Chapter 1

The Basics of Quantum Physics: Introducing State Vectors

In This Chapter

- Creating state vectors
- ▶ Using quantum physics operators
- Finding expectation values for operators
- ▶ Simplifying operations with eigenvalues and eigenvectors

f you want to hang out with the cool quantum physics crowd, you have to speak the lingo. And in this field, that's the language of mathematics. Quantum physics often involves representing probabilities in matrices, but when the matrix math becomes unwieldy, you can translate those matrices into the bra and ket notation and perform a whole slew of operations.

This chapter gets you started with the basic ideas behind quantum physics, such as the state vector, which is what you use to describe a multistate system. I also cover using operators, making predictions, understanding properties such as commutation, and simplifying problems by using eigenvectors. Here you can also find several problems to help you become more acquainted with these concepts.

Describing the States of a System

The beginnings of quantum physics include explaining what a system's *states* can be (such as whether a particle's spin is up or down, or what orbital a hydrogen atom's electron is in). The word *quantum* refers to the fact that the states are *discrete* — that is, no state is a mix of any other states. A quantum number or a set of quantum numbers specifies a particular state. If you want to break quantum physics down to its most basic form, you can say that it's all about working with multistate systems.

Don't let the terminology scare you (which can be a constant struggle in quantum physics). A *multistate system* is just a system that can exist in multiple states; in other words, it has different energy levels. For example, a pair of dice is a multistate system. When you roll a pair of dice, you can get a sum of 2, 3, 5, all the way up to 12. Each one of those values represents a different state of the pair of dice.



Quantum physics likes to spell everything out, so it approaches the two dice by asking how many ways they could be in the various states. For example, you have only one way to roll a 2 with two dice, but you have six ways to roll a total of 7. So if the relative probability of rolling a 2 is one, the relative probability of rolling a 7 is six.

With a little thought, you can add up all the ways to get a 2, a 3, and so on like this:

Sum of the Dice	Relative Probability of Getting That Sum
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

In this case, you can say that the total of the two dice is the quantum number and that each quantum number represents a different state. Each system can be represented by a *state vector* — a one-dimensional matrix — that indicates the relative probability amplitude of being in each state. Here's how to set one up:

1. Write down the relative probability of each state and put it in vector form.

You now have a one-column matrix listing the probabilities (though you can instead use a one-row matrix).

2. Take the square root of each number to get the probability amplitude.

State vectors record not the actual probabilities but rather the probability *amplitude*, which is the square root of the probability. That's because when you find probabilities using quantum physics, you multiply two state vectors together (sometimes with an *operator* — a mathematical construct that returns a value when you apply it to a state vector).

3. Normalize the state vector.

Because the total probability that the system is in one of the allowed states is 1, the square of a state vector has to add up to 1. To square a state vector, you multiply every element by itself and then add all the squared terms (it's just like matrix multiplication). However, at this point, squaring each term in the state vector and adding them all usually doesn't give you 1, so you have to *normalize* the state vector by dividing each term by the square root of the sum of the squares.

4. Set the vector equal to $|\psi\rangle$.

Because you may be dealing with a system that has thousands of states, you usually abbreviate the state vector as a Greek letter, using notation like this: $|\psi\rangle$ (or $\langle\psi|$ if you used a row vector). You see why this notation is useful in the next section.

Check out the following example problem and practice problems, which can help clarify any other questions you may have.



• What's the state vector for the various possible states of a pair of dice?

A.
$$|\psi\rangle =$$

1	
6	
ດ 1/2	
<u>4</u>	
0	
$3^{1/2}$	
6	
2	
<u>–</u>	
= 1/2	
5-/-	
6	
6 ^{1/2}	
6	
5 ^{1/2}	
6	
2	
6	
3 ^{1/2}	
6	
$2^{1/2}$	
6	
1	
6	

Start by creating a vector that holds the relative probability of each state — that is, the first value holds the relative probability (the number of states) that the total of the two dice is 2, the next item down holds the relative probability that

the total of the two dice is 3, and so on. That looks like this: 9



Convert this vector to probability amplitudes by taking the square root of each entry like this:

When you square the state vector, the square has to add up to 1; that is, the dice must show a 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. However, squaring each term in this state vector and adding them all up gives you 36, not 1, so you have to normalize the state vector by dividing each term by the square root of 36, or 6, to make sure that you get 1 when you square the state vector. That means the state vector looks like this:

$\frac{1}{6}$	
$2^{1/2}$	
6	
$2^{1/2}$	
$\frac{3}{6}$	
-0 -0	
$\frac{2}{6}$	
$5^{1/2}$	
6	
$6^{1/2}$	
6	
$5^{1/2}$	
6	
2	
$\overline{6}$	
$3^{1/2}$	
6	
$2^{1/2}$	
6	
$\frac{1}{6}$	

Now use the Greek letter notation to represent the state vector. So that's it; your state vector is

 $|\psi\rangle =$

1 $\overline{6}$ $2^{1/2}$ 6 $3^{1/2}$ 6 $\frac{2}{6}$ $5^{1/2}$ 6 $6^{1/2}$ 6 $5^{1/2}$ 6 $\frac{2}{6}$ $3^{1/2}$ 6 $2^{1/2}$ 6 1 $\overline{6}$

1. Assume you have two four-sided dice (in the shape of *tetrahedrons* — that is, mini pyramids). What are the relative probabilities of each state of the two dice? (*Note:* Four-sided dice are odd to work with — the value of each die is represented by the number on the *bottom* face, because the dice can't come to rest on the top of a pyramid!)

Solve It

2. Put the relative probabilities of the various states of the four-sided dice into vector form.



3. Convert the vector of relative probabilities in question 2 to probability amplitudes.

Solve It

4. Convert the relative probability amplitude vector you found for the four-sided dice in question 3 to a normalized state vector.



Becoming a Notation Meister with Bras and Kets

Instead of writing out an entire vector each time, quantum physics usually uses a notation developed by physicist Paul Dirac — the *Dirac* or *bra-ket* notation. The two terms spell braket, as in *bracket*, because when an operator appears between them, they bracket, or sandwich, that operator. Here's how write the two forms of state vectors:

$$arpsi$$
 Bras: $\langle \psi$

$$\checkmark$$
 Kets: $|\psi\rangle$



When you multiply the same state vector expressed as a bra and a ket together — the product is represented as $\langle \psi | \psi \rangle$ — you get 1. In other words, $\langle \psi | \psi \rangle$ = 1. You get 1 because the sum of all the probabilities of being in the allowed states must equal 1.

If you have a bra, the corresponding ket is the *Hermitian conjugate* (which you get by taking the transpose and changing the sign of any imaginary values) of that bra $-\langle \psi | \text{equals} | \psi \rangle^{\dagger}$ (where the [†] means the Hermitian conjugate). What does that mean in vector terms? Check out the following example.



What's the bra for the state vector of a pair of dice? Verify that $\langle \psi | \psi \rangle = 1$.

 $A. \quad \left| \frac{1}{6} \quad \frac{2^{1/2}}{6} \quad \frac{3^{1/2}}{6} \quad \frac{2}{6} \quad \frac{5^{1/2}}{6} \quad \frac{6^{1/2}}{6} \quad \frac{5^{1/2}}{6} \quad \frac{2}{6} \quad \frac{3^{1/2}}{6} \quad \frac{2^{1/2}}{6} \quad \frac{1}{6} \right|$

 $|\psi\rangle =$ 1 $\overline{6}$ $2^{1/2}$ 6 $3^{1/2}$ 6 $\frac{2}{6}$ $5^{1/2}$ 6 $6^{1/2}$ 6 $5^{1/2}$ 6 $\frac{2}{6}$ $3^{1/2}$ 6 $2^{1/2}$ 6 1 6

Start with the ket:

Now find the complex conjugate of the ket. To do so in matrix terms, you take the transpose of the ket and then take the complex conjugate of each term (which does nothing in this case because all terms are real numbers). Finding the transpose just involves writing the columns of the ket as the rows of the bra, which gives you the following for the bra:

$$\left|\frac{1}{6} \quad \frac{2^{1/2}}{6} \quad \frac{3^{1/2}}{6} \quad \frac{2}{6} \quad \frac{5^{1/2}}{6} \quad \frac{6^{1/2}}{6} \quad \frac{5^{1/2}}{6} \quad \frac{2}{6} \quad \frac{3^{1/2}}{6} \quad \frac{2^{1/2}}{6} \quad \frac{1}{6}\right|$$

To verify that $\langle \psi | \psi \rangle$ = 1, multiply the bra and ket together using matrix multiplication like this:

$$\begin{vmatrix} \frac{1}{6} & \frac{2^{1/2}}{6} & \frac{3^{1/2}}{6} & \frac{2}{6} & \frac{5^{1/2}}{6} & \frac{6^{1/2}}{6} & \frac{5^{1/2}}{6} & \frac{2}{6} & \frac{3^{1/2}}{6} & \frac{2^{1/2}}{6} & \frac{1}{6} \end{vmatrix} \begin{vmatrix} \frac{1}{6} \\ \frac{2^{1/2}}{6} \\ \frac{3^{1/2}}{6} \\ \frac{2}{6} \\ \frac{5^{1/2}}{6} \\ \frac{5^{1/2}}{6} \\ \frac{5^{1/2}}{6} \\ \frac{2}{6} \\ \frac{3^{1/2}}{6} \\ \frac{2^{1/2}}{6} \\ \frac{2^{1/2}}{6} \\ \frac{2}{6} \\ \frac{3^{1/2}}{6} \\ \frac{2^{1/2}}{6} \\ \frac{2}{6} \\ \frac{3^{1/2}}{6} \\ \frac{2^{1/2}}{6} \\ \frac{3^{1/2}}{6} \\ \frac{3^{1$$

Complete the matrix multiplication to give you

$$\left\langle \psi \left| \psi \right\rangle \right. = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = 1$$

5. Find the bra for the state vector of a pair of four-sided dice.

Solve It

6. Confirm that $\langle \psi | \psi \rangle$ for the bra and ket for the four-sided dice equals 1.

Solve It

Getting into the Big Leagues with Operators

What are bras and kets useful for? They represent a system in a stateless way — that is, you don't have to know which state every element in a general ket or bra corresponds to; you don't have to spell out each vector. Therefore, you can use kets and bras in a general way to work with systems. In other words, you can do a lot of math on kets and bras that would be unwieldy if you had to spell out all the elements of a state vector every time. Operators can assist you. This section takes a closer look at how you can use operators to make your calculations.

Introducing operators and getting into a healthy, orthonormal relationship

Kets and bras describe the state of a system. But what if you want to measure some quantity of the system (such as its momentum) or change the system (such as by raiding a hydrogen atom to an excited state)? That's where operators come in. You apply an *operator* to a bra or ket to extract a value and/or change the bra or ket to a different state. In general, an operator gives you a new bra or ket when you use that operator: $A|\psi\rangle = |\psi'\rangle$.

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Some of the most important operators you need to know include the following:

Hamiltonian operator: Designated as H, this operator is the most important in quantum physics. When applied to a bra or ket, it gives you the energy of the state that the bra or ket represents (as a constant) multiplied by that bra or ket again:

 $|\mathbf{H}|\psi\rangle = \mathbf{E}|\psi\rangle$

E is the energy of the particle represented by the ket $|\psi\rangle$.

Unity or identity operator: Designated as I, this operator leaves kets unchanged:

 $\mathbf{I}|\psi\rangle = |\psi\rangle$

 \checkmark Gradient operator: Designated as ∇ , this operator takes the derivative. It works like this:

$$\nabla |\psi\rangle = \frac{\partial}{\partial x} |\psi\rangle \mathbf{i} + \frac{\partial}{\partial y} |\psi\rangle \mathbf{j} + \frac{\partial}{\partial z} |\psi\rangle \mathbf{k}$$

Linear momentum operator: Designated as P, this operator finds the momentum of a state. It looks like this:

$$P|\psi\rangle = -i\hbar\nabla|\psi\rangle$$

✓ Laplacian operator: Designated as Δ , or ∇^2 , this operator is much like a second-order gradient, which means it takes the second derivative. It looks like this:

$$\nabla^{2} |\psi\rangle = \Delta |\psi\rangle = \frac{\partial^{2}}{\partial x^{2}} |\psi\rangle + \frac{\partial^{2}}{\partial y^{2}} |\psi\rangle + \frac{\partial^{2}}{2z^{2}} |\psi\rangle$$

In general, multiplying operators together is not the same independent of order, so for the operators A and B,

 $AB \neq BA$

You can find the complex conjugate of an operator A, denoted A^{\dagger} , like this:

$$\langle \psi | \mathbf{A}^{\dagger} | \phi \rangle = \langle \phi | \mathbf{A} | \psi \rangle$$



EMBER

When working with kets and bras, keep the following in mind:

 \checkmark Two kets, $|\psi\rangle$ and $|\phi\rangle$, are said to be *orthogonal* if

$$ig\langle\psiig|\phiig
angle$$

✓ Two kets are said to be *orthonormal* if all three of the following apply:

•
$$\langle \psi | \phi \rangle = 0$$

•
$$\langle \psi | \psi \rangle = 1$$

• $\left\langle \phi \middle| \phi \right\rangle = 1$



Find an orthonormal ket to the bra

$$igg| rac{1}{2^{1/2}} igg| 0 igg| 0 igg| rac{1}{2^{1/2}} igg|$$

$$\begin{array}{c|c} \textbf{4.} & \frac{1}{2^{1/2}} \\ & \textbf{0} \\ & \textbf{0} \\ -\frac{-1}{2^{1/2}} \end{array}$$

1

You know that to be orthonormal, the following relations must be true:

•
$$\langle \psi | \phi \rangle = 0$$

• $\langle \psi | \psi \rangle = 1$

So you need to construct a ket made up of elements A, B, C, D such that

$$\begin{vmatrix} \frac{1}{2^{1/2}} & 0 & 0 & \frac{1}{2^{1/2}} \\ B \\ C \\ D \end{vmatrix} = 0$$

Do the matrix multiplication to get

$$\frac{A}{2^{1/2}} + \frac{D}{2^{1/2}} = 0$$

So therefore, A = -D (and you can leave B and C at 0; their value is arbitrary because you multiply them by the zeroes in the bra, giving you a product of 0).

You're not free to choose just any values for A and D because $\langle \psi | \psi \rangle$ must equal 1.

So you can choose $A = \frac{1}{2^{1/2}}$ and $D = -\frac{1}{2^{1/2}}$, giving you the following ket:



7. Find an orthonormal ket to the bra

$$\left|\frac{1}{2^{1/2}} \quad \frac{1}{2^{1/2}}\right|$$

Solve It

8. Find the identity operator for bras and kets with six elements.



Grasping Hermitian operators and adjoints

Operators that are equal to their Hermitian adjoints are called *Hermitian operators*. In other words, an operator is Hermitian if

 $\mathbf{A}^{\dagger}=\mathbf{A}$

Here's how you find the Hermitian adjoint of an operator, A:

- 1. Find the transpose by interchanging the rows and columns, A^T.
- 2. Take the complex conjugate.

 $A^{T^{\star}} = A^{\dagger}$

In addition, finding the inverse is often useful because applying the inverse of an operator undoes the work the operator did: $A^{-1}A = AA^{-1} = I$. For instance, when you have equations like Ax = y, solving for x is easy if you can find the inverse of A: $x = A^{-1}y$. But finding the inverse of a large matrix usually isn't easy, so quantum physics calculations are sometimes limited to working with *unitary operators*, U, where the operator's inverse is equal to its Hermitian adjoint:

 $\mathbf{U}^{-1}=\mathbf{U}^{\dagger}$

Getting Physical Measurements with Expectation Values

Everything in quantum physics is done in terms of probabilities, so making predictions becomes very important. The biggest such prediction is the expectation value. The *expectation value* of an operator is the average value the operator will give you when you apply it to a particular system many times.

The expectation value is a weighted mean of the probable values of an operator. Here's how you'd find the expectation value of an operator A:

Expectation value = $\langle \psi | \mathbf{A} | \psi \rangle$

Because you can express $\langle \psi |$ as a row vector and $|\psi \rangle$ as a column vector, you can express the operator A as a square matrix.



Finding the expectation value is so common that you often find $\langle \psi | A | \psi \rangle$ abbreviated as $\langle A \rangle$.

The expression $|\phi\rangle\langle\psi|$ is actually a linear operator. To see that, apply $|\phi\rangle\langle\psi|$ to a ket, $|\chi\rangle$:

 $|\phi
angle\langle\psi|\chi
angle$

which is $\langle \psi | \chi \rangle | \phi \rangle$.

The expression $\langle \psi | \chi \rangle$ is always a complex number (which could be purely real), so this breaks down to $c | \phi \rangle$, where *c* is a complex number, so $| \phi \rangle \langle \psi |$ is indeed a linear operator.



What is the expectation value of rolling two dice?

• Seven. For two dice, the expectation value is a sum of terms, and each term is a value that the dice can display multiplied by the probability that that value will appear. The bra and ket handle the probabilities, so the operator you create for this problem, which I call the A operator for this example, needs to store the dice values (2 through 12) for each probability. Therefore, the operator A looks like this:

A :	=									
2	0	0	0	0	0	0	0	0	0	0
0	3	0	0	0	0	0	0	0	0	0
0	0	4	0	0	0	0	0	0	0	0
0	0	0	5	0	0	0	0	0	0	0
0	0	0	0	6	0	0	0	0	0	0
0	0	0	0	0	7	0	0	0	0	0
0	0	0	0	0	0	8	0	0	0	0
0	0	0	0	0	0	0	9	0	0	0
0	0	0	0	0	0	0	0	10	0	0
0	0	0	0	0	0	0	0	0	11	0
0	0	0	0	0	0	0	0	0	0	12

To find the expectation value of A, you need to calculate $\langle \psi | A | \psi \rangle$. Spelling that out in terms of components gives you the following:

$\langle \psi$	${ m A}ig \psiig angle$	=																				
$\frac{1}{6}$	$\frac{2^{1/2}}{6}$	$\frac{3^{1/2}}{6}$	$\frac{2}{6}$	$\frac{5^{1/2}}{6}$	$\frac{6^{1/2}}{6}$	$\frac{5^{1/2}}{6}$	$\frac{2}{6}$	$\frac{3^{1/2}}{6}$	$\frac{2^{1/2}}{6}$	$\frac{1}{6}$	2	0	0	0	0	0	0	0	0	0	0	$\frac{1}{6}$
										1	0	3	0	0	0	0	0	0	0	0	0	$\frac{2^{1/2}}{6}$
											0	0	4	0	0	0	0	0	0	0	0	$\frac{3^{1/2}}{6}$
											0	0	0	5	0	0	0	0	0	0	0	$\frac{2}{6}$
											0	0	0	0	6	0	0	0	0	0	0	$\frac{5^{1/2}}{6}$
											0	0	0	0	0	7	0	0	0	0	0	$\frac{6^{1/2}}{6}$
											0	0	0	0	0	0	8	0	0	0	0	$\frac{5^{1/2}}{6}$
											0	0	0	0	0	0	0	9	0	0	0	$\frac{2}{6}$
											0	0	0	0	0	0	0	0	10	0	0	$\frac{3^{1/2}}{6}$
											0	0	0	0	0	0	0	0	0	11	0	$\frac{2^{1/2}}{6}$
											0	0	0	0	0	0	0	0	0	0	12	$\frac{1}{6}$

Doing the matrix multiplication gives you

$$\langle \psi | \mathbf{A} | \psi \rangle = 7$$

So the expectation value of a roll of the dice is 7.

9. Find the expectation value of two four-sided dice.

Solve It

10. Find the expectation value of the identity operator for a pair of normal, six-sided dice (see the earlier section "Introducing operators and getting into a healthy, orthonormal relationship" for more on the identity operator).

Solve It

Commutators: Checking How Different Operators Really Are

In quantum physics, the measure of the difference between applying operator A and then B, versus B and then A, is called the operators' *commutator*. If two operators have a commutator that's 0, they *commute*, and the order in which you apply them doesn't make any difference. In other words, operators that commute don't interfere with each other, and that's useful to know when you're working with multiple operators. You can independently use commuting operators, whereas you can't independently use noncommuting ones.

Here's how you define the commutator of operators A and B:



$$[A, B] = AB - BA$$

Two operators commute with each other if their commutator is equal to 0:

[A, B] = 0

The Hermitian adjoint of a commutator works this way:

 $\left[\mathbf{A}, \mathbf{B} \right]^{\dagger} = \left[\mathbf{B}^{\dagger}, \mathbf{A}^{\dagger} \right]$

Check out the following example, which illustrates the concept of commuting.



Show that any operator commutes with itself.

A. [A, A] = 0. The definition of a commutator is [A, B] = AB – BA. And if both operators are A, you get

$$[A, A] = AA - AA$$

But AA - AA = 0, so you get [A, A] = AA - AA = 0 **11.** What is [A, B] in terms of [B, A]?

Solve It

12. What is the Hermitian adjoint of a commutator $[A, B]^{\dagger}$ if A and B are Hermitian operators?



When you apply an operator to a ket, you generally get a new ket. For instance, $A|\psi\rangle = |\psi'\rangle$. However, sometimes you can make matters a little simpler by casting your problem in terms of eigenvectors and eigenvalues (*eigen* is German for "innate" or "natural"). Instead of giving you an entirely new ket, applying an operator to its *eigenvector* (a ket) merely gives you the same eigenvector back again, multiplied by its *eigenvalue* (a constant). In other words, $|\psi\rangle$ is an eigenvector of the operator A if the number *a* is a complex constant and $A|\psi\rangle = a|\psi\rangle$.

So applying A to one of its eigenvectors, $|\psi\rangle$, gives you $|\psi\rangle$ back, multiplied by that eigenvector's eigenvalue, *a*. An eigenvalue can be complex, but note that if the operators are Hermitian, the values of *a* are real and their eigenvectors are orthogonal (see the earlier section "Grasping Hermitian operators and adjoints" for more on Hermitian operators).

To find an operator's eigenvalues, you want to find *a*, such that

$$A|\psi\rangle = a|\psi\rangle$$

You can rewrite the equation this way, where I is the identity matrix (that is, it contains all 0s except for the 1s running along the diagonal from upper left to lower right):

$$(\mathbf{A} - a\mathbf{I})|\psi\rangle = 0$$

For this equation to have a solution, the matrix determinant of (A - aI) must equal 0:

 $\det(\mathbf{A} - a\mathbf{I}) = 0$

Solving this relation gives you an equation for a — and the roots of the equation are the eigenvalues. You then plug the eigenvalues, one by one, into the equation $(A - aI)|\psi\rangle = 0$ to find the eigenvectors.



If two or more of the eigenvalues are the same, that eigenvalue is said to be *degenerate*.

Know that many systems, like free particles, don't have a number of set discrete energy states; their states are continuous. In such circumstances, you move from a state vector like $|\psi\rangle$ to a continuous wave function, $\psi(\mathbf{r})$. How does $\psi(\mathbf{r})$ relate to $|\psi\rangle$? You have to relate the stateless vector $|\psi\rangle$ to normal spatial dimensions, which you do with a state vector where the states correspond to possible positions, $\langle \mathbf{r} |$ (see my book *Quantum Physics For Dummies* [Wiley] for all the details). In that case, $\psi(\mathbf{r}) = \langle \mathbf{r} | \psi \rangle$.





What are the eigenvectors and eigenvalues of the following operator, which presents the operator for two six-sided dice?

2	0	0	0	0	0	0	0	0	0	0
0	3	0	0	0	0	0	0	0	0	0
0	0	4	0	0	0	0	0	0	0	0
0	0	0	5	0	0	0	0	0	0	0
0	0	0	0	6	0	0	0	0	0	0
0	0	0	0	0	7	0	0	0	0	0
0	0	0	0	0	0	8	0	0	0	0
0	0	0	0	0	0	0	9	0	0	0
0	0	0	0	0	0	0	0	10	0	0
0	0	0	0	0	0	0	0	0	11	0
0	0	0	0	0	0	0	0	0	0	12

The eigenvalues are 2, 3, 4, 5, ..., 12, A. and the eigenvectors are

1	0	0	
0	1	0	
0	0	0	
0	0	0	
0	0	 0	
0	0	0	
0	0	0	
0	0	0	
0	0	0	
0	0	0	
0	0	1	

Here's the operator you want to find the eigenvalues and eigenvectors of:

2	0	0	0	0	0	0	0	0	0	0
0	3	0	0	0	0	0	0	0	0	0
0	0	4	0	0	0	0	0	0	0	0
0	0	0	5	0	0	0	0	0	0	0
0	0	0	0	6	0	0	0	0	0	0
0	0	0	0	0	7	0	0	0	0	0
0	0	0	0	0	0	8	0	0	0	0
0	0	0	0	0	0	0	9	0	0	0
0	0	0	0	0	0	0	0	10	0	0
0	0	0	0	0	0	0	0	0	11	0
0	0	0	0	0	0	0	0	0	0	12

This operator operates in 11-dimensional space, so you need to find 11 eigenvectors and 11 corresponding eigenvalues.

This operator is already diagonal, so this problem is easy - just take unit vectors in the 11 different directions of the eigenvectors. Here's what the first eigenvector is:

0 0 And here's the second eigenvector:

And so on, up to the 11th eigenvector

What about the eigenvalues? The eigenvalues are the values you get when you apply the operator to an eigenvector, and because the eigenvectors are just unit vectors in all 11 dimensions, the eigenvalues are the numbers on the diagonal of the operator — that is, 2, 3, 4, and so on, up to 12.

13. What are the eigenvalues and eigenvectors of this operator?

$$\mathbf{A} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

Solve It

14. What are the eigenvalues and eigenvectors of this operator?

$$A = \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}$$

Solve It

Answers to Problems on State Vectors

The following are the answers to the practice questions presented earlier in this chapter. I first repeat the problems and give the answers in bold. Then you can see the answers worked out, step by step.

Assume you have two four-sided dice (in the shape of *tetrahedons* — that is, mini pyramids). What are the relative probabilities of each state of the two dice? **Here's the answer:**

1 = Relative probability of getting a 2

2 = Relative probability of getting a 3

- 3 = Relative probability of getting a 4
- 4 = Relative probability of getting a 5
- 3 = Relative probability of getting a 6
- 2 = Relative probability of getting a 7
- 1 = Relative probability of getting a 8

Adding up the various totals of the two four-sided dice gives you the number of ways each total can appear, and that's the relative probability of each state.

Put the relative probabilities of the various states of the four-sided dice into vector form.

Just assemble the relative probabilities of each state into vector format.

3 Convert the vector of relative probabilities in question 2 to probability amplitudes.

 $\begin{array}{c}
1\\
2^{1/2}\\
3^{1/2}\\
2\\
3^{1/2}\\
2^{1/2}\\
1
\end{array}$

To find the probability amplitudes, just take the square root of the relative probabilities.



Convert the relative probability amplitude vector you found for the four-sided dice in question 3 to a normalized state vector.



To normalize the state vector, divide each term by the square root of the sum of the squares of each term: $1^2 + (2^{1/2})^2 + (3^{1/2})^2 + (2^{1/2})^2 + (2^{1/2})^2 + 1^2 = 1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$, and $16^{1/2} = 4$, so divide each term by 4. Doing so ensures that the square of the state vector gives you a total value of 1.

5 Find the bra for the state vector of a pair of four-sided dice. **The answer is**

 $\left| \frac{1}{4} \quad \frac{2^{1/2}}{4} \quad \frac{3^{1/2}}{4} \quad \frac{1}{2} \quad \frac{3^{1/2}}{4} \quad \frac{2^{1/2}}{4} \quad \frac{1}{4} \right|$

To find the bra, start with the ket that you already found in problem 4:



Take the transpose of the ket and the complex conjugate of each term (which does nothing, because each term is real) to get

$$\frac{1}{4} \quad \frac{2^{1/2}}{4} \quad \frac{3^{1/2}}{4} \quad \frac{1}{2} \quad \frac{3^{1/2}}{4} \quad \frac{2^{1/2}}{4} \quad \frac{1}{4}$$

6 Confirm that $\langle \psi | \psi \rangle$ for the bra and ket for the four-sided dice equals 1. **Here's the answer:** $\langle \boldsymbol{\psi} | \boldsymbol{\psi} \rangle = \mathbf{1}$

To find $\langle \psi | \psi \rangle$, perform this multiplication:

$$\begin{vmatrix} \frac{1}{4} & \frac{2^{1/2}}{4} & \frac{3^{1/2}}{4} & \frac{1}{2} & \frac{3^{1/2}}{4} & \frac{2^{1/2}}{4} & \frac{1}{4} \end{vmatrix} \begin{vmatrix} \frac{1}{4} \\ \frac{2^{1/2}}{4} \\ \frac{3^{1/2}}{4} \\ \frac{1}{2} \\ \frac{3^{1/2}}{4} \\ \frac{1}{2} \\ \frac{3^{1/2}}{4} \\ \frac{1}{4} \\ \frac{2^{1/2}}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{vmatrix}$$

To find $\langle \psi | \psi \rangle$, perform this multiplication, giving you 1:

$$\left\langle \psi \left| \psi \right\rangle = \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16} + \frac{2}{16} + \frac{1}{16} = 1$$

7 Find an orthonormal ket to the bra
$$\left| \frac{1}{2^{1/2}} - \frac{1}{2^{1/2}} \right|$$
.
 $\left| \frac{1}{2^{1/2}} - \frac{1}{2^{1/2}} \right|$

You know that to be orthonormal, the following relations must be true:

$$\psi \left| \phi \right\rangle = 0$$

$$\psi \left| \psi \right\rangle = 1$$

So you need to construct a ket made up of elements A and B such that

$$\begin{vmatrix} \frac{1}{2^{1/2}} & \frac{1}{2^{1/2}} \end{vmatrix} \quad \begin{vmatrix} A \\ B \end{vmatrix} = 0$$

Do the matrix multiplication to get

$$\frac{A}{2^{1/2}} + \frac{D}{2^{1/2}} = 0$$

So therefore, A = –D. You're not free to choose just any values for A and D because $\langle \psi | \psi \rangle$ must equal 1. So you can choose A = $\frac{1}{2^{1/2}}$ and D = $\frac{-1}{2^{1/2}}$ to make the math come out right here, giving you the following ket:



Find the identity operator for bras and kets with six elements.

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

You need a matrix I such that

 $\mathbf{I}|\psi\rangle = |\psi\rangle$

In this problem, you're working with bras and kets with six elements:

A B C D E F

Therefore, you need a matrix that looks like this:

 $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$



Find the expectation value of two four-sided dice. The answer is 5.

For two four-sided dice, the expectation value is a sum of terms, and each term is a value that the dice can display, multiplied by the probability that that value will appear.

The bra and ket will handle the probabilities, so it's up to the operator you create for this — call it the A operator — to store the dice values (2 through 8) for each probability, which means that the operator A looks like this:

To find the expectation value of A, you need to calculate $\langle \psi | A | \psi \rangle$. Spelling that out in terms of components gives you the following:

$\langle \psi$	$ A \psi angle$	=												
$\left \frac{1}{4}\right $	$\frac{2^{1/2}}{4}$	$\frac{3^{1\!/\!2}}{4}$	$\frac{1}{2}$	$\frac{3^{1\!/2}}{4}$	$\frac{2^{1/2}}{4}$	$\frac{1}{4}$	2	0	0	0	0	0	0	$\left \begin{array}{c} \frac{1}{4} \end{array} \right $
							0	3	0	0	0	0	0	$\left \frac{2^{1/2}}{4}\right $
							0	0	4	0	0	0	0	$\left \frac{3^{1/2}}{4}\right $
							0	0	0	5	0	0	0	$\left \frac{1}{2} \right $
							0	0	0	0	6	0	0	$\left \frac{3^{1/2}}{4}\right $
							0	0	0	0	0	7	0	$\left \frac{2^{1/2}}{4}\right $
							0	0	0	0	0	0	8	$\left \frac{1}{4} \right $

Doing the matrix multiplication gives you

$$\langle \psi | \mathbf{A} | \psi \rangle = 5$$

So the expectation value of a roll of the pair of four-sided dice is 5.

10 Find the expectation value of the identity operator for a pair of normal, six-sided dice. **The answer is 1.**

For two dice, the expectation value is a sum of terms, and each term is a value that the dice can display, multiplied by the probability that that value will appear. The bra and ket will handle the probabilities. The operator A is the identity operator, so it looks like this:

A	=									
1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	1

To find the expectation value of I, you need to calculate $\langle \psi | A | \psi \rangle$. Spelling that out in terms of components gives you the following:

$\langle \psi$	$ert { m I} \psi angle$	=																				
$\frac{1}{6}$	$\frac{2^{1/2}}{6}$	$\frac{3^{1/2}}{6}$	$\frac{2}{6}$	$\frac{5^{1/2}}{6}$	$\frac{6^{1/2}}{6}$	$\frac{5^{1/2}}{6}$	$\frac{2}{6}$	$\frac{3^{1/2}}{6}$	$\frac{2^{1/2}}{6}$	$\frac{1}{6}$	1	0	0	0	0	0	0	0	0	0	0	$\left \frac{1}{6} \right $
											0	1	0	0	0	0	0	0	0	0	0	$\frac{2^{1/2}}{6}$
											0	0	1	0	0	0	0	0	0	0	0	$\frac{3^{1/2}}{6}$
											0	0	0	1	0	0	0	0	0	0	0	$\left \frac{2}{6}\right $
											0	0	0	0	1	0	0	0	0	0	0	$\frac{5^{1/2}}{6}$
											0	0	0	0	0	1	0	0	0	0	0	$\frac{6^{1/2}}{6}$
											0	0	0	0	0	0	1	0	0	0	0	$\frac{5^{1/2}}{6}$
											0	0	0	0	0	0	0	1	0	0	0	$\left \frac{2}{6}\right $
											0	0	0	0	0	0	0	0	1	0	0	$\frac{3^{1/2}}{6}$
											0	0	0	0	0	0	0	0	0	1	0	$\frac{2^{1/2}}{6}$
											0	0	0	0	0	0	0	0	0	0	1	$\left \frac{1}{6} \right $

Doing the matrix multiplication gives you

 $\langle \psi | \mathbf{I} | \psi \rangle = 1$

So the expectation value of the identity operator is 1.

What is [A, B] in terms of [B, A]? The answer is [A, B] = -[B, A].

The definition of a commutator [A, B] is [A, B] = AB - BA. And [B, A] = BA - AB, or [B, A] = -AB + BA. You can write this as

 $[\mathsf{B},\mathsf{A}]=-\!(\mathsf{A}\mathsf{B}-\mathsf{B}\mathsf{A})$

But AB - BA = [A, B], so [B, A] = -[A, B], or

[A, B] = -[B, A]

What is the Hermitian adjoint of a commutator $[A, B]^{\dagger}$ if A and B are Hermitian operators? The answer is $[A, B]^{\dagger} = -[A, B]$.

You want to figure out what the following expression is:

[A, B][†]

Expanding gives you

 $\left[A,B\right]^{\dagger} = \left(AB - BA\right)^{\dagger}$

And expanding this gives you

 $\left[A, B \right]^{\dagger} = \left(AB - BA \right)^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger}$

For Hermitian operators, $A = A^{\dagger}$, so

$$\left[A,B\right]^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = BA - AB$$

And BA – AB is just –[A, B], so you get

where A and B are Hermitian operators.

 $\left[\mathbf{A}, \mathbf{B} \right]^{\dagger} = -\left[\mathbf{A}, \mathbf{B} \right]$



Note that when you take the Hermitian adjoint of an expression and get the same thing back with a negative sign in front of it, the expression is called *anti-Hermitian*, so the commutator of two Hermitian operators is anti-Hermitian.

13 What are the eigenvalues and eigenvectors of this operator?

$$\mathbf{A} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

The eigenvalues of A are $a_1 = 1$ and $a_2 = 3$. The eigenvectors are

$$\begin{vmatrix} 1 \\ -1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

First find A – *a*I:

$$\mathbf{A} - \lambda \mathbf{I} = \begin{vmatrix} 2 - a & 1 \\ 1 & 2 - a \end{vmatrix}$$

Now find the determinant:

$$det(A - aI) = (2 - a)(2 - a) - 1$$
$$= a^2 - 4a + 3$$

Factor this into

$$\det(A - aI) = a^2 - 4a + 3 = (a - 1)(a - 3)$$

So the eigenvalues of A are $a_1 = 1$ and $a_2 = 3$.

To find the eigenvector corresponding to a_1 , substitute a_1 into A – aI:

$$\mathbf{A} - a\mathbf{I} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

Because (A - aI)x = 0, you have

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

Because every row of this matrix equation must be true, you know that $x_1 = -x_2$. And that means that up to an arbitrary constant, the eigenvector corresponding to a_1 is

$$\begin{vmatrix} c \\ -1 \end{vmatrix}$$

Drop the arbitrary constant, and just write this as

How about the eigenvector corresponding to a_2 ? Plugging a_2 in gives you

$$\mathbf{A} - a\mathbf{I} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

Then you have

1

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

So x_1 = 0, and that means that up to an arbitrary constant, the eigenvector corresponding to a_2 is

Drop the arbitrary constant and just write this as

14 What are the eigenvalues and eigenvectors of this operator?

$$A = \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}$$

The eigenvalues of A are $a_1 = 2$ and $a_2 = -1$. The eigenvectors are

$$\begin{vmatrix} 1 \\ 1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 \\ 4 \end{vmatrix}$$

First, find A – *a*I:

$$\mathbf{A} - a\mathbf{I} = \begin{vmatrix} 3 - a & -1 \\ 4 & -2 - a \end{vmatrix}$$

Now find the determinant:

$$det(A - aI) = (3 - a)(-2 - a) + 4$$
$$= a^2 - a - 2$$

Factor this into

$$\det(A - aI) = a^2 - a - 2 = (a + 1)(a - 2)$$

So the eigenvalues of A are $a_1 = 2$ and $a_2 = -1$.

To find the eigenvector corresponding to a_1 , substitute a_1 into A – aI:

$$\mathbf{A} - a\mathbf{I} = \begin{vmatrix} 1 & -1 \\ 4 & -4 \end{vmatrix}$$

Because (A - aI)x = 0, you have

$$\begin{vmatrix} 1 & -1 \\ 4 & -4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

Because every row of this matrix equation must be true, you know that $x_1 = x_2$. And that means that up to an arbitrary constant, the eigenvector corresponding to a_1 is

$$c \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

Drop the arbitrary constant and just write this as

How about the eigenvector corresponding to a_2 ? Plugging a_2 in gives you

$$\mathbf{A} - a\mathbf{I} = \begin{vmatrix} 4 & -1 \\ 4 & -1 \end{vmatrix}$$

Then you have

$$\begin{vmatrix} 4 & -1 \\ 4 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

So $4x_1 = x_2$, and that means that up to an arbitrary constant, the eigenvector corresponding to a_2 is

$$\begin{array}{c|c}c 1\\4\end{array}$$

Drop the arbitrary constant and just write this as

$$\begin{vmatrix} 1 \\ 4 \end{vmatrix}$$