his idea of a proper education, Plato wrote,

> Geometry will draw the soul towards the truth and create the spirit of philosophy . . . Nothing else will be more likely to have such an effect . . . in all departments of knowledge, as experience proves, any one who has studied geometry is infinitely quicker of apprehension than one who has not.[3]

Plato further describes geometry as "knowledge of what eternally exists."[4] He notes that a perfectly straight line or a perfect circle cannot physically exist. He was thus led to theorize the existence of another universe of ideal forms where our souls lived before birth. The reality we experience daily is a faint echo of these perfect forms, and geometry is the study of these ideal objects that inhabit the human mind. Learning geometry is essentially a process of awakening this inborn knowledge and thus prepares the mind for recovering other knowledge of goodness and justice, the ultimate goal of his philosophy. More than 2000 years later, the role of geometry as an innate product of the human mind formed the cornerstone of Immanuel Kant's philosophy. We will return to the influence of Kant in Chapters 5 and 6.

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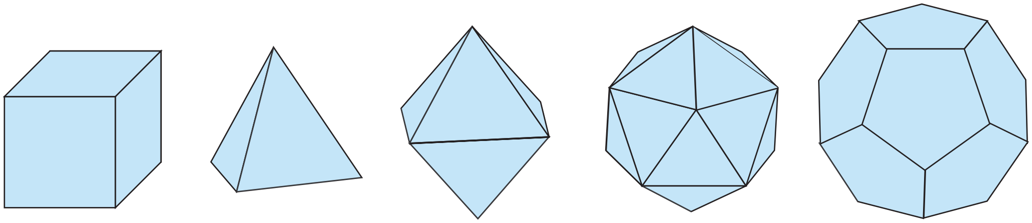
[3] Plato, *The Republic*, VII.
[4] Plato, *The Republic*, VII.

Plato and his vision of perfect geometric forms helped establish mathematics as a purely deductive discipline. He wrote with disdain of experimental science and applied mechanics and firmly designated the (unmarked) straightedge and compass as the proper tools for geometric constructions. All of these ideas can be seen to have influenced the development of Euclid's study of geometry.

The construction of the five regular polyhedra is the culmination of Euclid's *The Elements*, comprising the final chapter, Book XIII. They are known as the Platonic solids for their earliest mention, which occurred in a philosophy text, Plato's *Timaeus*. Timaeus, a fictional Pythagorean, discusses the nature of matter and the matter of nature. Following earlier writers such as Empedocles, Plato described nature as being built of four basic elements: water, earth, air, and fire, in differing combinations and proportions. Earth is represented by the stability of the cube. Fire is associated with the light and pointy tetrahedron. The octahedron represents air and the icosahedron water. Finally, the dodecahedron represents the universe. These associations were later revived by Johannes Kepler [1571–1630].



Plato writes,

> You are aware that students of geometry, arithmetic and the kindred sciences assume the odd and the even and the figures and three kinds of angles and the like in their several branches of science; these are their hypotheses, which they and everybody are supposed to know, and therefore they do not deign to give any account of them either to themselves or others; but they begin with them, and go on until they arrive at last, and in a consistent manner, at their conclusion.[5]

From this, it seems clear that while mathematics was treated as a deductive system, the foundation of definitions and axioms was not yet in place. A mathematician named Theudius wrote the geometry text used at the Academy during this period. This text no longer exists, having been superseded by Euclid.

One of Plato's students who later taught at the Academy was the brilliant mathematician Eudoxus [408–355 BC]. He developed a coherent general theory of proportions (later incorporated as Book V of Euclid's *The Elements*) that successfully dealt with both commensurable and incommensurable quantities by treating these quantities as line segments of appropriate lengths rather than as numbers. Eudoxus also developed the *method of exhaustion* which involves essentially taking successive approximations or limits. This theory was an important precursor to the calculus we use today.

The philosopher Aristotle [384–322 BC] also studied under Plato before founding his own school, the Lyceum. While not a mathematician, he made significant contributions to the foundations of mathematics, providing a systematic treatment of deductive logic and formal recognition of the role of axioms and

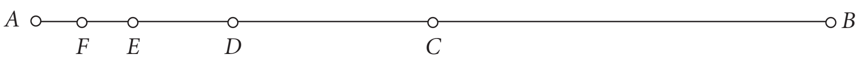
[5]Plato, *The Republic*, Book VI, sec. 510.

definitions in mathematics. In the works of Aristotle, many examples of logic taken from mathematics are given, and these are probably taken from Theudius's textbook.

Zeno [ca. 490–430 BC] studied with the Pythagoreans for a while. He is best remembered for several paradoxes, raising questions about infinity, which may also be the first examples of proofs by contradiction. These paradoxes, the difficulties they are designed to illustrate, and the attention they received from philosophers and geometers, may explain the wariness that Greek mathematicians used in dealing with infinite objects. Typical of Zeno's paradoxes is the one called the *Dichotomy*, and Aristotle discusses this:

> "Zeno's arguments against motion, which cause so much disquietude to those who try to solve the problems that they present, are four in number. The first asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the halfway stage before it arrives at the goal."[6]

In other words, in order to traverse line segment $\overline{AB}$, Zeno argues that one must first arrive at the halfway point C, but in order to get to C, one must first travel to the point D one quarter of the way from A to B, and in order to get to D, one must first get to the point E one-eighth of the distance, and so on. Thus, in order to traverse the line segment $\overline{AB}$, one must travel through an infinite number of points in a finite amount of time. Therefore, Zeno claimed the motion can never even begin.



All of Zeno's paradoxes attracted a fair amount of attention from mathematicians of the time. The result of these paradoxes was an extremely cautious attitude toward infinity. An immediate consequence of this that we will see reflected in Euclid's *The Elements* is that Euclid assumed that all line segments are finite in extent, though he also assumed that any line segment can be extended to any length required.

Mathematicians can now reply to Zeno's paradox by noting the convergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1.$$

The rigorous solution to such questions about convergence of infinite sequences and series was thousands of years in the making.

A recently rediscovered text, *The Method* or *The Method of Mechanical Theorems*, by Archimedes [ca. 287–212 BC], shows that in later years (after Euclid's *The Elements* was written), at least some Greek mathematicians handled infinite quantities much as modern mathematicians do. Archimedes shows a clear insight into the importance of the fundamental problems of calculus and applies some of the techniques that were much later refined in the development of that field.

♦ 1.3 Euclid

We know almost nothing of Euclid himself. He flourished (a term used when no one is quite sure when he was born or died, but knew he was alive at one particular time) around 300 BC. He may have studied at the Academy but is known to have lived and taught at the Museum in Alexandria, a Greek colony in Egypt. The only personal information we have consists of two anecdotes. Proclus [410–485], who led the

[6]Aristotle, *Physics*, VI, 9.

Academy several centuries later and who wrote a commentary on Euclid's *The Elements*, tells one of these about an exchange between Euclid and King Ptolemy I, though it should be noted that essentially the same story is told of other mathematicians and other kings.

> It is also reported that Ptolemy once asked Euclid if there was not a shorter road to geometry than through the *Elements*, and Euclid replied that there was no royal road to geometry.[7]

Another later commentator, Stobaeus, tells another legend:

> Some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him three obols [coins], since he must needs gain out of what he learns.'[8]

Euclid wrote a number of books, only a few of which have survived, but *The Elements of Geometry* is by far the best known and the most influential. The Greek word for "elements" also means a letter of the alphabet and was commonly used for similar texts, just as there are now a plethora of books entitled *Calculus*. It was written as a textbook, not a research document. It contains a selection of basic theorems of wide and general applicability. Most of these theorems were already known before Euclid's time, but the choice of theorems and their logical arrangement is due to Euclid alone.

The Elements consists of thirteen books or chapters. Roughly, the first of these, with parts of the third and fourth, comprises a standard high school geometry course. The contents are as follows:

I. Plane geometry through the Pythagorean Theorem
II. Geometric algebra, ending with the Law of Cosines
III. Circle geometry: chords and tangents
IV. Construction of the regular polygons of 3, 4, 5, 6, and 15 sides
V. Proportions
VI. Similarity
VII, VIII, and IX. Number theory
X. Irrational numbers
XI. Solid geometry
XII. Volume
XIII. Construction of the regular polyhedra

Euclid's *The Elements* is the most successful (as measured in number of editions printed) and influential mathematics book and has been listed, after the Bible and the Koran, as one of the three most important books of all time. Once considered essential reading for any educated person, it remains a testament to the beauty of mathematics.

The English political philosopher Thomas Hobbes [1588–1679] did not meet the euclidean method until later in life, but then it had a significant impact on his attitude towards mathematics and his approach to philosophy. His biographer describes the encounter:

> He was 40 years old before he looked on geometry; which happened accidentally. Being in a gentleman's library, Euclid's Elements lay open, and it was the 47 El. libri I [i.e., Proposition 47 of Book I of

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[7] Proclus, *A Commentary on the First Book of Euclid's Elements*.
[8] Stobaeus, *Eclogues*, II.31.

> *The Elements*: the Pythagorean Theorem]. 'By G—, said he, this is impossible!' (He would now and then swear, by way of emphasis.) So he reads the demonstration of it, which referred him back to such a proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps [And so back to the beginning], that at last he was demonstratively convinced of that truth. This made him in love with geometry.[9]

The noted mathematician and philosopher Bertrand Russell [1872–1970] describes his first encounter with Euclid:

> At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world. From that moment until I was thirty-eight, mathematics was my chief interest and my chief source of happiness.[10]

Albert Einstein [1879–1955] had a similar experience, with what he called a "holy geometry booklet":

> At the age of twelve I experienced a second wonder of a totally different nature—in a booklet dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year. Here were assertions, as for example the intersection of the three altitudes of a triangle at one point, that—though by no means evident—could nevertheless be proved with such certainty that any doubt appeared to be out of the question. This lucidity and certainty made an indescribable impression on me . . . it is marvelous enough that man is capable at all of reaching such a degree of certainty and purity in pure thinking as the Greeks showed us for the first time to be possible in geometry.[11]

♦ 1.4 *The Elements*

Euclid's *The Elements* begins straight off with a list of definitions. There is no prologue or encouraging notes to the the reader or motivational speeches. That the first of these definitions is somewhat impenetrable does not obscure the perfect simplicity and order of the universe we are about to enter.[12]

1. A point is that which has no part.

So what does this mean? Richard Trudeau in *The Non-Euclidean Revolution* gives an argument that this first definition means that points are to be thought of as indivisible: They have no length, width or thickness, unlike the points we draw on the blackboard. Otherwise, if points had physical size and if a line segment were made up of points, then any line segment would consist of a certain number of points. For example, a line segment a inches long might consist of 41,136,978 points, while a line segment b inches long might consist of 32,493,175 points. But then $\frac{a}{b} = \frac{41{,}136{,}978}{32{,}493{,}175}$, so the ratio of any two lengths would be rational and thus the two line segments are commensurable. Since it was known that incommensurable lengths exist, it must be true that points do not have size. So what Euclid seems to be saying is that

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[9] John Aubrey, *Brief Lives*.
[10] Bertrand Russell, *The Autobiography of Bertrand Russell*.
[11] Albert Einstein, *Autobiographical Notes*, translated and edited by Paul Schilpp.
[12] All statements attributed to Euclid are from Sir Thomas Heath's 1908 translation of *The Elements*.

a point has no size. That doesn't answer the question of what a point actually is. Nor does it answer the problem of how points, which have no size, can be put together to form a line segment which has positive length. However, in the context of the difficulties raised by the discovery of incommensurable quantities such as $\sqrt{2}$ and the Greeks' attempts to understand the nature of quantity and magnitude, Trudeau's argument makes it clear that this is Euclid's attempt to describe the properties that a point does *not* have.

2. A line is breadthless length.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.

The second definition is somewhat easier to swallow. Note that what Euclid calls a line might be curved: when he wants a straight line he calls specifically for a straight line. These lines or curves have length but not width, though that begs the question of what length and width mean. The third statement says that the ends of a line segment or curve are points, though there are also curves that do not have endpoints or extremities, such as the circle. The implication of this definition is that all of Euclid's lines and curves are finite in extent. As noted earlier, the Greek mathematicians of this time avoided dealing with infinity because of their inability to resolve the subtleties pointed out in Zeno's paradoxes. What Euclid calls a straight line modern mathematicians call a line segment. The fourth definition makes some sense if one thinks of it as a description: close one eye and hold a pencil up to the open eye so that it is pointed straight away from you. If you sight down the pencil, you should only see a circular cross-section. In this sense, all of the cross-sections are lined up, or "lie evenly with the points on itself." This property is not true for a curve.

In a modern deductive system, undefinable terms, called *primitive terms*, are allowed and are indeed required. This is our practical response to the recognition that one cannot define everything. (As an exercise, look up a random word in a dictionary, then look up the most important descriptive word used in the definition of that word, and so on. You will find that the words you look up will either lead you back in a circle to the original word or give an unending progression of terms needing definition.) While the distinction between primitive undefinable terms and definitions was not recognized in Euclid's time, Euclid does seem to instinctively hold to it. He never cites the more questionable definitions in the propositions that follow. Think of Euclid's early definitions of point and line as descriptions rather than as rigorous definitions, and use them to inform your intuition about the nature of these objects.

Following all of the definitions, there is a list of postulates, followed by some statements called common notions. After these, there are theorems (called propositions) with their proofs. In mathematics, axioms and postulates are things that are to be accepted without proof. As such, both ancient and modern mathematicians insist that the list of such axioms should be kept as short as possible. We should not assume anything we can possibly prove, though exceptions to this principle are occasionally made in the interest of expediency: In an introductory course, authors often adopt as axioms statements that could be proven in a more advanced course. In Euclid's day, it was further stated that axioms ought to be statements that anyone would accept as self-evident: things we all agree are true even without proof. Euclid's five geometric axioms or postulates are as follows.

Euclid's Postulate 1. [It is always possible] to draw a straight line from any point to any point.

Euclid's Postulate 2. [It is always possible] to produce [extend] a finite straight line continuously in a straight line.

Euclid's Postulate 3. [It is always possible] to describe [draw] a circle with any center and distance [radius].

Euclid's Postulate 4. All right angles are equal to one another.

Euclid's Postulate 5. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

In order to build a rigorous theory of geometry from these, we will need to modify almost all of them. In part this is because we have rejected some of Euclid's definitions and in part this is in response to changes in standards of rigor and in some conventions among mathematicians. In a modern approach, we need to explicate precisely the properties that points and lines, our primitive terms, must have. These properties are implicit in Euclid's definitions, postulates, and usage of the terms, but modern mathematics requires that we spell out the desired properties. In examining precisely how Euclid uses Postulate 1, for example, it is clear that he assumes the line between two points is unique, though he did not feel it necessary to state this. Modern standards demand that this be made explicit. Thus, in the next chapter, we will modify all of Euclid's postulates while trying to retain some historical flavor.

While the Greeks did not distinguish between undefinable and definable terms, they did make another distinction modern mathematicians no longer make. Euclid has two groups of axioms: those he calls postulates as given above and those he calls common notions. This is in agreement with Aristotle's writings: postulates are axioms pertinent to the field of study, while common notions are common to all fields of mathematics. Euclid's postulates all deal with geometric objects—lines, points, and circles—while the common notions deal with common algebraic properties. Euclid's common notions are listed below, with their interpretation in modern algebraic notation.

Common Notion 1. Things that are equal to the same thing are also equal to one another. [If $a = b$ and $c = b$, then $a = c$.]

Common Notion 2. If equals be added to equals, the wholes are equal. [If $a = b$ and $c = d$, then $a + c = b + d$.]

Common Notion 3. If equals be subtracted from equals, the remainders are equal. [If $a = b$ and $c = d$, then $a - c = b - d$.]

Common Notion 4. Things which coincide with one another are equal to one another.

Common Notion 5. The whole is greater than the part. [If $b > 0$, then $a + b > a$.]

However, Euclid does not list all of the common notions that he uses. For example, at some point we will want a statement like $a < b$ and $b < c$ implies $a < c$. This could be proved from the given common notions, but he never does. Also, he uses other simple algebraic properties which cannot be proved from this list, such as $a = b$ implies that $\frac{1}{2}a = \frac{1}{2}b$ (used in Proposition I.37).

Common Notion 4 is a little different. In practice, Euclid uses this, for example, to show that two lines are identical by showing that they have two points in common and then appealing to Postulate 1 and Common Notion 4. He also uses this to show two angles are equal, by showing that their vertex and sides coincide. This usage brings to light another difficulty: Euclid uses the word "equal" (or rather the Greek equivalent) in three senses: when things are identical (or coincide), when they have the same length, angle measure, or area, and when they are what we would call *congruent*.

Some modern axiomatic systems add postulates which associate a line with the real numbers and so guarantee points at every coordinate on the line. One can then use the natural ordering of the real numbers to specify the betweenness relationship among the points of a line. We have chosen not to pursue this approach, because one major issue in Euclid's *The Elements* is constructibility, and assuming all of the real numbers at the start makes this moot.

In the preceding paragraphs, we have indicated some of the problems in reading the text of Euclid's *The Elements* and have also outlined the course we will follow in Chapter 2. In that chapter, we will make the necessary modifications to Euclid's definitions and postulates so that we can recreate the basic foundations of plane geometry with due respect to Euclid's extraordinary accomplishment.

♦ 1.5 Projects

Project 1.1. Read and report on Charles L. Dodgson, "What the Tortoise Said to Achilles."[13] Note that Dodgson also wrote under the pseudonym Lewis Carroll of *Alice's Adventures in Wonderland* fame.

Project 1.2. Research the life of Pythagoras and the history of the Pythagoreans.

Project 1.3. Report on numerology and the Pythagoreans.[14]

Project 1.4. Research and report on the work of Eudoxus.

Project 1.5. The last major classical Greek geometer was Apollonios of Perga [ca. 240–174 BC]. Report on what is known of his work.

Project 1.6. Read *The Clouds* by Aristophanes and comment on his satire on Plato's Academy.

Project 1.7. Build models of the five platonic solids.

Project 1.8. Report on the influence of Euclid's work on modern philosophy.

Project 1.9. Find some other proofs of the irrationality of $\sqrt{2}$ and compare them to the one in this text and to each other.

[13] Reprinted in *Mind* 4, pp. 278–280.

[14] A reference is Dudley's *Numerology, or, what Pythagoras wrought.*