CHAPTER 1

Time Value of Money

A security is a package of cash flows. The cash flows are delivered across time with varying degrees of uncertainty. To value a security, we must determine how much this package of cash flows is worth today. This process employs a fundamental finance principle—the ***time value of money***. Simply stated, one dollar today is worth more than one dollar to be received in the future. The reason is that the money has a time value. One dollar today can be invested, start earning interest immediately, and grow to a larger amount in the future. Conversely, one dollar to be received one year from today is worth less than one dollar delivered today. This is true because an individual can invest an amount of money less than one dollar today and at some interest rate it will grow to one dollar in a year's time.

The purpose of this chapter is to introduce the fundamental principles of future value (i.e., compounding cash flows) and present value (i.e., discounting cash flows). These principles will be employed in every chapter in the remainder of the book. To be sure, no matter how complicated the security's cash flows become (e.g., bonds with embedded options, interest rate swaps, etc.), determining how much they are worth today involves taking present values. In addition, we introduce the concept of yield, which is a measure of potential return and explain how to compute the yield on any investment.

FUTURE VALUE OF A SINGLE CASH FLOW

Suppose an individual invests \$100 at 5% compounded annually for three years. We call the \$100 invested the ***original principal*** and denote it as P. In this example, the annual interest rate is 5% and is the compensation the investor receives for giving up the use of his or her money for one year's time. Intuitively, the interest rate is a bribe offered to induce an individual to postpone their consumption of one dollar until some time in the future. If interest is compounded annually, this means that interest is paid for use of the money only once per year.

We denote the interest rate as i and put it in decimal form. In addition, N is the number of years the individual gives up use of his or her funds and FV_N is the future value or what the original principal will grow to after N years. In our example,

P = \$100
i = 0.05
N = 3 years

So the question at hand is how much \$100 will be worth at the end of three years if it earns interest at 5% compounded annually?

To answer this question, let's first determine what the \$100 will grow to after one year if it earns 5% interest annually. This amount is determined with the following expression

$$FV_1 = P(1 + i)$$

Using the numbers in our example

$$FV_1 = \$100(1.05) = \$105$$

In words, if an individual invests \$100 that earns 5% compounded annually, at the end of one year the amount invested will grow to \$105 (i.e., the original principal of \$100 plus \$5 interest).

To find out how much the \$100 will be worth at the end of two years, we repeat the process one more time

$$FV_2 = FV_1(1 + i)$$

From the expression above, we know that

$$FV_1 = P(1 + i)$$

Substituting this in the expression and then simplifying, we obtain

$$FV_2 = P(1 + i)(1 + i) = P(1 + i)^2$$

Using the numbers in our example, we find that

$$FV_2 = \$100(1.05)^2 = \$110.25$$

Note that during the second year, we earn \$5.25 in interest rather than \$5 because we are earning interest on our interest from the first year. This example illustrates an important point about how securities' returns work; returns reproduce multiplicatively rather than additively.

To find out how much the original principal will be worth at the end of three years, we repeat the process one last time

$$FV_3 = FV_2(1 + i)$$

Like before, we have already determined FV_2, so making this substitution and simplifying gives us

$$FV_3 = P(1 + i)^2(1 + i)$$

$$FV_3 = P(1 + i)^3$$

Using the numbers in our example, we find that

$$FV_3 = \$100(1.05)^3 = \$115.7625$$

The future value of \$100 invested for three years earning 5% interest compounded annually is \$115.7625.

The general formula for the future value of a single cash flow N years in the future given an interest rate i is

$$FV_N = P(1 + i)^N \tag{1.1}$$

From this expression, it is easy to see that for a given original principal P the future value will depend on the interest rate (i) and the number of years (N) that the cash flow is allowed to grow at that rate. For example, suppose we take the same \$100 and invest it at 5% interest for 10 years rather than five years, what is the future value? Using the expression presented above, we find that the future value is

$$FV_N = \$100(1.05)^{10} = \$162.8894$$

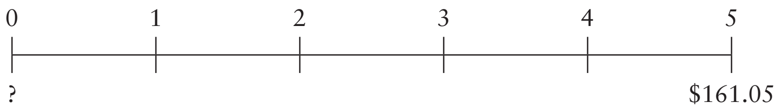
Now let us leave everything unchanged except the interest rate. What is the future value of \$100 invested for 10 years at 6%? The future value is now

$$FV_N = \$100(1.06)^{10} = \$179.0848$$

As we will see in due course, the longer the investment, the more dramatic the impact of even relatively small changes in interest rates on future values.

PRESENT VALUE OF A SINGLE CASH FLOW

The present value of a single cash flow asks the opposite question. Namely, how much is a single cash flow to be received in the future worth today given a particular interest rate? Suppose the interest rate is 10%, how much is \$161.05 to be received five years hence worth today? This question can be easily visualized on the time line presented below:



Alternatively, given the interest rate is 10%, how much would one have to invest today to have \$161.05 in five years? The process is called "discounting" because as long as interest rates are positive, the amount invested (the present value) will be less than \$161.05 (the future value) because of the time value of money.[1]

Since finding present values or discounting asks the opposite question from the future value, the mathematics should be opposite as well. We know the expression for the future value for a single cash flow is given by the expression:

$$FV_N = P(1 + i)^N$$

Let us plug in the information from the question above

$$\$161.05 = P(1.10)^5$$

In order to answer the question of how much we would have to invest today at 10% to have \$161.05 in five years, we must solve for P

$$P = \frac{\$161.05}{(1.10)^5} = \$100$$

So, the present value of \$161.05 delivered five years hence at 10% is \$100.

It is easy to see that the mathematics conform to our intuition. When we calculate a future value, we ask how much will the dollars invested today be worth in the future given a particular interest rate. So, the mathematics of future value involve multiplication by a value greater than one (i.e., making things bigger). Correspondingly, when we find present values, we ask how much a future amount of dollars is worth today given a particular interest

[1]The interest rates used to determine present values are often called "discount rates."

rate. Thus, the mathematics of present value involve division by a value greater than one (i.e., making things smaller).

The general formula for the present value (PV) of a single cash flow N years in the future given an interest rate i is

$$PV = \frac{FV_N}{(1+i)^N} \tag{1.2}$$

Note that we have replaced P with PV. In addition, PV does not have a subscript because we assume it is the value at time 0 (i.e., today).

It is instructive to write the expression for the present value of a single cash flow as follows

$$PV = FV_N\left[\frac{1}{(1+i)^N}\right]$$

The term in brackets is equal to the present value of one dollar to be received N years hence given interest rate i and is often called a ***discount factor***. The present value of a single cash flow is the product of the cash flow to be received (FV_N) and the discount factor. Essentially, the discount factor is today's value of one dollar that is expected to be delivered at some time in the future given a particular interest rate. An analogy will illustrate the point.

Suppose a U.S. investor receives cash payments of \$200,000, ¥500,000, and £600,000. How much does the investor receive? We cannot simply add up the cash flows since the three cash flows are denominated in different currencies. In order to determine how much the investor receives, we would convert the three cash flows into a common currency (say, U.S. dollars) using currency exchange rates. Similarly, we cannot value cash flows to be received at different dates in the future merely by taking their sum. The expected cash flows are delivered at different times and are denominated in different "currencies" (Year 1 dollars, Year 2 dollars, etc.). We use discount factors just like exchange rates to convert cash flows to be received across time into a "common currency" called the present value (i.e., Year 0 dollars).

To illustrate this, we return to the last example—what is the present value of \$161.05 to be received five years from today given that the interest rate is 10%? The present value can be written as

$$PV = \$161.05\left[\frac{1}{(1.10)^5}\right] = \$161.05(0.6209) = \$100$$

One dollar to be received in five years is worth $0.6209 today given the interest rate is 10%. We expect to receive $161.05 Year 5 dollars each worth 0.6209 dollars today. The present value is $100, which is the quantity ($161.05) multiplied by the price per unit ($0.6209).

As can be easily seen from the present value expression, the discount factor depends on two things. First, holding the interest rate constant, the longer the time until the cash flow is to be received, the lower the discount factor. To illustrate this, suppose we have $100 to be received 10 years from now and the interest rate is 10%. What is the present value?

$$PV = \$100\left[\frac{1}{(1.10)^{10}}\right] = \$100(0.3855) = \$38.55$$

Now suppose the cash flow is to be received 20 years hence instead, all else the same. What is the present value?

$$PV = \$100\left[\frac{1}{(1.10)^{20}}\right] = \$100(0.1486) = \$14.86$$

The discount factor falls 0.3855 to 0.1486. This is simply the time value of money at work. The present value is lower the farther into the future the cash flow will be received.

Why this occurs is apparent from looking at the present value equation. The numerator remains the same and is being divided by a larger number in the denominator as one plus the discount rate is being raised to ever higher powers. This is an important property of the present value: for a given interest rate, the farther into the future a cash flow is received, the lower its present value. Simply put, as cash flows move away from the present, they are worth less to us today. Intuitively, we can invest an even smaller amount now ($14.86) today and it will have more time to grow (20 years versus 10 years) to be equal in size to the payment to be received, $100.

The second factor driving the discount factor is the level of the interest rate. Specifically, holding the time to receipt constant, the discount factor is inversely related to the interest rate. Suppose, once again, we have $100 to be received 10 years from now at 10%. From our previous calculations, we know that the present value is $38.55. Now suppose everything is the same except that the interest rate is 12%. What is the present value when the interest rate increases?

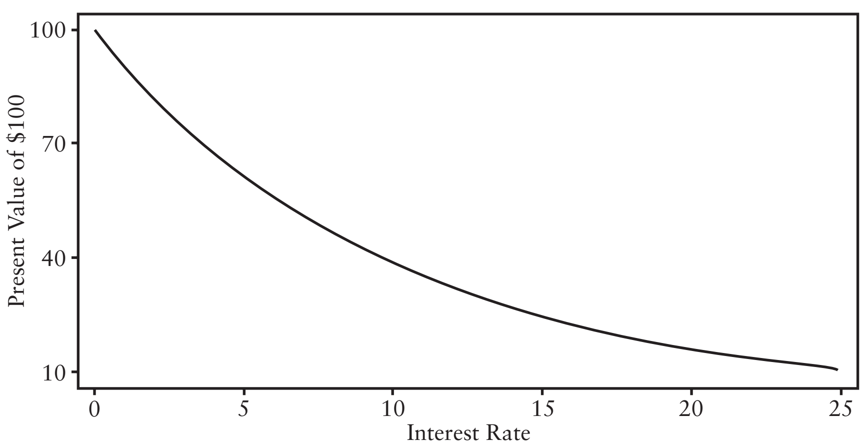
$$PV = \$100\left[\frac{1}{(1.12)^{10}}\right] = \$100(0.3220) = \$32.20$$

As the interest rate rises from 10% to 12%, the present value of $100 to be received 10 years from today falls from $38.55 to $32.20. The reasoning is equally straightforward. If the amount invested compounds at a faster rate (12% versus 10%), we can invest a smaller amount now ($32.20 versus $38.55) and still have $100 after 10 years.

The relationship between the present value of a single cash flow ($100 to be received 10 years hence) and the level of the interest rate is presented in Exhibit 1.1. For now, there are two things to note about present value/interest rate relationship depicted in the exhibit. First, the relationship is downward sloping. This is simply the inverse relationship between present values and interest rates at work. Second, the relationship is a curve rather than a straight line. In fact, the shape of the curve in Exhibit 1.1 is referred to as convex. By convex, it simply means the curve is "bowed in" relative to the origin.

This second observation raises two questions about the convex or curved shape of the present value/interest rate relationship. First, why is it curved? Second, what is the significance of the curvature? The answer to the first question is mathematical. The answer lies in the denominator of the present value formula. Since we are raising one plus the discount rate to powers greater than one, it should not be surprising that the relationship between the present value and the interest rate is not linear. The answer to the second question requires an entire chapter. Specifically, as we see in Chapter 12, this convexity or bowed shape has implications for the price volatility of a bond when interest rates change. What is important to understand at this point is that the relationship is not linear.

EXHIBIT 1.1 PV/Interest Rate Relationship



Note: Present value of $100 to be received in 10 years compounded semiannually.

COMPOUNDING/DISCOUNTING WHEN INTEREST IS PAID MORE THAN ANNUALLY

An investment may pay interest more frequently than once per year (e.g., semiannually, quarterly, monthly, weekly). If an investment pays interest compounded semiannually, then interest is added to the principal twice a year. To account for this, the future value and present value computations presented above require two simple modifications. First, the annual interest rate is adjusted by dividing by the number of times that interest is paid per year. The adjusted interest rate is called a ***periodic interest rate***. Second, the number of years, N, is replaced with the number of periods, n, which is found by multiplying the number of years by the number of times that interest is paid per year.

Future Value of a Single Cash Flow with More Frequent Compounding

The future value of a single cash flow when interest is paid m times per year is as follows:

$$FV_n = P(1 + i)^n \tag{1.3}$$

where

i = annual interest rate divided by m
n = number of interest payments (= $N \times m$)

To illustrate, suppose that a portfolio manager invests $500,000 in an investment that promises to pay an annual interest rate of 6.8% for five years. Interest is paid on this investment semiannually. What is the future value of this single cash flow given semiannual compounding? The answer is $698,514.45 as shown below:

PV = $500,000
m = 2
i = 0.034 (= 0.068/2)
N = 5
n = 10 (5 × 2)

Plugging this information into the future value expression gives us:

$$FV_{10} = \$500{,}000(1.034)^{10} = \$500{,}000(1.397029) = \$698{,}514.50$$

This future value is larger than if interest were compounded annually. With annual compounding, the future value would be \$694,746.34. The higher future value when interest is paid semiannually reflects the fact that the interest is being added to principal more frequently, which in turn earns interest sooner.

Lastly, suppose instead that interest is compounded quarterly rather than semiannually. What is the future value of \$500,000 at 6.8% compounded quarterly for five years? The future value is larger still, \$700,469, for the same reasoning as shown below:

PV = \$500,000
m = 4
i = 0.017 (= 0.068/4)
N = 5
n = 20 (5 × 4)

Plugging this information into the future value expression gives us:

$$FV_{20} = \$500{,}000(1.017)^{20} = \$500{,}000(1.400938) = \$700{,}469$$

Present Value of a Single Cash Flow Using Periodic Interest Rates

We must also adjust our present value expression to account for more frequent compounding. The same two adjustments are required. First, like before, we must convert the annual interest rate into a periodic interest rate. Second, we need to convert the number of years until the cash flow is to be received into the appropriate number of periods that matches the compounding frequency.

The present value of a single cash flow when interest is paid m times per year is written as follows:

$$PV = \frac{FV_n}{(1+i)^n} \tag{1.4}$$

where

i = annual interest rate divided by m
n = number of interest payments (= $N \times m$)

To illustrate this operation, suppose an investor expects to receive \$100,000, 10 years from today and the relevant interest rate is 8% compounded semiannually. What is the present value of this cash flow? The answer is $45,638.69 as shown below:

FV_{10} = $100,000
m = 2
i = 0.04 (= 0.08/2)
N = 10
n = 20 (10 × 2)

Plugging this information into the present value expression gives us:

$$PV = \frac{\$100{,}000}{(1.04)^{20}} = \$45{,}638.69$$

This present value is smaller than if interest were compounded annually. With annual compounding, the present value would be $46,319.35. The lower value when interest is paid semiannually means that for a given annual interest rate we can invest a smaller amount today and still have $100,000 in 10 years with more frequent compounding.

Moving to quarterly compounding, all else equal, should result in an even smaller present value. What is the present value of $100,000 to be received 10 years from today at 8% compounded quarterly? The present value is smaller still, $45,289.04, as shown below:

FV_{40} = $100,000
m = 4
i = 0.02 (= 0.08/4)
N = 10
n = 40 (10 × 4)

Plugging this information into the present value expression given by equation (1.4) gives

$$PV = \frac{\$100{,}000}{(1.02)^{40}} = \$45{,}289.04$$

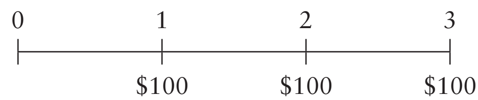
FUTURE AND PRESENT VALUES OF AN ORDINARY ANNUITY

Most securities promise to deliver more than one cash flow. As such, most of the time when we make future/present value calculations, we are working with multiple cash flows. The simplest package of cash flows is called an

annuity. An annuity is a series of payments of fixed amounts for a specified number of periods. The specific type of annuity we are dealing with in our applications is an ***ordinary annuity***. The adjective "ordinary" tells us that the annuity payments come at the end of the period and the first payment is one period from now.

Future Value of an Ordinary Annuity

Suppose an investor expects to receive \$100 at the end of each of the next three years and the relevant interest rate is 5% compounded annually. This annuity can be visualized on the time line presented below:



What is the future value of this annuity at the end of year 3? Of course, one way to determine this amount is to find the future value of each payment as of the end of year 3 and simply add them up. The first \$100 payment will earn 5% interest for two years while the second \$100 payment will earn 5% for one year. The third \$100 payment is already at the end of the year (i.e., denominated in year 3 dollars) so no adjustment is necessary. Mathematically, the summation of the future values of these three cash flows can be written as:

$$\begin{array}{ll}
\$100(1.05)^2 = \$100(1.1025) = & \$110.25 \\
\$100(1.05)^1 = \$100(1.0500) = & \$105.00 \\
\$100(1.05)^0 = \$100(1.0000) = & \underline{\$100.00} \\
\text{Total future value} & \$315.25
\end{array}$$

So, if the investor receives \$100 at the end of each of the next three years and can reinvest the cash flows at 5% compounded annually, then at the end of three years the investment will have grown to \$315.25.

The procedure for computing the future value of an annuity presented above is perfectly correct. However, there is a formula that can be used to speed up this computation. Let us return to the example above and rewrite the future value of the annuity as follows:

$$\$100(1.05)^2 + \$100(1.05)^1 + \$100(1.05)^0 = \$315.25$$

This expression can be rewritten as follows by factoring out the \$100 annuity payment:

$$\$100[(1.05)^2 + (1.05)^1 + (1.05)^0]$$

Since $[(1.05)^2 + (1.05)^1 + (1.05)^0] = 3.1525$,

$$\$100[3.1525] = \$315.25$$

The term in brackets is the ***future value of an ordinary annuity of \$1 per year***. Multiplying the future value of an ordinary annuity of \$1 by the annuity payment produces the future value of an ordinary annuity.

The general formula for the future value of an ordinary annuity of \$1 per year is given by

$$FV_N = A\left[\frac{(1+i)^N - 1}{i}\right] \tag{1.5}$$

where

A = amount of the annuity (\$)
i = annual interest rate (in decimal form)

Let us rework the previous example with the general formula where

A = \$100
i = 0.05
N = 3

therefore,

$$FV_N = \$100\left[\frac{(1.05)^3 - 1}{0.05}\right] = \$100(3.1525) = \$315.25$$

This value agrees with our earlier calculation.

Future Value of an Ordinary Annuity when Payments Occur More Than Once per Year

The future value of an ordinary annuity can be easily generalized to handle situations in which payments are made more than one time per year. For example, instead of assuming an investor receives and then reinvests \$100 per year for three years, starting one year from now, suppose that the investor receives \$50 every six months for three years, starting six months from now.

The general formula for the future value of an ordinary annuity when payments occur m times per year is

$$FV_N = A\left[\frac{(1+i)^N - 1}{i}\right] \tag{1.6}$$

where

A = amount of the annuity ($)
i = periodic interest rate, which is the annual interest rate divided by m (in decimal form)
n = $N \times m$

The value in brackets is the future value of an ordinary annuity of $1 per period.

Let us return to the example above and assume an annuity of $50 for six semiannual periods. The number line would appear as follows:

0	1	2	3	4	5	6
	$50	$50	$50	$50	$50	$50

Note the numbers across the top of the time line represent semiannual periods rather than years. The future value of six semiannual payments of $50 to be received plus the interest earned by investing the payments at 5% compounded semiannually is found as follows:

A = $50
m = 2
i = 0.025 (0.05/2)
N = 3
n = 6 (3 × 2)

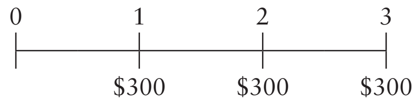
therefore,

$$FV_6 = \$50\left[\frac{(1.025)^6 - 1}{0.025}\right] = \$50(6.387737) = \$319.39$$

Although the total of the cash payments received by the investor over three years is $300 in both examples, the future value is higher ($319.39) when the cash flows are $50 every six months for six periods rather than $100 a year for three years ($315.25). This is true because of the more frequent reinvestment of the payments received by the investor.

Present Value of an Annuity

The coupon payments of a fixed rate bond are an ordinary annuity. Accordingly, in order to value a bond, we must be able to find the present value of an annuity. In this section, we turn our attention to this operation. Suppose we have an ordinary annuity of \$300 for three years. These cash flows are pictured on the time line below:



Suppose that the relevant interest rate is 12% compounded annually. What is the present value of this annuity? Of course, we can take the present value of each cash flow individually and then sum them up. The present value is \$720.57. To see this, we employ the present value of a single cash flow as follows:

$$
\begin{aligned}
PV &= \frac{\$300}{(1.12)^1} = \$267.87 \\
PV &= \frac{\$300}{(1.12)^2} = \$239.16 \\
PV &= \frac{\$300}{(1.12)^3} = \underline{\$213.54} \\
\text{Total} & \qquad\qquad\quad \$720.57
\end{aligned}
$$

We can rewrite the summation of these present values horizontally as shown below:

$$\frac{\$300}{(1.12)^1} + \frac{\$300}{(1.12)^2} + \frac{\$300}{(1.12)^3} = \$720.57$$

This expression can be rewritten by factoring out the \$300 annuity payment as follows:

$$
\begin{gathered}
\$300\left[\frac{1}{(1.12)^1} + \frac{1}{(1.12)^2} + \frac{1}{(1.12)^3}\right] \\
\$300[0.8929 + 0.7972 + 0.7118]
\end{gathered}
$$

Since the sum of the three terms in brackets is 2.4018, we can write

$$\$300(2.4018) = \$720.57$$

The term in brackets is the present value of an ordinary annuity of \$1 for three years at 12%.

Once again, there is a general formula for the present value of an ordinary annuity of \$1 for N years that can used to greatly simplify taking present values. The general formula is given below:

$$PV = A\left[\frac{1-\dfrac{1}{(1+i)^N}}{i}\right] \tag{1.7}$$

where

A = amount of the annuity (\$)
i = annual interest rate (in decimal form)
N = length of the annuity in years

Let us rework the previous example with the general formula where

A = \$300
i = 0.12
N = 3

therefore,

$$PV = \$300\left[\frac{1-\dfrac{1}{(1.12)^3}}{0.12}\right] = \$300(2.4018) = \$720.57$$

This value agrees with our earlier calculation.

Present Value of an Ordinary Annuity when Payments Occur More Than Once per Year

The present value of an ordinary annuity can be generalized to deal with cash payments that occur more frequently than one time per year. For example, instead of assuming an investor receives \$300 per year for three years, starting one year from now, suppose instead that the investor receives \$150 every six months for three years, starting six months from now.

The general formula for the present value of an ordinary annuity when payments occur m times per year is

$$PV = A\left[\frac{1-\dfrac{1}{(1+i)^N}}{i}\right] \tag{1.8}$$

where

A = amount of the annuity ($)
i = periodic interest rate, which is the annual interest rate divided by m (in decimal form)
n = $N \times m$

The value in brackets is the ***present value of an ordinary annuity of $1 per period.***

Let us return to the example above and assume an annuity of $150 for 6 semiannual periods. The time line would appear as follows:

0	1	2	3	4	5	6
	$150	$150	$150	$150	$150	$150

Note once again that the numbers across the top of the time line represent semiannual periods rather than years. The present value of six semiannual payments of $150 to be received plus the interest earned by investing the payments at 12% compounded semiannually is found as follows:

A = $300
m = 3
i = 0.06 (0.12/2)
N = 3
n = 6 (3 × 2)

therefore,

$$PV = \$150\left[\frac{1-\dfrac{1}{(1.06)^6}}{0.06}\right] = \$150(4.9173) = \$737.60$$

Although the total cash payments received by the investor over three years are $900 in both examples, the present value is higher ($737.60) when the cash flows are $150 every six months for six periods rather than $300 a year for three years ($720.57). This result makes sense because half the cash flows are six months closer when they are received semiannually so their present value should be higher.

Present Value of a Perpetual Annuity

We now consider the special case of an annuity that lasts forever, which is called a ***perpetual annuity***. The cash flow of some securities can be thought of as perpetual annuities (e.g., preferred stock). So, how do we take the present value of a stream of cash flows expected to last forever? The computation is surprisingly straightforward and is given by the expression:

$$PV = \frac{A}{i} \tag{1.9}$$

where

A = perpetual annuity payment
i = interest rate (in decimal form)

The reason equation (1.9) is so simple can be found in equation (1.8), which is the general formula for the present value of an ordinary annuity of \$1 per period. As the number of periods n gets very large, the numerator of the term in brackets in equation (1.8) collapses to 1 because the term $1/(1 + i)^n$ approaches zero producing equation (1.9), which is the present value of the perpetual annuity formula.

Let's use equation (1.9) to find the present value of a perpetual annuity. Suppose a financial instrument promises to pay \$350 per year in perpetuity. The investor requires an annual interest rate of 7% from this investment. What is the present value of this package of cash flows?

The present value of the \$350 perpetual annuity is equal to \$5,000, as shown below:

A = \$350
i = 0.07

$$PV = \frac{\$350}{0.07} = \$5{,}000$$

Present Value of a Package of Cash Flows with Unequal Interest Rates

To this point in our discussion, we have used the same interest rate to compute present values regardless of when the cash flows were to be delivered in the future. This will not generally be the case in practice. As we see in Chapter 2, the interest rates used to compute present values will depend on, among other things, the shape of the Treasury yield curve. Each cash flow will be discounted back to the present using a unique interest rate. Accordingly, the present value of a package of cash flows is the sum of the present values of each individual cash flow that comprises the package where each present value is computed using a unique interest rate.

As an illustration of this process, consider a 4-year 9% coupon bond with a $1,000 maturity value. Assume, for simplicity, the bond delivers coupon interest payments annually. The bond's cash flows and required interest rates are shown below:

Years from Now	Annual Cash Payments (in dollars)	Required Interest Rate (%)
1	$90	6.07
2	90	6.17
3	90	6.70
4	1,090	6.88

The present value of each cash flow is determined using the appropriate interest rate as shown below:

Years from Now	Annual Cash Payments (in dollars)	Required Interest Rate (%)	Discount Factor	Present Value of Payment (in dollars)
1	$90	6.07	0.942774	$84.84966
2	90	6.17	0.887149	79.84341
3	90	6.70	0.823203	74.08827
4	1,090	6.88	0.766327	835.29643
		Total Present Value		$1,074.07777

The present value of the cash flows is $1,074.07777.

Since the process of discounting cash flows with multiple interest rates is so important to our work in later chapters, let's work through another example. We demonstrate how to find the present value of the fixed rate payments in an interest rate swap. As explained in Chapter 13, in an interest rate swap, two counterparties agree to exchange periodic interest payments. The dollar amount of the interest payments exchanged is based on some notional principal. The dollar amount each counterparty pays to the other is the agreed-upon periodic interest rate multiplied by the notional principal.

To illustrate an interest rate swap, suppose that for the next five years party A agrees to pay party B 10% per year, while party B agrees to pay party A 6-month LIBOR (the reference rate). Party A is a fixed rate payer/floating rate receiver, while party B is a floating rate payer/fixed rate receiver. Assume the notional principal is \$50 million, and that payments are exchanged every six months for the next five years. This means that every six months, party A (the fixed rate payer/floating rate receiver) will pay party B \$2.5 million (10% × \$50 million × 0.5). The amount that party B (floating rate payer/fixed rate receiver) will be 6-month LIBOR × \$50 million × 0.5. For example, if 6-month LIBOR is 7%, party B will pay party A \$1.75 (7% × \$50 million × 0.5). Note that we multiply by 0.5 because one-half year's interest is being paid.[2]

Let's compute the present value of the fixed rate payments made by party A. As we see in Chapter 2, every cash flow should be discounted using its own interest rate. These interest rates are determined using Eurodollar futures contracts as described in Chapter 13. For now, we take the interest rates as given. The interest rate swap's fixed rate payments and required semiannual interest rates are shown below:

Periods from Now	Semiannual Fixed Rate Payments (in millions of dollars)	Required Semiannual Interest Rate (%)
1	\$2.5	3.00
2	2.5	3.15
3	2.5	3.20
4	2.5	3.30
5	2.5	3.38
6	2.5	3.42
7	2.5	3.45
8	2.5	3.50
9	2.5	3.53
10	2.5	3.54

The present value of this interest rate swap's fixed rate payments using the appropriate semiannual interest rates is shown below:[3]

[2]We see in Chapter 13 that the payments must be adjusted by the number of days in the payment period.

[3]The discount factor is

$$\frac{1}{(1 + \text{Required semiannual rate})^{\text{Periods from now}}}$$

Periods from Now	Semiannual Cash Flows (in millions of dollars)	Required Semiannual Interest Rate (%)	Discount Factor	Present Value of Payment (in millions of dollars)
1	$2.5	3.00	0.970874	2.427184
2	2.5	3.15	0.939856	2.349641
3	2.5	3.20	0.909831	2.274578
4	2.5	3.30	0.878211	2.195527
5	2.5	3.38	0.846871	2.117178
6	2.5	3.42	0.817284	2.043209
7	2.5	3.45	0.788654	1.971635
8	2.5	3.50	0.759412	1.898529
9	2.5	3.53	0.731820	1.829549
10	2.5	3.54	0.706185	1.765462
		Total Present Value		20.87249

The present value of the fixed rate payments in this interest rate swap is $20.87249 million.

YIELD (INTERNAL RATE OF RETURN)

Yield is a measure of potential return from an investment over a stated time horizon. We discuss several yield measures for both fixed rate and floating rate securities (e.g., yield-to-maturity, yield-to-call, discounted margin, etc.) in later chapters. In this section, we explain how to compute the yield on any investment.

Computing the Yield on Any Investment

The yield on any investment is computed by determining the interest rate or discount rate that will make the present value of an investment's cash flow equal to its price. Mathematically, the yield, y, on any investment is the interest rate that will make the following relationship hold:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_N}{(1+y)^N} \quad (1.10)$$

where

P = market price
C_t = cash flow in year t
N = number of years

The individual terms summed to produce the price are the present values of the cash flow. The yield calculated from the expression above is also termed the ***internal rate of return***.

There is no closed-form expression for determining an investment's yield given its price (except for investments with only one cash flow). The yield is, therefore, found by an iterative process. The objective is to find the interest rate that will make the present value of the cash flows equal to the price. The procedure is as follows:

Step 1 Select an interest rate.
Step 2 Compute the present value of each cash flow by using the interest rate selected in Step 1.
Step 3 Total the present value of the cash flows found in Step 2.
Step 4 Compare the total present value found in Step 3 with the price of the investment. Then, if the present value of the cash flows found in Step 3 is equal to the price of the investment, the interest rate selected in Step 1 is the yield. If the total present value of the cash flows found in Step 3 is more than the price of the investment, the interest rate selected is not the yield. Go back to Step 1 and use a higher interest rate. If the total present value of the cash flows found in Step 3 is less than the price of the investment, the interest rate used is not the yield. Go back to Step 1 and use a lower interest rate.

We illustrate how these steps are implemented.

Suppose a financial instrument offers the following annual payments for the next five years as displayed in Exhibit 1.2.

Suppose that the price of this financial instrument is \$1,084.25. What is the yield or internal rate of return offered by this financial instrument?

EXHIBIT 1.2 Cash Flows from a Financial Instrument

Years from Now	Annual Cash Payments (in dollars)
1	\$80
2	80
3	80
4	80
5	1,080

EXHIBIT 1.3 Present Value at 5%

Years from Now	Annual Cash Payments (in dollars)	Present Value of Cash Flow at 5%
1	$80	$76.1905
2	80	72.5624
3	80	69.1070
4	80	65.8162
5	1,080	846.2083
Total Present Value		$1,129.88

EXHIBIT 1.4 Present Value at 7%

Years from Now	Annual Cash Payments (in dollars)	Present Value of Cash Flow at 7%
1	$80	$74.7664
2	80	69.8751
3	80	65.3038
4	80	61.0316
5	1,080	770.0251
Total Present Value		$1,041.00

To compute the yield, we must compute the total present value of these cash flows using different interest rates until we find the one that makes the present value of the cash flows equal to $1,084.25 (the price). Suppose 5% is selected, the calculation is presented in Exhibit 1.3.

The present value using a 5% interest rate exceeds the price of $1,084.25, so a higher interest rate must be tried. If a 7% interest rate is utilized, the present value is $1,041.00 as seen in Exhibit 1.4.

At 7%, the total present value of the cash flows is less than the price of $1,084.25. Accordingly, the present value must be computed with a lower interest rate. The present value at 6% is presented in Exhibit 1.5.

The present value of the cash flows at 6% is equal to the price of the financial instrument when a 6% interest rate is used. Therefore, the yield is 6%.

Although the formula for the yield is based on annual cash flows, the formula can be easily generalized to any number of periodic payments delivered during a year. The generalized formula for computing the yield is

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n} \tag{1.11}$$

EXHIBIT 1.5 Present Value at 6%

Years from Now	Annual Cash Payments (in dollars)	Present Value of Cash Flow at 6%
1	$80	$75.4717
2	80	71.1997
3	80	67.1695
4	80	63.3675
5	1,080	807.0388
Total Present Value		$1,041.00

EXHIBIT 1.6 Yield Calculation with Semiannual Cash Flows

Annual Interest Rate (%)	Semiannual Interest Rate (%)	Total Present Value ($)
6	3.0	1,035.10
7	3.5	1,000.00
8	4.0	966.34
9	4.5	934.04

where

C_t = cash flow in period t
n = number of periods

It is important to bear in mind that the yield computed using equation (1.11) is now the yield for the period. If the cash flows are delivered semiannually, the yield is a semiannual yield. If the cash flows are delivered quarterly, the yield is a quarterly yield, and so forth. The annual rate is determined by multiplying the yield for the period by the number of periods per year (m).

As an illustration, suppose an investor is considering the purchase of a financial instrument that promises to deliver the following *semiannual* cash flows:

- Eight payments of $35 every six months for four years
- $1,000 eight semiannual periods from now

Suppose the price of this financial instrument is $934.04. What yield is this financial instrument offering? The yield is calculated via the iterative procedure explained before and the results are summarized in Exhibit 1.6.

When a semiannual rate interest rate of 4.5% is used to compute the total present value of the cash flows, the total present value is equal to the price of $934.04. Therefore, the semiannual yield is 4.5%. Doubling this yield gives an annual yield of 9%.

Yield Calculation When There is Only One Cash Flow

If a security delivers a single cash flow, it is possible to determine the yield analytically rather than using the iterative procedure. For example, suppose that a financial instrument can be purchased for $4,139.25 and delivers a single cash flow of $5,000 in three years. So, if the price is $4,139.25 and the future value is $5,000, at what yield must the money grow over the next three years? In other words, what value of y will satisfy the following relationship:

$$\$4{,}139.25(1 + y)^3 = \$5{,}000$$

We can solve this expression for y by first dividing both sides by $4,139.25:

$$(1 + y)^3 = \frac{\$5{,}000}{\$4{,}139.25} = 1.20795$$

Next, we take the third root of both sides, which is the same as raising both sides to (⅓) power:

$$(1 + y) = (1.20795)^{1/3} = 1.065$$

Finally, we subtract 1 from both sides:

$$y = 1.065 - 1 = 0.065$$

The yield on this investment is therefore 6.5%

Of course, once the process is well understood, the following formula that greatly simplifies the yield calculation can be used:

$$y = (\text{Future value per dollar invested})^{1/n} - 1 \tag{1.12}$$

where

n = number of periods until the cash flow will be received

$$\text{Future value per dollar invested} = \frac{\text{Cash flow from investment}}{\text{Price}}$$

As an illustration, suppose that a security can be purchased for \$71,298.62 today and promises to pay \$100,000 five years hence. What is the yield? The answer is 7% and the calculation is detailed below:

$$\text{Future value per dollar invested} = \frac{\$100{,}000}{\$71{,}298.63} = 1.40255$$

$$y = (1.40255)^{1/5} - 1 = 1.07 - 1 = 0.07 \text{ or } 7\%$$

Annualizing Yields

Up to this point in our discussion, we have converted periodic interest rates (semiannual, quarterly, monthly, etc.) into annual interest rates by simply multiplying the periodic rate by the frequency of payments per year. For example, we converted a semiannual rate into an annual rate by multiplying it by 2. Similarly, we converted an annual rate into a semiannual rate by dividing it by 2.

This simple rule for annualizing interest rates is not correct due to the mathematics of compound interest. A simple example will illustrate the problem. Suppose that \$1,000 is invested for 1 year at 10% compounded annually. At the end of the year, the interest earned will be \$100. Now suppose that same \$1,000 is invested at 10% compounded semiannually or 5% every six months. The interest earned during the year is determined by calculating the future value of \$100 one year hence at 10% compounded semiannually:

$$\$1{,}000(1.05)^2 = \$1{,}000(1.1025) = \$1{,}102.50$$

Interest is \$102.50 on a \$1,000 investment and the yield is 10.25% (\$102.50/\$1,000). The 10.25% is called the ***effective annual yield***.

The general expression for calculating the effective annual yield for a given periodic interest rate is given by:

$$\text{Effective annual yield} = (1 + \text{Periodic interest rate})^m - 1 \qquad (1.13)$$

where

m = frequency of payments

Using the numbers from the previous example, the periodic (semiannual) yield is 5% and the frequency of payments is twice per year. Therefore,

$$\text{Effective annual yield} = (1.05)^2 - 1 = 1.1025 - 1 = 0.1025 \text{ or } 10.25\%$$

If interest is paid quarterly, then the periodic interest rate is 2.5% and the frequency of payments per year is four. The effective annual yield is 10.38% as computed below:

$$\text{Effective annual yield} = (1.025)^4 - 1 = (1.1038) - 1 = 0.1038 \text{ or } 10.38\%$$

We can reverse the process and compute the periodic interest rate that will produce a given annual interest rate. For example, suppose we need to know what semiannual interest rate would produce an effective annual yield of 8%. The following formula is employed:

$$\text{Period interest rate} = (1 + \text{Effective annual yield})^{1/m} - 1 \qquad (1.14)$$

Using this expression, we find that the semiannual interest rate required to produce an effective annual yield of 8% is 3.9231%:

$$\text{Periodic interest rate} = (1.08)^{1/2} - 1 = (1.039231) - 1 = 0.039231 \text{ or } 3.9231\%$$

CONCEPTS PRESENTED IN THIS CHAPTER (IN ORDER OF PRESENTATION)

Time value of money
Original principal
Discount factor
Periodic interest rate
Annuity
Ordinary annuity
Future value of an ordinary annuity of $1 per year
Present value of an ordinary annuity of $1 per period
Perpetual annuity
Yield
Internal rate of return
Effective annual yield

APPENDIX: COMPOUNDING AND DISCOUNTING IN CONTINUOUS TIME

Most valuation models of derivative instruments (futures/forwards, options, swaps, caps, floors) utilize continuous compounding and discounting. Thus, in this section, we develop these important ideas. As we see, although the mathematics are somewhat more involved, the basic principles we have learned to this point are exactly the same.

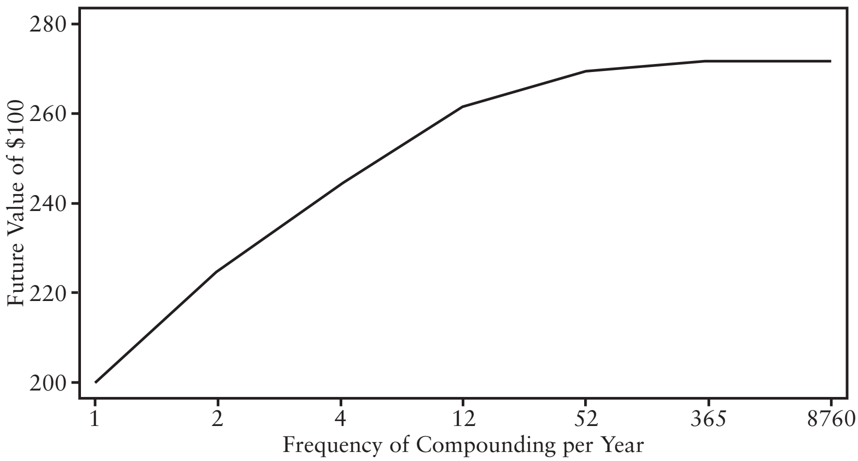
Normally, when computing present and future values, we assume that interest is added to the principal once each period, where the period may be one year, a month, a day, etc. Consider an extreme example: the future value of $100 one year hence, given a 100% interest rate and annual compounding is $200. This amount represents the present value ($100) plus the interest earned over the year ($100), which is added to the principal at the end of the year.

If the other factors remain unchanged, increasing the frequency with which interest is added to the principal (e.g., semiannually, quarterly, monthly, etc.) increases the future value. The future value of $100 one year hence, given a 100% rate and semiannual compounding is $225. Two steps are required to arrive at this amount. At the end of the first six months, the original $100 grows to $150, which represents the original principal ($100) plus the interest earned ($50) over the first six months at a periodic rate of 50%. The periodic rate is simply the annual rate (100%) divided by two, which is the number of times that interest is paid per year. During the second six months, although the account is still earning interest at a periodic rate of 50%, the principal is now $150. Accordingly, an additional $75 interest is added at the end of the period, bringing the total to $225. We earn $25 more in interest in the second six months (as opposed to the first six months) because our interest is also earning interest at a periodic rate of 50%.

So it goes with compound interest. The sooner interest is added to the principal, the sooner interest is earned on a larger balance at the same periodic rate. Therefore, it is not surprising that as annual periods are divided into even smaller increments of time (e.g., quarterly, monthly, daily, etc.), the future value of our $100 at the end of one year continues to grow.

Exhibit A1 depicts what happens to the future value of $100 one year hence given a 100% interest rate as we increase the number of times per year interest is added to the principal. The vertical axis measures the future value at year end; the horizontal axis measures the frequency of compounding per year. The “1” on the horizontal axis is annual compounding, the “2” semiannual compounding, and so forth to “8760,” which represents compounding interest every hour.

EXHIBIT A1 Future Value of $100 at 100%



The rate of increase in the future value is decreasing as we move from annual compounding ($200) to weekly compounding ($269.26) to hourly compounding ($271.81). As it turns out, no matter how frequently the interest is added to our account (every minute, every second, ...), the future value of $100 one year hence at 100% interest can be no more than $271.83. The amount $271.83 is the future value of $100 at 100% if interest is added to our balance continuously; interest is added to our account at literally each instant of time rather than once per period. The future value of $271.83 is the highest possible, given an interest rate of 100%. A future value computed when interest is compounded continuously represents a natural upper bound, similar to the speed of light.

This exercise usually engenders two questions. First, why is there an upper bound? Second, why is the upper bound $271.83? We consider each in turn.

Let's answer the first question by appealing to an analogy. Suppose you are going to fill a bathtub with water. You turn the faucet a quarter turn to the left and water begins to pour into the bathtub. This is analogous to how interest is added to the principal when interest is compounded continuously—the water tumbles out in a continuous stream. Suppose you are going to fill the bathtub for four minutes. Even though the water is coming out of the faucet in a continuous stream, the amount of water in the bathtub will only reach a certain level. The only way we can get more water in the bathtub in a given amount of time is to increase the water pressure. Similarly, if we invest \$100 for one year, the only way we can achieve a higher future value than \$271.83 is to increase the interest rate above 100%.[4]

The answer to the second question requires a brief mathematical interlude. The future value of \$1 when interest is compounded more than once per year is given by $(1 + i/m)^m$ where i is the annual interest rate and m is the frequency of compounding. When i is 100% and as m goes to infinity (i.e., continuous compounding), the future value of \$1 converges to 2.71828... This number, which is denoted by the letter e in honor of the famous Swiss mathematician Euler, is one of the most important numbers in mathematics. Among its many attributes, e is the base of natural logarithms (i.e., the natural logarithm of e is one).[5]

To this point, we have learned why the future value of \$1 one year hence given a particular interest rate has a limit. Moreover, when the interest rate is 100% and compounded continuously on a principal of \$1, the limit is \$2.71828 or \$$e$.[6] Now let's take up the general case and allow the interest rate to take on values other than 100%.

Let's define some terms. As before, let i be the annual interest rate. Let t denote the date to which we are computing the present value and T denote the terminal date of the investment. Accordingly, $(T - t)$ represents the number of periods for which one is investing a particular amount. Finally, let FV and PV denote the future value and present value, respectively. To compute a future value in continuous time, we need to evaluate the following expression:

$$FV = PV\, e^{i(T-t)} \tag{A.1}$$

From our discussion above, $e^{i(T-t)}$ represents the future value of \$1 at interest rate i for $(T - t)$ years. Consider a simple example. Suppose one invests \$1 continuously compounded at 10% for one year. What is the future value? In this case, $T = 1$, $t = 0$, and $i = 0.10$. Inserting these numbers into the expression we get

$$FV = 1e^{0.10(1-0)} = \$1.1052$$

[4]Compounding interest continuously is an example of the more general process of exponential growth, which can apply to a number of phenomena (e.g., population growth).
[5]The exponential function is transcendental so our value for e (2.71828) is only an approximation. In fact, e has an infinite number of decimal places that do not repeat.
[6]When the interest rate is 100% and compounded hourly on an original principal of \$1, the future value is \$2.71813 while continuous compounding gives the same number out to three decimal places.

From this example, it is apparent that a cash flow invested for one year at 10% compounded continuously, and one invested at 10.52% compounded annually will produce the same future value. In other words, the effective annual rate (or annual percentage rate) of 10% compounded continuously is 10.52%.

One brief aside is worth mentioning at this point. The preceding example takes a continuously compounded rate and tells us the equivalent simple interest rate. It is also quite easy to reverse the process. That is, given a simple rate, what is the equivalent continuously compounded rate? Since we use the exponential function to move from continuously compounded to simple interest rates, we use its inverse function (i.e., natural logarithmic function) to move in the other direction. Suppose we have a simple rate of 10.52%, what continuously compounded rate will give the same effective interest rate? To compute this, take the natural logarithm of one plus the simple interest rate, $\ln(1.1052) = 0.10$.

The only issue remaining is how to discount cash flows when interest is paid continuously. To do this, we must evaluate the following expression:

$$PV = FV\, e^{i(t-T)} \tag{A.2}$$

The quantity $(t - T)$ is a negative number and represents the number of years we are discounting the cash flow back in time. Let's rework the previous example: what is the present value of \$1.1052 to be received 1 year from today given continuous discounting at 10%? Just like before, $t = 0$, $T = 1$, and $i = 0.10$. Insert these numbers into the equation (A.2)

$$PV = 1.1052\, e^{0.10(0-1)} = 1$$

Two final points should be noted. First, the quantity $e^{-0.10}$ is equal to 0.9048 and represents the present value of \$1 discounted back one year given the continuously compounded interest rate of 10%. Second, discounting (or compounding) for more than one period is accomplished merely by increasing T.

QUESTIONS

1. Fred Derf found his lost passbook for a saving account that he had opened with a \$100 deposit 12 years ago. If the bank paid interest at a rate of 5% compounded annually over this period, what should be the balance in the account today?

2. You are planning to leave civilization to live in a heavily fortified bunker up in the mountains and be a survivalist in 10 years. You will be able to deposit $1,000 per year at the end of each of the first five years and $2,000 per year for the following five years. You start your savings plan today with a $5,000 deposit. The account pays 8% compounded annually. How much money will you be able to take to the mountains with you when you leave?
3. The grand prize for a lottery is $1,000 per year for 10 years and then $500 per year in perpetuity (i.e., the first $500 payment is at the end of year 11). If the relevant interest rate is 10%, what is the grand prize worth today?
4. What is the future value of $1,000 to be invested now for five years if the interest rate is 12% compounded

   a. annually?
   b. semiannually?
   c. quarterly?
   d. monthly?
5. What is the present value of $1,000 to be received five years from now if the interest rate is 12% compounded

   a. annually?
   b. semiannually?
   c. quarterly?
   d. monthly?
6. You are saving to retire with $1 million 30 years from today. You can start the savings plan with a $5,000 deposit today. Additionally, you can deposit $7,500 10 years from today, $10,000 20 years from today, and $15,000 upon retirement. You need to set up an ordinary annuity plan to reach your goal. What annual payment must you make in the plan to have $1 million upon retirement if you can invest at 12%?
7. Suppose an investor is considering the purchase of a financial instrument that promises to deliver the following semiannual cash flows: four payments of $40 every six months for two years and $1,000 delivered four semiannual periods from now. Suppose the price of this financial instrument is $982.0624. What yield is being offered by this financial instrument?
8. Consider a 4-year 8% coupon bond with a $1,000 maturity value. Assume the bond delivers coupon interest annually. What is the present value of the cash flows using the required interest rates shown below?

Years from Now	Annual Cash Payments	Required Interest Rate
1	$80	5.00%
2	80	5.20
3	80	5.30
4	1,080	5.38

9. What semiannual interest rate is required to produce an effective annual yield of 7.4%?