1

Vibrations of strings and bars

A one-dimensional continuous system, whose configuration at any time requires only one space dimension for description, is the simplest model of a class of continua with boundaries. Strings in transverse vibration, and bars of certain geometries in axial and torsional vibrations may be adequately described by one-dimensional continuous models. In this chapter, we will consider such models that are not only simple to study, but also are useful in developing the basic framework for analysis of continuous systems of one or more dimensions.

1.1 DYNAMICS OF STRINGS AND BARS: THE NEWTONIAN FORMULATION

1.1.1 Transverse dynamics of strings

A string is a one-dimensional elastic continuum that does not transmit or resist bending moment. Such an idealization may be justified even for cable-like components when the ratio of the thickness of the cable to its length (or wavelength of waves in the cable) is small compared to unity. In deriving the elementary equation of motion, it is assumed that the motion of the string is planar, and transverse to its length, i.e., longitudinal motion is neglected. Further, the amplitude of motion is assumed to be small enough so that the change in tension is negligible.

Consider a string, stretched along the x-axis to a length l by a tension T, as shown in Figure 1.1. Arbitrary distributed forces are assumed to act over the length of the string. The transverse motion of any point on the string at the coordinate position x is represented by the field variable $w(x, t)$ where t is the time. Consider the free body diagram of a small element of the string between two closely spaced points x and $x + \Delta x$, as shown in Figure 1.2. Let the element have a mass $\Delta m(x)$, and a deformed length Δs. The tensions at the two ends are $T(x, t)$ and $T(x + \Delta x, t)$, respectively, and the external force densities (force per unit length) are $p(x, t)$ in the transverse direction, and $n(x, t)$ in the longitudinal direction, as shown in the figure. Neglecting the inertia force in the longitudinal direction of the string, we can write the force balance equation for the small element in the longitudinal direction as

$$0 = T(x + \Delta x, t)\cos[\alpha(x + \Delta x, t)] - T(x, t)\cos[\alpha(x, t)] + n(x, t)\Delta s, \tag{1.1}$$

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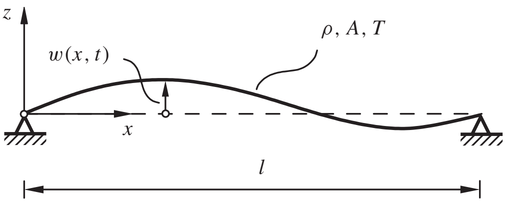

Figure 1.1 Schematic representation of a taut string

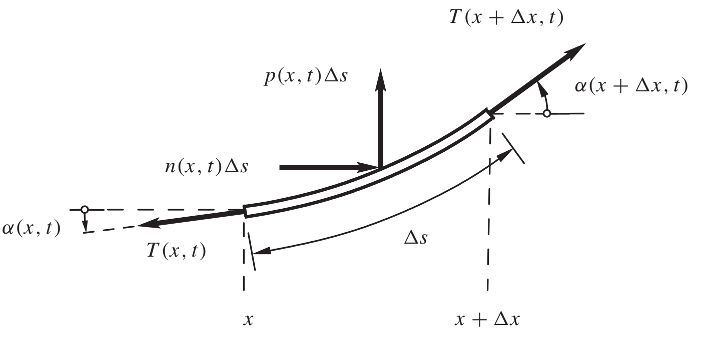

Figure 1.2 Free body diagram of a string element

where $\alpha(x,t)$ represents the angle between the tangent to the string at x and the x-axis, as shown in Figure 1.2. Dividing both sides of (1.1) by Δx and taking the limit $\Delta x \to 0$ yields

$$[T(x,t)\cos\alpha(x,t)]_{,x} = -n(x,t)\frac{\mathrm{d}s}{\mathrm{d}x}, \tag{1.2}$$

where $[\cdot]_{,x}$ represents partial derivative with respect to x. From geometry, one can write

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+w_{,x}^2}}, \qquad \text{and} \qquad \frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+w_{,x}^2}. \tag{1.3}$$

Substituting (1.3) in (1.2), and assuming $w_{,x} \ll 1$, yields on simplification

$$[T(x,t)]_{,x} = -n(x,t). \tag{1.4}$$

Therefore, when $n(x,t) \equiv 0$, (1.4) implies that the tension $T(x,t)$ is a constant. On the other hand, for a hanging string, shown in Figure 1.3, one has $n(x,t) = \rho A(x)g$, where ρ is the density, A is the area of cross-section, and g is the acceleration due to gravity. Then, using the boundary condition of zero tension at the free end, i.e., $T(l,t) \equiv 0$ (for constant ρA), (1.4) yields $T(x,t) = \rho Ag(l-x)$. In general, the tension in a string may also depend on time. However, in the following discussions, it will be assumed to depend at most on x.

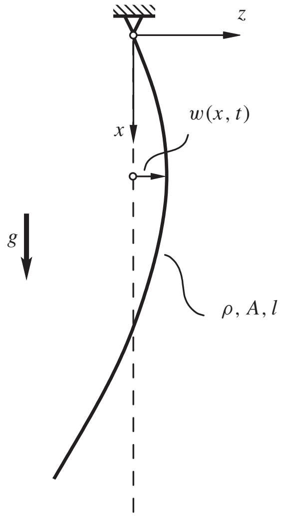

Figure 1.3 Schematic representation of a hanging string

Now, consider the transverse dynamics of the string element shown in Figure 1.1. The equation of motion of the small element in the transverse direction can be written from Newton's second law of motion as

$$\Delta m w_{,tt}(x+\theta\Delta x, t) = T(x+\Delta x)\sin[\alpha(x+\Delta x, t)] \\ -T(x)\sin[\alpha(x, t)] + p(x, t)\Delta s, \tag{1.5}$$

where Δm is the mass of the element, $\theta \in [0, 1]$, and $(\cdot)_{,tt}$ indicates double partial differentiation with respect to time. Again assuming $w_{,x} \ll 1$, one can write $\sin\alpha \approx \tan\alpha = w_{,x}$. Further, $\Delta m = \rho A(x)\Delta s$. Using these expressions in (1.5) and dividing by Δx on both sides, one can write after taking the limit $\Delta x \to 0$

$$\rho A(x) w_{,tt} - [T(x) w_{,x}]_{,x} = p(x, t), \tag{1.6}$$

where, based on the previous considerations, we have assumed $\mathrm{d}s/\mathrm{d}x \approx 1$. The linear partial differential equation (1.6), along with (1.4), represents the dynamics of a taut string. When the external force is not distributed but a concentrated force acting at, say $x = a$, the forcing function on the right hand side of (1.6) can be written using the *Dirac delta function* as

$$p(x, t) = f(t)\delta(x-a), \tag{1.7}$$

where $f(t)$ is the time-varying force, and $\delta(\cdot)$ represents the Dirac delta function.

Let us consider the hanging string shown in Figure 1.3 once again. The expression of tension derived earlier was $T(x) = \rho A g(l-x)$. Substituting this expression in (1.6) and assuming $p(x, t) \equiv 0$, one obtains on simplification

$$w_{,tt} - g[(l-x) w_{,x}]_{,x} = 0. \tag{1.8}$$

This case will be considered again later.

An important particular form of (1.6) is obtained for $p(x,t) \equiv 0$, and T and ρA not depending on x. We can rewrite (1.6) as

$$w_{,tt} - c^2 w_{,xx} = 0, \tag{1.9}$$

where $c = \sqrt{T/\rho A}$ is a constant having the dimension of speed. This represents the unforced transverse dynamics of a uniformly tensioned string. The hyperbolic partial differential equation (1.9) is known as the linear one-dimensional *wave equation*, and c is known as the wave speed. In the case of a taut string, c is the speed of transverse waves on the string, as we shall see later. This implies that a disturbance created at any point on the string propagates with a speed c. It should be clear that the wave speed c is distinct from the transverse material velocity (i.e., the velocity of the particles of the string) which is given by $w_{,t}(x,t)$. The solution and properties of the wave equation will be discussed in detail in Chapter 2.

The complete solution of the second-order partial differential equation (1.6) (or (1.9)) requires specification of two boundary conditions, and two initial conditions. For example, for a taut string shown in Figure 1.1, the appropriate boundary conditions are $w(0,t) \equiv 0$ and $w(l,t) \equiv 0$. For the case of a hanging string, the boundary conditions are $w(0,t) \equiv 0$ and $w(l,t)$ is finite. The initial conditions are usually specified in terms of the initial shape of the string, and initial velocity of the string, i.e., in the forms $w(x,0) = w_0(x)$, and $w_{,t}(x,0) = v_0(x)$, respectively. These will be discussed further later in this chapter.

Boundary conditions are classified into two types, namely *geometric* (or *essential*) boundary conditions, and *dynamic* (or *natural*) boundary conditions. A geometric boundary condition is one that imposes a kinematic constraint on the system at the boundary. The forces at such a boundary adjust themselves to maintain the constraint. On the other hand, a dynamic boundary condition imposes a condition on the forces, and the geometry adjusts itself to maintain the force condition. For example, in Figure 1.4, the right-end boundary condition is obtained from the consideration that the component of the tension in the transverse direction is zero, the roller being assumed massless. This implies $Tw_{,x}(l,t) \equiv 0$, which is a natural boundary condition. As a consequence of this force condition, the slope of the string remains zero. At the left-end boundary, the condition $w(0,t) \equiv 0$ is a geometric boundary condition, and the transverse force from the support point (which can be computed as $Tw_{,x}(0,t)$) will adjust itself appropriately to prevent any transverse motion of the right end of the string. Classification of boundary conditions based on their mathematical structure is discussed in Section 6.1.1.

When a string, in addition to the distributed mass, carries lumped masses (i.e., particles of finite mass) and is subjected to concentrated elastic restoring forces, these can be

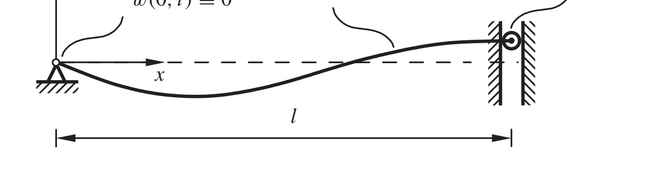

Figure 1.4 A taut string with geometric and natural boundary conditions

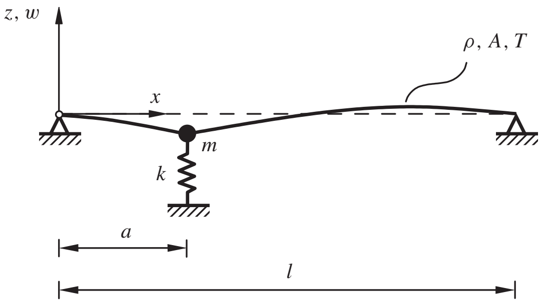

Figure 1.5 A taut string with lumped elements

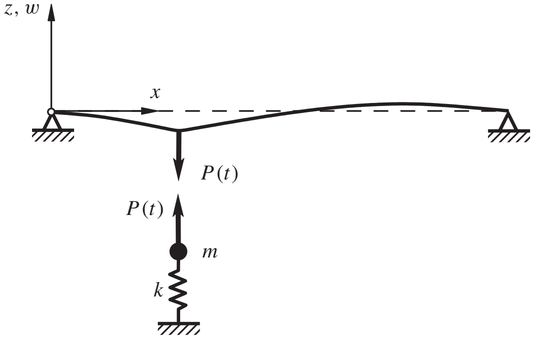

Figure 1.6 The interaction force diagram

easily incorporated into the equation of motion as follows. Consider the system shown in Figure 1.5, and the interaction force diagram shown in Figure 1.6. The force $P(t)$ at the interface between the string and the particle of mass m can be written from Newton's second law for the mass–spring system as $P(t) = mw_{,tt}(a, t) + kw(a, t)$, where $x = a$ is the location of the lumped system. Using the Dirac delta function, one can represent $P(t)$ as a distributed force

$$p(x, t) = mw_{,tt}(x, t)\delta(x - a) + kw(x, t)\delta(x - a). \tag{1.10}$$

Therefore, the equation of motion of the combined system can be written as

$$\rho A(x)w_{,tt} - [T(x)w_{,x}]_{,x} = -p(x, t),$$

or

$$[\rho A(x) + m\delta(x - a)]w_{,tt} - [T(x)w_{,x}]_{,x} + k\delta(x - a)w = 0. \tag{1.11}$$

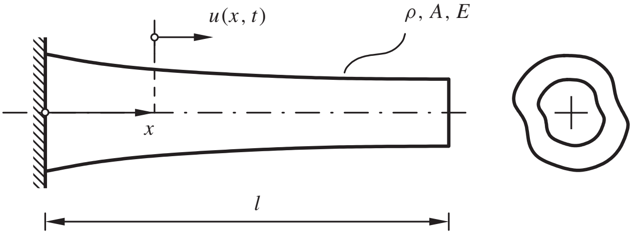

Figure 1.7 Schematic representation of a bar

1.1.2 Longitudinal dynamics of bars

Let us consider the longitudinal dynamics of a bar of arbitrary cross-section, as shown in Figure 1.7. We assume that the centroid of each cross-section lies on a straight line which is perpendicular to the cross-section. Under such assumptions, we can study the pure longitudinal motion of the bar. Such cases include bars which are solids of revolution (for example, cylinders and cones), and other standard structural elements.

Consider the free body diagram of an element of length Δx of the bar, as shown in Figure 1.8. We assume the displacement of any point of the bar to be along the x-axis, so that it can be represented by a single field variable $u(x, t)$. Using Newton's second law, one can write the equation of longitudinal motion of the element as

$$\rho A(x)\Delta x u_{,tt}(x + \theta\Delta x, t) = \sigma_x(x + \Delta x, t)A(x + \Delta x) - \sigma_x(x, t)A(x), \tag{1.12}$$

where ρ is the density, $A(x)$ is the cross-sectional area at x, $\theta \in [0, 1]$, and $\sigma_x(x, t)$ is the normal stress over the cross-section. Dividing (1.12) by Δx, and taking the limit $\Delta x \to 0$, yields

$$\rho A(x)u_{,tt}(x, t) = [\sigma_x(x, t)A(x)]_{,x}. \tag{1.13}$$

From elementary theory of elasticity (see [1]), we can relate the longitudinal strain $\epsilon_x(x, t)$ and the displacement field as $\epsilon_x(x, t) = u_{,x}(x, t)$. Using this strain–displacement relation and Hooke's law, one can write

$$\sigma_x(x, t) = E\epsilon_x(x, t) = Eu_{,x}(x, t), \tag{1.14}$$

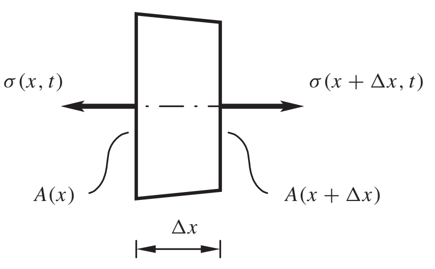

Figure 1.8 Free body diagram of a bar element

where E is the material's Young's modulus. Using (1.14) in (1.13) yields on rearrangement

$$\rho A(x)u_{,tt} - [EA(x)u_{,x}]_{,x} = 0. \tag{1.15}$$

If the bar is homogeneous and has a uniform cross-section, then (1.15) simplifies to

$$u_{,tt} - c^2 u_{,xx} = 0, \tag{1.16}$$

where $c = \sqrt{E/\rho}$ is the speed of the longitudinal waves in a uniform bar.

The boundary conditions for the bar can be written by inspection. For example, in Figure 1.7, the left-end boundary condition is $u(0, t) \equiv 0$, which is a geometric boundary condition. The right end of the bar is force-free, i.e., $EAu_{,x}(l, t) \equiv 0$. Hence, the right end of the bar has a dynamic boundary condition.

1.1.3 Torsional dynamics of bars

In this section, we make the same assumptions regarding the centroidal axis as made for the longitudinal dynamics of bars. The torsional dynamics of a bar depends on the shape of its cross-section. Complications arise due to warping of the cross-section during torsion in bars with non-circular cross-section (see [1]). In general, the torsional vibration of a bar is also coupled with its flexural vibration. Therefore, to keep the discussion simple, we will consider only torsional dynamics of bars with circular cross-section. As is known from the theory of elasticity, for bars with circular cross-section, planar sections remain planar for small torsional deformation. Further, an imaginary radial line on the undeformed cross-section can be assumed to remain straight even after deformation.

Consider a circular bar, as shown in Figure 1.9. A small sectional element of the bar between the centroidal coordinates x and $x + \Delta x$ is shown in Figure 1.10. Let $\phi(x, t)$ be the angle of twist at coordinate x, and $\phi + \Delta\phi(x, t)$ be the twist at $x + \Delta x$. From Figure 1.10, one can write, at any radius r, the kinematic relation

$$r\Delta\phi(x, t) = \Delta x\psi(r, t), \tag{1.17}$$

where $\psi(r, t)$ is the angular deformation of a longitudinal line at r, as shown in the figure. This angular deformation is the shear angle, as shown in Figure 1.11. Then, the shear stress $\tau_{x\phi}(r, t)$ is obtained from Hooke's law as

$$\tau_{x\phi}(r, t) = G\psi(r, t), \tag{1.18}$$

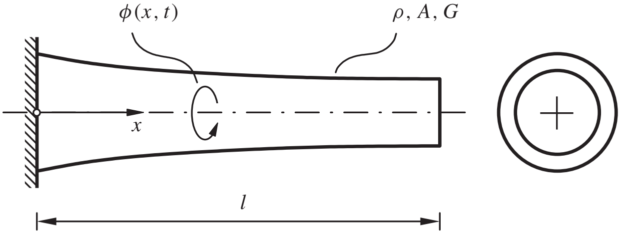

Figure 1.9 Schematic representation of a circular bar

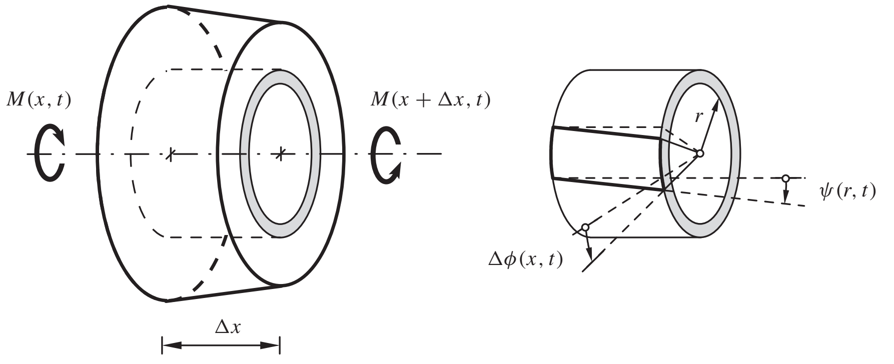

Figure 1.10 Deformation of a bar element under torsion

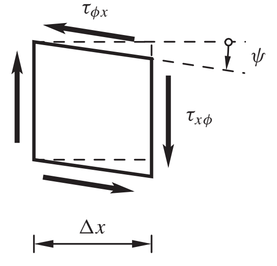

Figure 1.11 State of stress on a bar element under torsion

where G is the shear modulus. Substituting the expression of $\psi(r,t)$ from (1.17) in (1.18), one can write in the limit $\Delta x \to 0$

$$\tau_{x\phi}(r,t) = Gr\phi_{,x}. \tag{1.19}$$

Now, the torque at any cross-section x can be computed as

$$M(x,t) = \int_{A(x)} r\tau_{x\phi}(r,t)\,\mathrm{d}A = G\phi_{,x}\int_{A(x)} r^2\mathrm{d}A = GI_\mathrm{p}(x)\phi_{,x}, \tag{1.20}$$

where $A(x)$ represents the cross-sectional area, and $I_\mathrm{p}(x)$ is the polar moment of the area. Writing the moment of momentum equation for the element yields

$$\left[\int_{A(x+\theta\Delta x)} \rho r^2\Delta x\,\mathrm{d}A\right]\phi_{,tt}(x,t) = GI_\mathrm{p}(x+\Delta x)\phi_{,x}(x+\Delta x,t)$$
$$-GI_\mathrm{p}(x)\phi_{,x}(x,t) + n_\mathrm{E}(x,t)\Delta x, \tag{1.21}$$

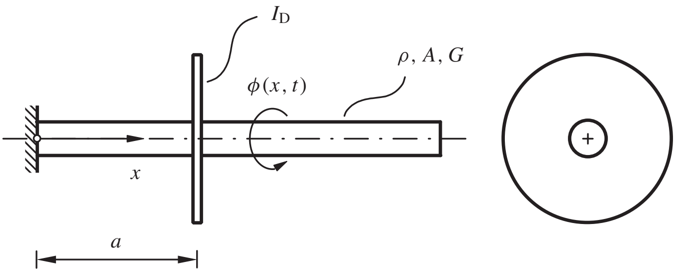

Figure 1.12 A circular bar with a disc

where $n_E(x, t)$ is an externally applied torque distribution. Dividing both sides in (1.21) by Δx and taking the limit $\Delta x \to 0$, we obtain

$$\rho I_p \phi_{,tt} - (G I_p \phi_{,x})_{,x} = n_E(x, t). \tag{1.22}$$

The partial differential equation (1.22) represents the torsional dynamics of a circular bar. For a bar with uniform cross-section (i.e., I_p independent of x), and $n_E(x, t) \equiv 0$, we obtain the wave equation

$$\phi_{,tt} - c^2 \phi_{,xx} = 0, \tag{1.23}$$

where $c = \sqrt{G/\rho}$ is the speed of torsional waves in the bar.

The boundary conditions for the fixed–free bar shown in Figure 1.9 can be written as $\phi(0, t) \equiv 0$, and $M(l, t) = G I_p \phi_{,x}(l, t) \equiv 0$. We can easily identify the first boundary condition as geometric, while the second is a natural boundary condition.

As an example, consider the torsional dynamics of a uniform circular bar with a massive disc at $x = a$, as shown in Figure 1.12. The disc can be considered as having a lumped rotational inertia. Therefore, the bar experiences an external torque due to the rotational inertia of the disc given by $n_E(x, t) = -I_D \phi_{,tt}(x, t)\delta(x - a)$, where I_D is the rotational inertia of the disc. Substituting this expression of external moment in (1.22), the complete equation of torsional dynamics of the bar can then be written as

$$[\rho I_p + I_D \delta(x - a)]\phi_{,tt} - G I_p \phi_{,xx} = 0. \tag{1.24}$$

1.2 DYNAMICS OF STRINGS AND BARS: THE VARIATIONAL FORMULATION

The variational formulation presents an elegant and powerful method of deriving the equations of motion of a dynamical system. Through this formulation, all the boundary conditions of a system are revealed. This is clearly an advantage especially for continuous mechanical systems. As will be discussed later, this approach also yields very useful methods of obtaining approximate solutions of vibration problems. The fundamentals of the variational approach for continuous systems is presented in Appendix A. In the following, we directly use the procedure discussed in Appendix A in deriving the equation of motion for strings and bars.

1.2.1 Transverse dynamics of strings

Consider a string of length l, as shown in Figure 1.1. The kinetic energy $\mathcal{T}$ of the string is

$$\mathcal{T} = \frac{1}{2}\int_0^l \rho A w_{,t}^2 \, \mathrm{d}x. \tag{1.25}$$

The potential energy can be written from the consideration that the unstretched length Δx is stretched to $\Delta s = \sqrt{1 + w_{,x}^2}\,\Delta x$ under a constant tension T. Therefore, the potential energy $\mathcal{V}$ stored in the string is given by

$$\begin{aligned}\mathcal{V} &= \int_0^l T(\mathrm{d}s - \mathrm{d}x) \approx \int_0^l T\left[\left(1 + \frac{1}{2}w_{,x}^2\right) - 1\right]\mathrm{d}x \\ &= \frac{1}{2}\int_0^l T w_{,x}^2 \, \mathrm{d}x.\end{aligned} \tag{1.26}$$

Defining the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$, Hamilton's principle can be written as

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t = 0 \tag{1.27}$$

or

$$\delta \int_{t_1}^{t_2} \frac{1}{2}\int_0^l \left[\rho A w_{,t}^2 - T w_{,x}^2\right] \mathrm{d}x. \tag{1.28}$$

As detailed in Appendix A, one obtains from (1.28)

$$\begin{aligned}&\int_0^l \rho A w_{,t} \delta w\Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} T w_{,x} \delta w\Big|_0^l \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2}\int_0^l [\rho A w_{,tt} - (T w_{,x})_{,x}]\, \delta w \, \mathrm{d}x \, \mathrm{d}t = 0.\end{aligned} \tag{1.29}$$

The first term in (1.29) is always zero since the variations of the field variable at the initial and final times are zero, i.e., $\delta w(x, t_0) \equiv 0$, and $\delta w(x, t_1) \equiv 0$. Following the arguments in Appendix A, the integrand of the third term in (1.29) has to be zero, i.e.,

$$\rho A w_{,tt} - (T w_{,x})_{,x} = 0, \tag{1.30}$$

which yields the equation of transverse dynamics of the string. The second term in (1.29) is zero if, for example,

$$Tw_{,x}(0,t) \equiv 0 \qquad \text{or} \qquad w(0,t) \equiv 0 \tag{1.31}$$

and

$$Tw_{,x}(l,t) \equiv 0 \qquad \text{or} \qquad w(l,t) \equiv 0, \tag{1.32}$$

which represent possible boundary conditions. For a fixed–fixed string, the conditions $w(0,t) \equiv 0$ and $w(l,t) \equiv 0$ hold, while for a fixed–sliding string (see Figure 1.4), $w(0,t) \equiv 0$ and $Tw_{,x}(l,t) \equiv 0$.

In the case of a string with discrete elements shown in Figure 1.5, the kinetic and potential energies can be written as, respectively,

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}\int_0^l \rho A w_{,t}^2(x,t)\,\mathrm{d}x + \frac{1}{2} m w_{,tt}^2(a,t)\\ &= \frac{1}{2}\int_0^l [\rho A + m\delta(x-a)] w_{,tt}^2(x,t)\,\mathrm{d}x,\end{aligned} \tag{1.33}$$

$$\begin{aligned}\mathcal{V} &= \frac{1}{2}\int_0^l T w_{,x}^2(x,t)\,\mathrm{d}x + \frac{1}{2} k w^2(a,t)\\ &= \frac{1}{2}\int_0^l [T w_{,x}^2(x,t) + k\delta(x-a) w^2(x,t)]\,\mathrm{d}x.\end{aligned} \tag{1.34}$$

Substituting $\mathcal{L} = \mathcal{T} - \mathcal{V}$ in the variational form (1.27) and taking the variation yields on simplification

$$\begin{aligned}&\int_0^l [\rho A + m\delta(x-a)] w_{,t}\delta w\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2} T w_{,x}\delta w\Big|_0^l \mathrm{d}t\\ &- \int_{t_1}^{t_2}\int_0^l [(\rho A + m\delta(x-a)) w_{,tt} - (T w_{,x})_{,x} + k\delta(x-a) w]\,\delta w\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

The equation of motion is obtained from the third term above which is the same as (1.11). The boundary conditions remain the same as in (1.31)–(1.32). When external forces are present, one can use the extended Hamilton's principle discussed in Appendix A to obtain the equations of motion.

1.2.2 Longitudinal dynamics of bars

In the case of longitudinal vibration of a bar, the kinetic energy is given by

$$\mathcal{T} = \frac{1}{2}\int_0^l \rho A u_{,t}^2\,\mathrm{d}x. \tag{1.35}$$

Defining σ_x and ϵ_x as the longitudinal stress and strain, respectively, the potential energy can be computed from the theory of elasticity as

$$\mathcal{V} = \frac{1}{2}\int_0^l \sigma_x \epsilon_x A \, \mathrm{d}x = \frac{1}{2}\int_0^l EA\epsilon_x^2 \, \mathrm{d}x$$
$$= \frac{1}{2}\int_0^l EAu_{,x}^2 \, \mathrm{d}x. \tag{1.36}$$

Writing the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$, Hamilton's principle assumes the form

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, \mathrm{d}t = 0,$$

or

$$\delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^l \left(\rho A u_{,t}^2 - EAu_{,x}^2\right) \mathrm{d}x \, \mathrm{d}t = 0,$$
$$\Rightarrow \int_0^l \rho A \delta u \Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} EAu_{,x} \delta u \Big|_0^l \mathrm{d}t$$
$$- \int_{t_1}^{t_2} \int_0^l [\rho A u_{,tt} - (EAu_{,x})_{,x}] \, \delta u \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{1.37}$$

Since by definition $\delta u(x, t_0) = \delta u(x, t_1) \equiv 0$, the first term in (1.37) vanishes identically. The third term in (1.37) yields the equation of motion

$$\rho A u_{,tt} - (EAu_{,x})_{,x} = 0, \tag{1.38}$$

and the boundary conditions are obtained from the second term. For example, the boundary conditions can be written as

$$EAu_{,x}(0, t) \equiv 0 \qquad \text{or} \qquad u(0, t) \equiv 0, \tag{1.39}$$

and

$$EAu_{,x}(l, t) \equiv 0 \qquad \text{or} \qquad u(l, t) \equiv 0. \tag{1.40}$$

It can be seen that the first condition in both (1.39) and (1.40) is the longitudinal force condition (natural boundary condition) at the two ends of the bar, while the second condition is the displacement condition (geometric boundary condition). Thus, for a fixed–fixed bar, $u(0, t) \equiv 0$, and $u(l, t) \equiv 0$, while for a fixed–free bar, $u(0, t) \equiv 0$ and $EAu_{,x}(l, t) \equiv 0$. In the case of a free–free bar, the boundary conditions are $EAu_{,x}(0, t) \equiv 0$, and $EAu_{,x}(l, t) \equiv 0$.

1.2.3 Torsional dynamics of bars

The kinetic energy of a circular bar undergoing torsional oscillations can be written in the notations used previously in Section 1.1.3 as

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}\int_0^l \int_0^R \int_0^{2\pi} \rho \phi_{,t}^2 r^3 \, \mathrm{d}\phi \, \mathrm{d}r \, \mathrm{d}x \\ &= \frac{1}{2}\int_0^l \rho I_\mathrm{p} \phi_{,t}^2 \, \mathrm{d}x. \end{aligned} \tag{1.41}$$

The potential energy can be written from elasticity theory as

$$\mathcal{V} = \frac{1}{2}\int_0^l \int_0^R \int_0^{2\pi} \tau_{x\phi} \psi r \, \mathrm{d}\phi \, \mathrm{d}r \mathrm{d}x. \tag{1.42}$$

Using the definitions of $\tau_{r\phi}$ and $\psi(x, t)$ from (1.17) and (1.18), respectively, in (1.42), we have

$$\begin{aligned}\mathcal{V} &= \frac{1}{2}\int_0^l \int_0^R \int_0^{2\pi} G \phi_{,x}^2 r^3 \, \mathrm{d}\phi \, \mathrm{d}r \, \mathrm{d}x \\ &= \frac{1}{2}\int_0^l G I_\mathrm{p} \phi_{,x}^2 \, \mathrm{d}x. \end{aligned} \tag{1.43}$$

Hamilton's principle can then be written as

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^l \left[\rho I_\mathrm{p} \phi_{,t}^2 - G I_\mathrm{p} \phi_{,x}^2\right] \mathrm{d}x = 0 \\ \Rightarrow &\int_0^l \rho I_\mathrm{p} \phi_{,t} \delta\phi \Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} G I_\mathrm{p} \phi_{,x} \delta\phi \Big|_0^l \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2} \int_0^l \left[\rho I_\mathrm{p} \phi_{,tt} - (G I_\mathrm{p} \phi_{,x})_{,x}\right] \delta\phi \, \mathrm{d}x = 0. \end{aligned} \tag{1.44}$$

The first term in (1.44) is zero by definition of the variational formulation. The third term in (1.44) yields the equation of motion

$$\rho I_\mathrm{p} \phi_{,tt} - (G I_\mathrm{p} \phi_{,x})_{,x} = 0, \tag{1.45}$$

while the second term provides information on the boundary conditions. For example, the possible boundary conditions could be

$$G I_\mathrm{p} \phi_{,x}(0, t) \equiv 0 \qquad \text{or} \qquad \phi(0, t) \equiv 0, \tag{1.46}$$

and

$$GI_{\mathrm{p}}\phi_{,x}(l,t) \equiv 0 \qquad \text{or} \qquad \phi(l,t) \equiv 0. \tag{1.47}$$

The first condition in (1.46) and (1.47) can be easily identified to be the torque condition (natural boundary condition) at the ends of the bar, while the second condition is on the angular displacement (geometric boundary condition).

1.3 FREE VIBRATION PROBLEM: BERNOULLI'S SOLUTION

Vibration analysis of a system almost always starts with the free or natural vibration analysis. This leads us to the important concepts of natural frequency and mode of vibration of the system. These two concepts form the starting point of any quantitative and qualitative analysis and understanding of a vibratory system.

It was observed in the above discussions that, under certain assumptions of uniformity, the one-dimensional wave equation represents the transverse dynamics of a string, and longitudinal and torsional dynamics of a bar. The wave equation is one of the most important equations that appear in the study of vibrations of continuous systems. The solution and properties of the wave equation are fundamental in understanding vibration and propagation of vibration in continuous media, and will be taken up in detail in later chapters. In this section, we will discuss a simple solution procedure for the one-dimensional wave equation and study some of the solution properties.

Consider the wave equation

$$w_{,tt} - c^2 w_{,xx} = 0, \qquad x \in [0,l], \tag{1.48}$$

with the boundary conditions

$$w(0,t) \equiv 0, \qquad \text{and} \qquad w(l,t) \equiv 0. \tag{1.49}$$

Such a problem corresponds to, for example, a fixed-fixed string or bar.

Let us first look for separable solutions of (1.48) in the form

$$w(x,t) = p(t)W(x). \tag{1.50}$$

Substituting (1.50) in (1.48) yields on rearrangement

$$\frac{\ddot{p}}{p} - c^2\frac{W''}{W} = 0. \tag{1.51}$$

It is easily observed that the first term in (1.51) is solely a function of t, while the second term is solely a function of x. Therefore, (1.51) will hold identically if and only if both the terms are constant, i.e.,

$$\frac{\ddot{p}}{p} = -\omega^2 \qquad \text{and} \qquad c^2\frac{W''}{W} = -\omega^2 \tag{1.52}$$

$$\Rightarrow \ddot{p} + \omega^2 p = 0 \tag{1.53}$$

and

$$W'' + \frac{\omega^2}{c^2} W = 0, \tag{1.54}$$

where ω is an arbitrary constant. It may be noted that the constant in (1.52) is chosen as $-\omega^2$, so that ω later will have the meaning of a circular frequency.

The general solutions of (1.53) and (1.54) can be written as, respectively,

$$p(t) = C \cos \omega t + S \sin \omega t \tag{1.55}$$

and

$$W(x) = D \cos \frac{\omega x}{c} + H \sin \frac{\omega x}{c}, \tag{1.56}$$

where C, S, D, and H are arbitrary constants of integration. The constants C and S are usually determined from the initial position and velocity conditions of the string/bar, while the determination of D and H requires the conditions at the two boundaries of the string/bar. Since the solution (1.50) must satisfy the boundary conditions (1.49), we must have

$$W(0) = 0 \qquad \text{and} \qquad W(l) = 0, \tag{1.57}$$

which can be written using (1.56) as

$$D + 0 \cdot H = 0,$$

and

$$\begin{aligned} &\left(\cos \frac{\omega l}{c}\right) D + \left(\sin \frac{\omega l}{c}\right) H = 0 \\ \Rightarrow &\begin{bmatrix} 1 & 0 \\ \cos \frac{\omega l}{c} & \sin \frac{\omega l}{c} \end{bmatrix} \begin{Bmatrix} D \\ H \end{Bmatrix} = 0. \end{aligned} \tag{1.58}$$

Therefore, for a non-trivial solution of D and H, the determinant of (1.58) must vanish, i.e.,

$$\sin \frac{\omega l}{c} = 0. \tag{1.59}$$

This equation is referred to as the *characteristic equation* of the system (1.48)–(1.49). The characteristic equation (1.59) is satisfied when ω takes any of the discrete values

$$\omega_k = \frac{k\pi c}{l}, \qquad k = 0, 1, \ldots, \infty \tag{1.60}$$

where ω_k is termed as the k^{th} *circular natural frequency* of the system. Thus, there are countably infinitely many natural frequencies of the continuous system (1.48)–(1.49). It may be noted that the negative solutions of ω have been dropped, as they do not yield any new solution to the vibration problem.

To every circular frequency ω_k, there corresponds a solution of (D_k, H_k). On substituting (1.60) in (1.58), one can conclude that $D_k = 0$ for all k, and H_k can be arbitrary. Thus, the solution of (D_k, H_k) can be determined up to an arbitrary (multiplicative) constant. Finally, from (1.56), we have

$$W_k(x) = H_k \sin \frac{\omega_k x}{c} = H_k \sin \frac{k\pi x}{l}, \qquad k = 1, 2, \ldots, \infty, \tag{1.61}$$

corresponding to the circular natural frequencies ω_k, $k = 1, 2, \ldots, \infty$. It may be observed that $k = 0$ has been dropped since $W_0(x) \equiv 0$, which is the trivial solution. We have therefore found infinitely many solutions of the form (1.50), which may be written as

$$\begin{aligned} w_k(x,t) &= p_k(t) W_k(x) \\ &= (C_k \cos \omega_k t + S_k \sin \omega_k t) \sin \frac{k\pi x}{l}, \qquad k = 1, 2, \ldots, \infty, \end{aligned} \tag{1.62}$$

where we have set all $H_k = 1$ without loss of generality, and C_k and S_k are arbitrary constants.

Let us assume that the system is oscillating according to any one of the infinite solutions given by (1.62). Corresponding to this solution, it is clear from (1.62) that all points of the string or bar oscillate with the same circular frequency ω_k. The system is then said to oscillate in the kth *mode*, and the solution $w_k(x, t)$ is known as the *modal solution* of the kth mode. The function $W_k(x)$ is known as the kth *eigenfunction* or *mode-shape-function*. The circular natural frequency ω_k is also referred to as the kth *circular modal frequency* of the system. It may be observed for any modal solution that, whenever $p_k(t) = 0$, the displacement $w(x, t)$ of all points of the string/bar is equal to zero. Thus, when the system is oscillating in a particular mode, all points pass through their equilibrium positions at the same time. Further, any two points on the string/bar have a phase difference of either 0 or π between their motion, i.e., either they move in phase, or in opposite phase. The modal solution has these characteristics only because it is separable.

Since the wave equation is a linear equation and the boundary conditions are assumed homogeneous, linear superposition of the individual modes also gives a solution. Therefore, the general solution of the free vibration problem is of the form

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t) W_k(x) = \sum_{k=1}^{\infty} (C_k \cos \omega_k t + S_k \sin \omega_k t) \sin \frac{k\pi x}{l}. \tag{1.63}$$

To obtain a unique solution, one has to determine the constants C_k and S_k. In practice, a system can vibrate freely when it is released from some non-equilibrium configuration, or started with some non-zero velocity, or both. These *initial conditions* determine the constants C_k and S_k. This problem of determining the free vibration solution uniquely is known as the *initial value problem*. The solution of the initial value problem relies on a very important property of the eigenfunctions, as discussed below.

From the theory of Fourier series (see [2]), (1.63) can be easily identified as the Fourier sine series with time-varying coefficients. It can be easily checked that the eigenfunctions of the string under consideration satisfy the *orthogonality property*

$$\langle W_j(x), W_k(x)\rangle := \int_0^l W_j(x)W_k(x)\,\mathrm{d}x \tag{1.64}$$

$$= \int_0^l \sin\frac{j\pi x}{l}\sin\frac{k\pi x}{l}\,\mathrm{d}x = \frac{l}{2}\delta_{jk}, \tag{1.65}$$

where $\langle W_j(x), W_k(x)\rangle$ is defined as an *inner product* (or *scalar product*) of the two functions $W_j(x)$ and $W_k(x)$, and δ_{jk} is the Kronecker delta symbol, i.e.,

$$\delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}.$$

Using the orthogonality property, one can *filter* the jth coefficient in (1.63) as

$$\begin{aligned}\langle w(x,t), W_j(x)\rangle &= \int_0^l w(x,t)\sin\frac{j\pi x}{l}\,\mathrm{d}x \\ &= \frac{l}{2}(C_j\cos\omega_j t + S_j\sin\omega_j t).\end{aligned} \tag{1.66}$$

The coefficients C_j and S_j can now be computed easily to match any initial shape and velocity of the string/bar. Let the initial shape and velocity be given by respectively,

$$w(x,0) = w_0(x), \qquad \text{and} \qquad w_{,t}(x,0) = v_0(x). \tag{1.67}$$

Then, from (1.66), one can obtain

$$C_j = \frac{2}{l}\langle w_0(x), W_j(x)\rangle, \qquad j = 1, 2, \ldots, \infty, \tag{1.68}$$

and

$$S_j = \frac{2}{l\omega_j}\langle v_0(x), W_j(x)\rangle, \qquad j = 1, 2, \ldots, \infty, \tag{1.69}$$

where (1.68) is obtained by setting $t = 0$ in (1.66), and (1.69) is obtained by differentiating (1.66) once with respect to t and then setting $t = 0$. This completes the solution (1.63) of the initial value problem of a string/bar defined by (1.48), with the boundary conditions (1.49), and the initial conditions (1.67).

1.4 MODAL ANALYSIS

As observed in the previous section, the solution of the initial value problem requires the mode-shape-functions and the associated circular modal frequencies. In this section, the problem of determination of the mode-shape-functions and the modal frequencies, usually termed *modal analysis*, is formulated as an eigenvalue problem. For convenience, we first introduce the complex notation for representing the general solution of the free vibration problem.

The general solution (1.63) can be compactly represented using the complex notation in the form

$$\begin{aligned} w(x,t) &= \sum_{k=1}^{\infty}\left[\frac{C_k}{2}(\mathrm{e}^{i\omega_k t}+\mathrm{e}^{-i\omega_k t})+\frac{S_k}{2i}(\mathrm{e}^{i\omega_k t}-\mathrm{e}^{-i\omega_k t})\right]W_k(x) \\ &= \sum_{k=1}^{\infty}\left[\frac{F_k}{2}\mathrm{e}^{i\omega_k t}+\frac{F_k^*}{2}\mathrm{e}^{-i\omega_k t}\right]W_k(x) \\ &= \sum_{k=1}^{\infty}\frac{F_k}{2}\mathrm{e}^{i\omega_k t}W_k(x)+\text{c.c.} \\ &= \sum_{k=1}^{\infty}\mathscr{R}\left[F_k\mathrm{e}^{i\omega_k t}W_k(x)\right], \end{aligned} \tag{1.70}$$

where $\mathscr{R}[\cdot]$ denotes real part of a complex number, $F_k = C_k - iS_k$, and the asterisk in the superscript denotes complex conjugate (c.c.). For notational convenience while obtaining the solution, we may write the solution as simply $w(x,t) = W(x)\mathrm{e}^{i\omega t}$, where the unknown mode-shape-function $W(x)$ may be complex in general. It is to be noted that the complex conjugate part can be dropped since the equations considered here are linear. In some cases, such as in the expression of kinetic energy $\rho A w_t^2(x,t)/2$, the complex conjugate part must be written explicitly. One further point to note is that we may also take the imaginary part of $F_k\mathrm{e}^{i\omega_k t}W_k(x)$ in (1.70) as the kth modal solution. The complex representation also allows us to treat problems with non-separable solution, and will be used for studying vibrations in translating strings later in this chapter.

1.4.1 The eigenvalue problem

The equation of motion for the systems discussed above can be represented in the general form

$$\mu(x)w_{,tt} + \mathcal{K}[w] = 0, \tag{1.71}$$

where $\mathcal{K}[\cdot]$ is a linear differential operator. For example, for a taut string

$$\mu(x) = 1 \qquad \text{and} \qquad \mathcal{K}[\cdot] = -c^2\frac{\partial^2}{\partial x^2}. \tag{1.72}$$

Consider a modal solution for the free vibration problem of (1.71) in the form

$$w(x, t) = W(x)\mathrm{e}^{i\omega t}, \tag{1.73}$$

where $W(x)$ is the mode-shape-function and ω is the modal frequency. Substituting (1.73) in (1.71) yields

$$-\omega^2\mu(x)W + \mathcal{K}[W] = 0. \tag{1.74}$$

Only for certain special values of ω, can (1.74) be solved for non-trivial solutions of $W(x)$ satisfying the boundary conditions of the problem. Hence, the differential equation (1.74) along with the boundary conditions on $W(x)$ is known as the *eigenvalue problem* for the system. For a taut string, it can be easily checked that the eigenvalue problem is defined by

$$\begin{aligned} &W'' + \frac{\omega^2}{c^2}W = 0, \\ &W(0) = 0 \qquad \text{and} \qquad W(l) = 0. \end{aligned}$$

This problem is solved in detail in Section 1.3, and the eigenvalues and eigenfunctions are given by

$$\omega_k = \frac{k\pi c}{l} \qquad \text{and} \qquad W_k(x) = \sin\frac{k\pi x}{l}, \qquad k = 1, 2, \ldots, \infty. \tag{1.75}$$

Thus, the solution of the eigenvalue problem yields the circular modal frequencies, and the corresponding mode-shape-functions, which are also known as the *circular eigenfrequencies* and *eigenfunctions*, respectively. In the following, we consider two slightly more complex eigenvalue problems.

1.4.1.1 The hanging string

Let us consider the unforced dynamics of a hanging string which is described by (1.8). The equation of motion can be represented in the form (1.71) where

$$\mu(x) = 1 \qquad \text{and} \qquad \mathcal{K}[\cdot] = -g\frac{\partial}{\partial x}\left[(l - x)\frac{\partial}{\partial x}\right]. \tag{1.76}$$

At the free end of the string, the transverse force is zero, i.e., $T(l)w_{,x}(l, t) \equiv 0$. However, since the tension at the free end $T(l) = 0$, it implies that $w_{,x}(l, t)$ can be arbitrary. As we will see shortly, a finiteness condition on the solution is required for the free end. The only boundary condition that can be specified is for the fixed end of the string which is given by

$$w(0, t) \equiv 0. \tag{1.77}$$

Now, the eigenvalue problem for the hanging string can be easily written from (1.74) as

$$\omega^2 W + g[(l - x)W']' = 0, \tag{1.78}$$

with the associated boundary condition obtained from (1.77) as

$$W(0) = 0. \tag{1.79}$$

In the following, the eigenvalue problem (1.78)–(1.79) is now solved to determine the circular eigenfrequencies ω and the corresponding eigenfunctions $W(x)$.

Consider the function

$$s(x) = 2\omega\sqrt{\frac{l - x}{g}}. \tag{1.80}$$

Then, defining $\tilde{W}(s)$ such that $\tilde{W}(s(x)) = W(x)$, one obtains using the chain rule of differentiation

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\mathrm{d}\tilde{W}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = -\tilde{W}'\frac{\omega}{\sqrt{g(l - x)}} \tag{1.81}$$

$$\begin{aligned}\frac{\mathrm{d}^2W}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2\tilde{W}}{\mathrm{d}s^2}\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}\tilde{W}}{\mathrm{d}s}\frac{\mathrm{d}^2s}{\mathrm{d}x^2} \\ &= \tilde{W}''\frac{\omega^2}{g(l - x)} - \tilde{W}'\frac{\omega}{2\sqrt{g(l - x)^3}},\end{aligned} \tag{1.82}$$

where the prime in $\tilde{W}'$ denotes differentiation with respect to s. Using (1.81) and (1.82) in (1.78) yields on simplification

$$\tilde{W}'' + \frac{1}{s}\tilde{W}' + \tilde{W} = 0, \qquad s \in [0, 2\omega\sqrt{l/g}] \tag{1.83}$$

$$\tilde{W}(2\omega\sqrt{l/g}) = 0. \tag{1.84}$$

The differential equation (1.83) is a special case of the Bessel equation (see [2])

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{n^2}{x^2}\right)y(x) = 0$$

with $n = 0$. Therefore, the general solution of (1.83) can be written as

$$\tilde{W}(s) = DJ_0(s) + EY_0(s), \tag{1.85}$$

where D and E are arbitrary constants, and $J_0(s)$ and $Y_0(s)$ are known as, respectively, zeroth-order Bessel functions of the first and second kind (or Neumann functions). The

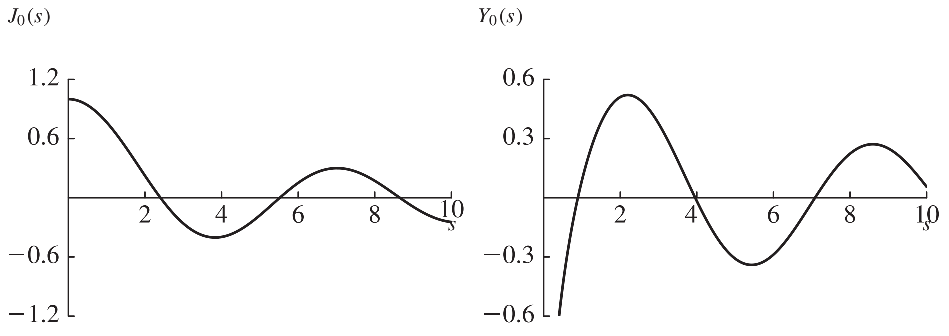

Figure 1.13 Bessel functions $J_0(s)$, and $Y_0(s)$

functions $J_0(s)$ and $Y_0(s)$ are plotted in Figure 1.13. Since $Y_0(s) \to -\infty$ as $s \to 0$ (i.e., $x \to l$), the condition of finiteness of the solution at the free end must imply $E = 0$. Therefore, the solution of (1.83) takes the form $\tilde{W}(s) = DJ_0(s)$. The boundary condition of the fixed end, $W(s(0)) = 0$, then implies, for non-triviality of $\tilde{W}(s)$,

$$J_0(2\omega\sqrt{l/g}) = 0, \tag{1.86}$$

which is the characteristic equation for the problem. The roots of $J_0(\gamma_k) = 0$, yield the eigenfrequencies

$$\omega_k = \frac{\gamma_k}{2}\sqrt{\frac{g}{l}}, \qquad k = 1, 2, \ldots, \infty, \tag{1.87}$$

where $\gamma_1 \approx 2.4048$, $\gamma_2 \approx 5.5201$, $\gamma_3 \approx 8.6537, \ldots$. It is interesting to note that the first eigenfrequency or fundamental frequency of a hanging string, $\omega_1 = 1.2024\sqrt{g/l}$, is about 1.2 times the small-amplitude oscillation frequency of a mathematical pendulum of length l. The kth eigenfunction can now be written as

$$W_k(x) = J_0\left(2\omega_k\sqrt{\frac{l-x}{g}}\right). \tag{1.88}$$

The first three mode-shapes of the hanging string are shown in Figure 1.14. From (1.70), we obtain the general solution of the initial value problem for the hanging string as

$$\begin{aligned} w(x,t) &= \sum_{k=1}^{\infty} \mathscr{R}\left[F_k \mathrm{e}^{i\omega_k t} W_k(x)\right] \\ &= \sum_{k=1}^{\infty}\left[(C_k\cos\omega_k t + S_k \sin\omega_k t) J_0\left(2\omega_k\sqrt{\frac{l-x}{g}}\right)\right]. \end{aligned} \tag{1.89}$$

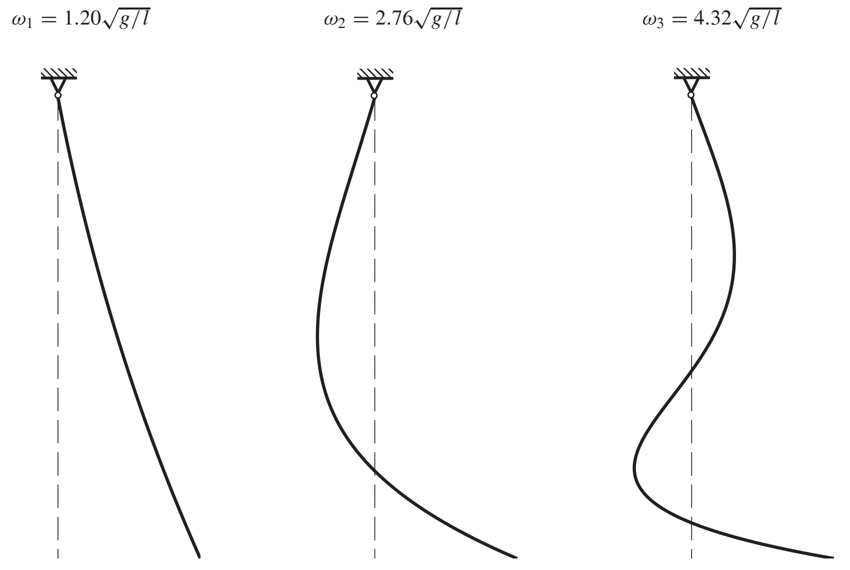

Figure 1.14 First three mode-shapes of a hanging string

The orthogonality property of the eigenfunctions $W_k(x)$ of the hanging string can be obtained from the theory of Bessel functions (see [2]) as

$$\int_0^l W_j(x)W_k(x)\,\mathrm{d}x = lJ_1^2(2\omega_j\sqrt{l/g})\delta_{jk}, \tag{1.90}$$

where $J_1(\cdot)$ is the Bessel function of order one, and δ_{jk} is the Kronecker delta symbol. Later, in Section 1.4.2, we shall see that the eigenfunctions of a large class of eigenvalue problems are always orthogonal in an appropriate sense.

1.4.1.2 Bar with varying cross-section

Consider the longitudinal dynamics of a fixed–free bar with varying cross-section described by (1.15). The equation of motion can be represented by (1.71) where

$$\mu(x) = \rho A(x) \qquad \text{and} \qquad \mathcal{K}[\cdot] = -\frac{\partial}{\partial x}\left(EA(x)\frac{\partial}{\partial x}\right).$$

The boundary conditions are given by

$$u(0,t) \equiv 0 \qquad \text{and} \qquad EA(l)u_{,x}(l,t) \equiv 0. \tag{1.91}$$

The eigenvalue problem in this case can be easily written as

$$A(x)\omega^2 U + c^2[A(x)U']' = 0, \tag{1.92}$$

where $c^2 = E/\rho$, along with the boundary conditions

$$U(0) = 0 \qquad \text{and} \qquad U'(l) = 0. \tag{1.93}$$

It is not possible to obtain an analytical solution of (1.92) for a general variation of the cross-sectional area $A(x)$. However, for a class of functions $A(x)$, one can solve (1.92) as follows.

Consider the transformation

$$W(x) = h(x)U(x), \tag{1.94}$$

where $h(x)$ is an unknown function. One can then write

$$[h^2U']' = hW'' - h''W. \tag{1.95}$$

Let us choose $h^2(x) = A(x)$. With this choice, one can rewrite (1.92) as

$$h^2\omega^2U + c^2[h^2U']' = 0,$$

or

$$\omega^2 W + c^2\left(W'' + \frac{h''}{h}W\right) = 0 \qquad \text{(using (1.94) and (1.95))}. \tag{1.96}$$

If the variation of the cross-section is such that $h''/h = \alpha$, where α is a constant, one can rewrite (1.96) as

$$W'' + \left(\frac{\omega^2}{c^2} + \alpha\right) W = 0.$$

This differential equation can be easily solved.

As a simple example, let

$$A(x) = A_0\left(1 - \frac{x}{2l}\right)^2.$$

Then, $h(x) = \sqrt{A_0}(1 - x/2l)$, and $h''(x) = 0$. Therefore, (1.96) simplifies to

$$W'' + \frac{\omega^2}{c^2}W = 0, \tag{1.97}$$

and the boundary conditions are given by

$$W(0) = 0 \qquad \text{and} \qquad W'(l) = \frac{h'(l)}{h(l)}W(l). \tag{1.98}$$

It is not difficult to show that the solutions of (1.97) satisfying the boundary conditions (1.98) can be written as

$$W_k(x) = D \sin \frac{\omega_k x}{c}, \qquad k = 1, 2, \ldots, \infty,$$

where ω_k are the eigenvalues obtained from the characteristic equation

$$\tan \frac{\omega l}{c} + \frac{\omega l}{c} = 0.$$

The first three eigenfrequencies are obtained as $\omega_1 = 2.029c/l$, $\omega_2 = 4.913c/l$, and $\omega_3 = 7.979c/l$. Finally, the eigenfunctions $U_k(x)$ are obtained from (1.94) as

$$U_k(x) = \frac{D \sin \dfrac{\omega_k x}{c}}{\sqrt{A_0}\left(1 - \dfrac{x}{2l}\right)}.$$

However, the orthogonality of the eigenfunctions may not be very obvious. Therefore, we need to have a general procedure to determine the orthogonality relations, which is discussed next.

1.4.2 Orthogonality of eigenfunctions

Consider a general eigenvalue problem formed by a differential equation of the type

$$-\lambda \mu(x) W + \mathcal{K}[W] = 0, \qquad x \in [0, l], \tag{1.99}$$

where $\lambda = \omega^2$, along with certain boundary conditions. If W_j and W_k $(j \neq k)$ are solutions of (1.99) corresponding to λ_j and λ_k, respectively, one can write

$$-\lambda_j \mu(x) W_j + \mathcal{K}[W_j] = 0, \tag{1.100}$$

and

$$-\lambda_k \mu(x) W_k + \mathcal{K}[W_k] = 0. \tag{1.101}$$

Multiplying (1.100) with W_k, and (1.101) with W_j, and integrating the difference of the two equations over the length of the string, one can write

$$-(\lambda_j - \lambda_k)\langle \mu(x) W_j, W_k \rangle + \langle W_k, \mathcal{K}[W_j] \rangle - \langle W_j, \mathcal{K}[W_k] \rangle = 0. \tag{1.102}$$

If the operator $\mathcal{K}[\cdot]$ is such that

$$\langle W, \mathcal{K}[\tilde{W}] \rangle = \langle \tilde{W}, \mathcal{K}[W] \rangle, \tag{1.103}$$

for any two functions $W(x)$ and $\tilde{W}(x)$ satisfying the boundary conditions, the operator $\mathcal{K}[\cdot]$ is called a *self-adjoint operator*. Since the eigenfunctions $W_j(x)$ and $W_k(x)$ satisfy the boundary conditions of the problem, one can write for a self-adjoint operator

$$\langle W_k, \mathcal{K}[W_j]\rangle = \langle W_j, \mathcal{K}[W_k]\rangle, \tag{1.104}$$

and (1.102) yields

$$-(\lambda_j - \lambda_k)\langle \mu(x) W_j, W_k\rangle = 0. \tag{1.105}$$

If $\lambda_j \neq \lambda_k$ (which is usually satisfied), we obtain the orthogonality relation

$$\langle \mu(x) W_j, W_k\rangle = 0 \qquad \Rightarrow \qquad \int_0^l \mu(x) W_j W_k \,\mathrm{d}x = 0. \tag{1.106}$$

It is always possible to normalize the eigenfunctions such that they are orthonormal, i.e,

$$\int_0^l \mu(x) W_j W_k \,\mathrm{d}x = \delta_{jk}, \tag{1.107}$$

where δ_{jk} is the Kronecker delta symbol. A consequence of orthonormality of the eigenfunctions can be obtained from (1.100) as

$$\int_0^l W_k \mathcal{K}[W_j] \,\mathrm{d}x = \lambda_j \delta_{jk}.$$

It can be concluded from the above that the eigenfunctions of a self-adjoint operator are orthogonal with respect to a suitably defined inner product, which may be determined using the above procedure.

1.4.3 The expansion theorem

Let us rework the solution procedure for the free vibration of a taut string presented in Section 1.4.1 in a slightly different manner. Based on our experience thus far, let us assume the solution of (1.71) as an expansion in terms of the eigenfunctions in the form

$$w(x,t) = \sum_{k=1}^{k=\infty} p_k(t) W_k(x), \tag{1.108}$$

where $W_k(x)$ is the kth eigenfunction given by (1.75) and $p_k(t)$ is the corresponding unknown *modal coordinate*. Substituting the expansion (1.108) in (1.71) yields

$$\sum_{k=1}^{k=\infty} \ddot{p}_k(t) W_k(x) + \mathcal{K}\left[\sum_{k=1}^{k=\infty} p_k(t) W_k(x)\right] = 0. \tag{1.109}$$

Using the linearity property of the operator $\mathcal{K}[\cdot]$, one can rewrite (1.109) as

$$\sum_{k=1}^{k=\infty} \mu(x)\ddot{p}_k(t)W_k(x) + \sum_{k=1}^{k=\infty} p_k(t)\mathcal{K}[W_k(x)] = 0,$$

or

$$\sum_{k=1}^{k=\infty} \mu(x)\ddot{p}_k(t)W_k(x) + \sum_{k=1}^{k=\infty} \mu(x)\omega_k^2 p_k(t)W_k(x) = 0 \qquad \text{(using (1.74))},$$

or

$$\sum_{k=1}^{k=\infty} \left[\ddot{p}_k(t) + \omega_k^2 p_k(t)\right] \mu(x)W_k(x) = 0. \tag{1.110}$$

Taking the inner product on both sides with W_j, $j = 1, 2, \ldots, \infty$, and using the orthogonality relations (1.106), one obtains the decoupled differential equations for the modal coordinates as

$$\ddot{p}_j(t) + \omega_j^2 p_j(t) = 0, \qquad j = 1, 2, \ldots, \infty. \tag{1.111}$$

It may be mentioned here that we have exchanged an integral and an infinite sum to arrive at the decoupled equations (1.111). The general solution of the jth modal coordinate is, therefore, obtained as

$$p_j(t) = C_j \cos\omega_j t + S_j \sin\omega_j t,$$

and the general solution of the free vibration problem can be written in the form

$$w(x,t) = \sum_{k=1}^{\infty} (C_k \cos\omega_k t + S_k \sin\omega_k t)W_k(x),$$

which is the same as (1.63) obtained by Bernoulli's method. Thus, we have reconstructed back the solution of the free vibration problem using the eigenfunction expansion (1.108) and the orthogonality relations (1.106).

The fundamental requirement for the expansion method to work is that any physically possible shape of the system, say a string, should be expandable as a linear combination of the eigenfunctions $W_k(x)$ in the form (1.108). In other words, $W_k(x)$ should form a basis of the space of all physically possible shapes of the string. The set of all eigenfunctions of the string is indeed a basis of the function space under consideration, and this follows from the self-adjointness of the differential operator $\mathcal{K}[\cdot]$ in the eigenvalue problem (1.71). This statement is referred to as the *expansion theorem*. The expansion theorem also provides a convenient method for solving forced vibration problems as discussed later.

1.4.4 Systems with discrete elements

A continuous system may interact with discrete elements as discussed in previous sections. For such hybrid systems, the modal analysis can be performed by analyzing the system in parts along with appropriate matching conditions and boundary conditions for each of the parts. Often in these systems, the boundary conditions themselves involve ordinary differential equations, as will be evident in this section.

Let us consider the modal analysis of longitudinal vibrations of a bar with a mass–spring system at the right boundary, as shown in Figure 1.15. This system can be described by one field variable $u(x,t)$ and one discrete variable $y(t)$. The equations of motion are

$$u_{,tt} - c^2 u_{,xx} = 0 \tag{1.112}$$

and

$$M\ddot{y} + Ky = Ku(l,t), \tag{1.113}$$

and the boundary conditions are given by

$$u(0,t) \equiv 0 \qquad \text{and} \qquad EAu_{,x}(l,t) \equiv K(y - u(l,t)). \tag{1.114}$$

As is evident, the second boundary condition in (1.114) involves the ordinary differential equation (1.113).

Assume a modal solution of the form

$$\left\{ \begin{array}{c} u(x,t) \\ y(t) \end{array} \right\} = \left\{ \begin{array}{c} U(x) \\ Y \end{array} \right\} \mathrm{e}^{i\omega t}. \tag{1.115}$$

It may be noted that the modal vector for this problem is given by $(U(x), Y)^T$. Substituting this solution in the equations of motion (1.112)–(1.113) and simplifying, we obtain the eigenvalue problem

$$U'' + \frac{\omega^2}{c^2} U = 0 \tag{1.116}$$

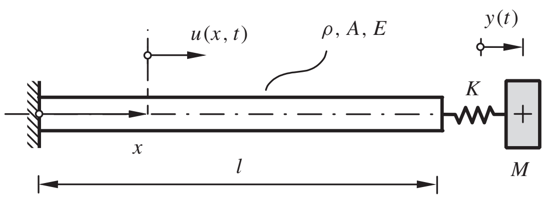

Figure 1.15 A hybrid system formed by a continuous sub-system and lumped elements

and

$$(-M\omega^2 + K)Y = KU(l), \tag{1.117}$$

with the associated boundary conditions given by (1.114) as

$$U(0) = 0 \tag{1.118}$$

and

$$EAU'(l) = K[Y - U(l)] = \frac{KM\omega^2}{K - M\omega^2}U(l) \quad \text{(using (1.117))}. \tag{1.119}$$

Note here that the boundary condition (1.119) also involves the circular frequency ω. Assuming a solution of (1.116) in the form

$$U(x) = C\cos\frac{\omega x}{c} + S\sin\frac{\omega x}{c} \tag{1.120}$$

we have from the boundary conditions (1.118)–(1.119)

$$\begin{bmatrix} 1 & 0 \\ \left(\dfrac{KM\omega^2}{K - M\omega^2}\cos\dfrac{\omega l}{c} + \dfrac{EA\omega}{c}\sin\dfrac{\omega l}{c}\right) & \left(\dfrac{KM\omega^2}{K - M\omega^2}\sin\dfrac{\omega l}{c} - \dfrac{EA\omega}{c}\cos\dfrac{\omega l}{c}\right) \end{bmatrix} \begin{Bmatrix} C \\ S \end{Bmatrix} = 0. \tag{1.121}$$

The non-triviality of the solution of (C, S) implies that the determinant of the matrix in (1.121) must vanish, which yields the characteristic equation

$$\tan\frac{\omega l}{c} - \frac{EA(K - M\omega^2)}{c\omega MK} = 0.$$

This transcendental equation yields infinitely many circular eigenfrequencies ω_k, $k = 1, 2, \ldots, \infty$. Substituting these eigenfrequencies in (1.121), one obtains $(C_k, S_k) = (0, 1)$, and correspondingly

$$U_k(x) = \sin\frac{\omega_k x}{c},$$

so that, using (1.117), the eigenvectors are obtained as

$$\begin{Bmatrix} U_k(x) \\ Y_k \end{Bmatrix} = \begin{Bmatrix} \sin\dfrac{\omega_k x}{c} \\ \dfrac{K\sin(\omega_k l/c)}{-M\omega_k^2 + K} \end{Bmatrix}, \qquad k = 1, 2, \ldots, \infty.$$

It is to be noted that these vectors are formed by the displacement field $U_k(x)$ in the rod, and the discrete coordinate Y_k. They are not vectors in two-dimensional Euclidean space, but rather in an $(\infty + 1)$-dimensional space. Since these infinitely many eigenvectors are all linearly independent, one can conveniently express the solution of (1.112)–(1.113) using the expansion theorem as

$$\left\{ \begin{array}{c} u(x,t) \\ y(t) \end{array} \right\} = \sum_{k=1}^{\infty} p_k(t) \left\{ \begin{array}{c} U_k(x) \\ Y_k \end{array} \right\},$$

where $p_k(t)$ is the modal coordinate corresponding to mode k.

The orthogonality relation for the above eigenvectors are obtained from the procedure discussed in Section 1.4.2 as follows. Consider the modes j and k which satisfy the following equations

$$U_j'' + \frac{\omega_j^2}{c^2} U_j = 0, \qquad Y_j = \frac{K U_j(l)}{-M\omega_j^2 + K}, \tag{1.122}$$

$$U_k'' + \frac{\omega_k^2}{c^2} U_k = 0, \qquad Y_k = \frac{K U_k(l)}{-M\omega_k^2 + K}, \tag{1.123}$$

along with appropriate boundary and matching conditions. Multiply the first equation in (1.122) by U_k and the first equation in (1.123) by U_j, and subtract the second product from the first and integrate over the length of the beam to obtain

$$\begin{aligned} &\int_0^l \left(U_k U_j'' + \frac{\omega_j^2}{c^2} U_k U_j \right) \mathrm{d}x \\ &\quad - \int_0^l \left(U_j U_k'' + \frac{\omega_k^2}{c^2} U_j U_k \right) \mathrm{d}x = 0 \\ \Rightarrow &\int_0^l \left(U_k U_j'' - U_j U_k'' + \frac{\omega_j^2 - \omega_k^2}{c^2} U_j U_k \right) \mathrm{d}x = 0. \end{aligned} \tag{1.124}$$

Integrating by parts the first term in (1.124) twice, and using the boundary and matching conditions from (1.118)–(1.119) yields on simplification

$$\begin{aligned} &(\omega_j^2 - \omega_k^2) \left[\frac{M}{EA} \left(\frac{K U_j(l)}{K - M\omega_j^2} \right) \left(\frac{K U_k(l)}{K - M\omega_k^2} \right) + \frac{1}{c^2} \int_0^l U_j U_k \, \mathrm{d}x \right] = 0 \\ \Rightarrow\ & M Y_j Y_k + \rho A \int_0^l U_j U_k \, \mathrm{d}x = 0, \qquad \text{for } j \neq k, \end{aligned}$$

where we have used (1.122) and (1.123). These are the orthogonality relations for the system.

1.5 THE INITIAL VALUE PROBLEM: SOLUTION USING LAPLACE TRANSFORM

The Laplace transform method is one of the standard methods of solving initial value problems. Consider the wave equation

$$w_{,tt} - c^2 w_{,xx} = 0, \tag{1.125}$$

with homogeneous boundary conditions $w(0, t) \equiv 0$ and $w(l, t) \equiv 0$, and initial conditions $w(x, 0) = w_0(x)$ and $w_{,t}(x, 0) = v_0(x)$. Taking the Laplace transform (see [2]) of both sides of (1.125) and the boundary conditions with respect to the variable t yields

$$\tilde{w}'' - \frac{s^2}{c^2}\tilde{w} = -\frac{1}{c^2}\left[s w_0(x) + v_0(x)\right], \tag{1.126}$$

$$\tilde{w}(0, s) \equiv 0 \qquad \text{and} \qquad \tilde{w}(l, s) \equiv 0, \tag{1.127}$$

where $\tilde{w}(x, s)$ represents the Laplace transform of $w(x, t)$, and is defined as

$$\tilde{w}(x, s) = \int_0^\infty w(x, t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{1.128}$$

The homogeneous solution of (1.126) is obtained as

$$\tilde{w}(x, s) = a\mathrm{e}^{sx/c} + b\mathrm{e}^{-sx/c}. \tag{1.129}$$

Using the boundary conditions (1.127) yields

$$\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{sl/c} & \mathrm{e}^{-sl/c} \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = 0.$$

For non-trivial solutions of (a, b), we must have

$$\mathrm{e}^{2sl/c} - 1 = 0 \qquad \Rightarrow \qquad s = \frac{in\pi c}{l}, \qquad n = 1, 2, \ldots, \infty.$$

For these values of s, one can easily obtain $(a, b) = (1, -1)$, and therefore, the general solution of (1.126)–(1.127) can be written using (1.129) as

$$\tilde{w} = \sum_{n=1}^{\infty} A_n(s) \sin\frac{n\pi x}{l}, \tag{1.130}$$

where $A_n(s)$ are arbitrary constants. Using this solution expansion in (1.126), and taking inner product with $\sin m\pi x/l$ yields on simplification

$$A_m(s) = \frac{s}{s^2+\alpha_m^2}\int_0^l w_0(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x + \frac{1}{s^2+\alpha_m^2}\int_0^l v_0(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x,$$

where $\alpha_m = m\pi c^2/l$. Substituting this expression in (1.130) and taking the inverse Laplace transform yields

$$w(x,t) = \sum_{n=1}^{\infty}(C_n\cos\alpha_n t + S_n\sin\alpha_n t)\sin\frac{n\pi x}{l}, \tag{1.131}$$

where

$$C_n = \int_0^l w_0(x)\sin\frac{n\pi cx}{l}\,\mathrm{d}x \qquad \text{and} \qquad S_n = \frac{1}{\alpha_n}\int_0^l v_0(x)\sin\frac{n\pi cx}{l}\,\mathrm{d}x.$$

The solution (1.131) is the same as obtained in (1.63) before.

1.6 FORCED VIBRATION ANALYSIS

The dynamics of one-dimensional continuous systems discussed above, subjected to an arbitrary distributed forcing $q(x,t)$, can be represented in a general form

$$\mu(x)w_{,tt} + \mathcal{K}[w] = q(x,t). \tag{1.132}$$

When a system is forced at a boundary, one can still convert the problem to the form (1.132), leaving the corresponding boundary condition homogeneous. For example, consider the bar shown in Figure 1.16, forced axially at the boundary. The equation of motion and boundary conditions are given by

$$\begin{aligned} &\rho A u_{,tt} - [EAu_{,x}]_{,x} = 0,\\ &-EAu_{,x}(0,t) = F(t), \qquad \text{and} \qquad u(l,t) \equiv 0. \end{aligned} \tag{1.133}$$

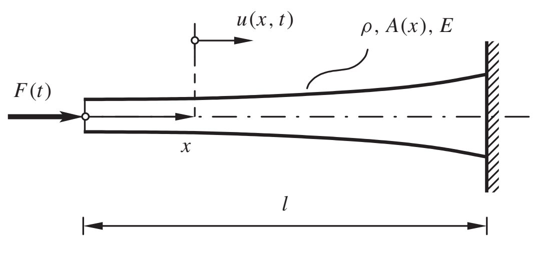

Figure 1.16 A bar forced axially at the boundary

It may be noted that the standard convention of taking compressive stress as negative has been used in (1.133). The dynamics of the bar can also be recast in the form (1.132) as

$$\rho A u_{,tt} - [EA u_{,x}]_{,x} = F(t)\delta(x),$$
$$EA u_{,x}(0,t) = 0, \qquad \text{and} \qquad u(l,t) \equiv 0.$$

Let the eigenvalue problem corresponding to the unforced dynamics in (1.132) be given by

$$-\omega^2 \mu(x) W(x) + \mathcal{K}[W(x)] = 0, \tag{1.134}$$

along with appropriate boundary conditions. We will represent the solutions of (1.134) as $(\omega_k, W_k(x))$, $k = 1, 2, \ldots, \infty$, where ω_k are the eigenvalues and $W_k(x)$ are the corresponding eigenfunctions. The pair (ω_k, W_k) is sometimes also called an *eigenpair*. In the following, we will discuss the solution of (1.132) for different forcing conditions.

1.6.1 Harmonic forcing

Consider a forcing $q(x,t)$ in (1.132) that is separable in space and time, and has a harmonic time function. In particular, consider $q(x,t) = Q(x)\cos\Omega t$, and let us represent the forced dynamics as

$$\mu(x) w_{,tt} + \mathcal{K}[w] = \mathscr{R}[Q(x)\mathrm{e}^{i\Omega t}], \tag{1.135}$$

where Ω is the circular forcing frequency, $Q(x)$ specifies the force distribution, and $\mathscr{R}[\cdot]$ represents the real part. Let us consider a solution of (1.135) in the form

$$\begin{aligned} w(x,t) &= w_{\mathrm{H}}(x,t) + w_{\mathrm{P}}(x,t) \\ &= \sum_{k=1}^{\infty} [C_k \cos\omega_k t + S_k \sin\omega_k t] W_k(x) + \mathscr{R}[X(x)\mathrm{e}^{i\Omega t}], \end{aligned} \tag{1.136}$$

where the first term $w_{\mathrm{H}}(x,t)$ represents the general solution of the homogeneous problem, also simply called the *homogeneous solution*, and the second term $w_{\mathrm{P}}(x,t)$ is a solution of the inhomogeneous problem, also simply called the *particular solution*. The amplitude function $X(x)$ in (1.136) is an unknown (real or complex), yet to be determined. It may be noted that we have assumed, based on the discussion in the previous sections, that the homogeneous solution is completely known except for the constants C_k and S_k, which will be determined from the initial conditions. Now, substituting the solution (1.136) in (1.135) yields on simplification

$$-\Omega^2 \mu(x) X(x) + \mathcal{K}[X(x)] = Q(x). \tag{1.137}$$

This equation along with the boundary conditions forms a *boundary value problem*. In the following, we discuss two methods of solving (1.137), namely the eigenfunction expansion method and Green's function method.

1.6.1.1 Eigenfunction expansion method

Assume the solution of (1.137) as the eigenfunction expansion

$$X(x) = \sum_{k=1}^{\infty} \alpha_k W_k(x), \tag{1.138}$$

where α_k are unknown coefficients. Substituting (1.138) in (1.137) yields

$$\begin{aligned}
&-\Omega^2 \mu(x) \sum_{k=1}^{\infty} \alpha_k W_k(x) + \mathcal{K}\left[\sum_{k=1}^{\infty} \alpha_k W_k(x)\right] = Q(x) \\
\Rightarrow &-\Omega^2 \mu(x) \sum_{k=1}^{\infty} \alpha_k W_k(x) + \sum_{k=1}^{\infty} \alpha_k \mathcal{K}[W_k(x)] = Q(x) \\
\Rightarrow &\sum_{k=1}^{\infty} (\omega_k^2 - \Omega^2)\alpha_k \mu(x) W_k(x) = Q(x), \qquad \text{(using (1.134)).}
\end{aligned} \tag{1.139}$$

Taking the inner product on both sides of (1.139) with $W_j(x)$, $j = 1, 2, \ldots, \infty$, and using the orthogonality property, we get

$$\begin{aligned}
&(\omega_j^2 - \Omega^2)\alpha_j \langle \mu(x) W_j(x), W_j(x)\rangle = \langle Q(x), W_j(x)\rangle, \qquad j = 1, 2, \ldots, \infty \\
\Rightarrow \alpha_j &= \frac{\int_0^l Q(x) W_j(x)\,\mathrm{d}x}{(\omega_j^2 - \Omega^2)\int_0^l \mu(x) W_j^2(x)\,\mathrm{d}x}, \qquad j = 1, 2, \ldots, \infty,
\end{aligned} \tag{1.140}$$

where it has been assumed that the forcing is non-resonant, i.e., $\Omega \neq \omega_j$ for all j. This completes the solution (1.138) of (1.137) for a non-resonant harmonic forcing.

In case $\Omega = \omega_j$ for some j, we have resonance, which is characterized by a very high response amplitude for the jth mode (infinite as far as the linear theory is concerned). To determine the response of the system at resonance, we use the method of variation of parameters in which the particular solution is assumed in the form

$$w_{\mathrm{P}}(x,t) = \mathscr{R}\left[\left(\alpha_j(t) W_j(x) + \sum_{\substack{k=1 \\ k\neq j}}^{\infty} \alpha_k W_k(x)\right) \mathrm{e}^{i\omega_j t}\right]. \tag{1.141}$$

It may be noted that the jth modal coordinate $\alpha_j(t)$ has been taken as a function of time. Substituting this solution form in (1.135) and proceeding as discussed above, one can easily obtain the equation of modal dynamics of the jth mode as

$$\ddot{\alpha}_j + 2i\omega_j \dot{\alpha}_j = \frac{\int_0^l Q(x) W_j(x)\,\mathrm{d}x}{\int_0^l \mu(x) W_j^2(x)\,\mathrm{d}x}.$$

Solving this and substituting in (1.141), the particular solution is finally obtained as

$$w_{\mathrm{P}}(x,t) = \frac{t}{2\omega_j}\frac{\int_0^l Q(x)W_j(x)\,\mathrm{d}x}{\int_0^l \mu(x)W_j^2(x)\,\mathrm{d}x}W_j(x)\sin\omega_j t + \sum_{\substack{k=1\\k\neq j}}^{\infty}\alpha_k W_k(x)\cos\omega_j t,$$

where the constants α_k are obtained from (1.140).

An interesting situation occurs for a resonant forcing with $\Omega = \omega_j$ if $Q(x)$ is such that

$$\int_0^l Q(x)W_j(x)\,\mathrm{d}x = 0,$$

i.e., the forcing amplitude distribution $Q(x)$ is orthogonal to $W_j(x)$. In such a case, the solution is still finite since the jth mode cannot be excited by the force. This situation is referred to as *apparent resonance*.

1.6.1.2 Green's function method

Boundary value problems may be conveniently solved using Green's function method (see [3]). In this method, the solution of (1.137) is obtained in an integral form as discussed in the following.

Let $G(x,\overline{x},\Omega)$ be the solution of (1.137) excited by a concentrated unit force at $x = \overline{x} \in [0,l]$, i.e.,

$$-\Omega^2\mu(x)G(x,\overline{x},\Omega) + \mathcal{K}[G(x,\overline{x},\Omega)] = \delta(x-\overline{x}), \tag{1.142}$$

with all the boundary conditions of (1.137), which are assumed homogeneous. In case the boundary conditions of (1.137) are not homogeneous, one can make them homogeneous by following the procedure discussed in Section 1.9.

Consider the function

$$X(x) = \int_0^l Q(\overline{x})G(x,\overline{x},\Omega)\,\mathrm{d}\overline{x}. \tag{1.143}$$

Substituting (1.143) in the left-hand side of (1.137), one obtains

$$\begin{aligned}
&-\Omega^2\mu(x)\int_0^l Q(\overline{x})G(x,\overline{x},\Omega)\,\mathrm{d}\overline{x} + \mathcal{K}\left[\int_0^l Q(\overline{x})G(x,\overline{x},\Omega)\,\mathrm{d}\overline{x}\right]\\
&\quad= -\Omega^2\mu(x)\int_0^l Q(\overline{x})G(x,\overline{x},\Omega)\,\mathrm{d}\overline{x} + \int_0^l Q(\overline{x})\mathcal{K}[G(x,\overline{x},\Omega)]\,\mathrm{d}\overline{x}\\
&\quad= \int_0^l Q(\overline{x})\big(-\Omega^2\mu(x)G(x,\overline{x},\Omega) + \mathcal{K}[G(x,\overline{x},\Omega)]\big)\,\mathrm{d}\overline{x}\\
&\quad= \int_0^l Q(\overline{x})\delta(x-\overline{x})\,\mathrm{d}\overline{x} \qquad \text{(using (1.142))}\\
&\quad= Q(x).
\end{aligned}$$

Thus, (1.143) is indeed the solution of (1.137) for a general $Q(x)$. It is to be noted that, if $\Omega \neq \omega_k$ for all k, the solution of (1.142) (and also (1.137)) is unique since the homogeneous problem (i.e., with zero right-hand side) has only the trivial solution. The function $G(x, \overline{x}, \Omega)$ is known as *Green's function* for the boundary value problem (1.137). The complete solution of (1.132) can, therefore, be written as

$$w(x,t) = \sum_{k=1}^{\infty} [C_k \cos\omega_k t + S_k \sin\omega_k t]\, W_k(x) + \mathscr{R}\left[\mathrm{e}^{i\Omega t}\int_0^l Q(\overline{x})G(x,\overline{x},\Omega)\,\mathrm{d}\overline{x}\right]. \tag{1.144}$$

We now compute Green's function for the forced vibration of a taut string for which $\mu(x) = 1$, and $\mathcal{K}[w] = -c^2 w_{,xx}$. In (1.142), we can consider two regions of the string as follows:

$$-\Omega^2 G - c^2 G_{,xx} = 0, \qquad 0 \leq x < \overline{x}, \tag{1.145}$$

and

$$-\Omega^2 G - c^2 G_{,xx} = 0, \qquad \overline{x} < x \leq l, \tag{1.146}$$

with appropriate matching conditions at $x = \overline{x}$, as discussed below. The solutions of (1.145) and (1.146) can be written as

$$G(x,\overline{x},\Omega) = \begin{cases} A_{\mathrm{L}} \sin\dfrac{\Omega x}{c} + B_{\mathrm{L}} \cos\dfrac{\Omega x}{c}, & 0 \leq x < \overline{x}, \quad (1.147) \\[2ex] A_{\mathrm{R}} \sin\dfrac{\Omega x}{c} + B_{\mathrm{R}} \cos\dfrac{\Omega x}{c}, & \overline{x} < x \leq l, \quad (1.148) \end{cases}$$

where A_{L}, B_{L}, A_{R}, and B_{R} are arbitrary constants. From the requirement of continuity of the solution at $x = \overline{x}$, and satisfaction of all the boundary conditions of the problem, we have

$$G(\overline{x}^-, \overline{x}, \Omega) = G(\overline{x}^+, \overline{x}, \Omega), \tag{1.149}$$

$$G(0, \overline{x}, \Omega) = 0, \qquad \text{and} \qquad G(l, \overline{x}, \Omega) = 0. \tag{1.150}$$

Further, integrating (1.142) over the domain of the string yields

$$\begin{aligned} & -\int_0^l [\Omega^2 G + c^2 G_{,xx}]\,\mathrm{d}x = 1 \\ & \Rightarrow \lim_{\epsilon\to 0}\int_{\overline{x}-\epsilon}^{\overline{x}+\epsilon} [\Omega^2 G + c^2 G_{,xx}]\,\mathrm{d}x = -1 \qquad \text{(using (1.145)–(1.146))} \\ & \Rightarrow \lim_{\epsilon\to 0}\left[\Omega^2 G(x+\theta\epsilon, \overline{x}, \Omega)\, 2\epsilon + c^2 G_{,x}\Big|_{\overline{x}-\epsilon}^{\overline{x}+\epsilon}\right] = -1 \qquad (1.151) \\ & \Rightarrow c^2 G_{,x}(\overline{x}^+, \overline{x}, \Omega) - c^2 G_{,x}(\overline{x}^-, \overline{x}, \Omega) = -1, \qquad (1.152) \end{aligned}$$

where, the first term in (1.151) is obtained from the mean value theorem. It can be easily observed that (1.152) provides the force equilibrium condition. Now, (1.149), (1.150), and (1.152) provide four conditions for finding the four constants in (1.148) and (1.148). Finally, after some algebra, Green's function for a taut string is obtained as

$$G(x,\overline{x},\Omega)=\begin{cases}\dfrac{\sin\frac{\Omega}{c}(l-\overline{x})\sin\frac{\Omega x}{c}}{\Omega c\sin\frac{\Omega l}{c}}, & 0\le x<\overline{x}, \quad (1.153)\\[2ex] \dfrac{\sin\frac{\Omega}{c}(l-x)\sin\frac{\Omega\overline{x}}{c}}{\Omega c\sin\frac{\Omega l}{c}}, & \overline{x}<x\le l. \quad (1.154)\end{cases}$$

Green's function can also be obtained from (1.142) using the eigenfunction expansion

$$G(x,\overline{x},\Omega)=\sum_{k=1}^{\infty}\alpha_k(\overline{x},\Omega)W_k(x),$$

where $\alpha_k(\overline{x},\Omega)$ are unknown functions. Following the procedure discussed in Section 1.6.1.1, it can be easily checked that

$$G(x,\overline{x},\Omega)=\sum_{k=1}^{\infty}\frac{W_k(\overline{x})W_k(x)}{(\omega_k^2-\Omega^2)\int_0^l\mu(x)W_k^2(x)\,\mathrm{d}x}. \tag{1.155}$$

A collection of Green's functions for various kinds of differential equations can be found in [4].

1.6.2 General forcing

For a general distributed forcing $q(x,t)$ which no longer needs to be separable, let us assume a solution of (1.132) in the form

$$w(x,t)=\sum_{k=1}^{\infty}p_k(t)W_k(x), \tag{1.156}$$

where $W_k(x)$ are the known eigenfunctions and $p_k(t)$ are the unknown modal coordinates. Substituting this solution form in (1.132) yields

$$\mu(x)\sum_{k=1}^{k=\infty}\ddot{p}_k(t)W_k(x)+\mathcal{K}\left[\sum_{k=1}^{k=\infty}p_k(t)W_k(x)\right]=q(x,t). \tag{1.157}$$

Using the linearity property of $\mathcal{K}[\cdot]$ and (1.134), one can simplify (1.157) as

$$\sum_{k=1}^{k=\infty}\left[\ddot{p}_k(t)+\omega_k^2p_k(t)\right]\mu(x)W_k(x)=q(x,t). \tag{1.158}$$

Now, taking the inner product with $W_j(x)$, on both sides of (1.158), and using the orthogonality property yields

$$\ddot{p}_j(t) + \omega_j^2 p_j(t) = f_j(t), \qquad j = 1, 2, \ldots, \infty, \tag{1.159}$$

where

$$f_j(t) = \frac{\langle W_j(x), q(x,t)\rangle}{\langle \mu(x) W_j(x), W_j(x,t)\rangle} = \frac{\int_0^l W_j(x) q(x,t)\,\mathrm{d}x}{\int_0^l \mu(x) W_j^2(x)\,\mathrm{d}x}, \qquad j = 1, 2, \ldots, \infty. \tag{1.160}$$

The second-order ordinary differential equation (1.159) with specified initial conditions $p_j(0) = p_{j0}$ and $\dot{p}_j(0) = v_{j0}$ represents an initial value problem. It can be easily solved using, for example, the method of Laplace transforms or the Duhamel convolution integral.

Consider the example of a constant point force F traveling with a speed v on a stretched string, as shown in Figure 1.17. The equation of motion is given by

$$\rho A w_{,tt} - T w_{,xx} = -F\delta(x - vt).$$

We assume the solution form (1.156), where, as we already know, $W_k(x) = \sin k\pi x/l$. Following the steps as discussed above, we obtain the modal dynamics from (1.159)–(1.160) as

$$\ddot{p}_j(t) + \omega_j^2 p_j(t) = -\frac{2F}{\rho A l} \sin\frac{j\pi vt}{l}, \qquad j = 1, 2, \ldots, \infty, \tag{1.161}$$

where $\omega_j = j\pi c/l$. When the forcing in (1.161) is non-resonant, i.e., $v \neq c$, the solution of the above differential equation can be written as

$$p_j(t) = C_j \cos\omega_j t + S_j \sin\omega_j t - \frac{2Fl}{\rho A j^2\pi^2(c^2 - v^2)} \sin\frac{j\pi vt}{l}, \tag{1.162}$$

where C_j and S_j are arbitrary constants of integration. Therefore, the solution for the string is obtained from (1.156) as

$$w(x,t) = \sum_{j=1}^{\infty} \left[C_j \cos\omega_j t + S_j \sin\omega_j t - \frac{2Fl}{\rho A j^2\pi^2(c^2 - v^2)} \sin\frac{j\pi vt}{l} \right] \sin\frac{j\pi x}{l}. \tag{1.163}$$

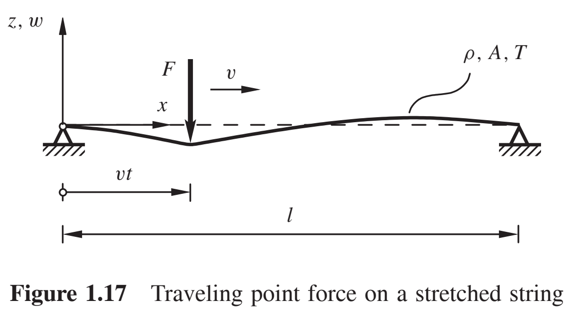

Figure 1.17 Traveling point force on a stretched string

Using the initial conditions of an undeformed string at rest, i.e., $w(x,0)=0$ and $w_{,t}(x,0)=0$, we get

$$C_j = 0 \qquad \text{and} \qquad S_j = \frac{2Flv}{\rho A c j^2 \pi^2 (c^2 - v^2)}, \qquad j = 1, 2, \ldots, \infty. \tag{1.164}$$

Finally, we can write the complete solution of the string as

$$w(x,t) = -\frac{2Fl}{\rho A \pi^2 (c^2 - v^2)} \sum_{j=1}^{\infty} \frac{1}{j^2} \left[\sin \frac{j\pi vt}{l} - \frac{v}{c} \sin \frac{j\pi ct}{l} \right] \sin \frac{j\pi x}{l}. \tag{1.165}$$

The above solution is valid as long as the force moves on the string, i.e., $0 \le t \le l/v$. Once the force leaves the string, the string undergoes free vibrations. The shape of the string and its velocity at the onset of the free vibrations can be obtained from (1.165) at $t_f = l/v$. Thus,

$$\begin{aligned} & w(x, t_f^+) = w(x, t_f^-) \qquad \text{and} \qquad w_{,t}(x, t_f^+) = w_{,t}(x, t_f^-) \\ \Rightarrow\ & p_j(t_f^+) = p_j(t_f^-) \qquad \text{and} \qquad \dot{p}_j(t_f^+) = \dot{p}_j(t_f^-), \qquad j = 1, 2, \ldots, \infty, \end{aligned} \tag{1.166}$$

where, for $t \ge t_f$, $p_j(t) = C'_j \cos \omega_j t + S'_j \sin \omega_j t$ with C'_j and S'_j as arbitrary constants. Using the matching conditions (1.166) at $t = t_f$, we obtain the free vibration solution as

$$\begin{aligned} w(x,t) = -\frac{2Flv}{\rho A c \pi^2 (c^2 - v^2)} \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} & \left[\left(\cos \frac{j\pi c}{v} - (-1)^j \right) \sin \omega_j t \right. \\ & \left. - \sin \frac{j\pi c}{v} \cos \omega_j t \right] \sin \frac{j\pi x}{l}, \end{aligned} \tag{1.167}$$

where $t \ge t_f$. In Figure 1.18, the shape of the string is represented graphically at selected times for a point force traveling at $v = 0.25c$.

Now, we consider the case when the force travels with the resonance speed, i.e., $v = c$ in (1.161). One can obtain the solution of (1.161) as

$$p_j(t) = -\frac{2F}{\rho A l \omega_j^2} \left[\sin \omega_j t - t\omega_j \cos \omega_j t \right]. \tag{1.168}$$

Therefore, the response of the string is obtained from (1.156) as

$$w(x,t) = -\frac{Fl}{T\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \left[\sin \omega_j t - t\omega_j \cos \omega_j t \right] \sin \frac{j\pi x}{l}. \tag{1.169}$$

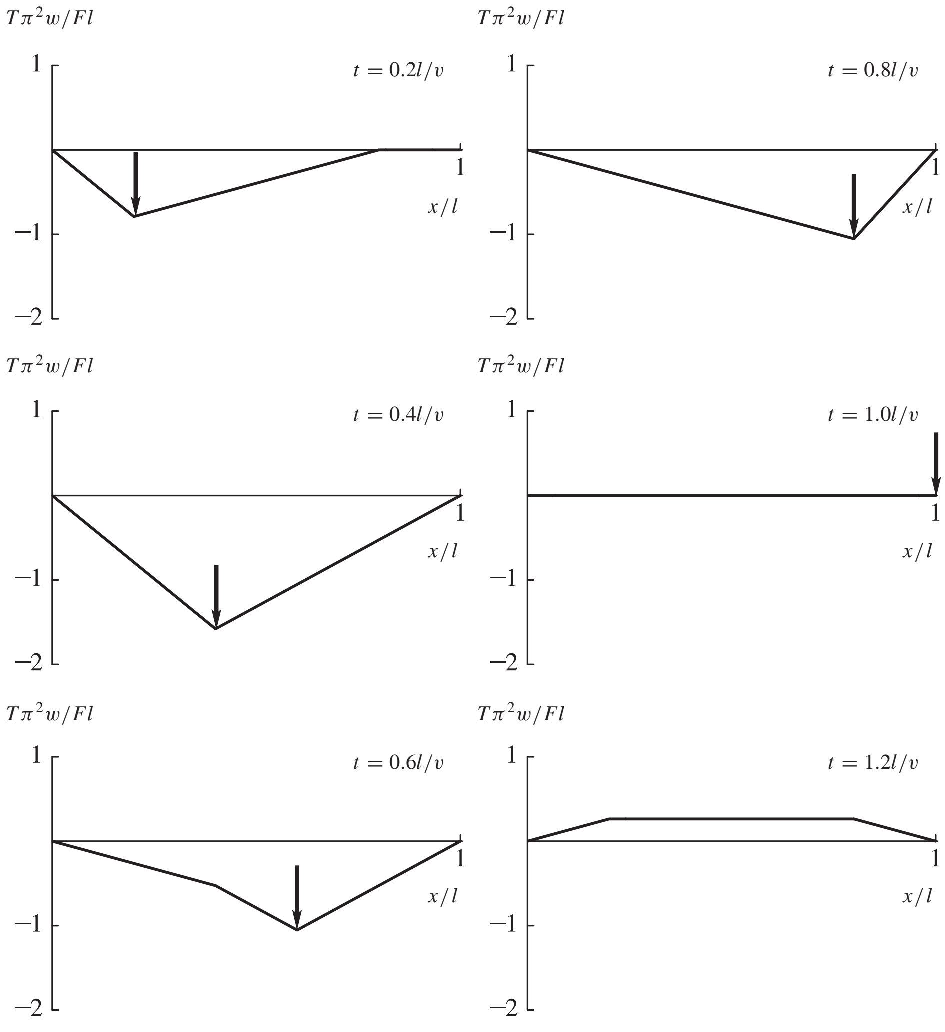

Figure 1.18 Shape of a string at selected times with a traveling point force: $v/c = 0.25$

For $t \geq t_f$, we get, following the procedure discussed above, the free response of the string as

$$w(x,t) = \frac{Fl}{T\pi} \sum_{j=1}^{\infty} \frac{1}{j} \cos \omega_j t \sin \frac{j\pi x}{l}. \tag{1.170}$$

The shape of the string at selected times with the force traveling at resonance speed is shown graphically in Figure 1.19.

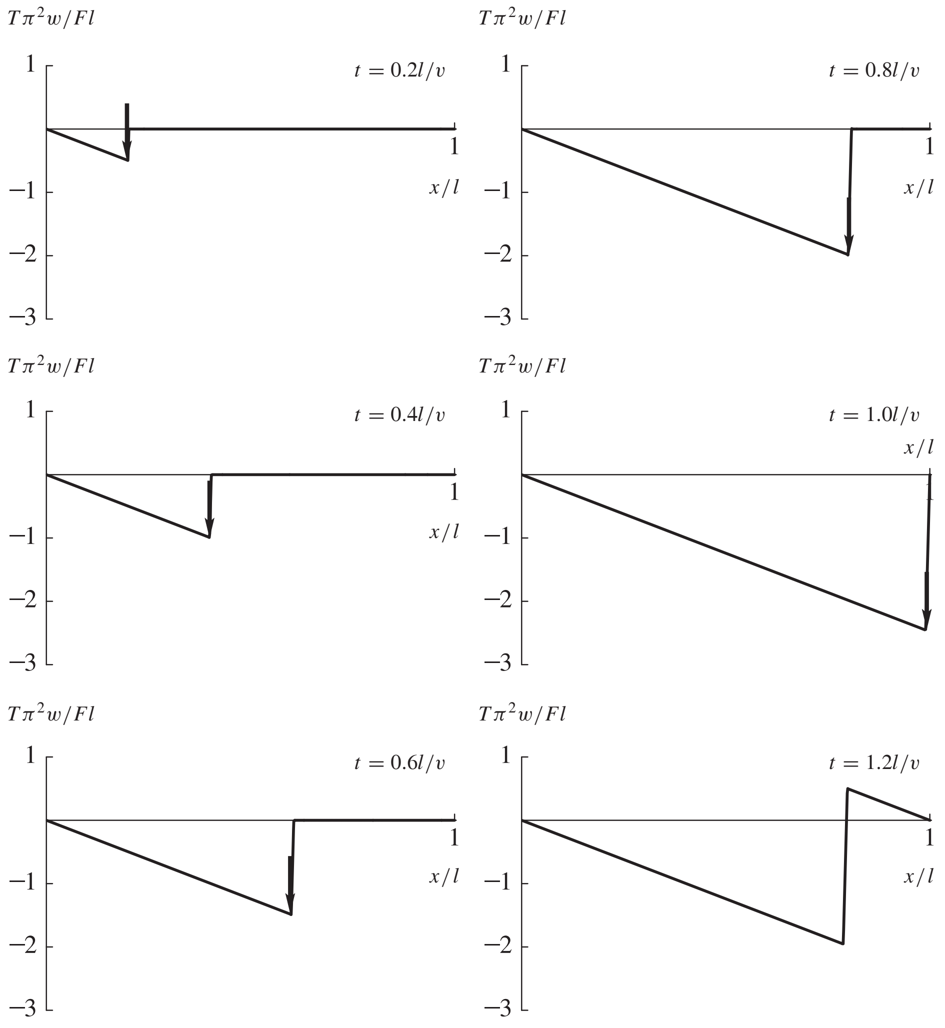

Figure 1.19 Shape of a string at selected times with a point force traveling at resonance speed

1.7 APPROXIMATE METHODS FOR CONTINUOUS SYSTEMS

In many problems of analysis of continuous systems, exact solutions are either not possible to obtain or become too cumbersome to use when a good estimate is all that is required. In certain cases, a given system may be close to an exactly solvable system, or a system with known solutions. In such situations, approximate methods provide sufficiently accurate results to serve the purpose. In the following, we discuss three approximate methods for analyzing continuous systems. Other approximate methods can be found in [5], [7], and [6] (see also Chapter 6).

1.7.1 Rayleigh method

The determination of the natural frequencies of vibration is of foremost importance in the analysis of vibratory systems. Rayleigh's method can be used to calculate or estimate the lowest (or fundamental) frequency of a self-adjoint (conservative) continuous system.

Consider the example of a bar of length l having a uniform cross-section of area A undergoing longitudinal vibrations. The total mechanical energy of the system comprising the kinetic and potential energies is given by

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \frac{1}{2}\int_0^l \rho A u_{,t}^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^l EA u_{,x}^2(x,t)\,\mathrm{d}x, \tag{1.171}$$

where ρ is the density of the material and E is Young's modulus. Since there is no dissipation, the total energy of the system is a constant. Assuming that the system is vibrating in one of its eigenmodes, we can write the solution as

$$u(x,t) = U(x)\cos\omega t, \tag{1.172}$$

where $U(x)$ is an eigenfunction of the system and ω the corresponding natural frequency. Substituting (1.172) in (1.171) yields on simplification

$$\mathcal{E} = \left[\frac{1}{2}\omega^2\int_0^l \rho A U^2\,\mathrm{d}x\right]\sin^2\omega_k t + \left[\frac{1}{2}\int_0^l EA U'^2\,\mathrm{d}x\right]\cos^2\omega_k t. \tag{1.173}$$

Now, $\mathcal{E}$ given by (1.173) is a constant (i.e., independent of time) for a non-trivial solution if and only if the amplitudes of the kinetic and potential energy terms are equal, i.e.,

$$\frac{1}{2}\omega^2\int_0^l \rho A U^2\,\mathrm{d}x = \frac{1}{2}\int_0^l EA U'^2\,\mathrm{d}x$$

$$\Rightarrow \omega^2 = \frac{\int_0^l EA U'^2(x)\,\mathrm{d}x}{\int_0^l \rho A U^2(x)\,\mathrm{d}x} := \mathcal{R}[U(x)]. \tag{1.174}$$

The ratio $\mathcal{R}[U(x)]$ defined in (1.174) is known as the *Rayleigh quotient*. If the eigenfunction $U_k(x)$ is known exactly, one can obtain the exact circular eigenfrequency from (1.174). However, if the eigenfunction is unknown, one can still use (1.174) to determine the fundamental circular frequency through the minimization problem

$$\omega_1^2 = \min_{\tilde{U}(x)\in\mathcal{U}} \mathcal{R}[\tilde{U}(x)] = \min_{\tilde{U}(x)\in\mathcal{U}} \frac{\int_0^l EA\tilde{U}'^2(x)\,\mathrm{d}x}{\int_0^l \rho A\tilde{U}^2(x)\,\mathrm{d}x}, \tag{1.175}$$

where the minimization is performed over the set $\mathcal{U}$ of all functions $\tilde{U}(x)$ that satisfy all the geometric boundary conditions of the problem, and are differentiable at least up to the highest order of space-derivative present in the energy integral. Such functions are known as

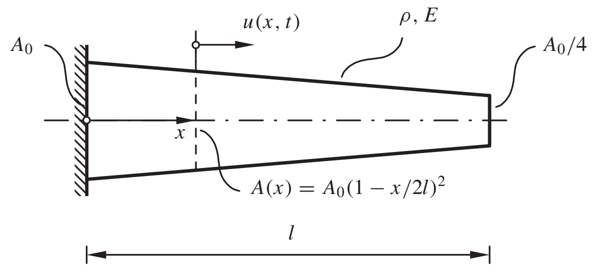

Figure 1.20 A tapered circular bar

admissible functions, and can be easily constructed using polynomials, trigonometric functions, or other elementary functions. The idea of using (1.174) (or (1.175)) for determining the fundamental frequency is due to Rayleigh, and is known as *Rayleigh's method*. The validity of (1.175) will be proved later in Section 1.7.2 in a more general context.

Assume the approximated fundamental mode-shape of longitudinal vibration of a fixed–free bar, as shown in Figure 1.20, in the form

$$F(x) = \left(\frac{x}{l}\right)^{\alpha}, \tag{1.176}$$

where α is a constant, and will be determined later. Computation of the numerator of the Rayleigh quotient in (1.174) yields

$$\begin{aligned}\int_0^l EAF'^2(x)\,\mathrm{d}x &= EA_0 \int_0^l \left(1 - \frac{x}{2l}\right)^2 \left(\frac{\alpha}{l}\right)^2 \left(\frac{x}{l}\right)^{2\alpha-2} \mathrm{d}x \\ &= \frac{EA_0}{l}\alpha^2 4 \frac{(2\alpha-1)2\alpha + 4(2\alpha+1)}{(2\alpha-1)2\alpha(2\alpha+1)},\end{aligned} \tag{1.177}$$

where it is required that $\alpha > 1/2$ for the definite integral in the above to exist. Further, if $\alpha < 1$, then as $x \to 0$, $F'(x) \to \infty$. The denominator of the Rayleigh quotient yields

$$\begin{aligned}\int_0^l \rho AF^2(x)\,\mathrm{d}x &= \rho A_0 \int_0^l \left(1 - \frac{x}{2l}\right)^2 \left(\frac{x}{l}\right)^{2\alpha} \mathrm{d}x \\ &= \rho A_0 l \frac{(2\alpha+1)(2\alpha+2) + 4(2\alpha+3)}{4(2\alpha+1)(2\alpha+2)(2\alpha+3)}.\end{aligned} \tag{1.178}$$

Therefore, the Rayleigh quotient is obtained as

$$\mathcal{R}[F(x),\alpha] = \frac{E}{\rho l} \frac{\alpha(2\alpha^2 + 3\alpha + 2)(2\alpha^2 + 5\alpha + 3)}{(2\alpha^2 + 7\alpha + 7)(2\alpha - 1)}. \tag{1.179}$$

We can now minimize $\mathcal{R}[F(x),\alpha]$ with respect to α, which yields $\alpha \approx 0.93$, and the fundamental circular frequency as $\omega_1 \approx 2.08303c/l$. The exact solution was obtained in

Section 1.4.1.2 as $\omega_1^{\text{exact}} = 2.029c/l$. It may be noted that the obtained shape-function $F(x)$ after minimization cannot be used to determine the strain at the fixed end since $F'(0) = \infty$. However, it gives the frequency estimate within 3% of the exact value.

1.7.2 Rayleigh–Ritz method

Though Rayleigh's method can be used in principle to determine all the natural frequencies of a vibratory system (see Section 6.1.3), it is most convenient for determining the fundamental frequency. In the following, we discuss an extension of Rayleigh's idea using an expansion technique due to Ritz. This method, usually known as the Rayleigh–Ritz method, can be used to determine the natural frequencies of a continuous system.

We consider the expansion of the mode-shape in terms of N linearly independent admissible functions $U_i(x)$, $i = 1, 2, \ldots, N$ in the form

$$U(x) = \sum_{i=1}^{N} \alpha_i U_i(x), \tag{1.180}$$

where α_i are unknown constants which are to be chosen suitably to minimize the Rayleigh quotient. Substituting (1.180) in the Rayleigh quotient (1.174) leads to

$$\omega^2 = \frac{\sum_{i,j=1}^{N} \alpha_i \alpha_j m_{ij}}{\sum_{i,j=1}^{N} \alpha_i \alpha_j k_{ij}} = \frac{\boldsymbol{\alpha}^{\mathrm{T}} \mathbf{M} \boldsymbol{\alpha}}{\boldsymbol{\alpha}^{\mathrm{T}} \mathbf{K} \boldsymbol{\alpha}}, \tag{1.181}$$

where

$$m_{ij} = \int_0^l EAU_i' U_j' \,\mathrm{d}x \qquad \text{and} \qquad k_{ij} = \int_0^l \rho A U_i U_j \,\mathrm{d}x, \tag{1.182}$$

$\boldsymbol{\alpha}$ is the vector of the α_i, and T in the superscript indicates transposition. Now the minimization condition of the Rayleigh quotient can be written as

$$\begin{aligned}
&\frac{\partial}{\partial \alpha_p}\left(\frac{\boldsymbol{\alpha}^{\mathrm{T}} \mathbf{M} \boldsymbol{\alpha}}{\boldsymbol{\alpha}^{\mathrm{T}} \mathbf{K} \boldsymbol{\alpha}}\right) = 0, \qquad p = 1, 2, \ldots, N, \\
&\Rightarrow \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{M} \boldsymbol{\alpha} \left(\frac{\partial \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{K} \boldsymbol{\alpha}}{\partial \boldsymbol{\alpha}}\right) - \boldsymbol{\alpha}^{T} \mathbf{K} \boldsymbol{\alpha} \left(\frac{\partial \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{M} \boldsymbol{\alpha}}{\partial \boldsymbol{\alpha}}\right) = \mathbf{0}, \\
&\Rightarrow 2\mathbf{K}\boldsymbol{\alpha} - 2\left(\frac{\boldsymbol{\alpha}^{\mathrm{T}} \mathbf{K} \boldsymbol{\alpha}}{\boldsymbol{\alpha}^{\mathrm{T}} \mathbf{M} \boldsymbol{\alpha}}\right) \mathbf{M}\boldsymbol{\alpha} = \mathbf{0}, \\
&\Rightarrow (\mathbf{K} - \omega^2 \mathbf{M})\boldsymbol{\alpha} = \mathbf{0} \qquad \text{(using (1.181))}.
\end{aligned} \tag{1.183}$$

Thus, it is observed that the extremization condition for Rayleigh's quotient (formed using a finite expansion) for a continuous system leads to the eigenvalue problem of a finite-dimensional system. It will be shown in Section 1.7.3 that this finite-dimensional system is actually the discretized version of the original continuous system. The eigenvalue problem (1.183) can now be solved to determine the first N approximate eigenvalues and eigenfunctions (from (1.180)) of the system. It must be mentioned, however, that the error is not uniform over all the eigenvalues. Convergence of the desired eigenvalues needs to be checked by increasing the number of terms in the expansion (1.180).

1.7.3 Ritz method

In this method, Ritz's idea of solution expansion is used in the variational formulation of system dynamics to obtain the discretized equations of motion of a continuous system.

Let us consider the example of a bar of varying cross-section in longitudinal vibration. The variational formulation of the problem can be stated in the form of Hamilton's principle as

$$\delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^l \left(\rho A u_{,t}^2 - EA u_{,x}^2\right) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{1.184}$$

Consider an approximate solution of this system in the form

$$u(x,t) = \sum_{k=1}^{N} p_k(t) H_k(x) = \mathbf{H}^{\mathrm{T}} \mathbf{p}, \tag{1.185}$$

where $\mathbf{H} = [H_1(x), \ldots, H_N(x)]^{\mathrm{T}}$ is a vector of N linearly independent admissible functions and $\mathbf{p} = [p_1(t), \ldots, p_N(t)]^{\mathrm{T}}$ is a vector of the corresponding unknown coordinate functions. Substituting (1.185) in (1.184), we get

$$\delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^l \left[\rho A \dot{\mathbf{p}}^{\mathrm{T}} \mathbf{H} \mathbf{H}^{\mathrm{T}} \dot{\mathbf{p}} - EA \mathbf{p}^{\mathrm{T}} \mathbf{H}' \mathbf{H}'^{\mathrm{T}} \mathbf{p}\right] \, \mathrm{d}x \, \mathrm{d}t = 0,$$

$$\Rightarrow \delta \int_{t_1}^{t_2} \frac{1}{2} \left[\dot{\mathbf{p}}^{\mathrm{T}} \mathbf{M} \dot{\mathbf{p}} - \mathbf{p}^{\mathrm{T}} \mathbf{K} \mathbf{p}\right] \, \mathrm{d}t = 0, \tag{1.186}$$

where

$$\mathbf{M} = \int_0^l \rho A \mathbf{H} \mathbf{H}^{\mathrm{T}} \, \mathrm{d}x, \qquad \text{and} \qquad \mathbf{K} = \int_0^l EA \mathbf{H}' \mathbf{H}'^{\mathrm{T}} \, \mathrm{d}x. \tag{1.187}$$

One can easily observe that (1.186) represents the variational formulation of dynamics of a discrete system with $\mathbf{p}$ as the vector of generalized coordinates. Therefore, one can directly use Lagrange's equations to obtain the discrete equations of motion

$$\mathbf{M} \ddot{\mathbf{p}} + \mathbf{K} \mathbf{p} = \mathbf{0}. \tag{1.188}$$

It may be observed from (1.187) that both **M** and **K** are symmetric. They are also positive definite since ρA and EA are positive functions, and the admissible functions chosen in the expansion (1.185) are linearly independent.

Consider the example of longitudinal vibration of the fixed–free tapered bar shown in Figure 1.20. We can choose the admissible functions as

$$H_j(x) = \frac{x}{l}\left(1 - \frac{x}{2l}\right)^{j-1}, \qquad j = 1, 2, \ldots, N, \tag{1.189}$$

since the geometric boundary conditions, $H_j(0) = 0$, are exactly satisfied. However, as can be checked, the natural boundary conditions, $H_j'(l) = 0$, are not satisfied. Considering only two admissible functions in the expansion (1.185), and following the steps discussed above, (1.188) takes the form

$$\rho A_0 l \begin{bmatrix} \dfrac{2}{15} & \dfrac{7}{80} \\[2ex] \dfrac{7}{80} & \dfrac{33}{560} \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \frac{EA_0}{l} \begin{bmatrix} \dfrac{7}{12} & \dfrac{17}{48} \\[2ex] \dfrac{17}{48} & \dfrac{31}{120} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = 0. \tag{1.190}$$

Assuming a modal solution $\mathbf{p}(t) = \mathbf{k}e^{i\omega t}$, (1.190) yields the eigenvalue problem

$$[-\omega^2\mathbf{M} + \mathbf{K}]\mathbf{k} = 0, \tag{1.191}$$

from which the characteristic equation is obtained as

$$\frac{81}{7}\omega^4 - 394\frac{c^2}{l^2}\omega^2 + 1455\frac{c^2}{l^2} = 0. \tag{1.192}$$

The first two approximate circular natural frequencies of longitudinal vibration are obtained from (1.192) as $\omega_1^{\mathrm{R}} = 2.053c/l$ and $\omega_2^{\mathrm{R}} = 5.462c/l$. The exact circular eigenfrequencies were obtained in Section 1.4.1.2 as $\omega_1^{\mathrm{exact}} = 2.029c/l$, and $\omega_2^{\mathrm{exact}} = 4.913c/l$. The eigenvectors are obtained from (1.191) as

$$\mathbf{k}_1 = \begin{Bmatrix} 1.0 \\ 1.475 \end{Bmatrix} \qquad \text{and} \qquad \mathbf{k}_2 = \begin{Bmatrix} 1.0 \\ -1.505 \end{Bmatrix}. \tag{1.193}$$

Using these eigenvectors, along with (1.189), in (1.185), we get the approximate eigenfunctions as

$$U_1(x) = \mathbf{H}^{\mathrm{T}}\mathbf{k}_1 = \frac{x}{l} + 1.457\frac{x}{l}\left(1 - \frac{x}{2l}\right) \tag{1.194}$$

and

$$U_2(x) = \mathbf{H}^{\mathrm{T}}\mathbf{k}_2 = \frac{x}{l} - 1.505\frac{x}{l}\left(1 - \frac{x}{2l}\right). \tag{1.195}$$

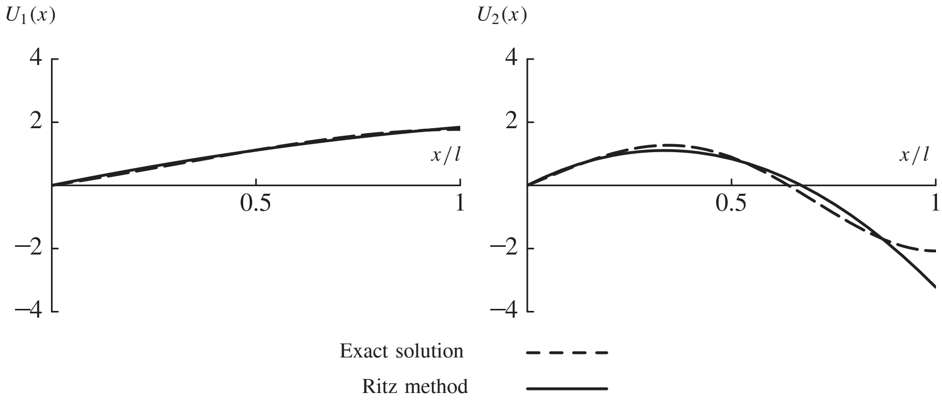

Figure 1.21 Comparison of first two mode-shapes from Ritz method and exact solution (mode-shapes normalized such that $\int_0^l \rho A U_i^2 \,\mathrm{d}x = \int_0^l \rho A \,\mathrm{d}x$)

The exact and approximate eigenfunctions are compared in Figure 1.21. It may be observed from the figure that the first mode-shape is determined reasonably accurately. However, the second mode-shape is in considerable error. For a better approximation of the second mode-shape, one must take more terms in the expansion (1.185). In general, the error in determination of the eigenfrequencies is less than that in the determination of the eigenfunctions. Further, the eigenfrequencies are always overestimated. In other words, we obtain an upper bound on the eigenfrequencies. This is expected since we are approximating an infinite degrees of freedom system by a finite degrees of freedom system, thereby increasing the stiffness of the system. The greatest advantage of the Ritz method is that only admissible functions are required to be constructed for the solution expansion.

When there are external forces, the variational principle (1.184) is modified to (see Appendix A)

$$\int_{t_1}^{t_2} \int_0^l \left[\delta \left(\frac{1}{2} \rho A u_{,t}^2 - \frac{1}{2} E A u_{,x}^2 \right) + q(x,t) \delta u \right] \mathrm{d}x \,\mathrm{d}t = 0, \tag{1.196}$$

where $q(x,t)$ is the generalized force. Substituting the solution form (1.185) in (1.196) and following the procedure presented in this section, we obtain the discretized equation of motion with forcing as

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}(t),$$

where

$$\mathbf{f}(t) = \int_0^l q(x,t) \mathbf{H}(x) \mathrm{d}x.$$

1.7.4 Galerkin method

Consider the dynamics of a continuous system governed by the equation of motion

$$\mu(x)u_{,tt} + \mathcal{K}[u] = 0. \tag{1.197}$$

Let us construct an approximate solution of $u(x,t)$ in the form

$$u(x,t) = \sum_{k=1}^{N} p_k(t)P_k(x) = \mathbf{P}^{\mathrm{T}}\mathbf{p}, \tag{1.198}$$

where $\mathbf{p} = [p_1(t), \ldots, p_N(t)]^{\mathrm{T}}$, $\mathbf{P} = [P_1(x), \ldots, P_N(x)]^{\mathrm{T}}$, and $P_k(x)$ satisfy all the geometric and natural boundary conditions of the problem, and are differentiable at least up to the highest order of space-derivative in the differential equation of motion. Such functions are known as *comparison functions*. It is clear that the approximate solution (1.198) will also satisfy all the boundary conditions, but in general will not satisfy (1.197) identically. Thus, there remains a *residue* defined by

$$e(x,t) := \mu(x)\mathbf{P}^{\mathrm{T}}\ddot{\mathbf{p}} + \mathcal{K}[\mathbf{P}^{\mathrm{T}}]\mathbf{p}, \tag{1.199}$$

where $\mathcal{K}[\mathbf{P}^{\mathrm{T}}] = (\mathcal{K}[P_1(x)], \ldots, \mathcal{K}[P_N(x)])$. Since we are now searching for an approximate solution in a finite N-dimensional space, we can force the residue to have a zero projection on the chosen basis functions $P_j(x)$, $j = 1, 2, \ldots, N$, of this space. Therefore, we put

$$\langle e(x,t), P_j(x)\rangle := \int_0^l e(x,t)P_j(x)\,\mathrm{d}x = 0, \qquad j = 1, 2, \ldots, N. \tag{1.200}$$

Substituting the expression of the residue from (1.199) in (1.200), and writing in a compact form, we have

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{0}, \tag{1.201}$$

where the elements of the matrices $\mathbf{M}$ and $\mathbf{K}$ are obtained as

$$\mathbf{M} = \int_0^l \mu(x)\mathbf{P}\mathbf{P}^{\mathrm{T}}\,\mathrm{d}x \qquad \text{and} \qquad \mathbf{K} = \int_0^l \mathbf{P}\mathcal{K}[\mathbf{P}^{\mathrm{T}}]\,\mathrm{d}x. \tag{1.202}$$

It is evident that $\mathbf{M}$ is a symmetric matrix. If $\mathcal{K}[\cdot]$ is self-adjoint, $\mathbf{K}$ is also a symmetric matrix, as can be easily checked. The equations (1.188) (obtained from the Ritz method) and (1.201) look similar. However, they differ in the computation of the matrix $\mathbf{K}$.

Galerkin's method can also be understood from the variational principle as follows. Consider once again the example of longitudinal vibration of a bar. The variational statement can be simplified and rewritten from (1.37) as

$$-\int_{t_1}^{t_2} EAu_{,x}\delta u\Big|_0^l\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^l [\rho Au_{,tt} - (EAu_{,x})_{,x}]\,\delta u\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{1.203}$$

Let us assume the solution of $u(x, t)$ in the form (1.198). Since $\mathbf{P}$ satisfies all the boundary conditions, the assumed solution makes the boundary terms in (1.203) identically zero. The quantity in square brackets in the second term in (1.203) yields the residue in the form (1.199). Writing the variation of $u(x, t)$ from (1.198) as

$$\delta u(x, t) = \mathbf{P}^{\mathrm{T}}\delta\mathbf{p} = \delta\mathbf{p}^{\mathrm{T}}\mathbf{P}, \tag{1.204}$$

one can rewrite (1.203) in the form

$$\int_{t_1}^{t_2}\int_0^l \delta\mathbf{p}^{\mathrm{T}}\mathbf{P}\mathrm{e}(x, t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

or

$$\int_{t_1}^{t_2} \delta\mathbf{p}^T \left[\int_0^l \mathbf{P}\mathrm{e}(x, t)\,\mathrm{d}x\right]\mathrm{d}t = 0. \tag{1.205}$$

Thus, the term in the square brackets in (1.205) must vanish identically for the above condition to hold for an arbitrary variation $\delta\mathbf{p}$. This again yields the condition (1.200), and the discretized equations (1.201). Since Galerkin's method works directly with the differential equation of motion, it offers certain advantages over the Ritz method. It is evident that one can in principle handle any kind of non-conservative and non-potential forces (non-self-adjoint problems) with Galerkin's method. However, generation of the comparison functions as required in Galerkin's method may be quite tedious for certain problems. Discretization of certain non-self-adjoint problems has been discussed in [6].

We consider the example of longitudinal vibration of the fixed–free tapered bar shown in Figure 1.20 once again. Two comparison functions are chosen as

$$P_k(x) = \left(1 - \frac{x}{l}\right)^{k+1} - 1, \qquad k = 1, 2. \tag{1.206}$$

It can be easily checked that these functions satisfy both the geometric and the natural boundary conditions of the problem. Following the steps discussed in this section, (1.201) takes the form

$$\rho A_0 l \begin{bmatrix} \dfrac{33}{140} & \dfrac{297}{1120} \\[2ex] \dfrac{297}{1120} & \dfrac{1517}{5040} \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \frac{EA_0}{l} \begin{bmatrix} \dfrac{31}{40} & \dfrac{49}{40} \\[2ex] \dfrac{49}{40} & \dfrac{213}{140} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = 0. \tag{1.207}$$

Assuming the solution in the form $\mathbf{p} = \mathbf{k}\mathrm{e}^{i\omega t}$ the eigenvalue problem is solved, and the circular eigenfrequencies are obtained as $\omega_1^{\mathrm{G}} = 2.2029c/l$, and $\omega_2^{\mathrm{G}} = 5.258c/l$. As compared to the eigenfrequencies obtained from the Ritz method, these are closer to the exact eigenfrequencies (see Section 1.4.1.2). This is because, in the case of Galerkin's method, the eigenfunctions satisfy all the boundary conditions of the problem. The eigenvectors are obtained as

$$\mathbf{k}_1 = \begin{Bmatrix} 1.0 \\ -0.472 \end{Bmatrix} \quad \text{and} \quad \mathbf{k}_2 = \begin{Bmatrix} 1.0 \\ -0.898 \end{Bmatrix}, \tag{1.208}$$

and we get the approximate eigenfunctions as

$$U_1(x) = \frac{x}{l}\left(\frac{x}{l} - 2\right) + 0.472\frac{x}{l}\left(\frac{x^2}{l^2} - 3\frac{x}{l} + 3\right) \tag{1.209}$$

and

$$U_2(x) = \frac{x}{l}\left(\frac{x}{l} - 2\right) + 0.898\frac{x}{l}\left(\frac{x^2}{l^2} - 3\frac{x}{l} + 3\right). \tag{1.210}$$

A comparison of the exact and the approximate eigenfunctions obtained in (1.209)–(1.210) is shown in Figure 1.22. On comparing the results in Figure 1.21 and Figure 1.22 it can be observed that Galerkin's method yields better results than the Ritz method. However, it must be remembered that the eigenfunctions satisfy different conditions in the two methods. As mentioned before, the difficulty of Galerkin's method lies in the construction of the comparison functions.

In the presence of an external forcing $q(x, t)$ in (1.197), Galerkin's method yields

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{f}(t),$$

where **M** and **K** are defined by (1.202), and

$$\mathbf{f}(t) = \int_0^l q(x, t)\mathbf{P}(x)\mathrm{d}x.$$

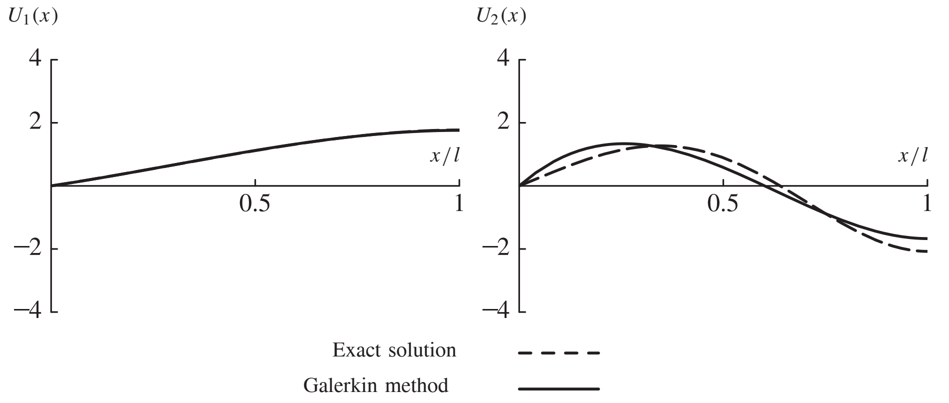

Figure 1.22 Comparison of first two mode-shapes from Galerkin method and exact solution (mode-shapes normalized such that $\int_0^l \rho A U_i^2 \,\mathrm{d}x = \int_0^l \rho A \,\mathrm{d}x$)

1.8 CONTINUOUS SYSTEMS WITH DAMPING

All vibratory systems experience energy dissipation, a phenomenon commonly known as *damping*. Damping forces may arise from external interactions of the system (external damping), or from within the system (internal damping). Damping from aerodynamic drag due to viscosity is the most common example of external damping, while internal damping occurs due to internal friction between the molecular layers as a result of differential straining. In these damping mechanisms, mechanical energy is converted irreversibly into thermal energy which flows out of the system. In Chapter 2, we will consider a damping mechanism in which energy is lost by a system through radiation.

Three damping models, namely viscous damping, Coulomb damping (or dry friction), and structural damping (or hysteretic damping) are usually used for engineering purposes. The viscous damping model, which is the most commonly used model, relates the damping forces with the time rate of change of the field variable, or its spatial derivatives. We will use only this model in our discussions below.

1.8.1 Systems with distributed damping

Consider the longitudinal oscillations of a uniform fixed–free bar. We assume that the internal damping in the material is such that the stresses are a linear function of both the strain and the strain rate. Thus, (1.14) is modified to

$$\sigma_x(x,t) = E\epsilon_x(x,t) + d_\mathrm{I}\epsilon_{x,t}(x,t) = Eu_{,x}(x,t) + d_\mathrm{I}u_{,xt}(x,t), \tag{1.211}$$

where $d_\mathrm{I} > 0$ is the coefficient of internal damping in the material. We also assume a distributed external damping force of the usual form $-d_\mathrm{E}u_{,t}(x,t)$, where $d_\mathrm{E} > 0$ is the coefficient of external damping. Then, proceeding similarly to what was done in Section 1.1.2, one obtains the equation of motion of the longitudinal dynamics of a bar with internal and external damping as

$$\rho A u_{,tt} - EAu_{,xx} - d_\mathrm{I}Au_{,xxt} + d_\mathrm{E}u_{,t} = 0, \tag{1.212}$$

instead of (1.15). The boundary conditions are not affected by these damping terms. One can define a damping operator

$$\mathcal{D}[\cdot] = \left(-d_\mathrm{I}A\frac{\mathrm{d}^2}{\mathrm{d}x^2} + d_\mathrm{E}\right)[\cdot], \tag{1.213}$$

and represent (1.212) in a compact form as

$$\rho A u_{,tt} + \mathcal{D}[u_{,t}] + \mathcal{K}[u] = 0, \tag{1.214}$$

where $\mathcal{K}[\cdot] = -EA[\cdot]_{,xx}$.

Multiplying both sides of (1.212) by $u_{,t}$ and integrating over the domain of the bar yields

$$\begin{aligned}
&\int_0^l (\rho A u_{,t} u_{,tt} - u_{,t} EA u_{,xx} - u_{,t} d_{\mathrm{I}} A u_{,xxt} + d_{\mathrm{E}} u_{,t}^2)\,\mathrm{d}x = 0 \\
\Rightarrow\ & \left[u_{,t} EA u_{,x} + u_{,t} d_{\mathrm{I}} A u_{,xt}\right]_0^l \\
&\quad + \int_0^l \left[\left(\frac{1}{2}\rho A u_{,t}^2\right)_{,t} + u_{,xt} EA u_{,x} + d_{\mathrm{I}} A u_{,xt}^2 + d_{\mathrm{E}} u_{,t}^2\right] \mathrm{d}x = 0.
\end{aligned} \tag{1.215}$$

Using the fixed–free boundary conditions, one can rewrite (1.215) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \left(\frac{1}{2}\rho A u_{,t}^2 + \frac{1}{2} EA u_{,x}^2\right) \mathrm{d}x = -\int_0^l (d_{\mathrm{I}} A u_{,xt}^2 + d_{\mathrm{E}} u_{,t}^2)\,\mathrm{d}x. \tag{1.216}$$

The integral on the left-hand side in (1.216) can be easily recognized to be the total mechanical energy of the bar. Since the right-hand side is always negative, (1.216) implies that the time rate of change of mechanical energy of the bar is always negative, i.e., mechanical energy monotonically decreases with time.

Consider now a system represented by

$$\mu(x) u_{,tt} + \mathcal{D}[u_{,t}] + \mathcal{K}[u] = 0. \tag{1.217}$$

We explore the possibility of a solution of (1.217) in the form

$$u(x,t) = \sum_{k=1}^{\infty} p_k(t) U_k(x), \tag{1.218}$$

where the shape-functions $U_k(x)$ are chosen to be the same as the eigenfunctions for the undamped case, i.e., they are solutions of the self-adjoint eigenvalue problem

$$-\lambda \mu(x) U + \mathcal{K}[U] = 0, \tag{1.219}$$

with appropriate boundary conditions. We will assume that these eigenfunctions are orthonormal with respect to $\mu(x)$, i.e., $\langle \mu(x) U_j, U_k \rangle = \delta_{jk}$. Substituting (1.218) in (1.217) and taking the inner product with $U_j(x)$ yields

$$\ddot{p}_j + \sum_{k=1}^{\infty} d_{jk} \dot{p}_k + \lambda_j p_j = 0, \qquad j = 1, 2, \ldots, \infty \tag{1.220}$$

where

$$d_{jk} = \langle \mu(x)(-d_I A U_{k,xx} + d_E U_k), U_j \rangle. \tag{1.221}$$

It is evident that, in general, the damping matrix $\mathbf{D} = [d_{jk}]$ will not be diagonal. Therefore, all the coordinates p_j of the system are coupled through $\mathbf{D}$.

Consider the special situation when

$$\mathcal{D}[U_k(x)] = d_k \mu(x) U_k(x), \tag{1.222}$$

where d_k are constants. Then, it can be easily checked that the resulting damping matrix $\mathbf{D}$ is diagonal. It can be observed that (1.222) represents an eigenvalue problem for the damping operator similar to (1.219). It then follows that if the operators $\mathcal{D}[\cdot]$ and $\mathcal{K}[\cdot]$ have the same eigenfunctions, the resulting damping matrix $\mathbf{D}$ is diagonal. We can determine the condition for the two operators to have the same eigenfunctions as follows. From (1.222), one can write

$$\begin{aligned}\mathcal{K}[\mu^{-1}(x)\mathcal{D}[U_k(x)]] &= \mathcal{K}[d_k U_k(x)] \\ &= d_k \lambda_k U_k(x) \qquad \text{(using (1.219))}.\end{aligned} \tag{1.223}$$

Similarly, from (1.219), it follows that

$$\begin{aligned}\mathcal{D}[\mu^{-1}(x)\mathcal{K}[U_k(x)]] &= \mathcal{D}[\lambda_k U_k(x)] \\ &= \lambda_k d_k U_k(x) \qquad \text{(using (1.222))}.\end{aligned} \tag{1.224}$$

From (1.223) and (1.224), we can conclude that when $\mathcal{K}[\cdot]$ and $\mathcal{D}[\cdot]$ have the same eigenfunctions they satisfy

$$\begin{aligned}&\mathcal{K}[\mu^{-1}(x)\mathcal{D}[U_k]] - \mathcal{D}[\mu^{-1}(x)\mathcal{K}[U_k]] = 0, \qquad k = 1, 2, \ldots, \infty \\ &\Rightarrow (\mathcal{K}[\mu^{-1}(x)\mathcal{D}] - \mathcal{D}[\mu^{-1}(x)\mathcal{K}])[\cdot] = 0,\end{aligned} \tag{1.225}$$

i.e., the two operators commute with respect to $\mu^{-1}(x)$. The converse of this result can also be easily established. Let the two operators commute, i.e., (1.225) is satisfied. From (1.219), one can easily obtain

$$\begin{aligned}&-\lambda\mathcal{D}[U] + \mathcal{D}[\mu^{-1}(x)\mathcal{K}[U]] = 0 \\ &\Rightarrow -\lambda\mathcal{D}[U] + \mathcal{K}[\mu^{-1}(x)\mathcal{D}[U]] = 0 \qquad \text{(using (1.225)},\end{aligned}$$

or

$$-\lambda\mu(x)V + \mathcal{K}[V] = 0, \tag{1.226}$$

where

$$V = \mu^{-1}(x)\mathcal{D}[U]. \tag{1.227}$$

It is evident that if V satisfies (1.226), in view of (1.219) it must be true that $V = \beta U$ for some constant factor β. Hence, from (1.227) we have

$$\mathcal{D}[U] = \beta\mu(x)U,$$

i.e., U must also be an eigenfunction of the damping operator $\mathcal{D}[\cdot]$. Therefore, (1.225) is the necessary and sufficient condition for $\mathcal{K}[\cdot]$ and $\mathcal{D}[\cdot]$ to have the same eigenfunctions, and hence for the damping matrix $\mathbf{D}$ to be diagonal. It is not difficult to show that the condition (1.225) implies that the operator $\mathcal{K}[\mu^{-1}(x)\mathcal{D}[\cdot]]$ is self-adjoint.

One clear advantage obtained if $\mathcal{D}[\cdot]$ satisfies (1.225) is that the discretized equations of motion are completely decoupled when the solution of the damped system is expanded in terms of the eigenfunctions of the undamped system. This decoupling allows us to solve the discretized equations in an easy manner. One special choice of the damping operator for which the commutation holds is

$$\mathcal{D}[\cdot] = \beta\mu(x) + \gamma\mathcal{K}[\cdot], \tag{1.228}$$

where β and γ are arbitrary constants. Such a damping is usually known as *classical damping* or *proportional damping*. The condition (1.228) is satisfied in the case of the damped bar described by (1.212). Therefore, the differential equation for the jth modal coordinate of the bar is given by

$$\ddot{p}_j + d_j\dot{p}_j + \lambda_j p_j = 0. \tag{1.229}$$

which can be easily solved for $p_j(t)$. Finally, the complete solution of the longitudinal vibration of the bar is obtained from (1.218).

1.8.2 Systems with discrete damping

In many practical situations, a continuous system may interact with discrete damping elements. For example, certain support points of a structure may provide substantially higher damping to the structure than its internal damping. In that case, the damping can be considered to be due to discrete dampers at such support points. Discrete damper elements are also routinely attached to structures for vibration control. Here we consider two specific cases, and discuss the effects of discrete damping.

Consider a uniform bar fixed at one end, and having an external damper at the other end, as shown in Figure 1.23. The equation of motion can be written as

$$u_{,tt} - c^2u_{,xx} = 0, \tag{1.230}$$

while the boundary conditions are

$$u(0,t) = 0, \qquad \text{and} \qquad EAu_{,x}(l,t) = -du_{,t}(l,t). \tag{1.231}$$

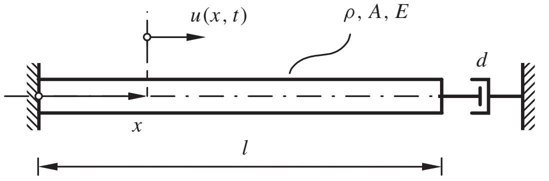

Figure 1.23 A uniform bar with boundary damping

Assuming a solution form

$$u(x, t) = U(x)\mathrm{e}^{st}, \tag{1.232}$$

we obtain the eigenvalue problem

$$U'' - \frac{s^2}{c^2}U = 0, \tag{1.233}$$

with

$$U(0) = 0 \qquad \text{and} \qquad U'(l) = -\frac{sd}{EA}U(l). \tag{1.234}$$

Consider the general solution of (1.233) in the form

$$U(x) = B\mathrm{e}^{sx/c} + C\mathrm{e}^{-sx/c}, \tag{1.235}$$

where B and C are constants of integration. Substituting this solution in the boundary conditions (1.234) yields on simplification

$$\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{\gamma}(1+a) & -\mathrm{e}^{-\gamma}(1-a) \end{bmatrix} \begin{Bmatrix} B \\ C \end{Bmatrix} = 0, \tag{1.236}$$

where $\gamma = sl/c$, and $a = cd/EA$. The condition of non-triviality of the solution of (1.236) yields the characteristic equation as

$$\mathrm{e}^{2\gamma} = \frac{a-1}{a+1}, \tag{1.237}$$

which can be solved for γ, and hence, the eigenvalues s of the system for $a \neq 1$. When $a = 1$, which occurs for the special value of boundary damping $d = EA/c$, it is observed from (1.237) that no eigenvalue exists. In this case, there is no solution of the assumed form (1.232). This will be discussed further in Chapter 2.

When $a \neq 1$, one can rewrite (1.237) using the definition $\gamma := \alpha + i\beta$ as

$$\begin{aligned} \mathrm{e}^{2(\alpha+i\beta)} &= \frac{a-1}{a+1}, \\ \Rightarrow \quad \alpha &= \frac{1}{2}\ln\left|\frac{a-1}{a+1}\right| \end{aligned}$$

and

$$\beta_k = \begin{cases} (2k-1)\pi/2, & 0 \leq a < 1 \\ k\pi, & a > 1 \end{cases} \qquad k = 1, 2, \ldots, \infty.$$

It can be easily checked that, when $d = 0$, this gives the eigenvalues of a fixed–free bar, while $d \to \infty$ yields the eigenvalues of a fixed–fixed bar. It is surprising to note that all the modes have the same decay rate, since α does not depend on k. Further, the transition in the imaginary part of the eigenvalues is discrete as a crosses unity. The locus of an eigenvalue with a as the parameter is depicted in Figure 1.24.

Consider next the case of a taut string with a discrete external damper, as shown in Figure 1.25. The equation of motion of the system can be written as

$$\rho A w_{,tt} + d w_{,t} \delta(x - x_d) - T w_{,xx} = 0, \tag{1.238}$$

where x_d is the location of the damper. Let us expand the solution in terms of the eigenfunctions of an undamped string as

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t) \sin\frac{k\pi x}{l}. \tag{1.239}$$

Substituting this solution in (1.238) and taking the inner product with $\sin j\pi x/l$ yields the jth modal coordinate equation as

$$\ddot{p}_j + \sum_{k=1}^{\infty}\left(\frac{d}{\rho A}\sin\frac{k\pi x_d}{l}\sin\frac{j\pi x_d}{l}\right)\dot{p}_k + \frac{T}{\rho A}p_j = 0. \tag{1.240}$$

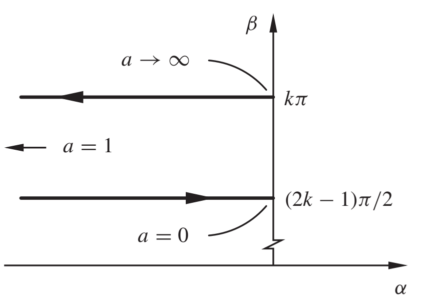

Figure 1.24 Locus of an eigenvalue with a as a parameter for a bar with boundary damping

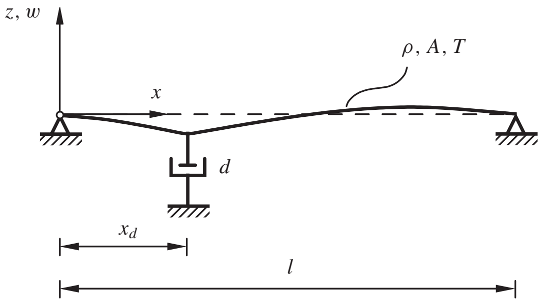

Figure 1.25 A string with discrete damping

It may be observed here that the damping matrix is positive semi-definite with rank one. Further, it couples all the modes of the undamped system. When x_d is chosen such that jx_d/l is never an integer for any j, it can be shown that all the modes are damped. In other words, the total mechanical energy of the string decreases monotonically in time. In this case, the damping is called *pervasive*. Such a damper location is most desirable when we want to damp any arbitrary string motion. In the case where jx_d/l is an integer for some j, the damping is not pervasive, and certain modes remain undamped since one of the nodes of such modes is at x_d. For example, if $x_d = l/3$, the 3rd, $6th, \ldots$ modes will remain undamped.

1.9 NON-HOMOGENEOUS BOUNDARY CONDITIONS

In all the preceding discussions, the boundary conditions were assumed to be homogeneous. However, there are situations where they are not. Non-homogeneity in boundary conditions occurs when either a motion or a force is prescribed at a boundary.

Consider a sliding–fixed string with a specified motion at the left boundary, as shown in Figure 1.26. The equation of motion and boundary conditions can be represented as

$$w_{,tt} - c^2 w_{,xx} = 0, \tag{1.241}$$

$$w(0, t) = h(t), \qquad \text{and} \qquad w(l, t) \equiv 0, \tag{1.242}$$

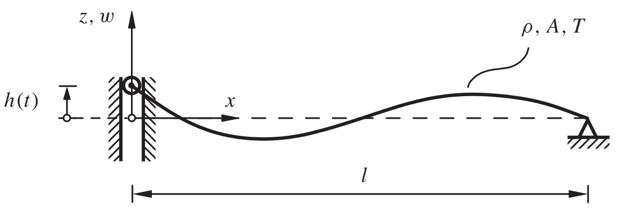

Figure 1.26 A string with a specified boundary motion

where $h(t)$ is an arbitrary function of time. For such non-homogeneous boundary conditions, the solution cannot be directly expanded in a series of eigenfunctions of a problem with homogeneous boundary conditions. However, the methods of integral transforms (such as Laplace transforms) may still be applicable. Alternatively, one may also convert a problem with non-homogeneous boundary conditions to an equivalent problem with homogeneous boundary conditions and an appropriate forcing in the equation of motion to take care of the boundary non-homogeneity. Once this is done, the modal expansion method becomes applicable. In the following, we shall pursue this method.

For the problem (1.241)–(1.242), let

$$w(x,t) = u(x,t) + h(t)\eta(x), \tag{1.243}$$

where $u(x,t)$ and $\eta(x)$ are unknown functions. Substituting this form in the boundary conditions (1.242), we have

$$w(0,t) = u(0,t) + h(t)\eta(0) = h(t) \qquad \text{and} \qquad w(l,t) = u(l,t) + h(t)\eta(l) = 0.$$

If we let

$$u(0,t) \equiv 0 \qquad \text{and} \qquad u(l,t) \equiv 0, \tag{1.244}$$

then $\eta(x)$ must be chosen such that $\eta(0) = 1$ and $\eta(l) = 0$. The simplest choice is then $\eta(x) = 1 - x/l$. Therefore, from (1.243),

$$w(x,t) = u(x,t) + h(t)\left(1 - \frac{x}{l}\right).$$

Substituting this in (1.241), one can write the equation of motion of the string using the field variable $u(x,t)$ as

$$u_{,tt} - c^2 u_{,xx} = -\left(1 - \frac{x}{l}\right)\ddot{h}(t),$$

along with the homogeneous boundary conditions (1.244). This transformed problem can be easily identified as a fixed–fixed string with distributed forcing, and can be solved using the modal expansion method.

1.10 DYNAMICS OF AXIALLY TRANSLATING STRINGS

Axially translating elastic continua are found in many situations of practical interest such as traveling threadlines in looms, rolling of rods in rolling mills, and traveling ropes in rope-ways. They exhibit very interesting dynamic characteristics which are not observed in non-translating continua. In this section, we will discuss the dynamics of a taut string translating along its length.

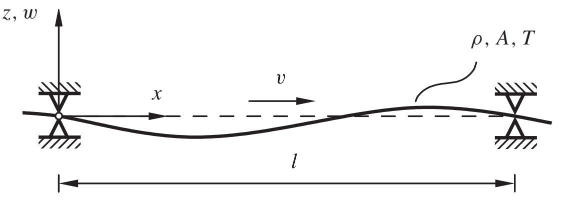

Figure 1.27 A translating string

1.10.1 Equation of motion

Consider a string under a tension T between two fixed supports, and translating along its length at a constant speed v, as shown in Figure 1.27. Let $w(x,t)$ denote the string displacement field variable which will be assumed to be small such that $w_{,x}(x,t) \ll 1$. Then, the Lagrangian of the system can be written as

$$\mathcal{L} = \frac{1}{2}\int_0^l \left[\rho A[(w_{,t} + v w_{,x})^2 + v^2] - T w_{,x}^2\right] \mathrm{d}x,$$

where ρ is the density and A is the area of cross-section of the string. Following the procedure discussed in Appendix A, the equation of motion is obtained as

$$\rho A[w_{,tt} + 2v w_{,xt} + v^2 w_{,xx}] - T w_{,xx} = 0$$

or

$$w_{,tt} + 2v w_{,xt} - (c^2 - v^2) w_{,xx} = 0 \tag{1.245}$$

where $c^2 = T/\rho A$, along with the boundary conditions

$$w(0,t) \equiv 0 \qquad \text{and} \qquad w(l,t) \equiv 0. \tag{1.246}$$

It may be mentioned here that the term $2v w_{,xt}$ in (1.245) is a result of Coriolis acceleration experienced by a string element moving at a speed v in a frame at the current location of the element and rotating at an angular speed $w_{,xt}$. This term is also known as the *gyroscopic term* for a reason that will become clear later. On the other hand, the term $v^2 w_{,xx}$ is due to the centripetal acceleration experienced by the element in the same rotating frame due to the tangential velocity v on a path of approximate curvature $w_{,xx}$.

1.10.2 Modal analysis and discretization

The next step is to study the free vibrations of a translating string. It can be easily verified that Bernoulli's solution procedure discussed in Section 1.3 cannot be applied in this case since the solution of (1.245) is non-separable. Let us assume a modal solution of the form

$$w(x,t) = \mathscr{R}[W(x)\mathrm{e}^{i\omega t}], \tag{1.247}$$

where $W(x)$ is the eigenfunction, ω is the circular frequency, and $\mathscr{R}[\cdot]$ denotes the real part. Substituting this solution form in (1.245) yields on rearrangement, the eigenvalue problem for the traveling string as

$$-(c^2 - v^2)W'' + 2i\omega v W' - \omega^2 W = 0, \tag{1.248}$$

$$W(0) = 0, \qquad \text{and} \qquad W(l) = 0. \tag{1.249}$$

It may be noted that the differential operator in the eigenvalue problem is not self-adjoint. Further, it is evident from (1.248) that the eigenfunction $W(x)$ is complex. Substituting the solution form $W(x) = B\mathrm{e}^{ikx}$ in (1.248), one obtains

$$\begin{aligned} &(c^2 - v^2)k^2 - 2\omega v k - \omega^2 = 0 \\ &\Rightarrow k = -\frac{\omega}{c+v} \qquad \text{or} \qquad k = \frac{\omega}{c-v}. \end{aligned} \tag{1.250}$$

Using (1.250), the general solution of (1.248) can be written as

$$W(x) = D\mathrm{e}^{-i\omega x/(c+v)} + E\mathrm{e}^{i\omega x/(c-v)} \tag{1.251}$$

where D and E are arbitrary constants. Using the boundary conditions (1.249), one obtains

$$\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{-i\omega l/(c+v)} & \mathrm{e}^{i\omega l/(c-v)} \end{bmatrix} \begin{Bmatrix} D \\ E \end{Bmatrix} = \mathbf{0}. \tag{1.252}$$

For non-trivial solution of $(D, E)^{\mathrm{T}}$, we must have

$$\mathrm{e}^{i\omega l[2c/(c^2-v^2)]} - 1 = 0, \tag{1.253}$$

which is the characteristic equation for the traveling string. Thus, the eigenvalues are obtained as

$$\omega_n = \frac{n\pi}{cl}(c^2 - v^2), \qquad n = 1, 2, \ldots, \infty. \tag{1.254}$$

The variation of the first three eigenvalues with speed of travel is shown in Figure 1.28. It is interesting to note that all the eigenvalues are zero when $v = c$, i.e., when the translation speed equals the wave speed in the string. Thus, when $v = c$, the string loses its stiffness completely and becomes neutrally (or marginally) stable. Hence, this speed is known as the *critical speed* of translation of the string. Next, the eigenfunctions are obtained using (1.251) and (1.252) as

$$W_n(x) = D_n\mathrm{e}^{in\pi vx/cl} \sin\frac{n\pi x}{l}, \tag{1.255}$$

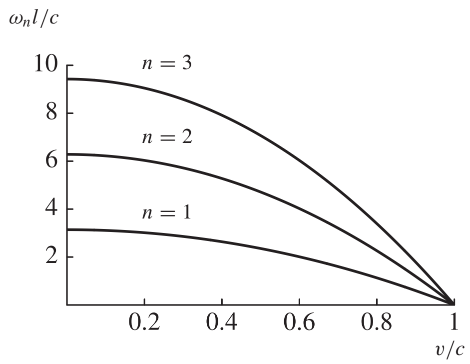

Figure 1.28 Variation of eigenvalues with speed of a translating string

where D_n is an arbitrary complex constant. Finally, the general solution of the string is obtained using (1.247) as

$$w(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\left[\frac{n\pi}{cl}\left(vx + (c^2 - v^2)t\right)\right] \right.$$
$$\left. + C_n \sin\left[\frac{n\pi}{cl}\left(vx + (c^2 - v^2)t\right)\right]\right) \sin\frac{n\pi x}{l}, \tag{1.256}$$

where we have put $D_n = B_n - iC_n$, and B_n and C_n are arbitrary real constants which can be determined from the initial conditions. It may be noted that the solution (1.256) is non-separable in time and space. Since the eigenvalue problem of a traveling string is not self-adjoint, the determination of orthogonality relations (more appropriately *biorthogonality* relations) among the eigenfunctions is more involved (see [8]), and will not be pursued here. More discussions on non-self-adjoint eigenvalue problems can also be found in [7]. In certain problems of traveling strings, it may be convenient to use the method of Laplace transforms (see Exercise 1.15).

Consider a one-term complex solution of the string in terms of the eigenfunction (1.255) as

$$w(x,t) = z_n(t) W_n(x), \tag{1.257}$$

where $z_n(t) = z_{n1}(t) + i z_{n2}(t)$ is the nth complex modal coordinate. Substituting (1.257) in the equation of motion (1.245), multiplying by $W_n^*(x)$ (the complex conjugate of $W_n(x)$), and integrating over the domain of the string yields

$$\ddot{z}_n + i\frac{2n\pi v^2}{cl}\dot{z}_n - (c^4 - v^4)\frac{n^2\pi^2}{c^2 l^2} z_n = 0$$
$$\Rightarrow \ddot{\mathbf{z}} + \mathbf{G}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = 0, \tag{1.258}$$

where $\mathbf{z} = (z_{n1}, z_{n2})^{\mathrm{T}}$,

$$\mathbf{G} = \begin{bmatrix} 0 & -2v^2 n\pi/cl \\ 2v^2 n\pi/cl & 0 \end{bmatrix},$$

and

$$\mathbf{K} = \begin{bmatrix} (c^4 - v^4)n^2\pi^2/c^2l^2 & 0 \\ 0 & (c^4 - v^4)n^2\pi^2/c^2l^2 \end{bmatrix}.$$

It is to be noted that $\mathbf{G}$ is skew-symmetric, i.e., $\mathbf{G}^{\mathrm{T}} = -\mathbf{G}$. The pair of ordinary differential equations in (1.258) represents a discrete *gyroscopic system* corresponding to the nth mode. It may be observed that the gyroscopic effect is due to the presence of the term $2vw_{,xt}$ in the equation of motion (1.245). An analysis of such discrete gyroscopic systems can be found in [9] (also see [7], and [10]).

1.10.3 Interaction with discrete elements

An ideal string, interacting with a discrete element or excited by a point force, can have a slope discontinuity at the interaction point due to its inability to transmit or resist moment. In the case of a traveling string, such a slope discontinuity at the interaction point brings in additional force terms due to abrupt change of momentum of the string in the transverse direction. As is well known in dynamics of mass flow systems (see [11]), a force term of the form

$$\mathbf{F}_{\mathrm{f}} = \dot{m}\mathbf{v}, \tag{1.259}$$

where $\dot{m}$ is the mass flow rate and $\mathbf{v}$ is the absolute velocity vector of the flow, appears in the equation of motion of such systems. In the following, we consider such a situation for a traveling string.

Let us consider a traveling string of length l interacting frictionlessly with a discrete spring at a location $x = a$, as shown in Figure 1.29. One can then write the equation of motion of the strings for the two regions separately as

$$\rho A w_{,tt} + 2\rho A v w_{,xt} - (T - \rho A v^2) w_{,xx} = 0, \qquad 0 \le x < a, \tag{1.260}$$

and

$$\rho A w_{,tt} + 2\rho A v w_{,xt} - (T - \rho A v^2) w_{,xx} = 0, \qquad a < x \le l, \tag{1.261}$$

where T is the tension in the string, which is the same in both the regions due to the assumption of no friction at the string–spring interface. The fixed end boundary conditions are

$$w(0, t) \equiv 0 \qquad \text{and} \qquad w(l, t) \equiv 0. \tag{1.262}$$

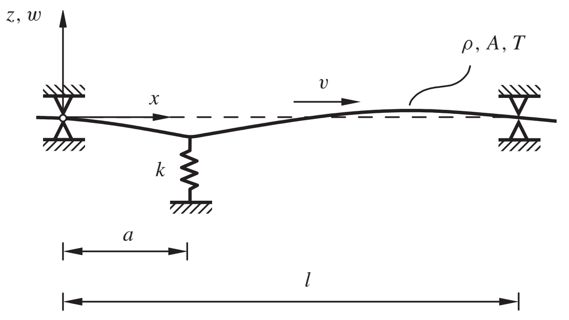

Figure 1.29 Traveling string interacting with a spring

At the point of attachment of the spring, we have the displacement condition

$$w(a^-, t) = w(a^+, t). \tag{1.263}$$

The transverse force condition at the string–spring interface consists of terms due to the tension in the two parts of the string, the extension of the spring, and the force due to mass flow. To determine the latter expression, we consider an infinitesimal control volume fixed to the spring at the interface point, and write down the net transverse force due to momentum flowing in from the region $x < a$ ($\dot{m}$ positive), and momentum flowing out into the region $x > a$ ($\dot{m}$ negative). With the help of (1.259), one can write

$$\begin{aligned} F_{\mathrm{f}} &= \rho A v(w_{,t} + v w_{,x})\big|_{x=a^-} - \rho A v(w_{,t} + v w_{,x})\big|_{x=a^+} \\ \Rightarrow F_{\mathrm{f}} &= \rho A v^2[w_{,x}(a^-, t) - w_{,x}(a^+, t)] \qquad \text{(using (1.263))}, \end{aligned} \tag{1.264}$$

where F_{f} is the net transverse force on the spring due to momentum flow. It may be mentioned that the force $-F_{\mathrm{f}}$ is responsible for changing the momentum of the string in the transverse direction. Now, one can write the force condition at the interaction point as

$$kw(a, t) = (\rho A v^2 - T)[w_{,x}(a^-, t) - w_{,x}(a^+, t)], \tag{1.265}$$

where k is the stiffness of the spring. The partial differential equations (1.260)–(1.261) along with the fixed–fixed boundary conditions, and matching conditions (1.263) and (1.265) completes the formulation of the problem of a traveling string interacting with a spring.

EXERCISES

1.1 Determine the eigenfrequencies and mode-shapes of transverse vibration of a taut string with a discrete mass, as shown in Figure 1.30. Discuss the cases when $m/\rho Al \to \infty$, and $m/\rho Al \to 0$.

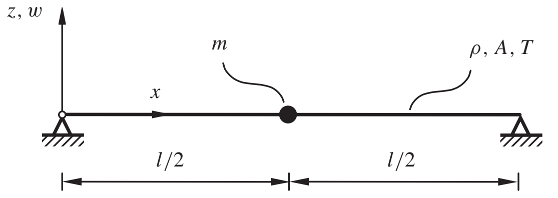

Figure 1.30 Exercise 1.1

1.2 A homogeneous bar is fixed at the left end, and sprung at the right end with a spring constant k, as shown in Figure 1.31.

(a) Using the variational principle, derive the equation of motion, and the boundary conditions of the system.

(b) For $k = EA/l$, determine the first two eigenfrequencies and the corresponding mode-shape-functions.

(c) Take $l = 2m, k = 5000$ N/m, m = 5 kg, and solve for two values of $m/\rho Al = 0.5$ and 2.

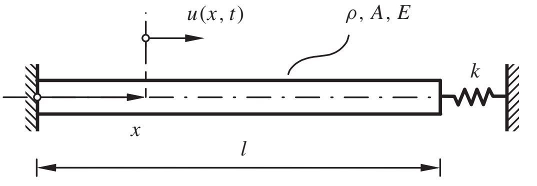

Figure 1.31 Exercise 1.2

1.3 Determine the eigenfrequencies and eigenfunctions for longitudinal vibrations of the system shown in Figure 1.32.

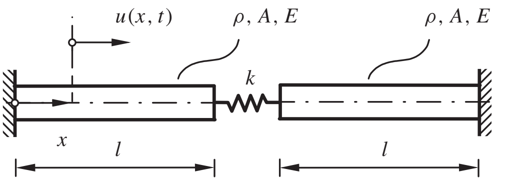

Figure 1.32 Exercise 1.3

1.4 A homogeneous uniform bar is kept under tension T with a string, as shown in Figure 1.33. If the string suddenly snaps, determine the response of the bar.

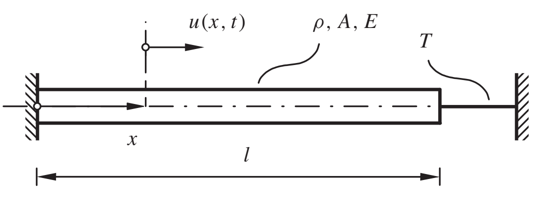

Figure 1.33 Exercise 1.4

1.5 The tapered bar shown in Figure 1.34 has a cross-sectional area that varies as $A(x) = A_0(1 - x/2l)^2$. The bar is excited at the center by a concentrated harmonic force $F(t) = F_0 \cos \Omega t$, as shown in the figure.

(a) Determine the exact solution of forced vibration of the bar.
(b) Determine the location of maximum normal stress in the bar.

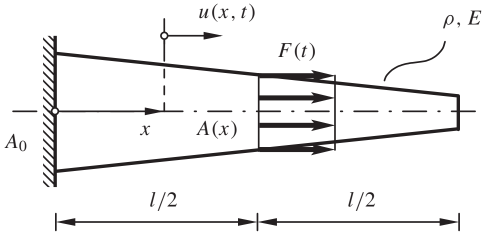

Figure 1.34 Exercise 1.5

1.6 For the systems shown in Figure 1.35, how should the parameters k and d be chosen so that the vibration of the mass m subsides the fastest? Discuss the result when $m \to 0$.

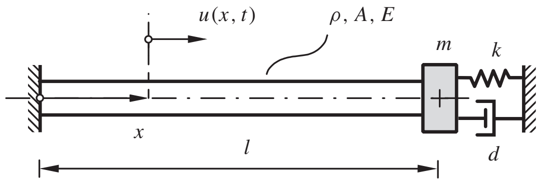

Figure 1.35 Exercise 1.6

1.7 A uniform string with an end-mass m is suspended, as shown in Figure 1.36.

(a) Determine the equation of small-amplitude motion of the string.
(b) Derive the exact characteristic equation, and determine the first three circular eigenfrequencies and the corresponding mode-shapes.

1.8 Using Galerkin's method, discretize the equation of motion of a hanging string. Use the comparison functions as $P_i(x) = x^i$, $i = 1, 2, \ldots, N$. For $N = 2$ determine the eigenfrequencies from the discretized system and compare with the exact solutions.

1.9 A homogeneous bar of circular cross-section with linearly varying radius is shown in Figure 1.37. Using Rayleigh's quotient, estimate the fundamental circular frequency of the bar in longitudinal vibration for the following choices of admissible functions:

(a) First eigenfunction for longitudinal vibration of a bar with constant cross-section.
(b) Admissible functions of the form $H_k(x) = (x/l)^k$, where k is an integer. What value of k yields the lowest value of the fundamental frequency?
(c) Using the static deflection function of a vertically hanging bar.

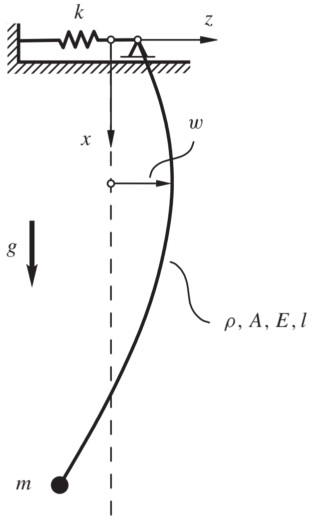

Figure 1.36 Exercise 1.7

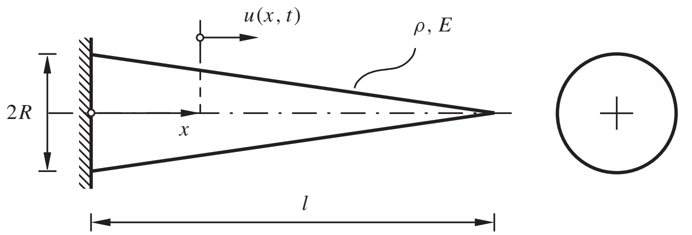

Figure 1.37 Exercise 1.9

1.10 Show that the initial value problem

$$\mu(x)w_{,tt} + \mathcal{K}[w] = 0, \qquad w(x,0) = w_0(x), \text{ and } w_{,t}(x,0) = v_0(x),$$

can be converted to the forced problem

$$\mu(x)w_{,tt} + \mathcal{K}[w] = \mu(x)w_0(x)\dot{\delta}(t) + \mu(x)v_0(x)\delta(t), \qquad w(x,0) = 0, \text{ and } w_{,t}(x,0) = 0.$$

1.11 A sliding–fixed string of length l is excited by a uniformly force $q(x,t) = Q_0 \cos \Omega t$, as shown in Figure 1.38. Determine the steady-state response of the string using: (a) Eigenfunction expansion method, and (b) Green's function method.

1.12 A bar of length l and varying cross-sectional area given by $A(x) = A_0(1 - x/2l)^2$ is fixed and $x = 0$ and free at $x = l$. The bar is harmonically forced by $F(t) = F_0 \sin \Omega t$ at $x = l/2$. Using the admissible functions $H_k(x) = (x/l)(1 - x/2l)^k$, $k = 1, 2, \ldots, N$, obtain the discretized equations of motion of the bar. With $N = 4$, determine the location and magnitude of the maximum response amplitude.

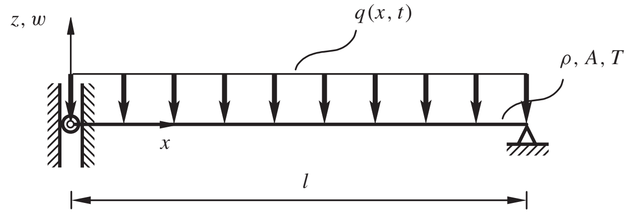

Figure 1.38 Exercise 1.11

1.13 Determine the response of the system shown in Figure 1.39 to a harmonic force $F(t) = B_0 \cos \Omega t$.

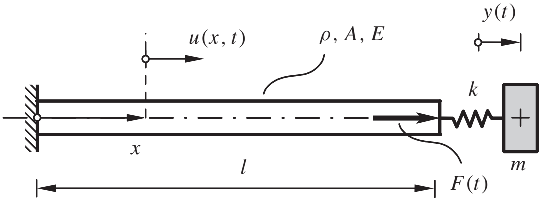

Figure 1.39 Exercise 1.13

1.14 A fixed–fixed string of length l carries a bead of mass m moving at a constant speed v. Determine the response of the string (a) during the transit of the bead, and (b) after the bead has left the span of the string. (One may refer to [12] for an alternative approach using methods described in Chapter 2.)

1.15 An axially translating string with an impulsive transverse point force at $x = \overline{x}$ is described by the equation of motion

$$\rho A[w_{,tt} + 2vw_{,xt} + v^2 w_{,xx}] - Tw_{,xx} = \delta(t - \tau)\delta(x - \overline{x}).$$

Show that the response (Green's function) of the string is given by

$$w(x, \overline{x}, t, \tau) = \mathcal{H}(t - \tau) \sum_{n=1}^{\infty} \frac{2}{n\pi\rho Ac} \sin\left[\frac{n\pi}{cl}\left\{(c^2 - v^2)(t - \tau) + v(x - \overline{x})\right\}\right] \sin\frac{n\pi\overline{x}}{l} \sin\frac{n\pi x}{l},$$

where $\mathcal{H}(\cdot)$ is the Heaviside step function and $c = \sqrt{T/\rho A}$. (*Hint*: Use Laplace transform, followed by a variable transformation $w(x, s) = e^{\alpha x} u(x, s)$, where $\alpha = vs/(c^2 - v^2)$, and s is the Laplace variable.)

1.16 A cable-car on a translating cable may be approximated by a constant point force W fixed to an axially translating string, where W is the weight of the car (see [8]). Using Green's function obtained in Exercise 1.15, determine the response of the string.

1.17 A traveling string is supported frictionlessly at the middle, as shown in Figure 1.40. If the middle support suddenly snaps leaving the string free, determine the subsequent motion of the string and plot its configurations. Assume the initial displacement a of the middle point in Figure 1.40 to be small so that the tension in the string does not change. (*Hint*: Use the idea of Exercise 1.10, and Green's function for the traveling string.)

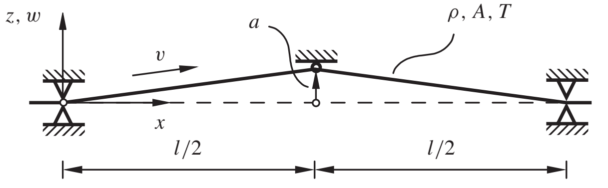

Figure 1.40 Exercise 1.17

REFERENCES

[1] Timoshenko, S.P., and Goodier, J.N., *Theory of Elasticity*, 3e, McGraw-Hill Book Co., Singapore, 1970.

[2] Kreyszig, E., *Advanced Engineering Mathematics*, 2e, Wiley Eastern Pvt. Ltd., New Delhi, 1969.

[3] Stakgold, I., *Boundary Value Problems of Mathematical Physics, Vol 1*, The Macmillan Co., New York, 1967.

[4] Butkovskiy, A.G., *Green's Functions and Transfer Functions Handbook*, Ellis Horwood Ltd., Chichester, UK, 1982.

[5] Hagedorn, P., *Technische Schwingungslehre: Lineare Schwingungen kontinuierlicher mechanischer Systeme*, Springer-Verlag, Berlin, 1989.

[6] Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[7] Meirovitch, L., and Hagedorn, P., A New Approach to the Modelling of Distributed Non-Self-Adjoint Systems, *J. of Sound and Vibration*, 178(2), 1994, pp. 227–241.

[8] Wickert, J.A., and Mote, Jr., C.D., Classical Vibration Analysis of Axially Moving Continua, *J. of Applied Mechanics, Trans. of ASME*, 57, 1990, pp. 738–744.

[9] Hagedorn, P., and Otterbein, S., *Technische Schwingungslehre*, Springer-Verlag, Berlin, 1982.

[10] Huseyin, K., *Vibrations and Stability of Multiple Parameter Systems*, Sijthoff & Noordhoff, Alphen aan den Rijn, 1978.

[11] Meriam, J.L., and Kraige, L.G., *Engineering Mechanics: Dynamics*, 4e, McGraw-Hill Book Co., Singapore, 1999.

[12] Smith, C. E., Motions of a Stretched String Carrying a Moving Mass Particle, *J. of Applied Mechanics, Trans. of ASME*, 31(1), 1964, pp. 29–37.