Chapter 1

Foundations of Maxwell's Equations

LEARNING OBJECTIVES

- Review selected chronological developments of electromagnetic concepts
- Appreciate the role of electromagnetic theory in electrical engineering
- Use fundamental electromagnetic field quantities, units, and universal constants
- Use statistical concepts for determining the precision of a measured number
- Understand and apply principles of complex variables and phasor notation

1.1 HISTORICAL OVERVIEW

Some credit the existence of electric charge to a discovery more than two and a half thousand years ago by a Greek astronomer and philosopher, Thales of Miletus. He found that an amber (*ήλεκτρον*) rod, after being rubbed with silk or wool, would attract straw and small pieces of parchment. The Greek word for amber is *éléktron*, from which the words *electron*, *electronics*, *electricity*, *electromagnetic*, and *electrical engineer* are derived.

The discovery of the magnetic polarities of lodestone (*μάγνηζ*), a natural material found in the Thessalian Magnesia, from which we derive[i] the name *magnetic*, by Pierre de Maricourt occurred around 1269. From that time through the early seventeenth century, progress in the study of magnetism was slow, but, during the seventeenth century, there were notable contributions by a number of scientists toward understanding magnetism. A. Kirchner demonstrated that the two poles of a magnet have equal strength, and Newton attempted to formulate the laws governing the forces between bar magnets.

The inverse square law of electric and magnetic forces was not postulated until John Michell proposed it in 1750 and Coulomb confirmed it in 1785. Coulomb's

Maxwell's Equations, by Paul G. Huray

law may be said to be the starting point of modern electromagnetic theory. Subsequent landmark developments in electromagnetic theory include the derivation by Laplace in 1782 and Poisson in 1813 of the famous equations that bear their names. Gauss published the divergence theorem, often called Gauss's law, in the same year.

Experiments with electric current could be performed only after invention of the battery by Volta in 1800. Having a source for generating a continuous current, Oersted, in 1820, was able to demonstrate the production of magnetic fields by electric currents. His discovery prompted others to investigate the relationship between electric current and magnetic fields. In 1820, Ampere announced a discovery relating to the forces between electric current-carrying conductors and magnets and the mutual attraction or repulsion of two electric currents. These experiments led to the formulation of what is now called Ampere's law. In 1820, Biot and Savart repeated Oersted's experiment to determine a law of force between current carrying conductors, giving us the so-called Biot–Savart law.

During the period of Oersted and Ampere, Faraday was also experimenting on the interaction between current-carrying conductors and magnetic fields and developed an electric motor in 1821. Faraday's experiments on developing induced currents by changing magnetic fields led to the law of electromagnetic induction in 1831. Faraday also proposed the concept of magnetic lines of force and laid the foundation of electromagnetic field theory.

In 1864, Maxwell proposed[ii] *A Dynamical Theory of the Electromagnetic Field* and thus unified the experimental researches of over a century through a set of equations known as Maxwell's equations. These equations were verified by Hertz in 1887 in a brilliant sequence of demonstrations. It is now generally accepted that all electromagnetic phenomena are governed by Maxwell's equations.

1.2 ROLE OF ELECTROMAGNETIC FIELD THEORY

Electromagnetic field theory is the study of the electric and magnetic phenomena caused by electric charges, q, at rest or in motion. There are two kinds of electric charges, positive and negative, following a definition given by Benjamin Franklin. Both positive and negative charges are sources of an electric field intensity,[1] $\boldsymbol{E}$ (or $\vec{E}$). Moving charges produce a current that can further give rise to a magnetic field intensity, $\boldsymbol{H}$ (or $\vec{H}$). A vector field is defined as a spatial distribution of a vector quantity, which may or may not be function of time. A time-varying electric field intensity is always accompanied by a magnetic field intensity and vice versa. In other words, time-varying electric and magnetic field intensities are intrinsically coupled and result in an electromagnetic field intensity. Time-dependent electromagnetic field intensities produce waves that radiate from their source toward an observation point. Many authors call this the causality principle because, they argue, the phenomenon does not work in the opposite direction. But, in our study of electromagnetic

[1] Other authors often use a bold type or a capital letter with an overhead vector to represent a vector quantity and interchange the two designations freely. This book will also color-code the electric field intensity and magnetic field intensity to make their representation clear in equations and drawings.

fields propagating in a waveguide, we will see that boundary conditions at a conducting surface boundary require a charge density distribution to support the electric and magnetic field intensities defined by Maxwell's equations. Because conduction electrons do not travel at velocities comparable with field propagation velocities, we can argue that the surface charges on the conductors must be induced by the field intensities. Such a picture of the physical universe gives symmetry to nature as defined by a "*principle of equivalence*": It is equivalent to view charges and currents as the source of electromagnetic fields or to view electromagnetic fields as the source of induced charges and currents.

The concept of propagating fields and waves is essential in the explanation of action at a distance. Satellite and mobile communications demonstrate that electric fields and magnetic fields propagate; that electromagnetic waves move in free space or in a medium such as air, water, resin fiberboard, or any other material. As we will see, they propagate *without* the presence of a luminiferous ether or "jelly."

Electromagnetic field theory is important in that it can explain many phenomena and solve complicated problems that conventional circuit theory cannot address. For instance, a mobile antenna can receive signals transmitted from base stations, where there are no physical connections between the transmitter and receiving antennas, and no free-space currents or voltages defined as in circuit theory. Another good example is the strong coupling that may exist between components printed some distance apart on circuit boards even though there are no identifiable resistance, capacitance, or inductance elements between them. By using computer techniques and electromagnetic theory, however, the intentional coupling between widely separated antennas and the unintentional coupling between nearby circuit components can be accurately predicted. In the discipline of Signal Integrity, the phenomenon is called "cross talk."

1.3 ELECTROMAGNETIC FIELD QUANTITIES

Historically, quantities in electromagnetic field theory are divided into two categories: source quantities and field quantities. The *source* of an electromagnetic field usually refers to electric charges at rest or in motion, while field quantities are usually observed or computed at an *observation* or *field* point. In this chapter, we will distinguish between classical view of cause and effect, at least for the purpose of discussion by routinely displaying source coordinates with a prime, for example, (x', y', z'); and field or observation coordinates as unprimed, for example, (x, y, z). However, we are mindful that it is equivalent to take the view that fields induce charges or charges induce fields and we shall see in the case of field propagation in a transmission line or waveguide that this duality can lead to a more complete understanding of power loss.

Electric Charges and Charge Densities

The symbol q or Q is used to denote electric charge, which is a fundamental property of matter and exists only in positive or negative integral multiples of the charge on an electron, $-e$, where

$$e = 1.60217653(14) \times 10^{-19}\ \text{C}. \tag{1.1}$$

C is the abbreviation for the meter–kilogram–second (or International System of Units [SI]) unit of charge, coulomb.[2] A coulomb is a very large unit for charge because it takes $1/1.60 \times 10^{-19}$ or 6.25×10^{18} electrons to make up 1 C. The quantity in parenthesis (14) is the standard deviation in all measurements that have been compiled by the National Institute of Standards and Technology to obtain an average of the measured values of e. This and other quantities described below can be found at http://physics.nist.gov/cgi-bin/cuu/Value?e.

The principle of conservation of electric charge is a fundamental postulate. The statement that electric charge is *conserved* simply means that it can neither be created nor destroyed. The principle of conservation of electric charge must be satisfied at all times and in all situations in electrical engineering.

Next, we define a volume charge density, ρ_v, as a source quantity as follows:

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \left(\text{C/m}^3\right), \tag{1.2}$$

where Δq is the amount of charge in a very small volume Δv. In many cases, an amount of charge Δq may be identified with an element of surface, Δs, or an element of line, Δl. In such cases, it will be more appropriate to define a surface charge density, Σ_s, or a line charge density, λ_l:

$$\Sigma_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \left(\text{C/m}^2\right) \tag{1.3}$$

$$\lambda_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \left(\text{C/m}\right). \tag{1.4}$$

In general, all charge densities are point functions of space coordinates and may also be time dependent. In some texts, the surface charge density, Σ_s, may be labeled σ_s or ρ_s, and the line charge density, λ_l, may be labeled ρ_l. Alternate labeling is necessary in preventing confusion when, in the same section or publication, we discuss electrical conductivity, traditionally labeled σ; and/or scattering cross section, traditionally labeled σ_s. Likewise, we often refer to the distance to the z-axis in cylindrical coordinates by the symbol ρ.

Current and Current Density

Electric current is the rate of transfer of charge across a reference surface with respect to time;[3] that is,

[2] One of the oddities of science and technology is that we traditionally do not capitalize the written unit that represents a person's name (like Coulomb) unless the symbol (e.g., C) is used for that unit.

[3] As mentioned earlier, a time-varying electric field intensity is always accompanied by a magnetic field intensity and vice versa. In this book, charges and electric field intensities will be colored red and currents and magnetic field intensities blue. Thus, a time derivative of a red quantity produces a blue quantity, as shown in Equation 1.5 and vice versa.

$$I = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt} \text{ (C/s or A)}, \tag{1.5}$$

where the unit of current is a coulomb per second (C/s) or ampere (A). A physical current must flow through a finite area; hence, it is not a point function but may be time dependent. However, in electromagnetic field theory, we define a vector point function, current density, $\vec{J}$, which measures the amount of current flowing through a unit area normal to the direction of current flow. The current density $\vec{J}$ is a vector whose magnitude and direction are the current per unit area (A/m^2), and the direction of current flow at a point in space, respectively. $\vec{J}$ may also be a time-dependent quantity.

Electromagnetic Field Quantities

An electromagnetic field can be described by four field quantities:

Electric field intensity	$\vec{E}$ (V/m);
Electric flux density or displacement	$\vec{D}$ (C/m^2);
Magnetic field intensity	$\vec{H}$ (A/m); and
Magnetic flux density	$\vec{B}$ (Wb/m^2 or T).

Here, the unit T stands for the tesla or volt-second per square meter and is named in honor of Nikola Tesla (1857–1943), who helped the understanding of rotating field poles in electric motors and transformers. The electric field intensity $\vec{E}$ is the vector field used in electrostatics when charge is at rest in free space and is defined as the electric force on a unit test charge. The electric displacement vector $\vec{D}$ (also called the electric flux density or displacement flux) is a vector field used in studying the electric fields inside material objects. Similarly, magnetic field intensity $\vec{H}$ is a vector needed in discussing magnetic phenomonen, that is the field generated at a point in free space by steady or time-varying electric currents in a source; it is related to the magnetic force acting on a moving charge. The magnetic flux density $\vec{B}$ is useful in the investigation of the magnetic fields within material objects where the material modifies the field intensity.

When there is no time variation in field quantities, the electric field quantities ($\vec{E}$, $\vec{D}$) are independent from the magnetic field quantities ($\vec{H}$, $\vec{B}$). In time-dependent cases, however, the electric and magnetic fields are coupled; that is, time-varying ($\vec{E}$, $\vec{D}$) will give rise to ($\vec{H}$, $\vec{B}$) and vice versa. The electromagnetic properties of materials are governed by the so-called *constitutive* relations between $\vec{E}$ and $\vec{D}$, and $\vec{H}$ and $\vec{B}$. The equations that represent these constitutive relations are called Maxwell's equations.

1.4 UNITS AND UNIVERSAL CONSTANTS

In this book, as in most contemporary engineering texts, we will adhere to the SI, often called the meter–kilogram–second system built from seven basic units, as shown in Table 1.1. All derived units can be expressed in terms of these quantities.

In the SI system, the speed of light is an exact quantity as a consequence of the definition of the meter adopted in 1983, the definition of the kilogram adopted in 1889, the definition of the second adopted in 1967:

1. **Meter** is the length of the path traveled by light in a vacuum during a time interval of 1/299,792,458 of a second.
2. **Kilogram** is the unit of mass; it is equal to the mass of the international prototype of the kilogram.
3. **Second** is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.
4. **Ampere** is that constant current that, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.
5. **Kelvin**, the unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.
6. **Mole** is the amount of substance of a system that contains as many elementary entities as there are atoms in 0.012 kg of carbon 12; its symbol is "mol." When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.
7. **Candela** is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} Hz and that has a radiant intensity in that direction of 1/683 W per steradian.

Table 1-1. Seven Basic Units

Quantity	Unit	Abbreviation
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

In electromagnetic field expressions, we frequently encounter three constants: the speed of light (and all other electromagnetic waves) in free space, c, the dielectric permittivity of free space, ε_0, and the magnetic permeability of free space, μ_0. Note that there is no uncertainty (no standard deviation) in any of these terms because they are defined exactly.

We define

$$c \equiv 299{,}792{,}458\,(\mathrm{m/s}) \approx 3\times 10^8\,(\mathrm{m/s}). \tag{1.6}$$

In addition, the magnetic permeability of free space, μ_0, is defined as

$$\mu_0 \equiv 4\pi\times 10^{-7}\,(\mathrm{H/m})\left(\text{or } \mathrm{N/A^2}\right)(\text{or } \Omega\mathrm{s/m}). \tag{1.7}$$

Thus, using an equality that we will later derive for free space that includes the electric permittivity of free space, ε_0, we can deduce the exact value

$$\varepsilon_0 \equiv 1/\mu_0 c^2 = 8.854187817\ldots\times 10^{-12}\,(\mathrm{F/m})\left(\text{or } \mathrm{C^2/N\,m^2}\right)(\text{or } \mathrm{s}/\Omega\mathrm{m}) \tag{1.8}$$

where the units H/m and F/m stand for henry per meter and farad per meter, respectively. We again note that, because they are defined, there is no uncertainty in any of the constants c, ε_0, or μ_0.

For convenience, we will often use the value 3×10^8 m/s for the speed of light because it is easier to recall than the defined figure, and, consistent with this approximation and Equation 1.8, we will often use the approximation $1/36\pi \times 10^9\,\mathrm{C^2/N\,m^2}$ for ε_0. This practice is common in the study of electromagnetic fields and seldom leads to significant error. Nonetheless, in critical computations, the more accurate values of c and ε_0 may be required.

In free space, the constants ε_0 and μ_0 are the proportionality constants between the electric field intensity, $\vec{E}$, and the electric flux density, $\vec{D}$, and the magnetic field intensity, $\vec{H}$, and the magnetic flux density, $\vec{B}$, respectively, such that

$$\vec{D} = \varepsilon_0 \vec{E}\ (\text{in free space}) \tag{1.9}$$

$$\vec{B} = \mu_0 \vec{H}\ (\text{in free space}). \tag{1.10}$$

Finally, we note that the force, $\vec{F}_{12}$, between two charges, q_1 and q_2, is given by the experimentally confirmed Coulomb's law, which is expressed as

$$\vec{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2}\hat{a}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{a}_{12}, \tag{1.11}$$

where k_e is Coulomb's constant and is approximately equal to $9 \times 10^9\,\mathrm{N\,m^2/C^2}$.

From Equation 1.11, we can see that a measurement of $\vec{F}_{12}$ (in kg m/s^2) and of r_{12}^2 (in m^2), with the derived quantity for ε_0 of Equation 1.8 for two identical charges, q, leads to a measured value of charge, q, in units of mass, length, and time.

The measured value of the ampere as defined by the SI is found from the force created by two parallel wires of length dl_1-carrying current I_1 and dl_2-carrying current I_2 respectively, by the Biot–Savart force law:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi}\frac{(I_1 dl_1)(I_2 dl_2)}{r_{12}^2}\hat{a}_{12}. \tag{1.12}$$

As we will see, the inverse square "law" of Coulomb (Equation 1.11) and the inverse square "force law" of Biot–Savart (Equation 1.12) also lead us to Maxwell's equations. Many researchers have tried, unsuccesfully to date, to measure any deviation from the inverse square law for these quantities. It is worthy of note that the gravitational force between two masses, m_1 and m_2, separated by r_{12} follows a mathematical expression similar to that of Coulomb's law or the Biot–Savart law:

$$\vec{F}_{12} = G\frac{m_1 m_2}{r_{12}^2}\hat{a}_{12} = \frac{1}{4\pi\varepsilon_g}\frac{m_1 m_2}{r_{12}^2}\hat{a}_{12}, \tag{1.13}$$

where $G = 6.673(10) \times 10^{-11}\,\mathrm{m^3/kg\,s^2}$ is the gravitational constant. In this equation, the author has chosen to define a new constant, ε_g, so that the gravitational force law looks the same as Coulomb's law.

The symmetry of the equations leads the casual observer to postulate[4] another force due to the *mass current* K_1 and K_2 in two parallel lengths dl_1 and dl_2, respectively:

$$\vec{F}_{12} = \frac{\mu_g}{4\pi}\frac{(K_1 dl_1)(K_2 dl_2)}{r_{12}^2}\hat{a}_{12}. \tag{1.14}$$

Here, another constant, μ_g, has been defined in order to make the force due to *mass currents* symmetric to the Biot–Savart force law. This force has been postulated by others and a measurement of it is being attempted by a group of researchers from Stanford and NASA.[iii]

Many researchers have also tried to measure deviations from an inverse square law for Equation 1.13 as well. It is partly the similarity of these forces that gives us confidence that the "laws" are correct. However, we should note that more powerful forces within the nucleus, the weak and strong forces, do not obey an inverse square law, so we should leave open the possibility that a future correction may need to be made to any one or all of these "laws."

EXERCISES

1.1 Compare the gravitational and electric forces[5] between a proton and an electron if they are separated by the same distance, as shown in Figure 1.1.

SOLUTION Suppose an electron at $\vec{r}_1$ and a proton at $\vec{r}_2$, so the distance between them is $r_{12} = |\vec{r}_{12}|$ where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$.

We know from Coulomb's law that $\vec{F}_{12} = k_e(q_1 q_2 / r_{12}^2)\hat{r}_{12}$ is the electrostatic force between the electron and the proton and from Newton's law

[4] Areas of speculation are often used in this text in shaded boxes; they are intended to stimulate thinking of the student on a topic she might not otherwise have considered.

[5] Gravitational force ≪ weak force ≪ electromagnetic force ≪ strong forces.

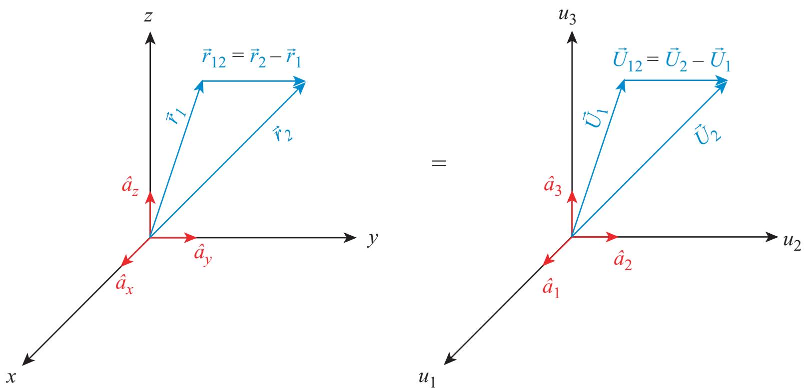

Figure 1.1 Vector representations of physical locations in space.

that $\vec{F}_{12} = G(m_1 m_2/r_{12}^2)\hat{r}_{12}$ is the gravitational force between the electron and the proton.

NOTE Both terms are attractive, both are proportional to the product of two measured quantities (charge and mass, respectively), and both are proportional to the inverse square of their separation. The gravitational constant G is $6.67 \times 10^{-11}\,\mathrm{m^3/kg\,s^2}$ and the electric constant k_e is $8.99 \times 10^9\,\mathrm{N\,m^2/C^2}$. The measured masses and charges are $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$, $q_e = -1.60 \times 10^{-19}\,\mathrm{C}$, $m_p = 1.67 \times 10^{-27}\,\mathrm{kg}$, $q_p = +1.60 \times 10^{-19}\,\mathrm{C}$. Thus,

$$\vec{F}_{\text{due to electrostatic charges}} = \left(8.99\times10^{9}\,\mathrm{N\,m^2/C^2}\right)\left(1.60\times10^{-19}\,\mathrm{C}\right)\left(1.60\times10^{-19}\,\mathrm{C}\right)\left(\hat{r}_{12}/r_{12}^2\right)$$

$$\vec{F}_{\text{due to gravity}} = \left(6.67\times10^{-11}\,\mathrm{N\,m^2/kg^2}\right)\left(1.67\times10^{-27}\,\mathrm{kg}\right)\left(9.11\times10^{-31}\,\mathrm{kg}\right)\left(\hat{r}_{12}/r_{12}^2\right)$$

and the ratio of these two forces is 2.27×10^{39}, independent of their separation.

SOLUTION The electrostatic force between an electron and a proton is so much larger than the gravitational force between an electron and a proton that we may ignore the gravitational forces.

1.2 Using classical arguments for an electron bound to a proton in a hydrogen atom with a circular radius of 1 Å, determine its tangential velocity.

SOLUTION From Exercise 1.1, $\vec{F}_{\text{due to electrostatic charges}} = 2.30\times10^{-28}\,\mathrm{Nm^2}(\hat{r}_{12}/r_{12}^2)$.

For $r_{12}^2 = (1\,\text{Å})^2 = 10^{-20}\,\mathrm{m^2}$, $\vec{F}_{\text{due to electrostatic charges}} = 2.30 \times 10^{-8}\,\mathrm{N}\hat{r}_{12}$. This force seems small until you use it to compute the acceleration of an electron:

$$\begin{aligned}|a_e| &= \left|\vec{F}_{\text{due to electrostatic charges}}\right|/m_e = 2.3\times10^{-8}\,\mathrm{N}/9.11\times10^{-31}\,\mathrm{kg}\\ &= 2.52\times10^{22}\,\mathrm{m/s^2} = 2.58\times10^{21}\,\mathrm{g},\end{aligned}$$

where g is the acceleration of gravity.

From our knowledge of centripetal forces, for an electron circling a proton at a radius of 1 Å, $a_e = v_t^2/r$, where v_t is the electron's tangential velocity. Thus

$$v_t^2 = ra_e = \left(10^{-10}\,\mathrm{m}\right)\left(2.52\times10^{22}\,\mathrm{m/s^2}\right) \text{ or } v_t = 1.58\times10^{6}\,\mathrm{m/s}.$$

CONCLUSION The tangential velocity of an electron circling a hydrogen nucleus (a proton) is approximately 0.5% the speed of light.[6] For larger Z atoms, the accelerations and tangential velocities will be even closer to the speed of light, so we must take into account relativistic effects when computing electron velocities for heavy atoms.

Electrical engineers prefer to express the properties of an electromagnetic wave[iv] via its frequency, f (in Hz), or its wavelength, λ (in m), which are related by

$$c = \lambda f. \tag{1.15}$$

Physicists and astronomers often express the properties of an electromagnetic wave via its energy, E, or its temperature, T, which are related by

$$E = hf, \tag{1.16}$$

where

$$h = 6.62606876(52)\times 10^{-34}\ \mathrm{J\,s}\ \left(\text{or } 4.13566727(16)\times 10^{-15}\ \mathrm{eV\,s}\right)$$

is Plank's constant.

$$E = k_B T, \tag{1.17}$$

where k_B = 1.3806503(24) × 10^{-23} J/K is Boltzmann's constant.

1.3 Find the wavelength, energy, and temperature of a 2.4-GHz wave.[7]

SOLUTION

$$\lambda = \frac{c}{f} = \frac{3.00\times 10^{8}\ \mathrm{m/s}}{2.40\times 10^{9}\ 1/\mathrm{s}} = 0.125\,\mathrm{m} = 12.5\,\mathrm{cm}$$

$$E = hf = \left(4.14\times 10^{-15}\ \mathrm{eV\,s}\right)\left(2.40\times 10^{9}\ 1/\mathrm{s}\right) = 9.94\times 10^{-6}\ \mathrm{eV} = 1.59\times 10^{-24}\ \mathrm{J}$$

$$T = \frac{E}{k_B} = \frac{1.59\times 10^{-24}\ \mathrm{J}}{1.38\times 10^{-23}\ \mathrm{J/K}} = 0.115\,\mathrm{K}$$

1.4 Find the wavelength, frequency, and characteristic temperature of a 1 keV x-ray.[8]

SOLUTION

$$T = \frac{E}{k_B} = \frac{\left(10^{3}\right)\left(1.60\times 10^{-19}\ \mathrm{C}\right)V}{1.38\times 10^{-23}\ \mathrm{J/K}} = 11.6\times 10^{6}\ \mathrm{K}$$

$$f = \frac{E}{h} = \frac{\left(10^{3}\right)\mathrm{eV}}{4.14\times 10^{-15}\ \mathrm{eV\,s}} = 2.42\times 10^{17}\ \mathrm{Hz}$$

[6] Had our answer come out closer to the speed of light, we would need a recalculation using the special theory of relativity for mass rather than the classical theory for rest mass.

[7] This frequency is common in computer central processing units (CPUs), cell phones, and microwave ovens.

[8] This energy is common at the face of a cathode ray tube (CRT) if electrons are accelerated by a 1 kV potential.

$$\lambda = \frac{c}{f} = \frac{3.00\times10^{8}\ \mathrm{m/s}}{2.42\times10^{17}\ 1/\mathrm{s}} = 1.24\times10^{-9}\ \mathrm{m} = 12.4\ \text{Å}$$

1.5 Because $1/\sqrt{\mu_0\varepsilon_0} = c$ for electromagnetic waves, it is not unreasonable to postulate that $1/\sqrt{\mu_g\varepsilon_g} = c$ for gravomagnetic waves. With this postulate, find the value of the constants ε_g and μ_g and compare the magnitude of the force caused by a 1 C/s electrical current with that of a 1 kg/s mass current if they are in the same lengths and have the same distance of separation.

SOLUTION
If $G = 6.673 \times 10^{-11}\ \mathrm{N\,m^2/kg^2} = 1/4\pi\varepsilon_g$, then $\varepsilon_g = 1.193 \times 10^{9}\ \mathrm{kg\,s^2/m^3}$.
If $1/\sqrt{\mu_g\varepsilon_g} = c$, then $\mu_g = 9.317 \times 10^{-27}$ m/kg.

$$\frac{F_{\text{charge current}}}{F_{\text{mass current}}} = \frac{\mu_0}{\mu_g} = 1.35\times10^{20}$$

1.6 If there is one "free electron" (conduction electron) per Cu atom,[9] compute the number of free electrons in a 1-m^3 block of Cu and find the average velocity (drift velocity) of electrons needed to produce a current of 1 C/s in one direction and the mass current of those same electrons.

SOLUTION The number of "free electrons" in a block of copper is

$$N = \frac{\text{density}}{\text{molar mass}}\text{Avogadro's number} = \frac{8.93\times10^{3}\ \mathrm{kg/m^3}}{64\times10^{-3}\ \mathrm{kg/mol}}\left(6.023\times10^{23}\ \mathrm{e/mol}\right)$$

and

$$q_N = N\left(1.60\times10^{-19}\ \mathrm{C/e}\right) = 1.34\times10^{10}\ \mathrm{C/m^3}.$$

is the "free electron" charge in a block of copper. If this charge in a 1-m^3 block is moving across one of the 1-m^2 faces at a velocity of 1 m/s, then it will produce a current $I = 1.34 \times 10^{10}$ C/s. Thus, to produce an electric current of 1 C/s, $\langle v \rangle$ need be only 7.44×10^{-11} m/s.

The mass of the "free electrons" in the block of copper is $m_N = N\,(9.11 \times 10^{-31}\ \mathrm{kg/e}) = 0.0766\ \mathrm{kg/m^3}$. If this mass is moving at an average velocity of 7.44×10^{-11} m/s, then the mass current across a 1-m^2 face of the block will be 5.70×10^{-11} kg/s.

[9] The designation "free electrons" is given by those electrons outside the bound core of an ion; these electrons interact with their neighbors to such an extent that they lose track of which one was their parent and thus are "free" to move in the conductor.

1.5 PRECISION OF MEASURED QUANTITIES

Standard Uncertainty and Relative Standard Uncertainty

Definition

The **standard uncertainty** σ_y of a measurement result, y, is the estimated standard deviation of y.

Meaning of Uncertainty

If the probability distribution characterized by the measurement result y and its standard uncertainty σ_y is approximately normal (Gaussian), and σ_y is a reliable estimate of the standard deviation of y, then the interval from $y - \sigma_y$ to $y + \sigma_y$ is expected to encompass approximately 68.26% of the distribution of values that could reasonably be attributed to the value of the quantity Y of which y is an estimate. This implies that it is believed with an approximate level of confidence of 68.26% that Y is greater than or equal to $y - \sigma_y$ and is less than or equal to $y + \sigma_y$, which is commonly written as $Y = y \pm \sigma_y$.

Use of Concise Notation

If, for example, $y = 1234.56789\,\mathrm{U}$ and $\sigma_y = 0.00011\,\mathrm{U}$, where U is the unit of y, then $Y = (1234.56789 \pm 0.00011)$ U. A more concise form of this expression, and one that is in common use, is $Y = 1234.56789(11)$ U, where it understood that the number in parentheses is the numerical value of the standard uncertainty referred to the corresponding last digits of the quoted result.

Appendix A contains a review of statistical definitions, examples, and interpretations. See http://physics.nist.gov/cuu/Uncertainty/index.html for additional information.

1.6 INTRODUCTION TO COMPLEX VARIABLES

Complex numbers are frequently used in the applications of electromagnetic applications. In this section, the definition and fundamental operations of complex numbers and complex variables will be reviewed.

A complex number, z, can be written as

$$z = \mathrm{Re}(z) + j\,\mathrm{Im}(z) = x + jy, \tag{1.18}$$

where x and y are both real numbers; x is said to be the real (Re) part of z, y is said to be the imaginary (Im) part of z, and $j = \sqrt{-1}$. z can be also be expressed in polar form by

$$z = |z|e^{j\theta}, \tag{1.19}$$

where $|z|$ and θ are both real and are called the *amplitude* and *phase* of z. With the use of Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta, \tag{1.20}$$

we obtain

$$z = |z|\cos\theta + j|z|\sin\theta. \tag{1.21}$$

Comparing Equations 1.20 and 1.21, we conclude

$$x = |z|\cos\theta, \tag{1.22a}$$
$$y = |z|\sin\theta, \tag{1.22b}$$

and, inversely,

$$|z| = \sqrt{x^2 + y^2} \tag{1.23a}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad 0 \le \theta \le 2\pi. \tag{1.23b}$$

The above relations can be graphically represented as shown in Figure 1.2.

The complex conjugate of z, designated with an asterisk (∗), is a complex number that replaces j with $-j$ in all places; that is,

$$z^* = (x + jy)^* = (x - jy) = |z|e^{-j\theta}. \tag{1.24}$$

The magnitude of z is the square root of the product of z and its complex conjugate:

$$|z| = \sqrt{z \cdot z^*} = \sqrt{(x + jy)(x + jy)^*} = \sqrt{x^2 + y^2} \tag{1.25a}$$

or

$$|z| = \sqrt{|z|e^{j\theta}|z|e^{-j\theta}} = \sqrt{x^2 + y^2}. \tag{1.25b}$$

Arithmetic Operations with Complex Numbers

Arithmetic with complex numbers is tedious when carried out by hand but otherwise is very much like arithmetic with real numbers.

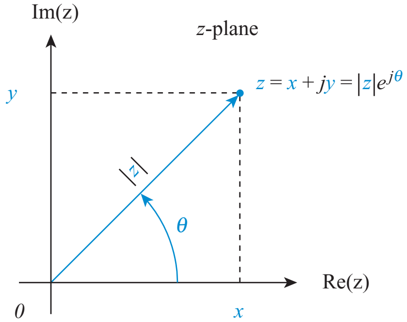

Figure 1.2 The relation between rectangular and polar coordinates.

1. Addition and subtraction:

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2) \tag{1.26a}$$

$$z_1 = z_2 = (x_1 + jy_1) - (x_2 + jy_2) = (x_1 - x_2) + j(y_1 - y_2) \tag{1.26b}$$

2. Multiplication:

$$z_1 \cdot z_2 = (x_1 + jy_1) \cdot (x_2 + jy_2) = (x_1 \cdot x_2 - y_1 \cdot y_2) + j(x_1 \cdot y_2 + x_2 \cdot y_1) \tag{1.27a}$$

In the polar form, the multiplication of two complex numbers can be written as

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| e^{j\theta_1} |z_2| e^{j\theta_2} = |z_1 z_2| e^{j(\theta_1 + \theta_2)} \\ &= |z_1 z_2| [\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)] \end{aligned}. \tag{1.27b}$$

3. Division: For any $z_2 \neq 0$,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(x_1 + jy_1)}{(x_2 + jy_2)} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} \frac{(x_2 - jy_2)}{(x_2 - jy_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 + x_1 y_2)}{\left(x_2^2 + y_2^2\right)} \end{aligned} \tag{1.28a}$$

or

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)} \\ &= \frac{|z_1|}{|z_2|} [\cos(\theta_1 - \theta_2) + j \sin(\theta_1 - \theta_2)] \end{aligned}. \tag{1.28b}$$

4. Power: For any positive or negative integer n, we have

$$\begin{aligned} z^n &= \left[|z| e^{j\theta}\right]^n = |z|^n e^{jn\theta} \\ &= |z|^n [\cos(n\theta) + j \sin(n\theta)] \end{aligned}. \tag{1.29}$$

Arithmetic Functions of Complex Numbers (Complex Variables)

A function of a complex number could be a combination of addition, multiplication, power, or other functions of a complex quantity, for example, $(z + 1/z)$, $\sin z$, e^z, $\tanh^{-1} z$. An excellent resource for the review of functions of complex numbers is given by Spiegel.[v]

In electrical engineering, it is common for the field vectors $\vec{E}$, $\vec{D}$, $\vec{H}$, $\vec{B}$, and the current density, $\vec{J}$, to be written as complex quantities. Furthermore, real and imaginary parts of the field vectors are likely to be functions of space and time. Such variable fields are called *complex variables*. The real part of the complex field vector (e.g., $\mathrm{Re}\,\vec{E}$) is typically labeled as $u(x, y, z, t)$, and the imaginary part of the complex field vectors (e.g., $\mathrm{Im}\,\vec{E}$) is typically labeled as $v(x, y, z, t)$ in Cartesian coordinates. It is shorthand to just write the electric field vector as $\vec{E}$ without denoting the fact that it is a complex quantity that depends on space and time coordinates. Some texts remind the student of this fact by expressing $\vec{E}$ as $\vec{E}(\vec{x}, t)$ no matter if the coordinate system is Cartesian, cylindrical, spherical, or other. Some texts even put a tilde over the vector field to remind the student that $\vec{E}$ is a complex variable, but we will not choose that complicated notation here. Our advice is to always assume that a quantity in question is a complex variable unless otherwise known or stated (e.g., x, y, z, r, θ, ϕ, and t are always real).

Fortunately, with computers, the tedious manipulation of complex numbers is very easy. Not infrequently, however, we will be required to carry out derivations using complex algebraic expressions. Although even symbolic simplification can be accomplished with computers, it will nonetheless be useful to become adept at doing complex algebra by hand.

1.7 PHASOR NOTATION

In electromagnetic engineering, electric and magnetic fields that vary sinusoidally with time play a large role. In the sense that an arbitrary but otherwise periodic field can be expanded into a Fourier series of sinusoidal components and a transient nonperiodic field can be expressed as a Fourier integral, we can concentrate on analyzing steady and sinusoidal fields with the confidence that our theory can be extended to the more general situation involving nonsinusoidal time dependence.

In this section, we first review the phasor notation and then represent Maxwell's equation with the phasors. Here, we would like to illustrate the uses of phasor notation by looking at some examples. Let us consider the series resistor, inductor, capacitor (RLC) circuit shown in Figure 1.3 with an applied voltage

$$V(t) = V_0 \cos(\omega t), \tag{1.30}$$

where V_0 is the amplitude of the voltage and ω is the angular frequency (rad/s), which is equal to $2\pi f$, with f being the frequency in Hz.

Our objective is to solve for the corresponding current $i(t)$, which, in general, can be expressed as

$$i(t) = I_0 \cos(\omega t + \phi), \tag{1.31}$$

where I_0 is the current amplitude and ϕ designates the current phase.

Using Kirchhoff's voltage law, we have

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)\,dt = V(t). \tag{1.32}$$

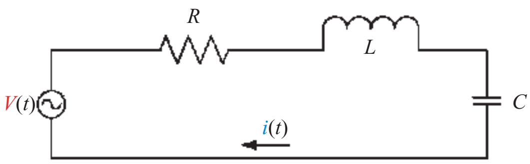

Figure 1.3 A series RLC circuit.

Using the phasor notation, we can express[10]

$$\begin{aligned} V(t) = V_0 \cos(\omega t) &= \mathrm{Re}\left[\left(V_0 e^{j0}\right) e^{j\omega t}\right] \\ &= \mathrm{Re}\left[V_S e^{j\omega t}\right] \end{aligned} \tag{1.33}$$

and

$$\begin{aligned} i(t) = I_0 \cos(\omega t + \phi) &= \mathrm{Re}\left[\left(I_0 e^{j\phi}\right) e^{j\omega t}\right] \\ &= \mathrm{Re}\left[I_S e^{j\omega t}\right] \end{aligned}. \tag{1.34}$$

With the phasor notation, we can deduce that

$$di/dt = \mathrm{Re}\left[j\omega I_S e^{j\omega t}\right] \tag{1.35}$$

$$\int i dt = \mathrm{Re}\left[\frac{1}{j\omega} I_S e^{j\omega t}\right]. \tag{1.36}$$

Substitution of Equations 1.35 and 1.36 into Equation 1.32 leads to

$$\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I_S = V_S \tag{1.37}$$

from which the phasor current, $I_S = V_S/Z_{Series\ RLC}$, can be easily obtained. We note that the phasor current, I_S, includes information about both the magnitude, I_0, and phase, φ, and that the corresponding instantaneous current, $i(t)$, then follows from Equation 1.34.

As seen in Equations 1.35 and 1.36, by using phasor notation, differentiation and integration in the time domain are converted to a simple algebraic operation symbolized by the following time-frequency conversion operations:

$$\frac{d}{dt} \Leftrightarrow j\omega \quad (\text{or } -i\omega \text{ in physics books}) \tag{1.38a}$$

$$\int dt \Leftrightarrow \frac{1}{j\omega} \quad \left(\text{or } \frac{1}{-i\omega} \text{ in physics books}\right). \tag{1.38b}$$

Because algebraic equations are much easier to solve than integral–differential equations, time-harmonic electromagnetic fields are much easier to analyze than time-varying fields.

NOTE Cheng and Hayt and Buck use a subscript S on the complex variable (e.g., I_S or V_S) to remind the reader that the variable is changing only with spatial quantities. Balanis and Inan and Indan use a script $\vec{\mathcal{E}}(x, y, z, t) = \vec{E}(x, y, z)e^{j\omega t}$ convention

[10] Electrical engineers almost always use the convention $e^{j\omega t}$ for the time dependence, while physicists and mathematicians conventionally use $e^{-i\omega t}$. In these cases, $j = \sqrt{-1}$ and $i = \sqrt{-1}$ so that, when we take the real part, $\cos(\omega t)$ results. However, if we take the derivative, $de^{j\omega t}/dt = j\omega e^{j\omega t}$ but $de^{-i\omega t}/dt = -i\omega e^{-i\omega t}$, we see that a negative sign occurs in equations with derivatives. We must take care to know which convention is being used in reading a given text when using Maxwell's equations.

to indicate that $\vec{E}$ varies only with spatial variables. Paul and Edminister use the convention $\vec{E}(x, y, z, t) = \vec{E}(x, y, z)e^{j\omega t}$ to indicate that $\vec{E}$ varies only with spatial variables. Pozar uses the convention that $\vec{E}(x, y, z, t) = \mathrm{Re}[\vec{E}(x, y, z)e^{j\omega t}]$, and Rao use the convention $\vec{E}(x, y, z, t) = \vec{E}(x, y, z)e^{j\omega t}$ to indicate that $\vec{E}$ varies only with spatial variables. We will adopt the Rao convention in the following sections and will color electric fields in red and magnetic fields in blue in equations and figures. When the mathematician's convention for time harmonics, $e^{-i\omega t}$, is used, we will highlight that fact by coloring the imaginary number. When $e^{j\omega t}$ is used, there will be no color for the imaginary number.

Time-Harmonic Electromagnetics

For a general time-varying electromagnetic field, the differential (or point) form of Maxwell's equations can be written as

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \tag{1.39a}$$

$$\nabla\times\vec{H} = \vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t} \tag{1.39b}$$

$$\nabla\cdot\vec{E} = \frac{\rho_v(t)}{\varepsilon} \tag{1.39c}$$

$$\nabla\cdot\vec{B} = 0. \tag{1.39d}$$

Now, considering a time-harmonic electromagnetic field with the time variation of $cos(\omega t)$, we can write the electric and magnetic fields as

$$\vec{E}(x, y, z, t) = \mathrm{Re}\left[\vec{E}(x, y, z)e^{j\omega t}\right] \tag{1.40a}$$

$$\vec{H}(x, y, z, t) = \mathrm{Re}\left[\vec{H}(x, y, z)e^{j\omega t}\right], \tag{1.40b}$$

where $\vec{E}(x, y, z)$ and $\vec{H}(x, y, z)$ are vector phasors that contain information on direction, magnitude, and phase. Using phasor relations in Equation 1.38a, we can simplify Equations 1.39a to 1.39d as

$$\nabla\times\vec{E} = -j\omega\mu\vec{H} \tag{1.41a}$$

$$\nabla\times\vec{H} = \vec{J} + j\omega\varepsilon\vec{E} \tag{1.41b}$$

$$\nabla\cdot\vec{E} = \rho_v/\varepsilon \tag{1.41c}$$

$$\nabla\cdot\vec{B} = 0, \tag{1.41d}$$

where the time variable has been eliminated from the differential form 1.39 of Maxwell's equations. Equations 1.41 are called the *time-harmonic* differential form of Maxwell's equations. Note that all fields in above equations are phasor quantities, and, to convert them to the time domain, we only have to use the relations 1.40a and 1.40b.

PROBLEMS

Using Complex Numbers

1.1 Two complex numbers are given as $z_1 = 5 - j3$ and $z_2 = 4 + j6$.

a. Express z_1 and z_2 in polar form.

b. Determine $z_1 \cdot z_2$ in rectangular and polar forms.

c. Determine z_1/z_2 in rectangular and polar form.

d. Determine $(z_1)^3$ and $(z_2)^5$ in polar form.

1.2 If $z = 3 + j4$, determine following quantities in polar form.

a. z^3

b. $|z^3|$

c. $1/|z^3|$

d. $\text{Re}(|z^3|)$

e. $\text{Im}(1/|z^3|)$

1.3 Complex numbers z_1 and z_2 are given as $z_1 = 10e^{-j\pi/4}$ and $z_2 = 5e^{j30°}$. In polar form, determine the following:

a. product $z_1 \cdot z_2$

b. ratio z_1/z_2

c. ratio z_1^*/z_2^*

d. value $\sqrt{z_1}$

1.4 If two complex number are given as $z_1 = 2 - j3$ and $z_2 = 4 + j5$, find the value of $\ln(z_1) \cdot \ln(z_2)$.

1.5 If two complex number are given as $z_1 = 4 - j3$ and $z_2 = 5 + j4$, find the value of $e^{z_1}e^{z_2}$.

Using Phasor Notation

1.6 A voltage source $V(t) = 100\cos(6\pi 10^9 t - 45°)(V)$ is connected to a series RLC circuit, as shown in Figure 1.3. If $R = 10\,\text{M}\Omega$, $C = 100\,\text{pF}$, and $L = 1\,\text{H}$, use phasor notation to find the following:

a. $i(t)$

b. $V_c(t)$, the voltage cross the capacitor

1.7 Find the phasors for the following field quantities:

a. $E_x(z, t) = E_0 \cos(\omega t - \beta z + \phi)(V/m)$

b. $E_y(z, t) = 100e^{-3z}\cos(\omega t - 5z + \pi/4)(V/m)$

c. $H_x(z, t) = H_0 \cos(\omega t + \beta z)(A/m)$

d. $H_y(z, t) = 120\pi e^{5z}\cos(\omega t + \beta z + \phi_h)(A/m)$

1.8 Find the instantaneous time domain sinusoidal functions corresponding to the following phasors:

a. $E_x(z) = E_0 e^{j\beta z}(\text{V/m})$

b. $E_y(z) = 100e^{-3z}e^{-j5z}(\text{V/m})$

c. $I_s(z) = 5 + j4(\text{A})$

d. $V_s(z) = j10e^{j\pi/3}(\text{V})$

1.9 Write the phasor expression I for the following current using a cosine reference.

a. $i(t) = I_0 \cos(\omega t - \pi/6)$

b. $i(t) = I_0 \sin(\omega t + \pi/3)$

1.10 Find the instantaneous $V(t)$ for the following phasors using a cosine reference.

a. $V_S = V_0 e^{j\pi/4}$

b. $V_S = [12 - j5](\text{V})$

1.8 QUATERNIONS

Because of its historical significance as the mathematical language of Maxwell, the subject of quaternions should be briefly known to students of electromagnetics. As mentioned in the "Introduction," Maxwell's equations were in the form of eight field equations that explicitly contained the magnetic vector potential and 12 quaternion equations that contained magnetic mass, magnetic charge, scalar magnetic potential, magnetic charge current, and magnetic conductivity of media. The complete set of equations is given in the next section. However, we must first understand the operations of the four-dimensional (4-D) complex numbers in which the formation exists. This formalization was devised by Sir William Rowan Hamilton in 1843. At that time, vector algebra and matrices had not yet been developed, but the vector dot and cross product were a result of Hamilton's work. It is said that Hamilton was walking across the Royal Canal in Dublin with his wife when the solution to quaternions came to him in the form of an equation, which he inscribed in stone on the bridge now called the Brougham or Broom Bridge. The original inscription has faded but a Quaternion plaque exists there today that reads, "Here as he walked by on the 16th of October 1843, Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.42}$$

and cut it on a stone of this bridge."[11]

In his formalism, Hamilton devised a four-vector form of a complex number that had the components of a 4-D space just as the two-dimensional (2-D) complex

[11] Equation 1.42 added for this book.

number ($a + ib$), where a and b are real and $i^2 = -1$. In quaternion language, a complex number would be written as

$$Q = a + ib + jc + kd, \tag{1.43}$$

where a, b, c, and d are real. The scalar part of the quaternion is a, and the vector part is $ib + jc + kd$. The appealing characteristics of the quaternions is that they obey the same rules of addition and multiplication as 2-D complex numbers:

$$\begin{aligned}&(a_1 + ib_1 + jc_1 + kd_1) + (a_2 + ib_2 + jc_2 + kd_2)\\ &\quad = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k\end{aligned} \tag{1.44}$$

and

$$\begin{aligned}&(a_1 + ib_1 + jc_1 + kd_1)(a_2 + ib_2 + jc_2 + kd_2)\\ &\quad = (a_1a_2) + (a_1b_2)i + (a_1c_2)j + (a_1d_2)k\\ &\qquad + (b_1a_2)i + (b_1b_2)i^2 + (b_1c_2)ij + (b_1d_2)ik\\ &\qquad + (c_1a_2)j + (c_1b_2)ji + (c_1c_2)j^2 + (c_1d_2)jk\\ &\qquad + (d_1a_2)k + (d_1b_2)ki + (d_1c_2)kj + (d_1d_2)k^2\end{aligned} \tag{1.45}$$

using the additional ring properties:

$$ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j,\ ik = -j \tag{1.46}$$

from Equation 1.42. Note that, unlike the commutative relations of 2-D complex numbers, Equation 1.46 shows that the 4-D quaternions do not commute (i.e., $ab \neq ba$).

In Appendix B, we have examined the roots of complex number equations like $z^2 + 1 = 0$ in 2-D space and found the roots to be at i and $-i$. Using the analogous equation in 4-D space, we would consider $Q^2 + 1 = 0$ and find an infinite number of solutions. We could draw the locus of these solutions in 3-D space when there was no real part ($a = 0$) for the quaternion with no real part, $Q = ib + jc + kd$ and $b^2 + c^2 + d^2 = 1$. These solutions form a unitary sphere centered on zero in the 3-D pure imaginary subspace of quaternions. We could then say that the locus of the solutions in 3-D space for a fixed real part ($a_1 = c\Delta t$) was a larger sphere with radius squared $b^2 + c^2 + d^2 = 1 + c^2\Delta t^2$ in 3-D space. Thus, the radius of the solution sphere is growing with time at a rate of $c\Delta t$. Sequencing the value of a to successively larger values would correspond to sequential spheres of larger radius. One can see that the appeal for saying the solutions in quaternion space is a movie of solutions with spheres of growing radius like the expansion of a spherical potential at constant velocity, c, in 3-D space (the scalar dimension corresponding to a multiple of c times time).

1.9 ORIGINAL FORM OF MAXWELL'S EQUATIONS

Maxwell originally introduced the following eight equations to represent the components of the electromagnetic field:

$$\vec{J} = \vec{j} + \partial\vec{D}/\partial t \tag{1.47}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{1.48}$$

$$\mu\vec{H} = \vec{\nabla} \times \vec{A} \tag{1.49}$$

$$\vec{E} = \mu\left(\vec{v} \times \vec{H}\right) - \partial\vec{A}/\partial t - \vec{\nabla}\varphi \tag{1.50}$$

$$\vec{D} = \varepsilon\vec{E} \tag{1.51}$$

$$\vec{j} = \sigma\vec{E} \tag{1.52}$$

$$\vec{\nabla} \cdot \vec{D} = -\rho_e \tag{1.53}$$

$$\vec{\nabla} \cdot \vec{j} = -\partial\rho_e/\partial t \tag{1.54}$$

While the original field equations do not exactly correspond to the Heavyside vector formulation, they will be addressed in the coming chapters. For example, the original field equations explicitly contain the magnetic vector potential, $\vec{A}$, which does not appear in the Heavyside vector formulation, but we will define $\mu\vec{H} = \vec{\nabla} \times \vec{A}$ as a mathematical convenience, and $\vec{E}_{Lorenz} = -\partial\vec{A}/\partial t - \vec{\nabla}\varphi$ as part of the Lorenz gauge, in which case the equations look alike. $\vec{E}_{motion} = \mu(\vec{v} \times \vec{H})$ is the one term that appears to be discarded. Hertz interpreted the velocity, v, as the (absolute) motion of charges relative to the luminiferous ether, but, if v is interpreted as relative velocity between charges, then the Maxwell Heavyside equations are defined for the case $v = 0$ (i.e., test charges do not move in the observer's reference frame).

Maxwell also described 12 quaternion equations by employing scalar and vector operators:

$$S \cdot Q = S \cdot (a + ib + jc + kd) = a \tag{1.55}$$

$$V \cdot Q = V \cdot (a + ib + jc + kd) = ib + jc + kd, \tag{1.56}$$

so that when he put S or V in front of a quaternion, he means that S is an operation that yields only the scalar part of the quaternion and V is an operation that yields only the vector part of a quaternion. The original equations are applied to isotropic media, normal letters imply a scalar quantity, and a capital letter implies a quaternion without the scalar:

$$\vec{B} = V \cdot \vec{\nabla}\vec{A} \tag{1.57}$$

$$\vec{E} = V \cdot v\vec{B} - \partial\vec{A}/\partial t - \vec{\nabla}\varphi \tag{1.58}$$

$$\vec{F} = V \cdot v\vec{B} + e\vec{E} - m\vec{\nabla}\Omega \tag{1.59}$$

$$\vec{B} = \vec{H} + 4\pi\vec{M} \tag{1.60}$$

$$4\pi\vec{J}_{tot} = V \cdot \vec{\nabla}\vec{H} \tag{1.61}$$

$$\vec{J} = C\vec{E} \tag{1.62}$$

$$\vec{D} = K\vec{E}/4\pi \tag{1.63}$$

$$\vec{J}_{tot} = \vec{J} + \partial\vec{D}/\partial t \tag{1.64}$$

$$\vec{B} = \mu\vec{H} \tag{1.65}$$

$$e = S \cdot \vec{\nabla}\vec{D} \tag{1.66}$$

$$m = S \cdot \vec{\nabla}\vec{M} \tag{1.67}$$

$$\vec{H} = -\vec{\nabla}\Omega. \tag{1.68}$$

The eight field Equations 1.47–1.54 and the 12 quaternions 1.57–1.68 constitute the original form of Maxwell's equations. The equations include the magnetic scalar potential, Ω, and the magnetic charge, m. The $\vec{\nabla} = id/dx_1 + jd/dx_2 + kd/dx_3$ is a quaternion operator without the scalar part. The factor of 4π came about as a result of using Gaussian or cgs units. As a result of the vector form of Maxwell's equations, we will deduce Equation 1.54, the so-called equation of continuity (or conservation of charge statement). It is also interesting that Maxwell included the equation of Lorentz force, Equation 1.59, as one of his quaternion equations. We will use this force law as the starting point for the development of magnetic field intensity as it pertains to two parallel current-carrying wires through the Biot–Savart formulation, an inverse square law, which is now the National Institute of Standards and Technology standard for measuring the unit of force (the newton).

In Maxwell's original formulation, Faraday's $\vec{A}$ field was central and had physical meaning. The magnetic vector potential was not arbitrary, as defined by boundary conditions and choice of gauge as we will discuss; they were said to be gauge invariant. The original equations are thus often called the Faraday–Maxwell theory. The centrality of the $\vec{A}$ field was abandoned in the later interpretation of Maxwell by Heavyside. In this interpretation, electromagnetic fields $\vec{E}$ and $\vec{D}$, $\vec{H}$ and $\vec{B}$ are the only physical entities, and the magnetic vector potential is considered a mathematical convenience. Some say this perception replaces action-at-a-distance, as defined by Newton; by contact-action, as defined by Descartes; that is, a theory accounting for both local and global effects was replaced by a completely local theory. The local theory can address global effects only with the aid of the Lorenz gauge. These concepts will be more meaningful when we address time-varying fields in Chapter 7.

ENDNOTES

i. James Clerk Maxwell, *A Treatise on Electricity & Magnetism*, Vol. 1, 3rd ed. (New York: Dover, 1954). An unabridged, slightly altered, republication of the third edition, published by the Clarendon Press in 1891.

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