Part A: A REVIEW OF ESSENTIALS IN MACROFLUIDICS

The review of macrofluidics repeats mostly undergraduate-level theory and provides solved examples of transport phenomena, i.e., traditional (meaning conventional macroworld) fluid mechanics, heat and mass transfer, with a couple of more advanced topics plus applications added. Internal flow problems dominate and for their solutions the differential modeling approach is preferred. Specifically, for any given problem the basic conservation laws (see Sect. A.5) are reduced based on physical understanding (i.e., system sketch plus assumptions), sound postulates concerning the dependent variables, and then solved via direct integration or approximation methods. Clearly, Part A sets the stage for most of the problems solved in Part B and Part C.

CHAPTER 1

Theory

Clearly, the general equations describing conservation of mass, momentum, and energy hold for transport phenomena occurring in all systems/devices from the macroscale to the nanoscale, outside quantum mechanics. However, for most real-world applications such equations are very difficult to solve and hence we restrict our analyses to special cases in order to understand the fundamentals and develop skills to solve simplified problems.

This chapter first reviews the necessary definitions and concepts in fluid dynamics, i.e., fluid flow, heat and mass transfer. Then the conservation laws are derived, employing different approaches to provide insight of the meaning of equation terms and their limitations.

It should be noted that Chapters 1 and 2 are reduced and updated versions of Part A chapters of the author's text *Biofluid Dynamics* (2006). The material (used with permission from Taylor & Francis Publishers) is now geared towards engineering students who already have had introductory courses in thermodynamics, fluid mechanics and heat transfer, or a couple of comprehensive courses in transport phenomena.

1.1 Introduction and Overview

Traditionally, "fluidics" referred to a technology where fluids were used as key components of control and sensing systems. Nowadays the research and application areas of "fluidics" have been greatly expanded. Specifically, *fluidics deals with transport phenomena, i.e., mass, momentum and heat transfer, in devices ranging in size from the*

macroscale down to the nanoscale. As it will become evident, this modern description implies two things:

(i) Conventional fluid dynamics (i.e., macrofluidics) forms a necessary knowledge base when solving most microfluidics and some nanofluidics problems.
(ii) Length scaling from the macroworld (in meters and millimeters) down to the micrometer or nanometer range (i.e., $1\ \mu m = 10^{-6}$ m while $1\ nm = 10^{-9}$ m) requires new considerations concerning possible changes in fluid properties, validity of the continuum hypothesis, modified boundary conditions, and the importance of new (surface) forces or phenomena.

So, to freshen up on *macrofluidics,* this chapter reviews undergraduate-level essentials in fluid mechanics and heat transfer and provides an introduction to porous media and mixture flows. Implications of *geometric scaling,* known as the "size reduction effect," are briefly discussed next.

The most important scaling impact becomes apparent when considering the area-to-volume ratio of a simple fluid conduit or an entire device:

$$\frac{\text{Device surface area}}{\text{Device volume}} = \frac{A}{\forall} \sim \frac{l^2}{l^3} = l^{-1} \tag{1.1}$$

Evidently, in the micro/nanosize limit the ratio becomes very large, i.e., $\lim_{l \to 0} l^{-1} \to \infty$, where $l \triangleq$ system length scale such as the hydraulic diameter, channel height, or width. This implies that *in micro/nanofluidics the system's surface-area-related quantities, e.g., pressure and shear forces, become dominant.* Other potentially important micro/nanoscale forces, rightly neglected in macrofluidics, are surface tension as well as electrostatic and magnetohydrodynamic forces. To provide a quick awareness of other size-related aspects, the following tabulated summary characterizes flow considerations in macrochannels versus microchannels. Specifically, it contrasts important flow conditions and phenomena in conduits of the order of meters and millimeters vs. those in microchannels being of the order of micrometers (see Table 1.1).

Brief Comments Regarding Table 1.1. Fortunately, the *continuum mechanics assumption* holds (i.e., a fluid is homogeneous and infinitely divisible) for most microchannel flows. Hence, reduced forms of the conservation laws (see Sect. 1.3) can be employed to solve fluid flow and heat/mass transfer problems in most device geometries (see Sect. 2.1 and Chapters 3 and 4). The boundary condition of "*no velocity slip at solid walls*" is standard in macrofluidics. However, microchannels fabricated with

hydrophobic material and/or having rough surfaces may exhibit liquid velocity slip at the walls. Considering *laminar* flow, the *entrance length of a conduit* can be estimated as:

$$L_{\text{entrance}} = \kappa D_h \,\text{Re} \tag{1.2}$$

where the hydraulic diameter is defined as $D_h = 4A/P$, with A being the cross-sectional area and P the perimeter, the Reynolds number $\text{Re} = u_{\text{mean}} D_h / \nu$, and $\kappa \approx 0.05$ for macroconduits and 0.5 for microchannels. For fully turbulent flow, $L_{\text{entrance}}^{\text{macro}} \approx 10 D_h$. Considering that typically $L_{\text{microchannel}} = \mathcal{O}(1\,\text{mm})$, entrance effects can be important. For example, if $L_{\text{microchannel}} \geq 0.2 L_{\text{conduit}}$ the favorite simplification "fully developed flow" cannot be assumed anymore (see Sect. 1.4). The *Reynolds number* is the most important dimensionless group in fluid mechanics. However, for microsystems employed in biochemistry as well as in biomedical and chemical engineering, the Reynolds number is usually very low, i.e., $\text{Re} \leq \mathcal{O}(1)$. In contrast, microscale cooling devices, i.e., heat exchangers, require high Reynolds numbers to achieve sufficient heat rejection. Onset to *turbulence*, mainly characterized by random fluctuations of all dependent variables, may occur earlier in microsystems than in geometrically equivalent macrosystems. In some cases, *surface roughness* over, say, 3% of the channel height may cause interesting flow phenomena near the wall, such as velocity slip and/or transition to turbulence. For microsystems with heavy liquids and high velocity gradients, energy dissipation due to *viscous heating* should be considered. The *temperature jump condition* at the wall may be applicable when dealing with convection heat transfer of rarefied gases (see Chapters 2 and 3). The last three entries in Table 1.1, i.e., *diffusion, surface tension, and electrokinetics,* are of interest almost exclusively in microfluidics and nanofluidics (see Part B and Part C).

Table 1.1 Comparison of Flows in Macrochannels vs. Microchannels

Condition/Phenomenon	*Flow in Macroconduits*	*Flow in Microconduits*
Continuum Mechanics Hypothesis	Any fluid is a continuum	Continuum assumption holds for most liquid flows when $D_h \geq 10\ \mu$m and for gases when $D_h \geq 100\ \mu$m
Type of Fluid (i.e., liquid versus gas)	• Special considerations for compressible and/or rarefied gases • Differentiate between Newtonian and non-Newtonian liquids	May have to treat gas flow and liquid flow differently because of the impact of a given fluid's molecular structure and behavioral characteristics

Table 1.1 *Continued*

Condition/Phenomenon	*Flow in Macroconduits*	*Flow in Microconduits*
No-Slip Condition	Can generally be assumed	Liquid-solid slip may occur on hydrophobic surfaces. Velocity slip and temperature jump may occur with rarefied gases
Entrance Effects	Entrance length is usually negligible when compared to the length of the conduit	Entrance length may be on the order of a microchannel length
Reynolds Number	Important to evaluate laminar vs. transitional vs. turbulent flows	Typically $\mathrm{Re}_{D_h} \leq \mathcal{O}(1)$ justifies Stokes flow and allows for nonmechanical pumps driving fluid flow
Turbulence	Transition varies with geometry of domain, but often requires larger Re numbers than in microchannel flow. Example, $\mathrm{Re}_D\big\vert_{\mathrm{pipe}} \approx 2000$	Transition to turbulence may occur earlier, e.g., at $\mathrm{Re}_{D_h} \geq 1200$
Surface Roughness	Is often negligible or included in the friction factor (see Moody chart in App. B)	May need to be considered due to manufacturing limitations at this small scale; roughness may be comparable to dimensions of the system and hence causes complex flow fields near walls
Viscous Heating	Is often small/negligible	May become a major player due to high velocity gradients in tiny channels with viscous fluids
Wall Temperature Condition	Usually thermodynamic equilibrium is assumed	For rarefied gas flows, there may be a temperature jump between the solid wall and the gas
Diffusion	Present, but often very slow; therefore, often negligible	Due to the small size of channels, diffusion is important and can be used for mixing
Surface Tension	Is often negligible	May become a major contributing force, and hence is being used for small fluid volume transfer
Electrohydrodynamic effects, such as electroosmosis (EO)	Negligible	In a liquid electrolyte an electric double layer (EDL) can be formed, which is set into motion via an applied electric field ($\vec{E}$): $E_x \Rightarrow \mathrm{EDL} \rightarrow f_{\mathrm{coulomb}} \rightarrow u_{\mathrm{EO}} \approx u_{\mathrm{slip}}$

Fluidics, as treated in this book, is part of Newtonian mechanics, i.e., dealing with deterministic, or statistically averaged, processes (see Branch A in Figure 1.1).

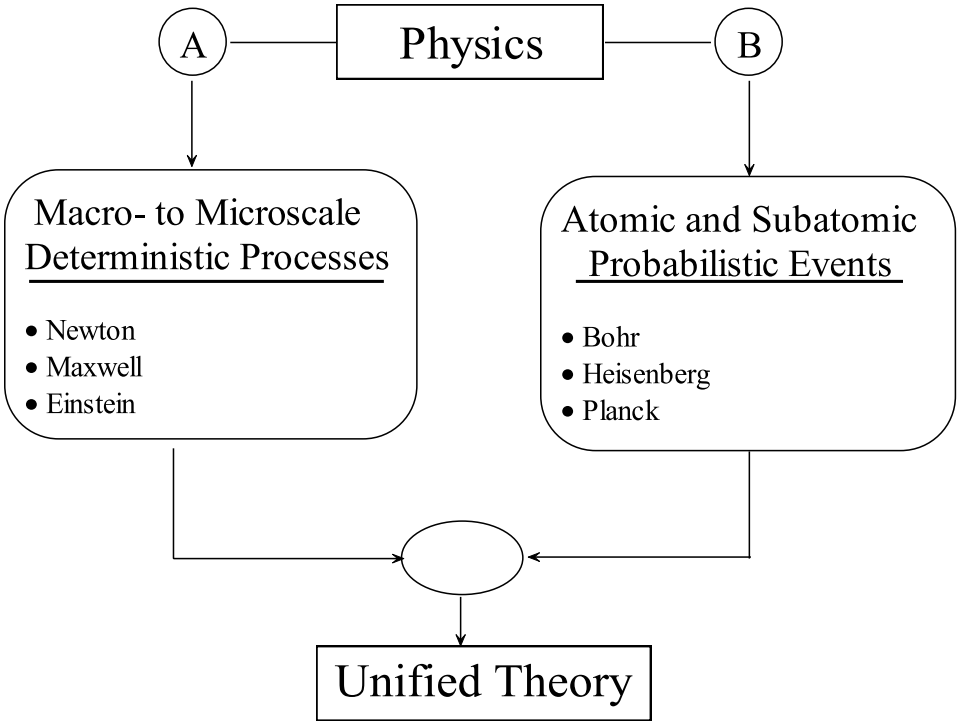


Figure 1.1 Branches of physics waiting for unification

For fluid flow in *nanoscale systems* the continuum mechanics assumption is typically invalid because the length scales of fluid molecules are on the order of nanochannel widths or heights. For example, the intermolecular distance for water molecules is 0.3-0.4 nm while for air molecules it is 3.3 nm, with a mean-free path of about 60 nm. Hence, for *rarefied gases*, not being in thermodynamic equilibrium, the motion and collision of packages of molecules have to be statistically simulated or measured. For *liquids* in nanochannels, molecular dynamics simulation, i.e., the solution of Newton's second law of motion for representative molecules, is necessary.

1.2 Definitions and Concepts

As indicated in Sect. 1.1, a solid knowledge base and good problem-solving skills in macroscale fluid dynamics, i.e., fluid flow plus heat and mass transfer, are important to model most transport phenomena in microfluidics and some in nanofluidics. So, we start out with a review of essential definitions and then revisit basic engineering concepts in macrofluidics. *The overriding goals are to understand the fundamentals and to be able to solve problems independently.*

1.2.1 Definitions

Elemental to transport phenomena is the description of fluid flow, i.e., the equation of motion, which is also called the momentum transfer equation. It is an application of Newton's second law, $\sum \vec{F}_{\text{ext.}} = m\vec{a}$, which Newton postulated for the motion of a particle. For most realistic engineering applications the equation of motion is three-dimensional (3-D) and *nonlinear*, the latter because of fluid inertia terms such as udu/dx, etc. (see App. A.5). However, it is typically independent of the *scalar* heat transfer and species mass equations, i.e., fluid properties are not measurably affected by changes in fluid temperature and species concentration, the latter in case of mixture flows. In summary, the major emphasis in Chapters 1 and 2 are on the description, solution, and understanding of the physics of fluid flow in conduits.

Here is a compilation of a few definitions:

- A *fluid* is an assemblage of gas or liquid molecules which deforms continuously, i.e., it *flows* under the application of a shear stress. Note: Solids do not behave like that; but what about borderline cases, i.e., the behavior of materials such as jelly, grain, sand, etc.?
- Key *fluid properties* are density ρ, dynamic viscosity μ, thermal conductivity k, species diffusivity $\mathcal{D}$, as well as heat capacities c_p and c_v. In general, all six are usually temperature dependent. Very important is the viscosity (see also kinematic viscosity $\nu \equiv \mu/\rho$) representing frictional (or drag) effects. Certain fluids, such as polymeric liquids, blood, food stuff, etc., are also shear rate dependent and hence called *non-Newtonian fluids* (see Sect. 2.3.4).
- *Flows, i.e., fluid in motion powered by a force or gradient,* can be categorized into:

Internal Flows	and	External Flows
- oil, air, water or steam in pipes and inside devices		- air past vehicles, buildings, and planes
- blood in arteries/veins or air in lungs		- water past pillars, submarines, etc.
- water in rivers or canals		- polymer coating on solid surfaces
- gas in pipelines		

- *Driving forces for fluid flow* include gravity, pressure differentials or gradients, temperature gradients, surface tension, electroosmotic or electromagnetic forces, etc.
- *Forces* appear either as body forces (e.g., gravity) or as surface forces (e.g., pressure). When acting on a fluid element they can be split into *normal* and *tangential forces* leading to pressure and normal/shear stresses. For example, on any surface element:

$$p \text{ or } \tau_{\text{normal}} = \frac{F_{\text{normal}}}{A_{\text{surface}}} \tag{1.3}$$

while

$$\tau_{\text{shear}} = \frac{F_{\text{tangential}}}{A_{\text{surface}}} \tag{1.4}$$

Recall: As Stokes postulated, the total stress depends on the spatial derivative of the velocity vector, i.e., $\vec{\vec{\tau}} \sim \nabla \vec{v}$ (see App. A.2). For example, shear stress τ_{shear} occurs due to relative frictional motion of fluid elements (or viscous layers). In contrast, the total pressure sums up three pressure forms, where the mechanical (or thermodynamic) pressure is experienced when moving with the fluid (and therefore labeled "static" pressure and measured with a piezometer). The dynamic pressure is due to the kinetic energy of fluid motion (i.e., $\rho \vec{v}^{\,2}/2$), and the hydrostatic pressure is due to gravity (i.e., $\rho g z$):

$$\begin{aligned} p_{\text{total}} &= p_{\text{static}} + p_{\text{dynamic}} + p_{\text{hydrostatic}} \\ &= p_{\text{static}} + \frac{\rho}{2}\vec{v}^{\,2} + \rho g z = \text{¢} \end{aligned} \tag{1.5a,b}$$

where

$$p_{\text{thermo (or static)}} + p_{\text{dynamic}} = p_{\text{stagnation}} = p\left(\vec{v} = 0\right) \tag{1.6a,b}$$

From the fluid statics equation for a *stagnant* fluid body (or reservoir), where h is the depth coordinate, we obtain:

$$p_{\text{hydrostatic}} = p_0 + \rho g h \tag{1.7}$$

Clearly, the hydrostatic pressure due to the fluid weight appears in the momentum equation as a body force per unit volume, i.e., $\rho\vec{g}$. On the *microscopic level,* fluid molecules are randomly moving in all directions. In the presence of a wall, collisions, i.e., impulse $m\vec{v}$ per time, cause a fluctuating force on the wall. This resulting push statistically averaged over time and divided by the impact area is the pressure.

In general:

- Any fluid flow is described by its *velocity* and *pressure* fields. The velocity vector of a fluid element can be written in terms of its three scalar components:

$$\vec{v} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} \qquad \text{<rectangular coordinates>} \tag{1.8a}$$

or

$$\vec{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \qquad \text{<cylindrical coordinates>} \tag{1.8b}$$

or

$$\vec{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_\varphi\hat{e}_\varphi \qquad \text{<spherical coordinates>} \tag{1.8c}$$

Its *total time derivative* is the fluid element acceleration (see Example 1.1 or Sect. A.1):

$$\left.\frac{d\vec{v}}{dt}\right|_{\substack{\text{solid}\\\text{particle}}} \hat{=} \left.\frac{D\vec{v}}{Dt}\right|_{\substack{\text{fluid}\\\text{element}}} = \vec{a}_{\text{total}} = \vec{a}_{\text{local}} + \vec{a}_{\text{convective}} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \tag{1.9}$$

where Eq. (1.9) is also known as the Stokes, material, or substantial time derivative.

- *Streamlines* for the visualization of flow fields are lines to which the local velocity vectors are tangential. In steady laminar flow streamlines and fluid-particle pathlines are identical. For example, for steady 2-D flow (see Sect. 1.4):

$$\frac{dy}{dx} = \frac{v}{u} \tag{1.10}$$

where the 2-D velocity components $\vec{v} = (u, v, 0)$ have to be given to obtain, after integration, the streamline equation $y(x)$.

- *Dimensionless groups,* i.e., ratios of forces, fluxes, processes, or system parameters, indicate the importance of specific transport phenomena. For example, the Reynolds number is defined as (see Example 1.2):

$$\mathrm{Re}_L \equiv \frac{F_{\text{inertia}}}{F_{\text{viscous}}} := \frac{vL}{\nu} \tag{1.11}$$

where v is an average system velocity, L is a representative system "length" scale (e.g., the tube diameter D), and $\nu \equiv \mu/\rho$ is the kinematic viscosity of the fluid.

Other dimensionless groups with applications in engineering include the Womersley number and Strouhal number (both dealing with oscillatory/transient flows), Euler number (pressure difference), Weber number (surface tension), Stokes number (particle dynamics), Schmidt number (diffusive mass transfer), Sherwood number (convective mass transfer) and Nusselt number, the ratio of heat conduction to heat convection (see Sect. A.3). The most common source (i.e., derivation) of these numbers is the nondimensionalization of partial differential equations describing the transport phenomena at hand, or alternatively via scale analysis (see Example 1.2).

Example 1.1: Derive the material (or Stokes) derivative, D/Dt operating on the velocity vector, describing the "total time rate of change" of a fluid flow field.

Hint: For illustration purposes, use an arbitrary velocity field, $\vec{v} = \vec{v}(x, y, z; t)$, and form its total differential.
Recall: The total differential of any continuous and differentiable function, such as $\vec{v} = \vec{v}(x, y, z; t)$, can be expressed in terms of its infinitesimal contributions in terms of changes of the independent variables:

$$d\vec{v} = \frac{\partial \vec{v}}{\partial x}dx + \frac{\partial \vec{v}}{\partial y}dy + \frac{\partial \vec{v}}{\partial z}dz + \frac{\partial \vec{v}}{\partial t}dt$$

Solution:

Dividing through by dt and recognizing that $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$ are the local velocity components, we have:

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial x}u + \frac{\partial \vec{v}}{\partial y}v + \frac{\partial \vec{v}}{\partial z}w + \frac{\partial \vec{v}}{\partial t}$$

Substituting the "particle dynamics" differential with the "fluid element" differential yields:

$$\frac{d\vec{v}}{dt} \hat{=} \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \vec{a}_{\text{local}} + \vec{a}_{\text{conv.}}$$

Example 1.2: Generation of Dimensionless Groups

(a) Scale Analysis

As outlined in Sect. 1.3, the Navier-Stokes equation (see Eq. (1.63)) describes fluid element acceleration due to several forces per unit mass, i.e.,

$$\vec{a}_{\text{total}} \equiv \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\substack{\text{transient}\\ \text{term}}} + \underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{\substack{\text{inertia force}\\ \text{term}}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\substack{\sim\text{pressure}\\ \text{force}}} + \underbrace{\nu\nabla^2\vec{v}}_{\substack{\sim\text{viscous}\\ \text{force}}} + \underbrace{\vec{g}}_{\sim\text{gravity}}$$

Now, by definition:

$$\text{Re} \equiv \frac{\text{inertia force}}{\text{viscous force}} := \frac{(\vec{v}\cdot\nabla)\vec{v}}{\nu\nabla^2\vec{v}}$$

Employing the scales $\vec{v} \sim v$ and $\nabla = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z) \sim 1/L$ where v may be an average velocity and L a system characteristic dimension, we obtain:

$$\text{Re} = \frac{\left(v\cdot\frac{1}{L}\right)v}{vL^{-2}v} = \frac{vL}{\nu}$$

Similarly, taking

$$\frac{\text{Local acceleration}}{\text{Convective acceleration}} \equiv \frac{\text{transient term}}{\text{inertia term}} = \frac{\partial\vec{v}/\partial t}{(\vec{v}\cdot\nabla)\vec{v}}$$

we can write with system time scale T (e.g., cardiac cycle: $T = 1$ s)

$$\frac{v/T}{vL^{-1}v} = \frac{L}{vT} = \text{Str}$$

which is the *Strouhal number*. For example, when $T >> 1$, $\text{Str} \to 0$ and hence the process, or transport phenomenon, is quasi-steady.

(b) Nondimensionalization of Governing Equations

Taking the transient boundary-layer equations (see Sect. 1.3, Eq. (1.63)) as an example,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

we nondimensionalize each variable with suitable, constant reference quantities. Specifically, approach velocity U_0, plate length ℓ, system time T, and atmospheric pressure p_0 are such quantities. Then,

$$\hat{u} = u/U_0\,,\ \hat{v} = v/U_0\,;\ \hat{x} = x/\ell\,,\ \hat{y} = y/\ell\,;\ \hat{p} = p/p_0 \ \text{ and } \hat{t} = t/T$$

Note: Commonly, $\hat{y}$ is defined as $\hat{y} = y/\delta(x)$, where $\delta(x)$ is the varying boundary-layer thickness.

Inserting all variables, i.e., $u = \hat{u}U_0$, $t = \hat{t}T$, etc., into the governing equation yields

$$\frac{\rho U_0}{T}\frac{\partial \hat{u}}{\partial \hat{t}} + \left[\frac{\rho U_0^2}{\ell}\right]\left(\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}\right) = -\left[\frac{p_0}{\ell}\right]\frac{\partial \hat{p}}{\partial \hat{x}} + \left[\frac{\mu U_0}{\ell^2}\right]\frac{\partial^2 \hat{u}}{\partial y^2}$$

Dividing the entire equation by, say, $\left[\rho U_0^2/\ell\right]$ generates:

$$\underbrace{\left[\frac{\ell}{Tu_0}\right]}_{\text{Strouhal \#}}\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\underbrace{\left[\frac{p_0}{\rho U_0^2}\right]}_{\text{Euler \#}}\frac{\partial \hat{p}}{\partial \hat{x}} + \underbrace{\left[\frac{\mu}{\rho U_0 \ell}\right]}_{\substack{\text{inverse} \\ \text{Reynolds \#}}}\frac{\partial^2 \hat{u}}{\partial \hat{y}^2}$$

Comments:

In a way three goals have been achieved:

- the governing equation is now dimensionless;
- the variables vary only between 0 and 1; and
- the overall fluid flow behavior can be assessed by the magnitude of three groups, i.e., Str, Eu, and Re numbers.

1.2.2 Flow Field Description

Any flow field can be described at either the microscopic or the macroscopic level. The *microscopic* or molecular models consider the position, velocity, and state of every molecule, or representative packages of molecules, of a fluid at all times. In contrast, averaging discrete-particle information (i.e., position, velocity, and state) over a local fluid volume yields *macroscopic* quantities, e.g., the velocity field $\vec{v}(\vec{x},t)$ in time and space of the flow field. The advantages of the molecular approach include general applicability, i.e., no need for submodels (e.g., for the stress tensor, heat flux, turbulence, wall conditions, etc.) and an absence of numerical instabilities (e.g., due to steep flow field gradients). However, considering myriads of molecules, atoms, or nanoparticles requires enormous computer resources, and hence only simple channel or stratified flows with a finite number of interacting molecules (assumed to be solid spheres) can be presently analyzed.

For example, in a 1-mm cube there are about 34 billion water molecules (about a million air molecules at STP); these high numbers make molecular dynamics simulation prohibitive but, on the other hand, intuitively validate the continuum assumption if the flow domain is sufficiently large.

Continuum Mechanics Assumption. As alluded to in Sect. 1.2.1 (see Table 1.1), fundamental to the description of all transport phenomena are the conservation laws, concerning mass, momentum, and energy, as applied to continua. In general, *solid structures and fluid flow fields are continua* as long as the local material properties can be defined as averages computed over material elements/volumes sufficiently large when compared to microscopic length scales of the solid or fluid but small relative to the (macroscopic) structure. Variations in solid-structure or fluid flow quantities can be obtained via solutions of differential equations describing the interactions between forces (or gradients) and motion. Specifically, the continuum mechanics method is an effective tool to physically explain and mathematically describe various transport phenomena without detailed knowledge of their internal molecular structures. In summary, *continuum mechanics* deals with three aspects:

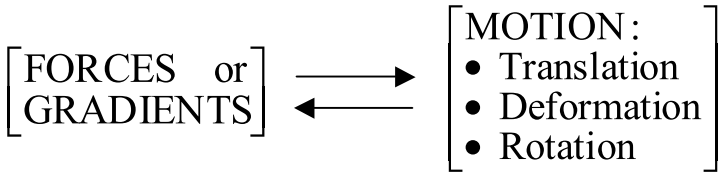
- *Kinematics,* i.e., fluid element motion regardless of the cause
- *Dynamics,* i.e., the origin and impact of forces and fluxes generating fluid motion and waste heat, e.g., the stress tensor or heat flux vector, leading to entropy increase
- *Balance principles,* i.e., the mass, momentum, and energy conservation laws

Also, usually all flow properties are in *local thermodynamic equilibrium*, implying that the macroscopic quantities of the flow field can adjust swiftly to their surroundings. This local adjustment to varying conditions is rapidly achieved if the

fluid has very small characteristic length and time scales of molecular collisions, when compared to the macroscopic flow variations. However, as the channel (or tube) size, typically indicated by the hydraulic diameter D_h, is reduced to the *microscale*, the surface-area-to-volume ratio becomes larger because $A/V \sim D_h^{-1}$ and wall surface effects may become important, as mentioned in Sect. 1.1.

Flow Dynamics and Fluid Kinematics. Here, the overall goal is to find and analyze the interactions between *fluid forces*, e.g., pressure, gravity/buoyancy, drag/friction, inertia, etc., and *fluid motion*, i.e., the velocity vector field and pressure distribution from which everything else can be directly obtained or derived (see Figure 1.2). In turn, scalar transport equations, i.e., convection mass and heat transfer, can be solved based on the velocity field to obtain critical magnitudes and gradients (or fluxes) of species concentrations and temperatures, respectively.

(a) Cause-and-effect dynamics:



(b) Kinematics of a 2-D fluid element (Lagrangian frame):

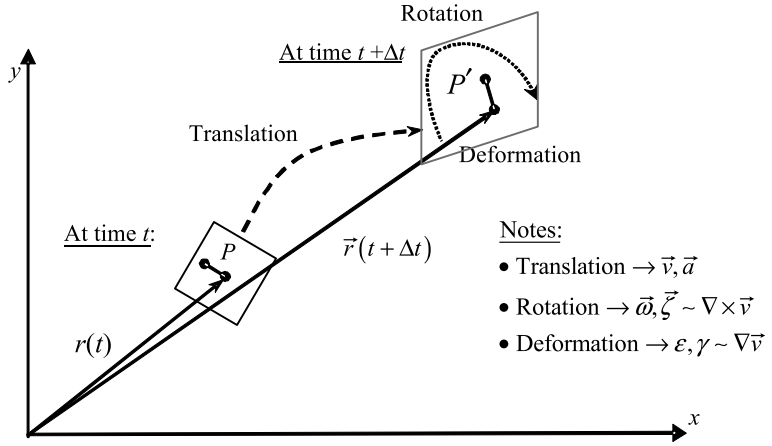


Figure 1.2 Dynamics and kinematics of fluid flow: (a) force-motion interactions; and (b) 2-D fluid kinematics

In summary, *unbalanced surface/body forces and gradients cause motion in the form of fluid translation, rotation, and/or deformation, while temperature or concentration gradients cause mainly heat or species mass transfer.* Note that flow visualization CDs

plus web-based university sources provide fascinating videos of complex fluid flow, temperature, and species concentration fields. Please check out the following links:
http://en.wikipedia.org/wiki/Flow_visualization
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.100.6782&rep=rep1&type=pdf

Balance Principles. The fundamental laws of mass, energy, and momentum conservation are typically derived in the form of mass, energy, and force balances for a differential fluid volume (i.e., a representative elementary volume) generating partial differential equations (PDEs). When a much larger *open system* is considered for mass, momentum, and/or energy balances, integral expressions result. Such balances in integral form, known as the Reynolds Transport Theorem (RTT), can be readily transformed into PDEs by employing the Divergence Theorem (see Sect. 1.3 and App. A).

Within the continuum mechanics framework, two basic flow field descriptions are of interest, i.e., the *Lagrangian viewpoint and the Eulerian (or control volume) approach* (see Figure 1.3, where C.∀. ≙ control volume and C.S. ≙ control surface).

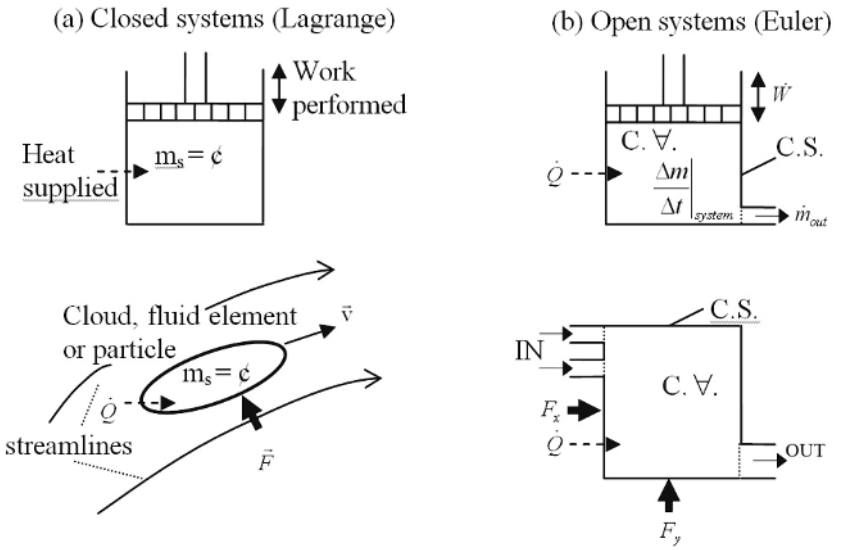


Figure 1.3 Closed versus open systems

For the *Lagrangian description* (see Figure 1.3a) one could consider first just a particle moving on a pathline with respect to a fixed Cartesian coordinate system. Initially, the particle position is at $\vec{r}_0 = \vec{r}_0(\vec{x}_0, t_0)$ and a moment later at $\vec{r} = \vec{r}(\vec{r}_0, t_0 + \Delta t)$ where based on vector algebra $\vec{r} = \vec{r}_0 + \Delta\vec{r}$. Following the particle's motion for $t > t_0$, the position vector is in general:

$$\vec{r} = \vec{r}\left(\vec{r}_0, t\right) \tag{1.12}$$

In the limit the time rate of change in location is the particle (or fluid element) velocity, i.e.,

$$\frac{d\vec{r}}{dt} = \vec{v} \tag{1.13}$$

and after a second time derivation:

$$\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{a} \tag{1.14}$$

Now, the *"material point concept"* is extended to a *material volume with constant identifiable mass, forming a "closed system"* that moves and deforms with the flow but no mass crosses the material volume surface, because it is closed (see Figure 1.3a, second example). Again, the system is tracked through space, and as time expires it is of interest to know what the changes in system mass, momentum, and energy are. This can be expressed in terms of the system's extensive property $\boldsymbol{B}_{\text{system}}$ which could be mass m, momentum $m\vec{v}$, or total energy E. Thus, the key question is: "How can we express the fate of the $\boldsymbol{B}_{\text{system}}$" or, in mathematical shorthand, what is "$D\boldsymbol{B}_s / Dt$"? Clearly, the material time (or Stokes) derivative $D/Dt \equiv \partial/\partial t + \vec{v} \cdot \nabla$ (see Example 1.1) follows the closed system and records the total time rate of change of whatever is being tracked.

In order to elaborate on the material derivative (see Example 1.1) as employed in the Lagrangian description, a brief illustration of the various *time derivatives* is in order, i.e., $\partial/\partial t$ (local), d/dt (total of a material point or solid particle), and D/Dt (total of a fluid element). Their differences can be illustrated using acceleration (see Example 1.1 and App. A):

- $a_{x,\text{local}} = \partial u/\partial t$, where u is the fluid element velocity in the x-direction,
- $\vec{a}_{\text{particle}} = d\vec{v}/dt$ is employed in solid particle dynamics,

whereas

- $\vec{a}_{\substack{\text{fluid}\\\text{element}}} = \dfrac{D\vec{v}}{Dt} = \underbrace{\dfrac{\partial \vec{v}}{\partial t}}_{\vec{a}_{\text{local}}} + \underbrace{\left(\vec{v} \cdot \nabla\right)\vec{v}}_{\vec{a}_{\text{convective}}}$ is the total fluid element acceleration.

In the *Eulerian frame,* an "open system" is considered where mass, momentum, and energy may readily cross boundaries, i.e., being convected across the control volume surface and local fluid flow changes may occur within the control volume over time (see Figure 1.3b). The fixed or moving control volume may be a large

system/device with inlet and outlet ports, it may be small finite volumes generated by a computational mesh, or it may be in the limit a "point" in the flow field. In general, the Eulerian observer fixed to an inertial reference frame records temporal and spatial changes of the flow field at all "points" or, in case of a control volume, transient mass, momentum, and/or energy changes inside and fluxes across its control surfaces.

In contrast, the Lagrangian observer stays with each fluid element or material volume and records its basic changes while moving through space. Section 1.3 employs both viewpoints to describe mass, momentum, and heat transfer in integral form, known as the Reynolds Transport Theorem (RTT). *Thus, the RTT simply links the conservation laws from the Lagrangian to the Eulerian frame*. In turn, a surface-to-volume integral transformation then yields the conservation laws in differential form (i.e., PDEs) in the Eulerian framework, also known as the *control volume approach*.

1.2.3 Flow Field Categorization

Exact *flow problem identification*, especially in industrial settings, is one of the more important and sometimes the most difficult first task. After obtaining some basic information and reliable data, it helps to think and speculate about the physics of the fluid flow, asking:

(i) What category does the given flow system fall into, and how does it respond to normal as well as extreme changes in operating conditions? Figure 1.4 may be useful for categorization of real fluids and types of flows.
(ii) What variables and system parameters play an important role in the observed transport phenomena, i.e., linear or angular momentum transfer, fluid mass or species mass transfer, and heat transfer?
(iii) What are the key dimensionless groups and what are their expected ranges?

Answers to these questions assist in grouping the flow problem at hand. For example, with the exception of "superfluids," all others are viscous, some more (e.g., syrup) and some less (e.g., rarefied gases). However, with the advent of *Prandtl's boundary-layer concept* the flow field, say, around an airfoil has been traditionally divided into a very thin (growing) viscous layer and beyond that an unperturbed *inviscid region* (see Schlichting & Gersten, 2000). This paradigm helped to better understand actual fluid mechanics phenomena and to simplify velocity and pressure as well as drag and lift calculations. Specifically, at sufficiently high approach velocities a fluid layer adjacent to a submerged body experiences steep gradients due to the "no-slip" condition and hence constitutes a *viscous flow region*, while outside the boundary layer frictional effects are negligible (see Prandtl equations versus Euler equation in Sect. 1.3.3.3). Clearly, with the prevalence of powerful

CFD software and engineering workstations, such a fluid flow classification is becoming more and more superfluous for practical applications.

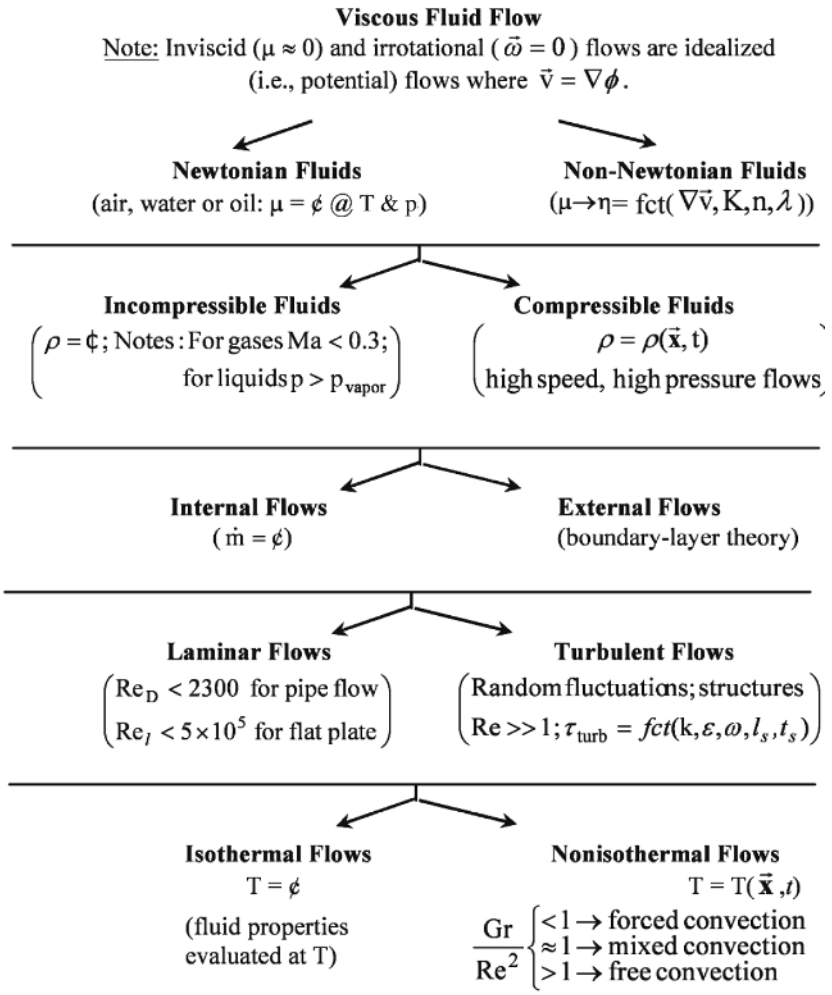


Figure 1.4 Special cases of viscous fluid flows

While, in addition to air, water and almost all oils are *Newtonian*, some synthetic motor oils are shear rate dependent and that holds as well for a variety of new (fluidic) products. This implies that modern engineers have to cope with the analysis and computer modeling of *non-Newtonian fluids* (see Sect. 2.3.4). For example, Latex paint is shear thinning, i.e., when painting a vertical door rapid brush strokes induce high shear rates ($\dot{\gamma} \sim dw/dz$) and the paint viscosity/resistance is very low.

When brushing stops, locally thicker paint layers (due to gravity) try to descend slowly; however, at low shear rates the paint viscosity is very high and hence "tear-drop" formation is avoided and a near-perfect coating can dry on the vertical door.

All natural phenomena change with time and hence are *unsteady* (i.e., *transient*) while in industry it is mostly desirable that processes are steady, except during production line start-up, failure, or shut-down. For example, turbines, compressors, and heat exchangers operate continuously for long periods of time and hence are labeled "steady-flow devices"; in contrast, pacemakers, control systems, and drink dispensers work in a time-dependent fashion. In some cases, like a heart valve, devices change their orientation periodically and the associated flows oscillate about a mean value. In contrast, it should be noted that the term *uniform* implies "no change with system *location*," as in uniform (i.e., constant over a cross section) velocity or uniform particle distribution, which all could still vary with time.

Mathematical flow field descriptions become complicated when *laminar flow* turns unstable due to high speed and/or geometric irregularities ranging from surface roughness to complex conduits. The deterministic laminar flow turns *transitional* on its way to become *fully turbulent*, i.e., chaotic, transient 3-D with random velocity fluctuations, which help in mixing but also induce high apparent stresses. As an example of "flow transition," picture a group (on bikes or skis) going faster and faster down a mountain while the terrain gets rougher. The initially quite ordered group of riders/skiers may change swiftly into an unbalanced, chaotic group. So far no *universal model for turbulence*, let alone for the transitional regime from laminar to turbulent flow, has been found. Thus, major efforts focus on direct numerical simulation (DNS) of turbulent flows which are characterized by relatively high Reynolds numbers and chaotic, transient 3-D flow pattern.

Basic Flow Assumptions and Their Mathematical Statements. Once a given fluid dynamics problem has been categorized (Figure 1.4), some justifiable assumptions have to be considered in order to simplify the equations describing the flow system's transport phenomena. The three most important ones are time dependence, dimensionality, and flow (or Reynolds number) regime. Especially, if justifiable, steady laminar 1-D (*parallel or unidirectional or fully developed*) flow simplifies a given problem analysis (see Sect. 1.3.3.3 with Table 1.2 as well as Table 2.1).

1.2.4 Thermodynamic Properties and Constitutive Equations

Thermodynamic Properties. Examples of thermodynamic properties are mass and volume (extensive properties) as well as velocity, pressure, and temperature (intensive properties), all essential to characterize a general system, process, or device. In addition, there are *transport properties*, such as viscosity, diffusivity, and thermal

conductivity, which are all temperature dependent and may greatly influence, or even largely determine, a fluid flow field. Any extensive, i.e., mass-dependent, property divided by a unit mass is called a *specific property*, such as the specific volume $\upsilon = \forall / m$ (where its inverse is the fluid density) or the specific energy $e = E/m$ (see Sect. 1.3). An *equation of state* is a correlation of independent intensive properties, where for a simple compressible system just two describe the state of such a system. A famous example is the ideal-gas relation, $p\forall = mRT$, where $m = \rho\forall$ and R is the gas constant.

At the *microscopic level* (based on kinetic theory), the fluid temperature is directly proportional to the kinetic energy of the fluid's molecular motion (Probstein, 1994). Specifically, $3kT = \langle m\vec{v} \cdot \vec{v} \rangle$, where k is the Boltzmann constant, m is the molecular mass, and $\vec{v}$ is the fluctuating velocity vector. The pressure, as indicated, is the result of molecular bombardment. The density depends macroscopically on both pressure and temperature and microscopically on the number of molecules per unit volume; for example, there are 2.69×10^{19} air molecules in 1 cm^3. Comparing the compressibility of liquids vs. gases, it takes $\Delta p_{\text{water}} / \Delta p_{\text{air}} = 2.15 \times 10^4$ to achieve the same fractional change in density.

Constitutive Equations. Looking ahead (see Sect. 1.3), when considering the conservation laws for fluid flow and heat transfer, it is apparent that additional relationships must be found in order to solve for the variables: velocity vector $\vec{v}$, fluid pressure p, fluid temperature T, and species concentration c as well as stress tensor $\vec{\vec{\tau}}$, heat flux vector $\vec{q}$, and species flux $\vec{j}_c$. Thus, this is necessary for reasons of (i) mathematical closure, i.e., a number of unknowns require the same number of equations, and (ii) physical evidence, i.e., additional material properties other than the density ρ are important in the description of system/material/fluid behavior. These additional relations, or *constitutive equations,* are fluxes which relate via "material properties" to gradients of the principal unknowns. Specifically, for basic *linear* proportionalities we recall:

Stokes' postulate, i.e., the fluid stress tensor

$$\vec{\vec{\tau}} = \mu\left(\nabla\vec{v} + \nabla\vec{v}^{\,T}\right) \left[\mathrm{N/m^2}\right] \tag{1.15}$$

where μ is the dynamic viscosity;

Fourier's law, i.e., the heat conduction flux

$$\vec{q} = -k\nabla T \left[\mathrm{J/m^2 \cdot s}\right] \tag{1.16}$$

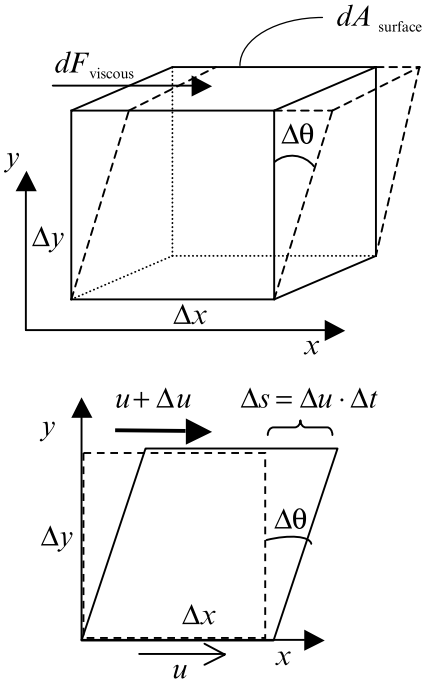
where k is the thermal conductivity; and

Fick's law for the species mass flux

$$\vec{j}_c = -\mathcal{D}_{AB} \nabla c \ \left[\mathrm{kg/m^2 \cdot s}\right] \tag{1.17}$$

where $\mathcal{D}_{AB}$ is the binary diffusion coefficient.

Of these three constitutive equations, the total stress tensor is in macrofluidics the most important and complex one (see Sect. 1.3.3.2 for more details). For, say, one-dimensional (1-D) cases to move fluid elements relative to each other, a shear force $F_{\text{tangential}} = \tau_{\text{shear}} A_{\text{interface}}$ is necessary. Thus, for simple shear flows (see Figure 1.5) $\tau_{yx} = \mu\, du/dy$. In general, the shear stress is proportional to $\nabla \vec{v}$ and the dynamic viscosity is just temperature dependent for Newtonian fluids (e.g., air, water, and oil) or shear rate dependent for polymeric liquids, paints, blood (at low shear rates), food stuff, etc.



Physics:

$$\tau_{ij} = \lim_{\delta A_j \to 0} \frac{\delta F_i}{\delta A_j} \triangleq \frac{\text{surface force}}{\text{unit area}}$$

$$\tau = \frac{dF_v}{dA_s} \sim \frac{\Delta\theta}{\Delta t}$$

Geometry:

$$\tan \Delta\theta \approx \Delta\theta = \frac{\Delta s}{\Delta y} = \frac{\Delta u \cdot \Delta t}{\Delta y}$$

Combining both:

$$\tau \sim \frac{\Delta u}{\Delta y}$$

where in the limit with the proportionality factor, μ, for unidirectional flow:

$$\tau_{yx} = \mu \frac{du}{dy}$$

Figure 1.5 Illustration of the shear stress derivation for simple shear flow

1.3 Conservation Laws

After proper problem recognition and classification (see Sect. 1.2.3), central to engineering analysis are the tasks of realistic, accurate, and manageable modeling followed by analytical (or numerical) solution. While in most cases a given system's conservation laws are known, to solve the equations, often subject to complex boundary conditions and closure models, is quite a different story. So, after highlighting different approaches to derive conservation laws for mass, linear momentum, and energy, several examples in this section as well as in Chapter 2 illustrate basic phenomena and solution techniques.

1.3.1 Derivation Approaches

Derivation of the conservation laws describing all essential transport phenomena is very important because they provide a deeper understanding of the underlying physics and implied assumptions, i.e., the power and limitations of a particular mathematical model. Of course, derivations are regarded by most as boring and mathematically quite taxing; however, for those, it's time to become a convert for the two beneficial reasons stated. Furthermore, one should not forget the power of *dimensional analysis* (DA) which requires only simple algebra when nondimensionalizing governing equations and hence generating dimensionless groups. Alternatively, *scale analysis* (SA) is a nifty way of deriving dimensionless groups as demonstrated in this chapter (see Example 1.2). Both DA and SA are standard laboratory/computational tools for estimating dominant transport phenomena, graphing results, to evaluate engineering systems, and to test kinematic/dynamic similarities between a physical model and the actual prototype.

Outside the cutting-edge research environment, fluid mechanics problems are solved as special cases, i.e., the conservation equations are greatly reduced based on justifiable assumptions on a case-by-case basis (see Sect. 1.3.3.3). Clearly, the simplest case is *fluid statics* where the fluid mass forms a "whole body," either stationary or moving without any *relative* velocities (see Eq. (1.7)). The popular (because very simple) *Bernoulli equation,* for frictionless fluid flow along a representative streamline, balances kinetic energy ($\sim \rho \bar{v}^2$), flow work ($\sim \Delta p$), and potential energy ($\sim \rho g z$) and hence in some cases provides useful pressure-velocity-elevation correlations. The most frequently used equations in macrofluidics and microfluidics are the *Navier-Stokes (N-S) equations*, describing momentum and heat transfer for constant-property fluids, assuming that the flow is a continuum. After some basic fluid flow applications, relatively new material is introduced to broaden the student's knowledge base and provide a higher skill level to cope with today's engineering problems encountered in industry or graduate school.

There are basically four ways of obtaining specific equations expressing the conservation laws:

(i) *Molecular Approach:* Fluid properties and transport equations can be obtained from kinetic theory and the Boltzmann equation, respectively, employing statistical means. Alternatively, $\sum \vec{F} = m\vec{a}$ is solved for each molecule using direct numerical integration.

(ii) *Integral Approach:* Starting with the RTT for a fixed open control volume (Euler), specific transport equations in integral form can be obtained (see Sect. 1.3.2).

(iii) *Differential Approach:* Starting with 1-D balances over an REV (representative elementary volume) and then expanding them to 3-D, the mass, momentum, and energy transfer equations in differential form can be formulated. Alternatively, the RTT is transformed via the divergence theorem, where in the limit the field equations in differential form are obtained (see Sect. 1.3.3).

(iv) *Phenomenological Approach:* Starting with balance equations for an open system, transport phenomena in complex transitional, turbulent, or multiphase flows are derived largely based on empirical correlations and dimensional analysis considerations. A very practical example is the description of transport phenomena with fluid compartment models. These "compartments" are either well-mixed, i.e., *transient lumped-parameter models* without any spatial resolution, or transient with a one-dimensional resolution in the axial direction.

Especially for the (here preferred) differential approach (iii), the system-specific fluid flow assumptions have to be carefully stated and justified.

1.3.2 Reynolds Transport Theorem

Consider $\boldsymbol{B}_{\text{system}} \equiv \boldsymbol{B}$ to be an arbitrary extensive quantity of a *closed system,* say, a moving material volume. In general, such a system could be an ideal piston-cylinder device with enclosed (constant) gas mass, a rigid tank without any fluid leaks, or an identifiable pollutant cloud—all subject to forces and energy transfer (see Figure 1.3a). In any case, $\boldsymbol{B}$ represents the system's mass, momentum, or energy.

Task 1 is to express in the Lagrangian frame the fate of $\boldsymbol{B}$ in terms of the material derivative, $D\boldsymbol{B}/Dt$, i.e., the total time rate of change of $\boldsymbol{B}$ (see Sect. 1.2.2 reviewing the two system approaches and Example 1.1 discussing the operator D/Dt). Specifically, based on:

- Conservation of mass

$$\boldsymbol{B} \equiv m_{\text{system}} = \text{¢} \rightarrow \frac{Dm}{Dt} = 0 \tag{1.18a}$$

- Conservation of momentum (or Newton's second law)

$$\boldsymbol{B} \equiv (m\vec{v})_{\text{system}} \rightarrow m\frac{D\vec{v}}{Dt} = m\vec{a}_{\text{total}} = \sum \vec{F}_{\text{external}} = \sum \vec{F}_{\text{surface}} + \sum \vec{F}_{\text{body}} \qquad (1.18b)$$

- Conservation of energy or first law of thermodynamics

$$\boldsymbol{B} \equiv E_{\text{system}} \rightarrow \frac{DE}{Dt} = \dot{Q} - \dot{W} \qquad (1.18c)$$

In *Task 2* the conservation laws, in terms of $D\boldsymbol{B}/Dt$, are related to an *open system*, i.e., in the Eulerian frame. Here, for a fixed control volume (*C*.∀.) with material streams flowing across the control surface (C.S.), and possibly accumulating inside C.∀., we observe with specific quantity $\beta \equiv \boldsymbol{B}/m$, or $\rho\beta = \boldsymbol{B}/\forall$:

$$\left\{\begin{array}{l}\textit{Total}\text{ time rate of}\\ \text{change of system}\\ \text{property } \boldsymbol{B}\end{array}\right\} = \left\{\begin{array}{l}\textit{Local}\text{ time rate of}\\ \text{change of } \boldsymbol{B}/\forall \equiv \rho\beta\\ \text{within the C.}\forall.\end{array}\right\} + \left\{\begin{array}{l}\textit{Net efflux}\text{ of } (\rho\beta),\\ \text{i.e., net material}\\ \text{convection across}\\ \text{control surface C.S.}\end{array}\right\}$$

or in mathematical shorthand:

$$\left.\frac{D\boldsymbol{B}}{Dt}\right|_{\text{closed system}} = \frac{\partial}{\partial t}\iiint_{\text{C.}\forall.} (\rho\beta)\, d\forall + \iint_{\text{C.S.}} (\rho\beta)\vec{v}\cdot d\vec{A} \qquad (1.19)$$

Equation (1.19), which is formally derived in any undergraduate fluids text, is the RTT for a fixed control volume. Clearly, the specific quantity β can be expressed as:

$$\beta \triangleq \frac{\boldsymbol{B}}{m} := \begin{cases} 1 & \text{mass per unit mass} \\ \vec{v} & \text{momentum per unit mass} \\ e & \text{energy per unit mass} \end{cases} \qquad (1.20a\text{-}c)$$

Extended Cases. For a *moving* control volume the fluid velocity $\vec{v}$ is replaced by $\vec{v}_{\text{relative}} = \vec{v}_{\text{fluid}} - \vec{v}_{\text{C.}\forall.}$ (see Example 1.6). The operator $\partial/\partial t$, acting on the first term on the right-hand-side (RHS) of Eq. (1.19), has to be replaced by d/dt when the control volume is *deformable*, i.e., the C.S. moves with time (see Example 1.5).

For a *noninertial* coordinate system, for example, when tracking an accelerating system such as a rocket, $\sum \vec{F}_{\text{external}}$ of Eq. (1.18b) is expressed as:

$$m\vec{a}_{\text{abs}} = m\left(\frac{d\vec{v}}{dt} + \vec{a}_{\text{rel}}\right) \tag{1.21a}$$

where $m\vec{a}_{\text{rel}}$ accounts for noninertial effects (e.g., arbitrary C.∀. acceleration):

$$\vec{a}_{\text{rel}} = \frac{d^2\vec{R}}{dt^2} \tag{1.21b}$$

In case of C.∀.. rotation,

$$\vec{a}_{\text{rel}} = \frac{d\vec{w}}{dt} \times \vec{r} \tag{1.21c}$$

Specifically, for a *rotating* material volume, the fluid angular momentum per unit volume $\beta \equiv \rho(\vec{r} \times \vec{v})$ has to be considered. The law of *conservation of angular momentum* states that the rate of change of angular momentum of a material volume is equal to the resultant moment on the volume (see any undergraduate fluids text for more details).

Setting Up the Reynolds Transport Theorem. There are a few sequential steps necessary for tailoring the general RTT toward a specific flow system description and solving the resulting integral equations:

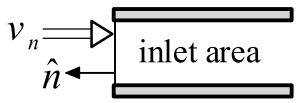
(1) Identify the extensive quantity $\boldsymbol{B}_{\text{system}}$, e.g., the mass of the identifiable material, linear (or angular) momentum, or total energy. As a result, the specific property of the closed system $\beta = \boldsymbol{B}/m_{\text{system}}$ is known (see Eqs. (1.18 and 1.20)).
(2) Determine $D\boldsymbol{B}/Dt\big|_{\text{system}}$ for each conservation case, i.e., mass, momentum, or energy (see Eq. (1.18)).
(3) Select a "smart" control volume and determine *if*:

- the control volume is fixed, or moving at $\vec{v}_{C.\forall.} = ¢$, or accelerating at $\vec{a}_{\text{rel}} = ¢$, or accelerating and rotating (see "Extended Cases" above);
- the flow problem is steady or transient, i.e., is $\partial/\partial t = 0$ or $\partial/\partial t \neq 0$ (note, $\partial/\partial t \iiint_{C.\forall.} (\rho\beta) d\forall \cong 0$ when the rate of change of $(\rho\beta)$ inside the C.∀. is negligible);
- a control surface moves, i.e., we have a deformable C.∀. where $\partial/\partial t \to d/dt$;
- the fluid properties are constant or variable;

- the inflow/outflow velocity fields are constant, i.e., uniform, or a function of inlet/exit space variables; and
- the resulting integral balance equations for mass and momentum (or energy) are decoupled or not.

(1) Set up the momentum, i.e., force balance, equation for each coordinate direction.
(2) Solve the volume and/or surface integrals (use integration tables, if necessary).
(3) Follow the inflow/outflow sign convention (see Figure 1.6), i.e., IN $\hat{=}$ "$-$" and OUT $\hat{=}$ "$+$"
(4) Check the results for correctness, i.e., apply common sense.

Inflow: $\vec{v}\ d\vec{A} = -v_n dA$



Outflow: $\vec{v}\ d\vec{A} = v_n dA$

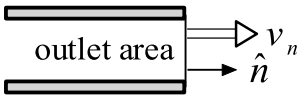


Figure 1.6 Sign convention for the "net efflux" RTT term (Recall: $\cos 180° = -1$; and $\cos 0° = 1$)

1.3.2.1 Fluid Mass Conservation in Integral Form

Conservation of mass is very intuitive and standard in daily-life observations. A given mass of a fluid may change its thermodynamic state, i.e., liquid or gaseous, but it can neither be destroyed nor created. This, as the other two conservation laws, can be expressed in *integral form* for a control volume of any size and shape or derived in *differential form*.

In order to track within the Lagrangian frame an identifiable constant mass of fluid, we set (see Figure 1.3a and Eq. (1.20a))

$$\boldsymbol{B}_{\text{system}} \equiv m \text{ and hence } \beta \equiv 1$$

The conservation principle requires that, with $m = \text{¢}$, $Dm/Dt = 0$ and hence Eq. (1.19) reads:

$$0 = \frac{\partial}{\partial t}\iiint_{\text{C.V.}} \rho d\forall + \iint_{\text{C.S.}} \rho \vec{v} \cdot d\vec{A} \tag{1.22}$$

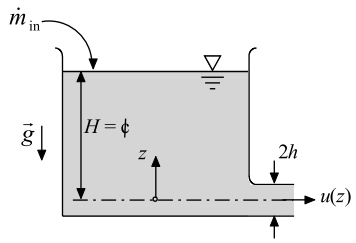
Thus, we just completed Steps (i) and (ii) of the "**setting-up-the-RTT**" procedure. Aspects of Step (iii) are best illustrated with a couple of examples.

Example 1.3: Volumetric Flow Rate

Consider a liquid-filled tank (depth H) with a horizontal slot outlet (height $2h$ and width w) where the short-tube locally varying outlet velocity can be expressed as:

$$u \approx \sqrt{2g(H-z)}$$

A constant fluid mass flow rate, $\dot{m}_{\text{in}}$, is added to maintain the liquid depth H. The z-coordinate indicates the location of the center of the outlet. Find Q_{outlet} as a function of H and h.

Sketch:	*Assumptions:*	*Concepts:*
	• Steady incompressible flow • Outflow velocity as $u(z) = \sqrt{2g(H-z)}$ based on Torricelli's law	• Mass RTT • Fixed, nondeforming C.∀., i.e., H=const. • Torricelli's law

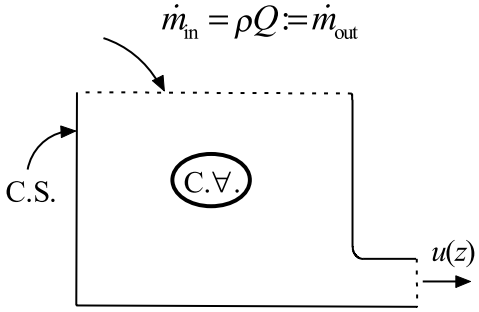
Solution:

The given $u(z)$-equation is a special case of Bernoulli's equation. Specifically, with $\partial/\partial t \iiint \rho d\forall = 0$ (steady state because no system parameter changes with time), we have with $\rho = ¢$ (incompressible fluid):

$$0 = \iint_{\text{C.S.}} \vec{v} \cdot d\vec{A} \tag{E1.3-1}$$

Fluid mass crosses the control surface at two locations (see graph). Recalling that "inflow" is negative and "outflow" positive (see Figure 1.6), Eq. (E1.3-1) reads, with $\dot{m}_{\text{in}} / \rho = Q_{\text{in}}$,

Graph:



$$Q_{in} + \iint_{A_{slot}} \vec{v} \cdot d\vec{A} = 0$$

or

$$Q_{in} = \int_A v_n dA \qquad \text{(E1.3-2)}$$

Here $Q_{in} = Q_{outlet}$ because H = constant;

$v_n = u(z) = \sqrt{2g(H-z)}$ and $dA = wdz$ so that

$$Q_{outlet} = w\sqrt{2g} \int_{-h}^{h} \sqrt{H-z}\, dz$$

which yields

$$Q_{outlet} = K\left[(H+h)^{3/2} - (H-h)^{3/2}\right] \qquad \text{(E1.3-3)}$$

where $K \equiv 2/3\, w\sqrt{2g}$.

For $h << H$, as it is often the case, Eq. (E1.3-3) can be simplified to (see Sect. 1.4):

$$Q_{outlet} \approx 2\, wh\sqrt{2gH} \qquad \text{(E1.3-4)}$$

Comment: Equation (E1.3-4) is known as *Torricelli's law.*

What can be deduced from Example 1.3 is that for *incompressible* fluid flow through a conduit,

$$\iint_{C.S.} \vec{v} \cdot d\vec{A} = 0 \qquad (1.23)$$

or $-Q_{in} + \int_{A_{outlet}} \vec{v} \cdot d\vec{A} = 0$, i.e., in general:

$$\boxed{Q = \int_A \vec{v} \cdot d\vec{A} = \int_A v_n\, dA = \overline{v}A} \qquad (1.24)$$

where $\overline{v} = Q/A$ is the cross-sectionally averaged velocity. We also recall that the mass flow rate at any point in a conduit is:

$$\boxed{\dot{m} = \rho \bar{v} A = \rho Q = \text{constant}} \tag{1.25}$$

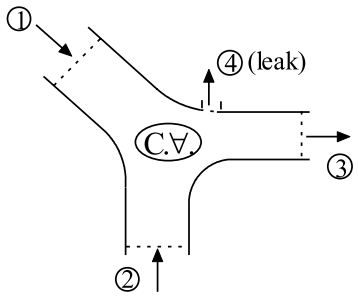
which holds for any fluid and is a key "internal flow" condition, reflecting conservation of mass. Clearly, for a C.∀. with multiple inlets and outlets:

$$\sum Q_{\text{in}} = \sum Q_{\text{out}} \tag{1.26}$$

as illustrated in Example 1.4.

Example 1.4: Multiport Flow Junction

Consider a feed pipe (A, v_1) bifurcating into two outlet pipes (A_2, v_2 and A_3, v_3) where a small hole (A_4) has been detected in the junction area. Develop an equation for the leak Q_4.

Sketch:	*Assumptions*:	*Approach*:
	• Steady incompressible flow • Fixed, nondeformable C.∀. • Constant velocities	• Reduced mass RTT, i.e., direct use of Eq. (1.26) • Sign convention (Figure 1.6b)

Solution:

The fact that the inlet/outlet velocities are all constant simplifies Eq. (1.23) to Eq. (1.26), i.e.,

$$-Q_1 - Q_2 + Q_3 + Q_4 = 0 \text{ where } Q_i = v_i A_i$$

Thus,

$$Q_{\text{leak}} \equiv Q_4 = v_1 A_1 + v_2 A_2 - v_3 A_3$$

The next example considers a *deforming* control volume inside a tank due to a single outflow in terms of a variable velocity. Thus, Eq. (1.22) has to be rewritten as:

$$0 = \frac{d}{dt}\iiint_{\text{C.}\forall.} \rho d\forall + \iint_{\text{C.S.}} \rho \vec{v} \cdot d\vec{A} \tag{1.27a}$$

For incompressible fluid flow, we have:

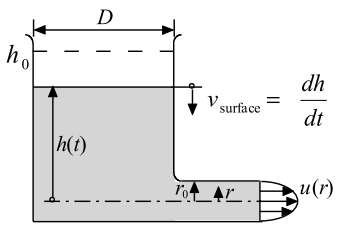
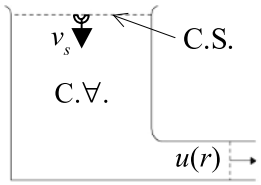
$$\left.\frac{d\forall}{dt}\right|_{\text{C.}\forall.} = -\iint_{\text{C.S.}} \vec{v} \cdot d\vec{A} \tag{1.27b}$$

Example 1.5: Draining of a Tank: A "deformable C.∀." because the fluid level decreases and hence we have a shrinking C.∀. with moving C.S.

Consider a relatively small tank of diameter D and initially filled to height h_0. The fluid drains through a pipe of radius (r_0) according to (see Example 2.1):

$$u(r) = 2\overline{u}\left[1 - \left(r/r_0\right)^2\right]$$

where $\overline{u} = \sqrt{2gh}$ (see Example 1.3), r_0 is the outlet pipe radius, and r is its variable radius, $0 \le r \le r_0$. The fluid depth was h_0 at time $t = 0$. Find $h(t)$ for a limited observation time Δt.

Sketch:	*Assumptions*:	*Control Volume*:
	• Transient incompressible flow • Stationary tank but "deforming" liquid volume C.∀.	

Solution:

Equation (1.27b) can be expanded for this problem with $\forall_{\text{C.}\forall.} = \left(D^2\pi/4\right)h$ to

$$\left(\frac{D^2\pi}{4}\right)\frac{dh}{dt} = -\int_0^{r_0} u(r)\,dA \tag{E1.5-1}$$

where $dA = 2\pi r dr$ *is the variable ring element* as part of the cross-sectional area of the outlet pipe. Thus,

$$\frac{dh}{dt} = -\frac{16\sqrt{2gh}}{D^2}\int_0^{r_0}\left[1-\left(\frac{r}{r_0}\right)^2\right]rdr \tag{E1.5-2a}$$

or

$$\frac{dh}{dt} = -4\sqrt{2g}\left(\frac{r_0}{D}\right)^2\sqrt{h} \tag{E1.5-2b}$$

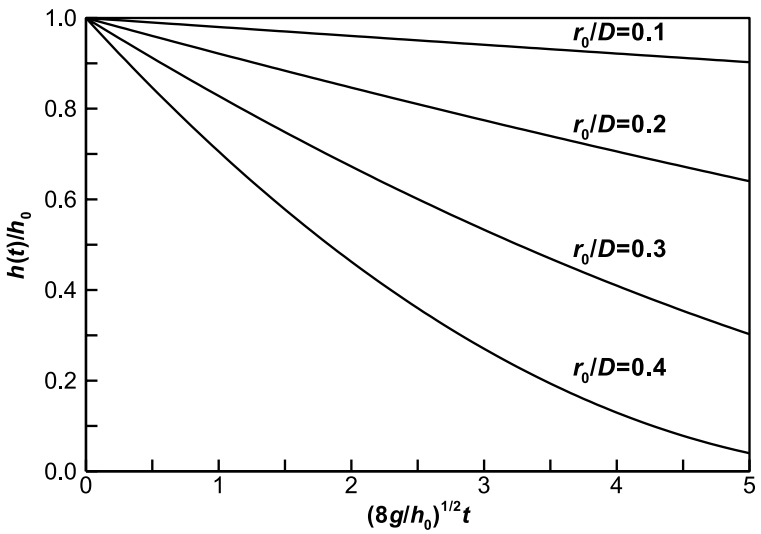
subject to $h(t=0)=h_0$. Separation of variables and integration yield:

$$\sqrt{h_0}-\sqrt{h} = \sqrt{8g}\left(\frac{r_0}{D}\right)^2 t \tag{E1.5-3a}$$

or

$$\frac{h(t)}{h_0} = \left[1-\sqrt{\frac{8g}{h_0}}\left(\frac{r_0}{D}\right)^2 t\right]^2 \tag{E1.5-3b}$$

Graph:



Comments:

- The standard assumption that $h = ¢$, i.e., being a reservoir, is only approximately true when $r_0 / D < 0.1$.
- A variable speed dh/dt, i.e., accelerated tank draining, occurs when $r_0 / D > 0.1$.

1.3.2.2 Momentum Conservation in Integral Form

Focusing on linear momentum transfer (in contrast to angular momentum transfer) the momentum conservation law is again derived via the integral.

Forces acting on an identifiable *fluid* element accelerate it, as it is well known from Newton's second law of motion $(m\vec{a})_{\text{fluid}} = \Sigma \vec{F}_{\text{external}}$. Specifically, we set (see Eq. 1.18b)):

$$B_{\text{system}} \equiv m\vec{v} \text{ and hence } \beta \equiv \vec{v}.$$

As previously indicated, with $m = ¢$, $m\,D\vec{v}/Dt = \Sigma\vec{F}_{\text{ext}}$ and hence Eq. (1.19) reads:

$$m\frac{D\vec{v}}{Dt} = m\vec{a}_{\text{total}} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = \frac{\partial}{\partial t}\int_{\text{C.}\forall.} \rho\vec{v}d\forall + \int_{\text{C.S.}} \vec{v}\rho\vec{v}\cdot d\vec{A} \qquad (1.28a)$$

As discussed, for control volumes *accelerating* without rotation relative to inertial coordinates, an additional force $\vec{F}_{\text{inertia}} \sim \vec{a}_{\text{relative}}$ appears, where $\vec{a}_{\text{rel}}$ is the C.$\forall$. acceleration relative to the fixed frame of reference *X*-*Y*-*Z*. Thus, with

$$\vec{v}_{XYZ} = \vec{v}_{xyz} + \vec{v}_{\text{relative}} \text{ or } \vec{a}_{XYZ} = \vec{a}_{\text{absolute}} = \frac{d\vec{v}}{dt} + \vec{a}_{\text{rel}}$$

we can express the sum of all forces as:

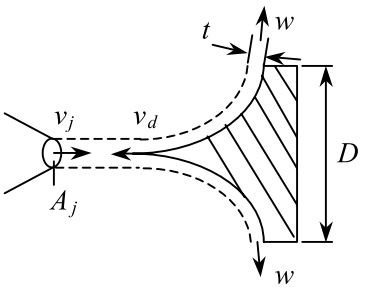
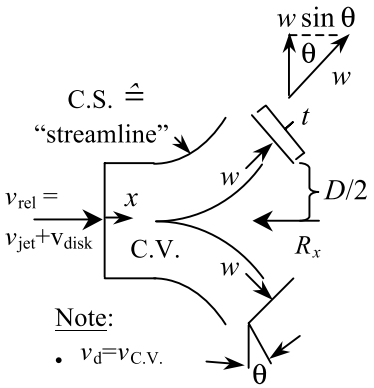
$$\sum \vec{F}_{\text{total}} = m\,\vec{a}_{\text{abs}} = m\left(\frac{D\vec{v}}{Dt} + \vec{a}_{\text{rel}}\right)$$

i.e., for an accelerating and deforming control volume:

$$m\frac{D\vec{v}}{Dt} = \Sigma\vec{F}_B + \Sigma\vec{F}_S - \int_{\text{C.}\forall.} \vec{a}_{\text{rel}}dm = \frac{d}{dt}\int_{\text{C.}\forall.} \rho\vec{v}d\forall + \int_{\text{C.S.}} \vec{v}\,\rho\,\vec{v}_{\text{rel}}\cdot dA \qquad (1.28b)$$

Example 1.6: Force on a Disk Moving into an Axisymmetric Jet

A steady uniform round jet impinges upon an approaching conical disk as shown. Find the force exerted on the disk, where v_{jet}, A_{jet}, v_{disk}, diameter D, angle θ, and fluid layer thickness t are given.

Sketch:	*Control volume:*
	

Assumptions: as stated; constant averaged velocities and properties
Approach: RTT (mass balance and 1-D force balance)
Solution:

(a) Mass Conservation:

$$0 = \int_{\text{C.S.}} \rho \vec{v} \cdot d\vec{A} = -\int_{A_{\text{jet}}} \rho v_{\text{rel}} dA + \int_{A_{\text{exit}}} \rho w dA \tag{E1.6-1}$$

where $v_{\text{rel}} = v_{\text{jet}} - \left(-v_{\text{disk}}\right) = v_{\text{j}} + v_{\text{d}}$ and $A_{\text{exit}} \approx \pi D t$.

$$\therefore -\rho A_j \left(v_j + v_d\right) + \rho\left(\pi D t\right) w = 0$$

Hence,

$$w = \frac{\left(v_j + v_d\right) A_j}{\pi D t} \tag{E1.6-2}$$

(b) Momentum Conservation:

$$B = (m\vec{v})_s\,;\ \beta = \vec{v};\ DB/Dt := \vec{F}_{\text{surf}} = -R_x$$

$$\Sigma\vec{F}_{\text{surf}} = \int_{\text{C.S.}} \vec{v}\rho\vec{v}_{\text{rel}} \cdot d\vec{A} \xrightarrow{\text{x-component}} -R_x = \int_{\text{C.S.}} u\rho v_{\text{rel}}\, dA \qquad \text{(E1.6-3a.b)}$$

Thus,

$$-R_x = -v_{\text{rel}}\rho v_{\text{rel}} A_{\text{jet}} + w\ \sin\ \theta\rho w A_{\text{exit}}$$

where with Eq. (E1.6-2):

$$R_x = \rho A_j \left(v_j + v_d\right)^2 \left[1 - \frac{A_j \sin\theta}{\pi t D}\right] \qquad \text{(E1.6-4)}$$

Comment:

The resultant fluid structure force (E1.6-4) can be rewritten as:

$$R_x = \left(\dot{m}v_x\right)_{\text{out}} - \left(\dot{m}v_x\right)_{\text{in}} \qquad \text{(E1.6-5)}$$

which is the result of a change in fluid flow momentum. If the disk would move away with $v_d = v_j$, i.e., escaping the jet, $v_{\text{rel}} = 0$ and hence $R_x = 0$.

Expanding on Eq. (E1.6-5), we can generalize that the net momentum flux due to applied forces is:

$$\sum \vec{F} = \dot{m}\left(\alpha_2\vec{v}_2 - \alpha_1\vec{v}_1\right) \qquad (1.29)$$

where the correction factor α accounts for the variation of v^2 across the inlet ① or outlet ② duct section. Specifically,

$$\rho\int_A v^2 dA = \alpha\dot{m}v_{\text{average}} = \alpha\rho A v_{\text{av}}^2$$

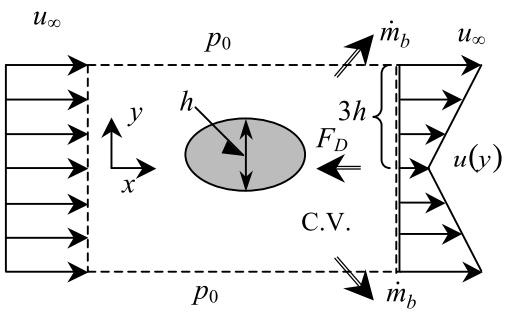
so that

$$\alpha \equiv \frac{1}{A}\int\left(\frac{v}{v_{\text{av}}}\right)^2 dA \tag{1.30}$$

Example 1.7: Force on a Submerged Body

Find the drag on a submerged elliptic rod of characteristic thickness h, based on a measured velocity profile downstream from the body, say,

$$u(y) = u_\infty / 4\left[1 + y/h\right];\ 0 \le y \le 3h$$

Sketch:	*Approach:*
	• Integral mass and momentum balances (i.e., RTT) *Assumption:* • Steady 2-D flow • Constant properties • Fixed C.∀., infinite extent in the z-direction

Solution:

(a) Mass Conservation:

$$\dot{m}_{\text{in}} = \dot{m}_{\text{boundary}} + \dot{m}_{\text{out}} \tag{E1.7-1}$$

$$0 = \int_{\text{C.S.}} \rho\vec{v}\cdot d\vec{A} = \rho u_\infty 2(3h) + \dot{m}_{\text{boundary}} + 2\rho\frac{u_\infty}{4}\int_0^{3h}\left(1+\frac{y}{h}\right)dy \tag{E1.7-2a}$$

$$\therefore\quad \dot{m}_{\text{boundary}} = 6\rho h u_\infty - \frac{15}{4}\rho h u_\infty = \frac{9}{4}\rho h u_\infty \tag{E1.7-2b}$$

(b) Momentum Conservation (x-momentum):

$$-F_D = \int_{\text{C.S.}} v_x \rho\vec{v}\cdot d\vec{A}: = \left(\dot{m}v_x\big|_{\text{exit}} + \dot{m}v_x\big|_b\right) - \dot{m}v_x\big|_{\text{inlet}} \tag{E1.7-3a}$$

$$-F_D = -u_\infty \rho u_\infty (6h) + u_\infty \left(\frac{9}{4}\rho h u_\infty\right) + 2\rho \int_0^{3h} \left[u(y)\right]^2 dy \qquad \text{(E1.7-3b)}$$

$$F_D = \frac{9}{8}\rho h u_\infty^2 \qquad \text{(E1.7-3c)}$$

Comment:

The fluid flow field inside the C.∀., especially behind the submerged body, is very complex. The RTT treats it as a "black box" and elegantly obtains $F_D = F_{\text{fixation}}$ via "velocity defect" measurements, indicated with the given velocity profile $u(y)$. Note that the given system is 2-D with $z \to \infty$; but, after integration the result, $F_x = -F_D$, is obtained in 1-D.

1.3.2.3 Conservation Laws of Energy and Species Mass

Although many natural and industrial fluid flow problems are nonisothermal, fluid mechanics education typically only deals with constant-temperature flows, leaving thermal flows for separate thermodynamics and convection heat transfer courses. Species mass transfer is almost entirely left for chemical and biomedical engineers. Being more comprehensive, we highlight the energy RTT plus the resulting heat transfer equation with an analogy to mass transfer, to lay the groundwork for some interesting engineering applications in remaining sections plus Chapters 3, 5 and 6.

Two approaches are considered which highlight *global energy balance* for closed and open systems, as well as convection heat transfer described by the *energy RTT*.

Global Energy Balance. The first law of thermodynamics states that energy forms can be converted but the total energy is constant, i.e., conserved:

$$E_{\text{total}} = \underbrace{E_{\text{kinetic}} + E_{\text{potential}} + E_{\text{internal}}}_{=E_{\text{system}}} + \underbrace{E_{\text{mass flow}} + E_{\substack{\text{work}\\\text{performed}}} + E_{\substack{\text{heat}\\\text{transfered}}}}_{=E_{\text{surrounding}}} \ldots = ¢ \qquad (1.31)$$

Taking the total time derivative in an "engineering approach" yields:

$$\frac{dE_{\text{total}}}{dt} \approx \frac{\Delta E_{\text{total}}}{\Delta t} = 0 = \left.\frac{\Delta E}{\Delta t}\right|_{\text{system}} + \sum \dot{E}_{\text{net exchange}} \qquad (1.32a)$$

Expressing the net energy rate efflux as $\sum \dot{E}_{\text{out}} - \sum \dot{E}_{\text{in}}$, we have:

$$\left.\frac{\Delta E}{\Delta t}\right|_{\text{system}} = \sum \dot{E}_{\text{in}} - \sum \dot{E}_{\text{out}} \tag{1.32b}$$

For *observation time* Δt the *global energy balance* for any system can be written as:

$$\underbrace{\sum E_{\text{in}} - \sum E_{\text{out}}}_{\substack{\text{Net energy transfer by heat,}\\ \text{work, and/or mass flow in/out}}} = \underbrace{\Delta E_{\text{system}} := \left[\Delta kE + \Delta pE + \Delta U\right]_{\text{system}}}_{\substack{\text{Change of total energy inside the system}\\ \text{(e.g., kinetic, potential, and internal energies)}}} \tag{1.33}$$

where the internal energy U is the sum of all microscopic energy forms, i.e., mainly due to molecular vibration.

For example, for a *closed system* with heat transferred to the system and work done by the system (see Figure 1.7):

$$\dot{Q} - \dot{W} = \frac{\Delta E}{\Delta t} \tag{1.34}$$

or during Δt considering only an internal energy change from State 1 <initial> to State 2 <final>:

$$Q - W = \Delta E \approx \Delta U = U_2 - U_1 \tag{1.35}$$

Heat can be transferred via conduction, convection, and/or thermal radiation, while work done by the closed system, such as a piston-cylinder device, is typically boundary or electric work:

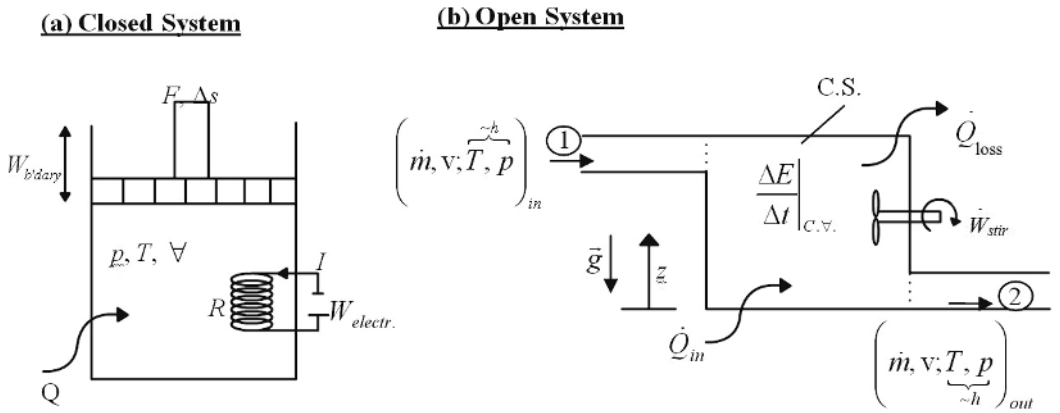


Figure 1.7 Energy transfer for closed and open systems

$$W_b = \int_1^2 F ds := \int_1^2 p d\forall \quad \text{while} \quad W_{\text{el}} = \int_1^2 VI dt \tag{1.36a,b}$$

For an *open system*, i.e., control volume, energy forms flowing in and out of the system have to be accounted for. Thus, Eq. (1.33) can be written for a fixed control volume with uniform streams entering and leaving (see Figure 1.7b):

$$\left.\frac{\Delta E}{\Delta t}\right|_{\text{C.}\forall\text{.}} = \Sigma\left[\dot{Q} + \dot{W}\right]_{\text{in}} - \Sigma\left[\dot{Q} + \dot{W}\right]_{\text{out}} + \Sigma\left[\dot{m}\left(h + \frac{v^2}{2} + gz\right)\right]_{\text{in}} - \Sigma\left[\dot{m}\left(h + \frac{v^2}{2} + gz\right)\right]_{\text{out}} \tag{1.37}$$

where for constant fluid properties $h \equiv \tilde{u} + p\upsilon$ is the enthalpy per unit mass; h combines internal energy and flow work due to pressure p moving specific volume υ, i.e., $p\upsilon$; $e_{\text{kin}} = v^2/2$ is the specific kinetic energy, and $e_{\text{pot}} = gz$ is the potential energy per unit mass.

As an aside, for steady, single-inlet/outlet, frictionless flows without internal energy changes, heat transferred, and work done, the *Bernoulli equation* appears:

$$\left[\frac{p}{\rho} + \frac{v^2}{2} + gz\right]_{\text{in}} = \left[\frac{p}{\rho} + \frac{v^2}{2} + gz\right]_{\text{out}} \tag{1.38}$$

Energy Conservation in Integral Form. The global energy balance (see Eq. (1.33) or Eq. (1.37)) is a special case of the energy RTT (see Eq. (1.40)). Specifically, taking $\boldsymbol{B}_{\text{system}} \equiv E_{\text{total}}$ and hence $\beta \equiv e_t$, the energy RTT reads [*Recall*: Eqs. (1.18) and (1.34)]:

$$\left.\frac{DE_t}{Dt}\right|_{\substack{\text{close}\\ \text{system}}} = \sum \dot{W} + \sum \dot{Q} = \frac{\partial}{\partial t}\int_{\text{C.V.}} \rho e_t d\forall + \int_{\text{C.S.}} \rho e_t \dot{v} \cdot d\vec{A} \tag{1.39}$$

Typically, $E_{\text{total}} = E_{\text{kinetic}} + E_{\text{internal}} := \left(m/2\right)\left|\vec{v}\right|^2 + m\tilde{u}$, i.e., the left-hand-side (LHS) has to be separately expressed. The physical meaning of each term in Eq. (1.39) can be summarized as:

$$\left\{\begin{array}{l}\text{Time rate of change}\\ \text{of } E_{\text{total}} \text{ in material}\\ \text{volume moving with}\\ \text{the flow}\end{array}\right\} = \left\{\begin{array}{l}\text{Rate of work } \dot{W}\\ \text{done on the C.}\forall\text{. by}\\ \text{surface forces plus}\\ \text{body forces}\end{array}\right\} + \left\{\begin{array}{l}\text{Net heat flux } \vec{q}_t\\ \text{across the C.S.}\\ \text{of the}\\ \text{control volume}\end{array}\right\}$$

where the total heat flux $\vec{q}_t = \vec{q}_{\text{cond.}} + \vec{q}_{\text{conv.}} + \vec{q}_{\text{rad.}}$. In some cases, in order to complete the energy conservation law, a distributed internal heat source term, e.g., $\int_{\forall} \rho \hat{q}_{\text{int}} d\forall$, may have to be added. Finally, using Eq. (1.39), the energy RTT reads for a stationary control volume:

$$\begin{aligned} \int_{\text{C.}\forall.} \rho\left(\vec{v}\cdot\vec{f}_b\right)d\forall + \int_{\text{C.S.}} \left(\vec{v}\cdot\vec{\vec{\tau}}\right)dA + \int_{\text{C.S.}} \vec{q}_t \cdot d\vec{A} + \int_{\text{C.}\forall.} \rho\hat{q}_{\text{int}} d\forall \\ = \frac{\partial}{\partial t}\int_{\text{C.}\forall.} \rho e_t d\forall + \rho e_t \vec{v}\cdot d\vec{A} \end{aligned} \tag{1.40}$$

Employing the *Divergence Theorem* (see App. A), the RTTs (i.e., Eqs. (1.22), (1.28), and (1.40)) can be transformed into PDEs as shown in Sects. 1.3.3.1 to 1.3.3.4. To connect Eqs. (1.37) and (1.40) is left as a homework assignment.

1.3.3 Conservation Equations in Differential Form

As discussed and illustrated in Sect. 1.3.2, the RTT provides integral system quantities such as flow rate, fluxes, and forces. In microfluidics and nanofluidics, high-resolution results in terms of *local* velocities, pressures, temperatures, concentrations as well as shear rates and fluxes are most desirable. Hence, starting with the RTT, fluid mass continuity, linear momentum, heat transfer, and species mass transfer equations are derived in *differential form*. Alternatively, the conservation laws are then also derived starting with mass, momentum, and energy balances for a fluid element.

1.3.3.1 Fluid Mass Conservation

Continuity Equation in Differential Form Based on the RTT. As discussed, the RTT is great for computing global (or integral) quantities, such as flow rates and mass fluxes or forces and energies without knowledge of the *detailed* fluid flow field *inside* the open system (i.e., the control volume). However, if it is necessary to find point-by-point density variations as well as velocity and pressure distributions in order to analyze fluid flow patterns, the conservation laws in *differential* rather than integral form have to be solved.

The conservation equations can be readily derived from the RTT, e.g., Eq. (1.19), by considering an infinitesimally small control volume $d\forall$ and then expressing each term in the form of a volume integral. For example, Eq. (1.22) contains a surface integral which has to be transformed into a volume integral, employing *Gauss' Divergence Theorem* (App. A):

$$\iint_S \vec{v} \cdot dS \equiv \iiint_{\forall} \nabla \cdot \vec{v} d\forall \tag{1.41}$$

where $\vec{v}$ is a vector in surface S, ∇ is the del operator (see App. A), and $(\nabla \cdot \vec{v})$ is the divergence of the vector field.

Using Eq. (1.41) to express the surface integral in Eq. (1.22) as a volume integral, Eq. (1.22) can be written as:

$$\frac{\partial}{\partial t} \iiint \rho d\forall + \iiint \nabla \cdot (\rho \vec{v}) d\forall = 0$$

or, following Leibniz's rule (App. A) and switching the operation for the first term, we have

$$\iiint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] d\forall = 0$$

Clearly, either $d\forall = 0$ (not physical) or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.42}$$

Equation (1.42) is known as the *continuity equation*, stating fluid mass conservation on a differential basis. Note the special cases:

- For steady flow $\left(\frac{\partial}{\partial t} \equiv 0\right)$: $\nabla \cdot (\rho \vec{v}) = 0$ (1.43)
- For incompressible fluids $(\rho = ¢)$: $\nabla \cdot \vec{v} = 0$ (1.44)

It should be noted that the widely applicable Eq. (1.44) holds for transient flow as well.

Continuity Derived from a Mass Balance. In order to gain more physical insight, Eq. (1.42) is now derived based on a 3-D mass balance. A fluid mass balance over an open system, say, a cube of volume $\Delta\forall = \Delta x \Delta y \Delta z$, yields:

$$\sum \dot{m}_{\text{in}} - \sum \dot{m}_{\text{out}} = \left. \frac{\partial m}{\partial t} \right|_{\Delta\forall} \tag{1.45}$$

from a global perspective. Using Eq. (1.45) in 1-D on a differential basis (see Figure 1.8):

$$\left[(\rho u)\big|_{x} - (\rho u)\big|_{x+\Delta x}\right]\Delta y \Delta z = \frac{\partial \rho}{\partial t}\Delta \forall \tag{1.46}$$

Now, Taylor's truncated series expansion (see App. A) states:

$$f\big|_{x+\Delta x} = f\big|_{x} + \frac{\partial f}{\partial x}\Delta x + \cdots \tag{1.47}$$

so that

$$-\frac{\partial(\rho u)}{\partial x}\Delta x \Delta y \Delta z = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z \tag{1.48}$$

Adding the other two net fluxes ρv and ρw in the y- and z-direction, respectively, and dividing by the arbitrary volume, $\Delta \forall$, yield:

$$-\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} = \frac{\partial \rho}{\partial t} \tag{1.49}$$

Thus,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.50}$$

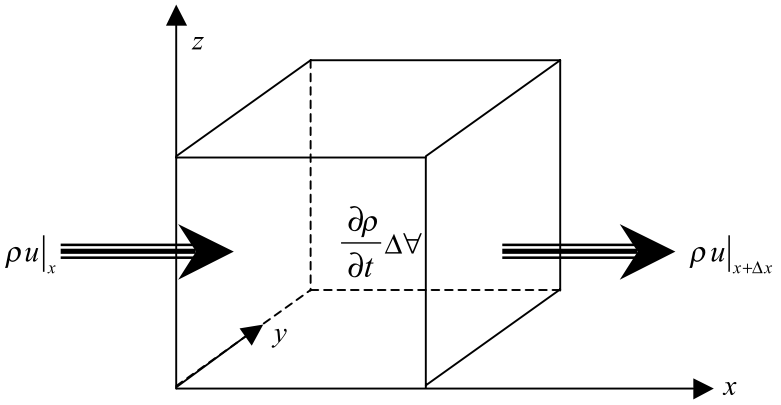


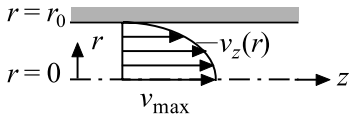
Figure 1.8 One-dimensional fluid mass balance for a 3-D control volume

Example 1.8: Use of the Continuity Equation (Two Problems: A&B)

(a) For steady laminar fully developed pipe flow of an incompressible fluid, the axial flow is:

$$v_z(r) = v_{\max}\left[1-\left(\frac{r}{r_0}\right)^2\right] \tag{E1.8-1}$$

Show that the radial (or normal) velocity $v_r = 0$.

Sketch:	*Assumptions:*
 Note: $v_{\max} = 2v_{\text{average}}$ (see Example 2.2)	• Steady implies $\partial/\partial t \equiv 0$ • Incompressible fluid: $\rho = ¢$ • Axisymmetric pipe: $\partial/\partial\,\Theta \equiv 0$ • Fully developed flow: $\partial/\partial z \equiv 0$

Solution:

Based on the assumptions, Eq. (1.42) is appropriate and reads in cylindrical coordinates (App. A):

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\Theta}{\partial \Theta}+\frac{\partial v_z}{\partial z}=0 \tag{E1.8-2}$$

Clearly, the given velocity profile $v_z(r)$ (see Eq. (E1.8-1)) is not a function of z, i.e., $\partial v_z/\partial z = 0$, which implies with $\partial/\partial\Theta = 0$ (axisymmetry) that Eq. (E1.8-2) reduces to:

$$\frac{\partial(rv_r)}{\partial r}=0 \tag{E1.8-3}$$

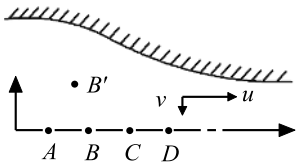
Partial integration yields:

$$v_r = f(z)/r \tag{E1.8-4}$$

where $0 \le r \le r_0$, i.e., r could be zero. That fact and the boundary condition $v_r(r = r_0) = 0$, i.e., no fluid penetrates the pipe wall, force the physical solution $v_r \equiv 0$.

Indeed, if $v_r \neq 0$, such a radial velocity component would alter the axial velocity profile to $v_z = v_z(r, z)$, which implies *developing* flow; that happens, for example, in the pipe's entrance region or due to a porous pipe wall through which fluid can escape or is being injected.

(b) Consider 2-D steady laminar symmetric flow in a smooth converging channel where axial velocity values were measured at five points (see sketch). Estimate the fluid element acceleration a_x at point C as well as the normal velocity v at point B'. All distances are 2 cm and the centerline velocities are 5 m/s at A; 7 m/s at B; 10 m/s at C; and 12 m/s at D.

Sketch:	*Concept:*
	Approximate via finite differencing the reduced acceleration and continuity equations.

Solution:

From Sect. 1.2.2 (Eq. (1.14)) the axial acceleration can be written as:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{E1.8-5}$$

Based on the stated assumptions $(\partial/\partial t = 0, \partial/\partial y = 0, w = 0)$:

$$a_x = u\frac{\partial u}{\partial x} \approx u\frac{\Delta u}{\Delta x} = u_c \frac{u_D - u_B}{x_D - x_B} = 10\frac{12-7}{0.04}$$

$\therefore$

$$\boxed{a_x|_C \approx 1250 \text{ m/s}^2}$$

In order to find $v|_B$, we employ the 2-D continuity equation in rectangular coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{E1.8-6}$$

which can be approximated as:

$$\frac{\Delta v}{\Delta y} = -\frac{\Delta u}{\Delta x}$$

or

$$\Delta v = -\frac{10-5}{0.04}0.02 = -2.5 \text{ m/s}$$

Recall that $\Delta v = v|_{B'} - v|_{B}$ where $v_B \equiv 0$ <symmetry> so that

$$\boxed{v|_{B'} = -2.5 \text{ m/s}}$$

Comment:

This is a very simple example of finite differencing where derivatives are approximated by finite differences of all variables. Discretization of the governing equations describing the conservation laws is the underlying principle of CFD (computational fluid dynamics) software (see Chapter 7).

1.3.3.2 Linear Momentum Conservation

Momentum Equation in Differential Form Based on the RTT. In order to obtain the equation of motion describing any *point* in a fluid flow field, all terms in the RTT have to be again converted to volume integrals, employing Gauss' Divergence Theorem (see Eq. (1.39)).

The Equation of Motion. First, body forces in Eq. (1.28) are logically expressed in terms of volume integrals, i.e.,

$$\vec{F}_B = \iiint_{\text{C.V.}} \rho \vec{f}_B \, d\forall$$

and surface forces in terms of surface integrals, i.e.,

$$\vec{F}_S = \iint_{\text{C.S.}} \vec{\vec{T}} \cdot d\vec{A}$$

where $\vec{f}_B$ is a body force per unit mass and $\vec{\vec{T}}$ is the total (or Cauchy) stress tensor. Now, for a stationary control volume the linear momentum equation in integral form reads:

$$\int_{\text{C.V.}} \rho \vec{f}_B d\forall + \int_{\text{C.S.}} \vec{\vec{T}} \cdot d\vec{A} = \frac{\partial}{\partial t} \int_{\text{C.V.}} \rho \vec{v} d\forall + \int_{\text{C.S.}} \vec{v} \rho \vec{v} \cdot d\vec{A} \tag{1.51}$$

Recall: This is a (3-component) vector equation in principle for the velocity field $\vec{v}$. It contains $\vec{\vec{T}} \equiv -p\vec{\vec{I}} + \vec{\vec{\tau}}$, i.e., the (9-component) total stress tensor (see App. A), as an additional unknown because in most cases $\vec{f}_B$ is simply weight ($\vec{g}$) per unit mass. Thus, in order to solve this closure problem, we have to know the *thermodynamic pressure p* and an expression for the *stress tensor* $\vec{\vec{\tau}}$. Recall that $\vec{\vec{I}}$ is the unit tensor, i.e., only ones on the diagonal and zeros everywhere else in the 3×3 matrix, elevating the product $p\vec{\vec{I}}$ to a "pseudotensor" because p is just a scalar.

Now, converting all surface integrals of Eq. (1.51) into volume integrals yields:

$$\iiint_{\text{C.V.}} \left[\frac{\partial(\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) + \nabla \cdot \left(p\vec{\vec{I}} - \vec{\vec{\tau}} \right) - \rho \vec{g} \right] d\forall = 0 \tag{1.52}$$

or

$$\frac{\partial(\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \vec{\vec{\tau}} + \rho \vec{g} \tag{1.53}$$

Equation (1.53) is the Cauchy equation of motion (or linear momentum equation) for any fluid and with gravity as the body force. In order to reduce it in complexity and provide some physical meaning, let's consider *constant fluid properties* and express the unknown stress tensor in terms of the principal variable. Employing Stokes' hypothesis, we have in vector notation:

$$\vec{\vec{\tau}} = \mu \left(\nabla \vec{v} + \nabla \vec{v}^{\,\text{tr}} \right) \equiv \mu \vec{\vec{\dot{\gamma}}} \tag{1.54a}$$

and in index (or tensor) notation:

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \equiv \mu \dot{\gamma}_{ij} \tag{1.54b}$$

where $\dot{\gamma}_{ij}$ is the shear rate tensor.

Stress Tensors and Stress Vectors. Physically τ_{ij} represents a force field per unit area (see Sect. 1.2.4 and Example 1.9) as a result of the resistance to the rate of deformation of fluid elements, i.e., internal friction. This insight leads for Newtonian fluids, such as air, water, and typical oils, to the postulate:

$$\tau_{ij} = \text{fct}\left(\varepsilon_{ij}\right) \tag{1.55}$$

where $\varepsilon_{ij} = \left(v_{i,j} + v_{j,i}\right)/2$ is the strain rate tensor, familiar from solid mechanics. Now, Stokes suggested that $\vec{\vec{\tau}}$ is a *linear function* of $\vec{\vec{\varepsilon}}$, which is not the case for non-Newtonian fluids and rarefied gases, as well as some fluid flow in microscale devices. Specifically, for Newtonian fluids:

$$\vec{\vec{\tau}} = \lambda\left(\nabla \cdot \vec{v}\right)\vec{\vec{I}} + 2\mu\vec{\vec{\varepsilon}} \tag{1.56}$$

where the viscosity coefficients λ and μ depend only on the thermodynamic state of the fluid. For incompressible flow $\nabla \cdot \vec{v} = 0$ (see Eq. (1.44)) and the total stress tensor reduces to

$$\tau_{ij} = 2\mu\varepsilon_{ij} \tag{1.57}$$

where again

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\dot{\gamma}_{ij} \tag{1.58}$$

so that

$$\tau_{ij} = \mu\left(\nabla\vec{v} + \nabla\vec{v}^{|\text{tr}}\right) \equiv \mu\dot{\gamma}_{ij} \tag{1.59}$$

The Cauchy or *total stress tensor* $\vec{\vec{T}} = -p\vec{\vec{I}} + \vec{\vec{\tau}}$ being an unknown in Eq. (1.53) constitutes a closure problem, i.e., $\vec{\vec{T}}$ has to be related to the principal variable $\vec{v}$ or its derivatives. As mentioned, p is the *thermodynamic pressure*, $\vec{\vec{I}}$ is the necessary unit tensor for homogeneity, and $\vec{\vec{\tau}}$ is the stress tensor. For any coordinate system, the stress vector $\vec{\tau}$ relates to the *symmetric* second-order tensor $\vec{\vec{T}}$ as:

$$\vec{\tau} = \hat{n} \cdot \vec{\vec{T}} = \vec{\vec{T}} \cdot \hat{n} \tag{1.60}$$

where $\hat{n}$ is the normal (unit) vector. Note, without tensor symmetry, i.e., $T_{ij} \neq T_{ji}$, angular momentum would not be conserved (see Batchelor, 1967). It is more insightful to write $\vec{\vec{T}} = -p\vec{\vec{I}} + \vec{\vec{\tau}}$ in *tensor (or index) notation* so that the total stress tensor reads:

$$T_{ij} = -p\delta_{ij} + \tau_{ij} \tag{1.61}$$

where $-p\delta_{ij}$ is interpreted as the isotropic part (e.g., fluid statics and inviscid flow) and τ_{ij} is the deviatoric part for which a constitutive equation has to be found. Physically, $\tau_{ij} \hat{=} \tau_{(i=\text{surface-normal})(j=\text{stress-direction})}$ represents a force field per unit area as a result of the resistance to the rate of deformation of fluid elements, i.e., $\tau_{ij} \sim \varepsilon_{ij}$ (see Figure 1.5).

The relation between τ_{ij} and ε_{ij} (plus vorticity tensor ζ_{ij}) can be more formally derived, starting with a fluid element displacement from point P (with $\vec{v}$ at t) to point P' ($\vec{v}+d\vec{v}$ at $t+dt$) a distance ds apart. Expanding the total derivative in Cartesian coordinates:

$$d\vec{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} d\vec{s} = \nabla\vec{v}\, d\vec{s} \tag{1.62}$$

The spatial changes, or deformations, the fluid element is experiencing can be expressed as the "rate-of-deformation" tensor (see dyadic product $\nabla\vec{v}$ in App. A):

$$\frac{d\vec{v}}{d\vec{s}} = \nabla\vec{v} := \frac{\partial v_i}{\partial x_j} \tag{1.63a,b}$$

Equation (1.63) can be decomposed into the strain rate tensor ε_{ij} (symmetrical part) and the vorticity (or rotation tensor) ζ_{ij}:

$$\frac{\partial v_i}{\partial x_j} = \varepsilon_{ij} + \zeta_{ij} \tag{1.64}$$

It can be readily shown that:

- $\zeta_{yx} = -\zeta_{xy} = \omega_z$, $\zeta_{xz} = -\zeta_{zx} = \omega_y$, and $\zeta_{zy} = -\zeta_{yz} = \omega_x$, where $\omega_z = \frac{1}{2}\left(\partial v/\partial x - \partial u/\partial y\right)$, etc.; thus, the tensor ζ_{ij} collapses to the *vorticity vector:*

$$2\vec{\omega} = \nabla \times \vec{v} = \vec{\zeta} \tag{1.65a,b}$$

- $\varepsilon_{ii} := \dot{\gamma}_{ii} \equiv \partial v_i / \partial x_i$ indicates volume change (dilation)
- $\varepsilon_{ij} := \dot{\gamma}_{ij}/2$, $i \neq j$, represents element distortion

- The shear rate tensor $\dot{\gamma}_{ij} \equiv \partial v_i / \partial x_j + \partial v_j / \partial x_i$ and hence the $1/2$ in $\varepsilon_{ij} = \dot{\gamma}_{ij} / 2$ is mathematically necessary in order to match Eq. (1.64).

As mentioned, Stokes suggested that $\vec{\vec{\tau}}$ is a *linear* function of $\vec{\vec{\varepsilon}}$, which is not the case for non-Newtonian fluids, rarefied gases, and some fluid flows in microscale devices, e.g., bioMEMS. Specifically, for Newtonian fluids, i.e., air, water, and most oils:

$$\vec{\vec{\tau}} = \lambda(\nabla \cdot \vec{v})\vec{\vec{I}} + 2\mu\vec{\vec{\varepsilon}} \tag{1.66}$$

where the viscosity coefficients λ and μ depend only on the thermodynamic state of the fluid. For incompressible flow $\nabla \cdot \vec{v} = 0$ and the total stress tensor reduces to:

$$T_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} \tag{1.67}$$

where

$$2\mu\varepsilon_{ij} \equiv \tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \equiv \mu\dot{\gamma}_{ij} \tag{1.68}$$

Here, $\dot{\gamma}_{ij} \equiv 2\varepsilon_{ij}$ is called the shear rate tensor (see App. A for all stress and shear rate components in rectangular and cylindrical coordinates).

Of great importance is the wall shear stress vector ($\vec{\tau}_{\text{wall}} \equiv \text{WSS}$) as a result of frictional (or viscous) effects and the *no-slip condition* for macroscale systems, i.e., at any solid surface:

$$\vec{v}_{\text{fluid}} = \vec{v}_{\text{wall}} \tag{1.69}$$

Typically, $\vec{v}_{wall} = 0$, i.e., the wall is stationary and impermeable. The experimentally verified no-slip condition in macrofluidics generates velocity gradients normal to the wall at all axial flow speeds. As illustrated in Figure 1.5,

$$\tau_{\text{wall}} \equiv \text{WSS} \sim \begin{cases} \partial u/\partial n & \text{(due to the no-slip condition)} \\ \mu & \text{(due to viscous fluid effects)} \\ F_{\text{tang}}/A & \text{(tangential force per unit surface area)} \end{cases} \tag{1.70a-c}$$

Very high or low WSS values have been related to device malfunctions in mechanical and arterial diseases in biomedical engineering (Kleinstreuer, 2006).

Integration of $\vec{\tau}_{\text{wall}}$ over the entire surface of a submerged body or inside a conduit yields the frictional drag:

$$F_{\text{viscous}} = \int_A \tau_{\text{wall}}\, dA \tag{1.71}$$

Viscous drag (frictional effect) plus form drag (pressure effect) make up the total drag:

$$F_{\text{drag}} = \int_A (\tau_w + p)\, dA \tag{1.72}$$

which, for most cases, would require elaborate CFD (computational fluid dynamics) analysis to evaluate the WSS and pressure distributions on the submerged body surface and then integrating.

Force Balance Derivation. A more physical approach for deriving the (linear) momentum equation starts with a force balance for a representative elementary volume (REV). Employing rectangular coordinates and an incompressible fluid, external surface and body forces accelerate an REV of mass m, so that we can write Newton's second law of motion per unit volume as (cf. Figure 1.9):

$$\rho \frac{D\vec{v}}{Dt} = \sum \vec{f}_{\text{surface}} + \sum \vec{f}_{\text{body}} \tag{1.73}$$

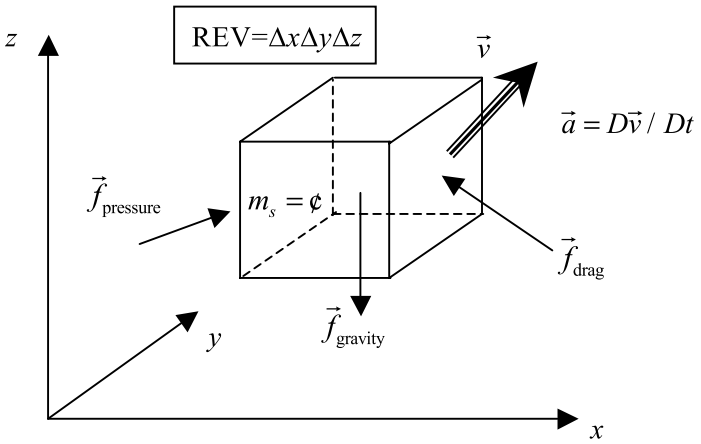


Figure 1.9 Closed system, i.e., accelerating material volume (REV)

Applying Newton's second law of motion, the REV is a control volume (i.e., fluid element) for which we record local and convective momentum changes due to *net* pressure, viscous, and gravitational forces per unit volume, viz.:

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=\vec{f}_{\underset{\text{pressure}}{\text{net}}}+\vec{f}_{\underset{\text{viscous}}{\text{net}}}+\vec{f}_{\underset{\text{buoyancy}}{\text{net}}} \tag{1.74}$$

$$f_{\underset{\text{pressure}}{\text{net}}}=f_p\big|_x-f_p\big|_{x+\Delta x}=-\frac{\partial f_p}{\partial x}\Delta x \tag{1.75}$$

and with

$$f_p\equiv\frac{p\Delta A}{\Delta V},\ f_{\underset{\text{pressure}}{\text{net}}}=-\frac{\partial p}{\partial x}\frac{\Delta y\Delta z}{\Delta x\Delta y\Delta z}\Delta x=-\frac{\partial p}{\partial x},\ \text{etc.}$$

In 3-D:

$$\vec{f}_{\underset{\text{pressure}}{\text{net}}}=-\nabla p=-\left(\frac{\partial p}{\partial x}\vec{i}+\frac{\partial p}{\partial y}\vec{j}+\frac{\partial p}{\partial z}\vec{k}\right) \tag{1.76}$$

Similarly, the net viscous force per unit volume in the x-direction reads (see Figure 1.10):

$$f_{\underset{\text{viscous}}{\text{net}}}=f_v\big|_z-f_v\big|_{z+\Delta z}=-\frac{\partial f_v}{\partial z}\Delta z \tag{1.77}$$

and with

$$f_v\equiv\frac{\tau\Delta A}{\Delta V},\ f_{\underset{\text{viscous}}{\text{net}}}=-\frac{\partial\tau}{\partial z}\frac{\Delta x\Delta z}{\Delta x\Delta y\Delta z}\Delta z=-\frac{\partial\tau}{\partial z},\ \text{etc.}$$

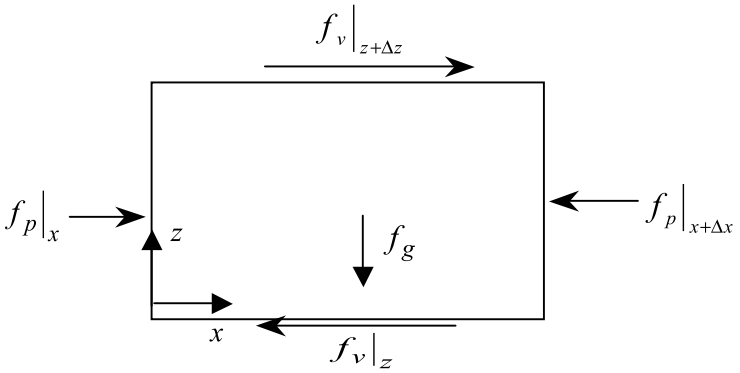


Figure 1.10 Control volume for 1-D force balances

In 3-D, the net frictional force can be expressed as:

$$\vec{f}_{\substack{\text{net}\\\text{viscous}}} = \nabla \cdot \vec{\vec{\tau}} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \vec{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \vec{j} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \vec{k} \tag{1.78}$$

As discussed, with Stokes' hypothesis for incompressible Newtonian fluids, we have (see Eq. (1.59)):

$$\vec{\vec{\tau}} = \mu \left(\nabla \vec{v} + \nabla \vec{v}^{\,\text{tr}} \right) \tag{1.79}$$

Taking the divergence of the tensor field, i.e., $\nabla \cdot \vec{\vec{\tau}} = \mu \nabla^2 \vec{v}$ (see App. A) allows expressing Eq. (1.73) as:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla \right) \vec{v} \right] = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \tag{1.80}$$

This linear momentum equation and the continuity equation $\nabla \cdot \vec{v} = 0$ are the N-S equations.

At the *molecular level for gas flow*, momentum transfer can be explained via the net shear stress as the result of mean momentum flux in the normal direction of flow (see Wilcox, 2007):

$$\tau_{yx} = 0.5 \rho v_{\text{thermal}}, \lambda_{\text{mfp}} \, du/dy = \mu \, du/dy \tag{1.81a,b}$$

where v_{thermal} (approximately 4/3 times the speed of sound in air) is the average molecular velocity and λ_{mfp} is the mean free path, i.e., the average distance traveled by a gas molecule before collision.

<u>*Example 1.9*</u>: Wall Boundary Conditions and Shear Stress in Simple Couette Flow

<u>*Note:*</u> *This detailed solution illustrates the differential approach for solving the reduced Navier-Stokes equations.*

Consider Couette flow, i.e., a viscous fluid between two parallel plates a small gap h apart, where the upper plate moves with a constant velocity u_o, in general due to an external tangential force, F_{pull}.

Note: The experimentally observed boundary condition for a conventional fluid at any solid surface demands that:

$$\vec{v}_{\text{fluid}} = \vec{v}_{\text{wall}}$$

where in rectangular coordinates $\vec{v} = (u,v,w)$ or $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$.

Applying for the present case (see system sketch), we have for a stationary solid wall $(y = 0)$:

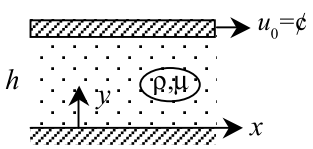
$$u_{\text{fluid}} = 0 \qquad \text{<“no-slip” condition>}$$

For the moving wall $(y = h)$:

$$u_{\text{fluid}} = u_0 \qquad \text{<“no-slip” condition>}$$

and the normal velocity component at both walls is:

$$v_{\text{fluid}} = 0 \qquad \text{<“no-penetration” condition>}$$

Sketch:	*Assumptions:*	*Approach:*
	• Steady laminar fully developed (unidirectional) flow • Constant fluid properties	• Reduced N-S equations based on assumptions and boundary conditions

Solution:

Translating the problem statement plus assumptions into mathematical shorthand, we have (see Sect. 1.2.3):

Movement of the upper plate (u_0 = constant keeps the viscous fluid between the plates in motion via frictional effects propagating normal to the plate; hence, the usual “driving force” $\partial p/\partial x \equiv 0$.

Steady flow⇒all time derivatives are zero, i.e., $\partial/\partial t \equiv 0$.

Laminar unidirectional flow⇒only one velocity component dependent on one dimension (1-D) is nonzero, i.e., $\vec{v} = (u,0,0)$, where $u = u(y)$ only. This implies parallel or fully developed flow where $\partial/\partial x \equiv 0$ and hence $v = 0$.

In summary, we can postulate that

$$u = u(y),\ v = w = 0$$

$$\frac{\partial u}{\partial x} = 0;\ \frac{\partial p}{\partial x} = 0;\ g_x = 0$$

Checking Eqs. (1.82a-c) with these postulates, we realize the following using Sect. A.5:

$$\underline{\text{Continuity equation confirms:}}\ 0 + \frac{\partial v}{\partial y} = 0 \succ \underline{v = 0} \tag{E1.9-1}$$

or better, $v = 0 \Rightarrow \partial u/\partial x = 0$; i.e., fully developed flow:

$$\underline{x\text{-Momentum yields:}}\ 0 + 0 = 0 + v\left(0 + \frac{\partial^2 u}{\partial y^2}\right) + 0 \tag{E1.9-2}$$

and

$$\underline{y\text{-Momentum collapses to:}}\ \frac{\partial p}{\partial y} = \rho g_y \ \text{<fluid statics>} \tag{E1.9-3}$$

Thus, Eq. (E1.9-2) can be written as

$$\frac{d^2 u}{dy^2} = 0 \tag{E1.9-4a}$$

subject to the "no-slip" conditions

$$u(y = 0) = 0 \ \text{ and } \ u(y = h) = u_0 \tag{E1.9-4b,c}$$

Double integration of (E1.9-4a) and inserting the boundary conditions (BCs) (E1.9-4b,c) yields:

$$u(y) = u_0 \frac{y}{h} \tag{E1.9-5}$$

Of the stress tensor (Eq. (1.54b)), $\tau_{ij} = \mu\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$, only τ_{yx} is nonzero, i.e.,

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{E1.9-6}$$

With $v = 0$ and Eq. (E1.9-5)

$$\tau_{xy} = \frac{\mu u_0}{h} = \text{¢}$$

Of the vorticity tensor $\omega_{ij} = \partial u_i / \partial x_j - \partial u_j / \partial x_i$ (Sect. A5), only ω_{yx} is nonzero, i.e.,

$$\omega_{yx} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} := \frac{u_0}{h} = \text{¢} \tag{E1.9-7}$$

which implies that the fluid elements between the plates rotate with constant angular velocity ω_{yx}, while translating with $u(y)$.

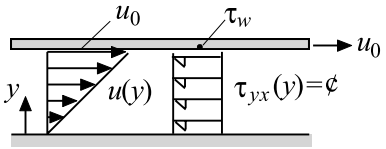
Note: The wall shear stress at the upper (moving) plate is also constant, i.e.,

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=h} = \frac{\mu u_0}{h}$$

so that

$$F_{\text{drag}} = -\int \tau_w dA_{\text{plate}} = F_{\text{pull}} = \frac{\mu u_0}{h} A_{\text{surface}} = \text{constant}$$

Profiles:



Comments:

In the absence of a pressure gradient, only viscous effects set the fluid layer into (linear) motion. The necessary "pulling force" is inversely proportional to the gap height, i.e., the thinner the fluid layer, the larger is the shear stress and hence F_{pull}.

1.3.3.3 Reduced Forms of the Momentum Equation

Returning to Eq. (1.53), which is generally known as Cauchy's *equation of motion,* we now introduce simplifications of increasing magnitude. Fluid properties, i.e.,

density ρ and dynamic viscosity μ, are typically constant; but, in general, ρ and μ are functions of temperature T, pressure p, and species concentration c. Thus, the underlying assumptions for $\rho = \text{¢}$ and $\mu = \text{¢}$ are that:

- only relatively small temperature variations occur;
- the Mach number $M \equiv v_{\text{fluid}} / a_{\text{sound}} < 0.3$, which may be only violated by gases;
- pressure drops in gas flow are relatively small, and cavitation in liquid flow is avoided;
- concentration variations of components in mixture flows are small.

(i) *The Navier-Stokes Linear Momentum Equation*:

Dividing Eq. (1.53) through by the constant density ρ and recalling that $\mu/\rho \equiv \nu$, the kinematic viscosity, we have with Stokes' hypothesis (Eq. (1.59)):

$$\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}}_{\frac{D\vec{v}}{Dt} = \vec{a}_{\text{total}}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\vec{f}_{\text{net pressure}}} + \underbrace{\nu \nabla^2 \vec{v}}_{\vec{f}_{\text{net viscous}}} + \underbrace{\vec{g}}_{\vec{f}_{\text{body}}} \tag{1.80}$$

Clearly, Eq. (1.80) is Newton's particle dynamics equation applied to fluid elements.

For example, for steady 2-D flows, the N-S equations read in rectangular coordinates (see App. A):

- (Continuity) $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$
- (x-Momentum) $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + g_x$ (1.82a-c)
- (y-Momentum) $u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + g_y$

On a professional level this set of four PDEs, subject to appropriate boundary conditions, is now being routinely solved for the four unknowns, u, v, w, and p, using numerical software packages on desktop workstations, HPC clusters, and supercomputers (see Chapter 8). In a classroom environment, only reduced forms of Eqs. (1.82a-c) can be solved.

(ii) *Prandtl's Boundary-Layer Equations:*

As indicated in Example 1.9, the fluid velocity is zero at a stationary wall. Now, considering relatively high-speed fluid flow past a (horizontal) solid surface, the

quasi-uniform high velocity suddenly has to reduce, within a narrow region, to zero at the stationary wall. This region of high-velocity gradients is called a "thin shear layer," or more generally a *boundary layer*. For example, Figure 1.11 depicts such a (laminar) boundary layer of thickness $\delta(x)$, formed along a horizontal stationary flat plate (e.g., a giant razor blade) which is approached by a uniform fluid stream of velocity u_∞ with $10^3 < \mathrm{Re} = u_\infty \ell/\nu < 10^5$. It can be readily demonstrated that the $v\partial^2 u/\partial x^2$ term (axial momentum diffusion) of Eq. (1.82b) is negligible and that the y-momentum equation collapses to $-(1/\rho)\partial p/\partial y = 0$; i.e., $p = p(x)$ only. As a result, Eqs. (1.82a-c) reduce to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.83}$$

and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{1.84}$$

inside the boundary layer $0 \le y \le \delta(x)$ and $0 \le \mathrm{x} \le \ell$.

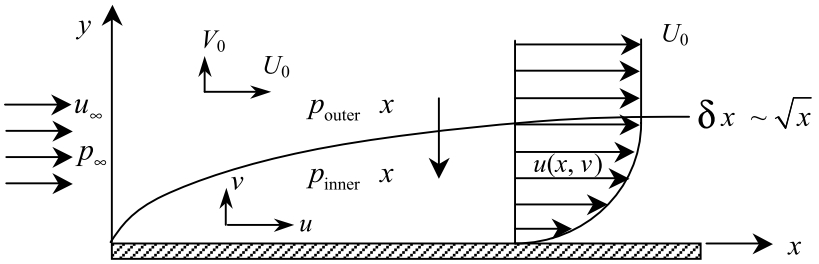


Figure 1.11 Laminar flat-plate boundary layer

(iii) <u>*Stokes' Creeping Flow Equation*</u>:

When the viscous forces are dominant, the Reynolds number (Re = inertial forces/viscous forces) is very small, i.e., the term $(\vec{v}\cdot\nabla)\vec{v}$ in Eq. (1.80) is negligible. As a result, the *Stokes equation* is *obtained* which holds for "creeping" flows Eq. (1.80) (see Example 2.13 for an application):

$$\rho\frac{\partial \vec{v}}{\partial t} = -\nabla p + \mu\nabla^2\vec{v} \tag{1.85}$$

(iv) <u>*Euler's Inviscid Flow Equation*</u>:

For frictionless flows $(\mu \equiv 0)$, Eq. (1.80) reduces to:

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + p\vec{g} \tag{1.86}$$

which is the *Euler equation*. Although ideal fluids, i.e., inviscid flows, hardly exist, the second-order term also vanishes when $\nabla^2\vec{v} \approx 0$; for example, outside boundary layers as indicated with the velocity profile in Figure 1.11. In fact, aerodynamics people employ Eq. (1.86) to find the pressure field around airfoils (see p_{outer} in Figure 1.11).

(v) *Bernoulli's Equation:*

Equation (1.86) applied in 2-D to a representative streamline along coordinate "*s*" yields (see Figure 1.12):

$$\frac{\partial v_s}{\partial t} + v_s\frac{\partial v_s}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} + g\underbrace{\sin\theta}_{\partial z/\partial s} = 0 \tag{1.87a}$$

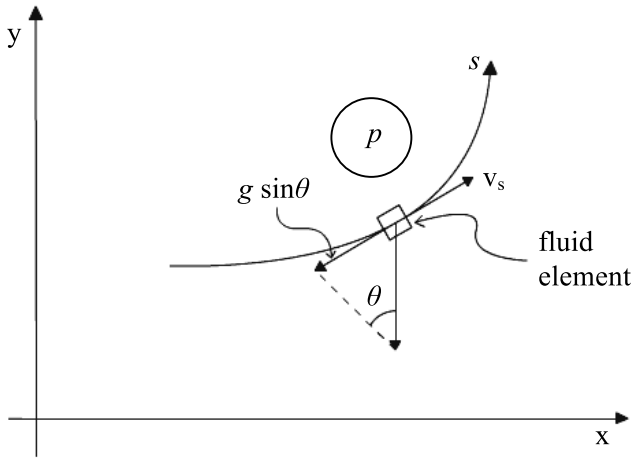


Figure 1.12 A fluid element along a representative streamline

which leads to the *Bernoulli equation*. Multiplying Eq. (1.87a) through by ∂s and integrating yield for *steady incompressible inviscid flows*:

$$\frac{v^2}{2} + \frac{p}{\rho} + gz = C \tag{1.87b}$$

where v and p are locally averaged quantities along a representative streamline and the z-coordinate extends against the direction of gravity. Thus, for two points on a

representative streamline the total energy per unit mass is balanced, i.e. (see also Eq. (1.38)):

$$\frac{v_1^2}{2}+\frac{p_1}{\rho}+gz_1=\frac{v_2^2}{2}+\frac{p_2}{\rho}+gz_2 \tag{1.88}$$

For example, for a given system where $v_2 = 0$ (e.g., point 2 of a streamline is on the front of a submerged body) and $g\Delta z \approx 0$, we have:

$$p_2 = p_1 + \frac{\rho v_1^2}{2} \tag{1.89}$$

where p_2 is the total or *stagnation point pressure*, p_1 is the *thermodynamic* (or a *static*) *pressure* at point 1, and $\rho v_1^2/2$ is the *dynamic pressure* at point 1. One application of Eq. (1.89) is the Pitot-static tube, which measures $\Delta p = p_2 - p_1$, so that $v_1 = \sqrt{2(p_2 - p_1)/\rho}$ (see Figure 1.13).

An *extended, i.e., more realistic form of Bernoulli's equation* adds a frictional loss term to the RHS of Eq. (1.88). For example, multiplying Eq. (1.88) through by ρ yields an energy balance per unit volume:

$$\underbrace{\frac{\rho}{2}v_1^2+p_1+\rho gz_1}_{\sim E_{\text{total}}}=\underbrace{\frac{\rho}{2}v_2^2}_{\sim E_{\text{kin.}}}+\underbrace{p_2}_{\sim E_{\text{pr.}}}+\underbrace{\rho gz_2}_{\sim E_{\text{pot.}}}+\underbrace{H_f}_{E_{\text{loss}}} \tag{1.90}$$

where H_f represents an energy loss between stations ① and ②.

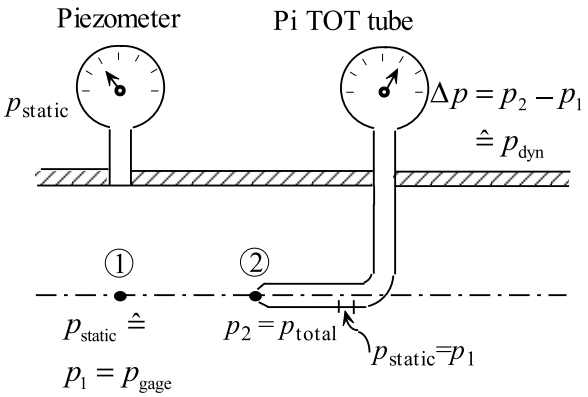


Figure 1.13 Different manometers to measure different pressures

Naturally, Eq. (1.90) can also be expressed in terms of heights (or "heads"):

$$\frac{v_1^2}{2g}+\frac{p_1}{\gamma}+z_1=\frac{v_2^2}{2g}+\frac{p_2}{\gamma}+z_2+h_f \tag{1.91}$$

The specific weight $\gamma \equiv \rho g$ and the frictional loss term $h_f \sim \tau_{\text{wall}} \sim \Delta p$ is usually expressed as a portion of the kinetic energy. Clearly, while Eq. (1.91) is based on the conservation of energy, Eq. (1.88) is based on the conservation of linear momentum.

1.3.3.4 Energy and Species Mass Conservation

Employing again the divergence theorem (see Eq. (1.41)), we can rewrite Eq. (1.40) as:

$$\rho\frac{De_t}{Dt}\equiv\rho\left[\frac{\partial e_t}{\partial t}+(\vec{v}\cdot\nabla)e_t\right]=\rho\left(\vec{f}_b\cdot\vec{v}E\right)+\nabla\cdot\left(\vec{\vec{T}}\cdot\vec{v}\right)+\nabla\cdot q_t \tag{1.92}$$

where the specific total energy is simply $e_t=|\vec{v}|^2/2+\tilde{u}\cong|\vec{v}|^2/2+c_vT$ and $\vec{q}_t=-k\nabla T+\ldots$. Clearly, T is now the temperature.

Another derivation of the energy equation, resulting in a directly applicable form, starts with $h=\tilde{u}+p/\rho$ <enthalpy per unit mass> as the principal unknown, and considering $\vec{q}$ <diffusive heat flux> and $\nu\Phi$ <energy dissipation due to viscous stress>, we obtain:

$$\frac{\partial}{\partial t}(\rho h)+\nabla\cdot(\rho\vec{v}h)=-\nabla\cdot\vec{q}+\mu\Phi \tag{1.93}$$

With $dh\equiv c_p dT$, or simplified to $h=c_pT$ when $c_p=$ constant, $\vec{q}=-k\nabla T$ after Fourier, and $\mu\Phi=\tau_{ij}\partial v_i/\partial x_j$, we obtain for thermal flow with constant fluid properties the *heat transfer equation*:

$$\frac{\partial T}{\partial t}+(\vec{v}\cdot\nabla)T=\alpha\nabla^2T+\frac{\mu}{\rho c_p}\Phi \tag{1.94}$$

where $\alpha\equiv k/(\rho c_p)$ is the thermal diffusivity and Φ is given in Sect. A.5.

It is interesting to note that when contracting Eq. (1.94),

$$\frac{DT}{Dt}=a\nabla^2T+S_T \tag{1.95}$$

has the same form as the *species mass transfer* equation:

$$\frac{Dc}{Dt} = \mathcal{D}\nabla^2 c + S_c \tag{1.96}$$

where $\mathcal{D}$ is the binary diffusion coefficient and S_c denotes possible species sinks or sources. Clearly, momentum diffusivity ν (Eq. (1.80)), thermal diffusivity a (Eq. (1.94)), and mass diffusivity $\mathcal{D}$ (Eq. (1.96)) have the same dimensions [length^2/time].

Section 1.3 is summarized in compact form via Table 1.2 and Figure 1.14. Specifically, Table 1.2 highlights the governing equations plus solution methods needed for the remainder of the book. Figure 1.14 conveys that nowadays all transport phenomena with constant fluid properties are described by the Navier-Stokes system of equations, and it provides an overview of the fluidics topics and associated equations of interest.

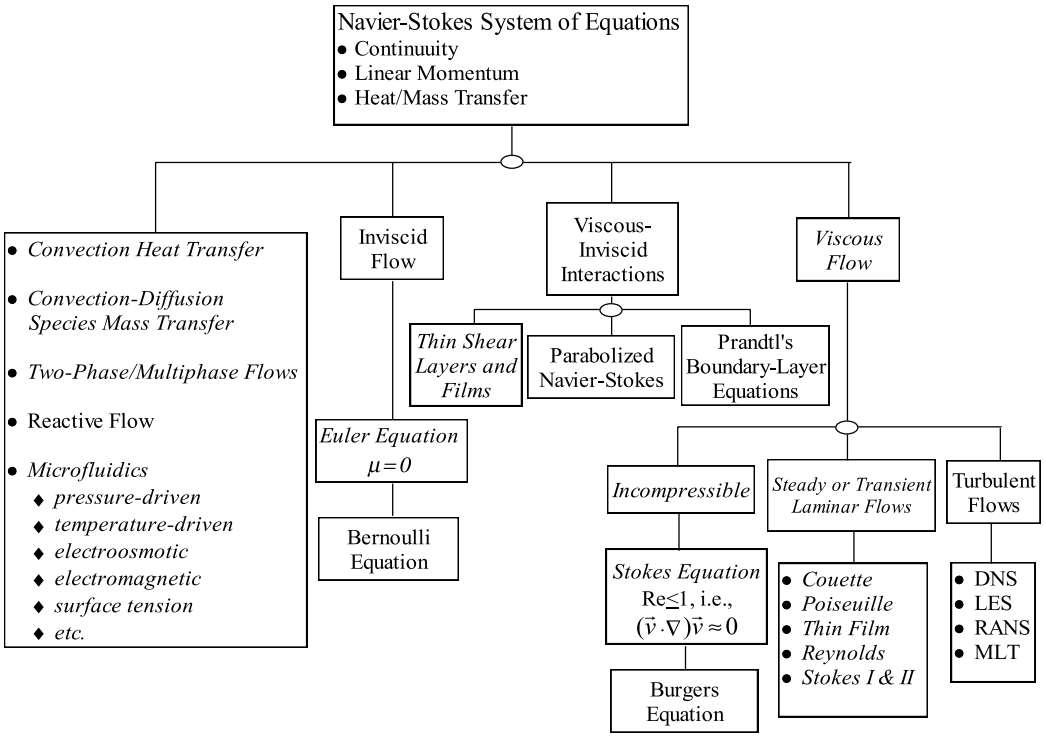


Figure 1.14 Navier-Stokes system of equations (Note: Topics and equations in italics are the main focus in this book)

Table 1.2 Solutions of Special Cases of the N-S Equations

Notes: • Laminar flow with constant fluid properties
Slip velocity plus temperature jump BCs may be needed in micro/nanofluidics (see Sect. 3.2.3)

(i) Continuity Plus Linear Momentum:

$$\nabla\cdot\vec{v}=0\ ;\ \rho\frac{D\vec{v}}{Dt}=\sum\vec{f}_i=-\nabla p+\mu\nabla^2\vec{v}+\rho\vec{g}+\vec{f}_{\text{electric}}+\vec{f}_{\text{particle}}+\ldots$$

and

(ii) Scalar Transport Equations:

$$\frac{D\Phi}{Dt}\equiv\Gamma\nabla^2\Phi\pm S_\Phi;\ \Phi=\begin{cases}T\\ c\end{cases},\Gamma=\begin{cases}\alpha\\ \mathcal{D}\end{cases},\ \alpha\equiv\frac{k}{\rho c_p};\ \text{binary diffusivity } \mathcal{D}$$

(see Stokes-Einstein equation)

where the Stokes or material derivative is:

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+(\vec{v}.\nabla)\ ;\ \text{e.g.,}\ \frac{D\vec{v}}{Dt}\equiv\vec{a}_{\text{total}}=\frac{\partial\vec{v}}{\partial t}+(\vec{v}.\nabla)\vec{v}$$

Exact Solutions:
1-D: Simple geometry + fully developed flow
Couette, Poiseuille, thin-film
Direct integration $u''=K;\ K\sim\Delta p/L$
$Q=\Delta p/R_{\text{flow}}$; hydraulic resistance
$R_{\text{flow}}\sim\mu$
2-D: Simple geometry
Combined variable $\eta=\eta(x,t)$
Examples: • Stokes Problems I & II
• Von Karman Problem
• Blasius Problem

Approximate Solutions:

- Prandtl's Boundary-Layer (BL) Theory:

$$\text{Re}>>1,\frac{\delta}{l}<<1$$

- Euler's Equation $(\mu=0)$
- Stokes' Equation (Re <1.0)

Numerical Solutions

- Transient 3-D FE, FV, or FD solver
- Submodels for Exotic Fluids
- Submodels for Turbulence
- OpenFOAM: free access software (http://www.wikki.co.uk/)
- DNS (Direct Numerical Simulation)
- LBM (Lattice Boltzmann Method)
- MD (Molecular Dynamics) Simulation
- DSMC (Direct-Simulation Monte Carlo) Method

Example 1.10: Thermal Pipe Flow ($q_{\text{wall}} = \not{c}$)

Consider Poiseuille flow where a uniform heat flux, q_w, is applied to the wall of a pipe with radius r_0.

Set up the governing equations for the fluid temperature assuming thermally fully developed flow, i.e.,

$$\frac{T_w - T}{T_w - T_m} \equiv \Theta\left(\frac{r}{r_0}\right) \tag{E1.10-1}$$

where $T_w(x)$ is the wall temperature, $T(r,x)$ is the fluid temperature, and $T_m(x)$ is the cross-sectionally averaged temperature, i.e.,

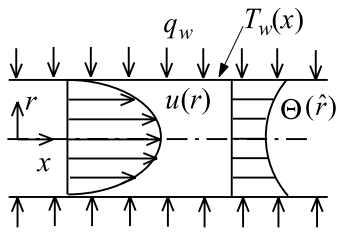
$$T_m = \frac{1}{\bar{u}A}\int_A uTdA \tag{E1.10-2}$$

Note that $\Theta = \Theta(r)$ only, describing thermally fully developed flow.

Solve a reduced form of the heat transfer equation (1.75) and develop an expression for the Nusselt number, defined as:

$$\text{Nu} = \frac{2r_0}{k}\frac{q_w}{T_w - T_m} := \frac{hD}{k} \tag{E1.10-3a,b}$$

where k is the fluid conductivity and D is the pipe diameter.

Sketch:	*Assumptions:*	*Concept:*
	• As stated, i.e., $u(r) = u_{\max}(1 - (\frac{r}{r_0})^2)$ $\partial T / \partial t = 0$ <steady state> $(\vec{v} \cdot \nabla)T \Rightarrow u\frac{\partial T}{\partial x}$ $\alpha \nabla T \Rightarrow \frac{\alpha}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$	• Reduced heat transfer equation (Eq. (1.75)) based on assumptions

Solution:

With the reduced heat transfer equation in cylindrical coordinates from App. A (see also list of assumptions), we have:

$$\frac{u(r)}{\alpha}\frac{\partial T}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{E1.10-4}$$

Employing the dimensionless temperature profile $\Theta\left(r/r_0\right)\equiv\Theta\left(\hat{r}\right)$ given as Eq. (E1.10-1), we can rewrite Eq. (E1.10-4) as

$$-2\frac{hD}{k}\left(1-\hat{r}^2\right)=\frac{d^2\Theta}{d\hat{r}^2}+\frac{1}{\hat{r}}\frac{d\Theta}{d\hat{r}} \tag{E1.10-5}$$

Specifically,

$$\text{for } q_w=\text{¢},\ \frac{\partial T}{\partial x}=\frac{dT_w}{dx}=\frac{dT_m}{dx}=\frac{2}{r_0}\frac{q_w}{\rho c_p\overline{u}}=\text{¢} \tag{E1.10-6a-c}$$

as stated,

$$\frac{hD}{k}\equiv Nu_D:=\frac{D}{k}\frac{q_w}{T_w-T_m}=\text{¢} \tag{E1.10-7a, b}$$

and with $d\Theta/d\hat{r}$ being finite at $\hat{r}=0$, we obtain

$$T-T_w=-\left(T_w-T_m\right)\mathrm{Nu}_D\left(\frac{3}{8}-\frac{\hat{r}^2}{2}+\frac{\hat{r}^4}{8}\right) \tag{E1.10-8}$$

Now, by definition

$$T_w-T_m=\frac{2\pi}{\pi r_0^2\overline{u}}\int_0^{r_0}\left(T_w-T\right)u(r)rdr \tag{E1.10-9}$$

so that, when combining both equations and integrating, we have

$$1 = 4\mathrm{Nu}_D \int_0^1 \left(\frac{3}{8} - \frac{\hat{r}^2}{2} + \frac{\hat{r}^4}{8} \right) \left(1 - \hat{r}^2 \right) \hat{r}\, d\hat{r} \tag{E1.10-10}$$

from which we finally obtain:

$$\mathrm{Nu}_D = \frac{48}{11} = 4.36 \tag{E1.10-11}$$

Comment:

It is interesting to note that for hydrodynamically and thermally fully developed flow in a tube, subject to a constant wall heat flux, the Nusselt number (or the heat transfer coefficient) is constant. The same holds for the isothermal wall condition; however, the Nu value is lower (see Kleinstreuer, 1997; or Bejan, 2002), as summarized in Table 1.3.

1.3.4 Entropy Generation Analysis

Background Information. The second law of thermodynamics is the "increase of entropy" principle for any real process, i.e.,

$$\Delta S_{\text{total}} = \Delta S_{\text{system}} + \Delta S_{\text{surrounding}} \equiv S_{\text{gen}} > 0 \tag{1.97a-c}$$

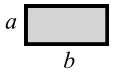
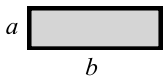
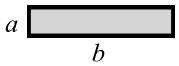
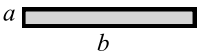
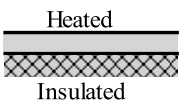
An entropy value S [kJ/K] indicates the degree of molecular activity or *randomness* and the condition $S_{\text{gen}} > 0$ is necessary for a process to proceed or a device to work. The source of entropy change is heat transferred from different sources. As Clausius stated, $dS > \delta Q / T$, implying all irreversibilities are contributing, e.g., due to heat exchange with internal and/or external sources as well as internal friction (or viscous effects) and net influx of entropy carried by fluid streams. The inequality (1.97c) can be recast as an entropy "balance" by recasting Eq. (1.97b) as:

$$\underbrace{\sum_{\text{in}} m \cdot s - \sum_{\text{out}} m \cdot s + \sum \frac{Q}{T_{\text{ambient}}}}_{\Delta S_{\text{surrounding}}} + \underbrace{S_{\text{gen}}}_{\substack{\text{system} \\ \text{irreversibilities}}} = \Delta S\big|_{\text{C.}\forall.} \tag{1.98a}$$

where

$$\Delta S\big|_{\text{C.}\forall.} = \underbrace{(ms)_{\text{final}} - (ms)_{\text{initial}}}_{\Delta S_{\text{system}}} \tag{1.98b}$$

Table 1.3 Nusselt Numbers and Poiseuille Numbers for Fully Developed Flows in Different Conduits

Cross Section	*b/a Ratios*	$\mathrm{Nu} = hD_h / k$		$\mathrm{Po} = f\,\mathrm{Re}_{D_h}$
		$q_{\text{wall}} = ¢$	$T_{\text{wall}} = ¢$	
	——	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
	∞	5.39	4386	96
	——	3.11	2.49	53

Notes: (i) The Poiseuille number $\mathrm{Po} = f\,\mathrm{Re}_{D_h}$, where the Darcy friction factor $f = 8\tau_w / \rho v^2$, (ii) The Nusselt number $\mathrm{Nu} = hD_h / k$.

Clearly, the larger S_{gen}, the more inefficient a process, device, or system is, i.e., S_{gen} is equivalent to "amount of waste generated." In convection heat transfer this "energy destruction" appears as viscous dissipation and random disorder due to heat input:

$$S_{\text{gen}} \sim \mu\Phi \text{ and } k(\nabla T)^2 \tag{1.99a}$$

or

$$S_{\text{gen}}^{\text{total}} = S_{\text{gen}}^{\text{friction}} + S_{\text{gen}}^{\text{thermal}} \tag{1.99b}$$

Entropy Generation Derivation. For optimal system/device design it is important to find for a given objective the *best possible system geometry and operational conditions* so that S_{gen} is a minimum. Thus, within the framework of convection heat transfer with Newtonian fluids, it is of interest to derive an expression for

$$S_{gen} = S_{gen}(\text{thermal}) + S_{gen}(\text{friction}) \tag{1.100}$$

Clearly, Eq. (1.100) encapsulates the irreversibilities due to heat transfer ($S_{gen,thermal}$) and viscous fluid flow ($S_{gen,friction}$).

Considering a point (x,y,z) in a fluid with convective heat transfer, the fluid element *dx-dy-dz* surrounding this point is part of a thermal flow system. Thus, the small element *dx-dy-dz* can be regarded as an open thermodynamic system, subject to mass fluxes, energy transfer, and entropy transfer interactions that penetrate the fixed control surface formed by the *dx-dy-dz* box of Figure 1.15. Hence, the local volumetric rate of entropy generation $\left(S_{gen} \text{ in } \left[\text{kW}/\left(\text{m}^3 \cdot \text{K}\right)\right]\right)$ is considered inside a viscous fluid with convective heat transfer without internal heat generation. The second law of thermodynamics for the *dx-dy-dz* box as an open system experiencing fluid flow and convective heat transfer then reads, based on the Clausius definition $dS = \delta Q/T\big|_{\text{reversible}}$ and Figure 1.15:

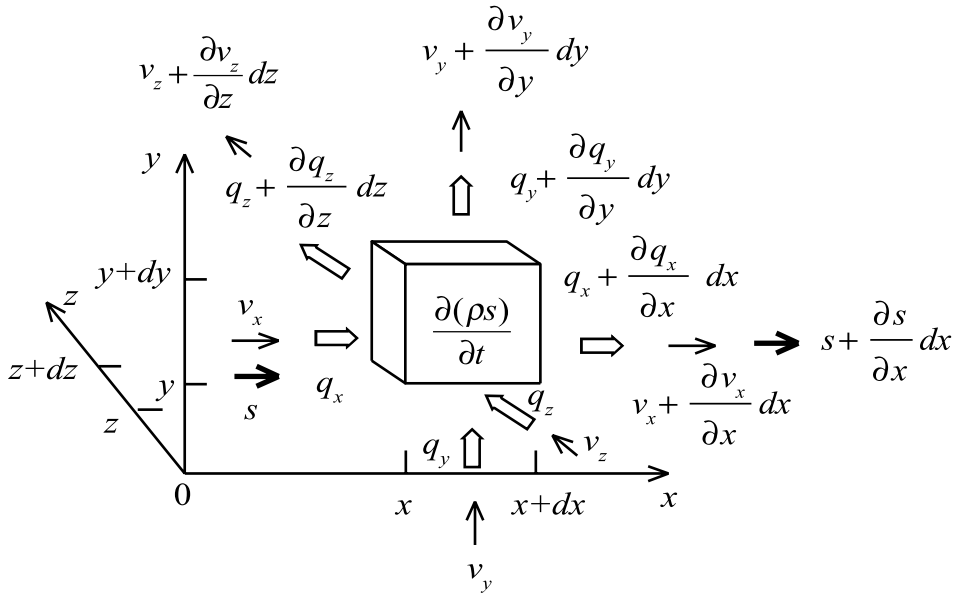


Figure 1.15 The local generation of entropy in a flow with a viscous fluid and conductive heat transfer

$$S_{\text{gen}}dxdydz = \frac{q_x + \frac{\partial q_x}{\partial x}dx}{T + \frac{\partial T}{\partial x}dx}dydz - \frac{q_x}{T}dydz + \frac{q_y + \frac{\partial q_y}{\partial y}dy}{T + \frac{\partial T}{\partial y}dy}dxdz - \frac{q_y}{T}dxdz$$
$$+ \frac{q_z + \frac{\partial q_z}{\partial z}dz}{T + \frac{\partial T}{\partial z}dz}dxdy - \frac{q_z}{T}dxdy + \frac{\partial(\rho s)}{\partial t}dxdydz$$
$$+ \left(s + \frac{\partial s}{\partial x}dx\right)\left(v_x + \frac{\partial v_x}{\partial x}dx\right)\left(\rho + \frac{\partial \rho}{\partial x}dx\right)dydz - sv_x\rho dydz$$
$$+ \left(s + \frac{\partial s}{\partial y}dy\right)\left(v_y + \frac{\partial v_y}{\partial y}dy\right)\left(\rho + \frac{\partial \rho}{\partial y}dy\right)dxdz - sv_y\rho dxdz$$
$$+ \left(s + \frac{\partial s}{\partial z}dz\right)\left(v_z + \frac{\partial v_z}{\partial z}dz\right)\left(\rho + \frac{\partial \rho}{\partial z}dz\right)dxdy - sv_z\rho dxdy \qquad (1.101a)$$

The first six terms on the right side of Eq. (1.101a) account for the entropy transfer associated with heat transfer. Combining terms 1 and 2, 3 and 4, and 5 and 6 and dividing by *dx*-*dy*-*dz* and taking the limit, the former six terms in Eq. (1.101a) can be reduced to:

$$\frac{T\frac{\partial q_x}{\partial x} - q_x\frac{\partial T}{\partial x}}{T\left(T + \frac{\partial T}{\partial x}dx\right)} + \frac{T\frac{\partial q_y}{\partial y} - q_y\frac{\partial T}{\partial y}}{T\left(T + \frac{\partial T}{\partial y}dy\right)} + \frac{T\frac{\partial q_z}{\partial z} - q_z\frac{\partial T}{\partial z}}{T\left(T + \frac{\partial T}{\partial z}dz\right)}$$
$$= \frac{1}{T}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial y}\right) - \frac{1}{T^2}\left(q_x\frac{\partial T}{\partial x} + q_y\frac{\partial T}{\partial y} + q_z\frac{\partial T}{\partial z}\right) \qquad (1.101b)$$

Terms 7 to 12 in Eq. (1.101a) represent the entropy convected into and out of the system, while the last term is the time rate of entropy accumulation in the *dx*-*dy*-*dz* control volume. Decomposing and combining the last seven terms as well as considering the limit, the last seven terms can be rearranged as:

$$\rho\left(\frac{\partial s}{\partial t} + v_x\frac{\partial s}{\partial x} + v_y\frac{\partial s}{\partial y} + v_z\frac{\partial s}{\partial z}\right)$$
$$+ s\left[\frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\right] \qquad (1.101c)$$

Combining Eq. (1.101b) and Eq. (1.101c), the local rate of entropy generation becomes:

$$\begin{aligned} S_{\text{gen}} = & \frac{1}{T}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-\frac{1}{T^2}\left(q_x\frac{\partial T}{\partial x}+q_y\frac{\partial T}{\partial y}+q_z\frac{\partial T}{\partial z}\right) \\ & +\rho\left(\frac{\partial s}{\partial t}+v_x\frac{\partial s}{\partial x}+v_y\frac{\partial s}{\partial y}+v_z\frac{\partial s}{\partial z}\right) \\ & +s\left[\frac{\partial \rho}{\partial t}+v_x\frac{\partial \rho}{\partial x}+v_y\frac{\partial \rho}{\partial y}+v_z\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\right] \end{aligned} \tag{1.102}$$

Note that the last term of Eq. (1.102) (in square brackets) vanishes identically based on the mass conservation principle. Equation (1.102) can be recast, so that in vector notation the volume rate of entropy generation reads:

$$S_{\text{gen}} = \frac{1}{T}\nabla\cdot\vec{q}-\frac{1}{T^2}\vec{q}\cdot\nabla T+\rho\frac{Ds}{Dt} \tag{1.103}$$

According to the first law of thermodynamics, the rate of change in internal energy per unit volume is equal to the net heat transfer rate by conduction, plus the work transfer rate due to compression, plus the work transfer rate per unit volume associated with viscous dissipation, i.e.,

$$\rho\, D\tilde{u}/Dt = -\nabla\cdot\vec{q}-p\left(\nabla\cdot\vec{v}\right)+\mu\Phi \tag{1.104}$$

Employing the Gibbs relation in the form $d\tilde{u} = Tds - pd\left(1/r\right)$ and using the substantial derivative notation Eq. (1.104), we obtain:

$$\rho\frac{Ds}{Dt} = \frac{\rho}{T}\frac{D\tilde{u}}{Dt}-\frac{p}{\rho T}\frac{D\rho}{Dt} \tag{1.105}$$

Combining Eq. (1.105) with $\rho\, Ds/Dt$ given by Eq. (1.103) and $\rho\, D\tilde{u}/Dt$ given by Eq. (1.104), the volumetric entropy generation rate can be expressed as:

$$S_{\text{gen}} = -\frac{1}{T^2}\vec{q}\cdot\nabla T+\frac{\mu}{T}\Phi \tag{1.106}$$

If the Fourier law of heat conduction for an isotropic medium applies, i.e.,

$$\vec{q} = -k\nabla T \tag{1.107}$$

the rate of volumetric entropy generation (S_{gen}) in three-dimensional Cartesian coordinates is then (see also Bejan, 1996):

$$S_{\text{gen}} \equiv S_G = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right] + \frac{\mu}{T}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} \tag{1.108}$$

where u, v, and w are velocity vectors in the x-, y-, and z-directions, respectively; T is the temperature, k is the thermal conductivity, and μ is the dynamic viscosity.

Specifically, the dimensionless entropy generation rate induced by fluid friction and heat transfer can be defined as follows:

$$S_{G,F} = S_{\text{gen}}(\text{frictional}) \cdot kT_0{}^2/q^2 \tag{1.109}$$

where

$$S_{\text{gen}}(\text{frictional}) = \frac{\mu}{T}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} \tag{1.110}$$

while for the thermal entropy source,

$$S_{G,Y} = S_{\text{gen}}(\text{thermal}) \cdot \frac{kT_0^2}{q^2} \tag{1.111}$$

where

$$S_{\text{gen}}(\text{thermal}) = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right] \tag{1.112}$$

Finally,

$$S_{G,\text{total}} = S_{\text{gen}} \frac{kT_0^2}{q^2} = S_{G,F} + S_{G,T} \tag{1.113}$$

where T_0 is the fluid inlet temperature and q is the wall heat flux.

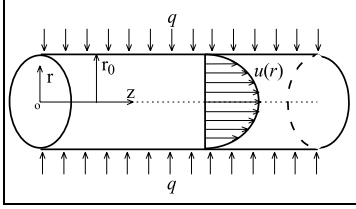
Example 1.11: Thermal Pipe Flow with Entropy Generation

Deriving the irreversibility profiles for Hagen-Poiseuille (H-P) flow through a smooth tube of radius r_0 with uniform wall heat flux $q\left[\mathrm{W/m^2}\right]$ at the wall, the velocity and temperature for the fully developed regime are given by:

$$u = 2U\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{E1.11-1}$$

and

$$T - T_s = -\frac{qr_0}{k}\left[\frac{3}{4} - \left(\frac{r}{r_0}\right)^2 + \frac{1}{4}\left(\frac{r}{r_0}\right)^4\right] \tag{E1.11-2}$$

Sketch:	*Assumptions:*	*Conceptions:*
	• Fully developed H-P flow • Constant properties and parameters	• Volumetric entropy generation rate Eq. (1.91) • Thermal entropy generation • Frictional entropy generation

Solution:

The wall temperature $T_s = T\left(r = r_0\right)$ can be obtained from the condition

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{2q}{\rho c_p U r_0} = \text{constant} \tag{E1.11-3}$$

$$\frac{\partial T}{\partial r} = \frac{q}{k}\left[2\frac{r}{r_0} - \left(\frac{r}{r_0}\right)^3\right] \tag{E1.11-4}$$

$$\frac{\partial u}{\partial r} = \frac{-4Ur}{r_0^2} \tag{E1.11-5}$$

Hence, the dimensionless entropy generation for fully developed tubular H-P flow can be expressed as:

$$S_{\text{gen}}\frac{kT_0^{\,2}}{q^2}=\frac{4k^2}{\left(\rho c_p\bar{U}r_0\right)^2}\frac{T_0^{\,2}}{T^2}+\left(2R-R^3\right)^2\frac{T_0^{\,2}}{T^2}+\frac{16kT_0^{\,2}\mu\bar{U}^2}{q^2Tr_0^{\,2}}R^2$$
$$=\underbrace{\frac{16}{\text{Pe}^2}\left(\frac{T_0}{T}\right)^2+\left(2R-R^3\right)^2\frac{T_0^{\,2}}{T^2}}_{\text{heat transfer}}+\underbrace{\varphi\frac{T_0}{T}R^2}_{\text{fluid friction}} \tag{E1.11-6}$$

with

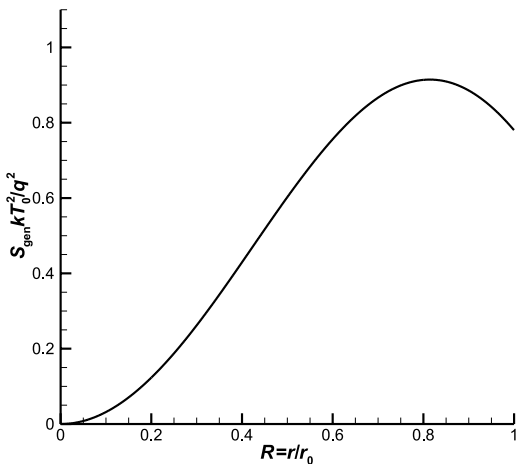
$$\text{Pe}=\text{Re}\cdot\text{Pr}=\frac{2r_0\rho c_pU}{k} \tag{E1.11-7}$$

and

$$R=\frac{r}{r_0}\ \text{and}\ \varphi=\frac{16kT_0\mu U^2}{q^2r_0^{\,2}} \tag{E1.11-8a,b}$$

Here, R is the dimensionless radius, T_0 is the inlet temperature which was selected as the reference temperature. On the right side of Eq. (E1.11-6), the first term represents the entropy generation by axial conduction, the second term is the entropy generated by heat transfer in the radial direction, and the last term is the fluid friction contribution. Parameter φ is the irreversibility distribution ratio ($S_{\text{gen}}\left(\text{fluid friction}\right)/S_{\text{gen}}\left(\text{heat transfer}\right)$).

Graph:



Comments:

As expected, according to Eq. (E1.11-6), at the center point, i.e., $R = 0$, only the first term on the right side contributed to the dimensionless entropy generation rate; however, for $\mathrm{Pe} >> 1$, the irreversibility due to axial conduction is negligible in the fully developed range. In contrast, in the wall region both thermal and frictional effects produce entropy with a maximum at $R \approx 0.8$ generated by dominant heat transfer induced entropy generation.

1.4 Homework Assignments

The suggested homework assignments either try to illuminate the text material via questions and tasks (Category I), probing basic understanding and "physical insight," or pose typical fluid dynamics problems whose solutions illustrate the chapter theory. These "text problems" (Category II) were accumulated selectively in revised and extended form over years of undergraduate teaching at North Carolina State University with books by Cengel & Cimbala, Fox et al., Potter & Wiggert, and White. Again, their contributions are gratefully acknowledged.

1.4.1 Physical Insight

1.1 Discuss additional implications of fluidic-device miniaturization down to the molecular level, i.e., provide examples of new (or unusual) material/fluid properties, fluid flow behavior, and device/system applications.

1.2 A force per unit area is labeled "pressure" or "stress"; however, one is a scalar and the other, in general, a tensor of rank 2: Explain the differences mathematically and physically.

1.3 Fluid flow through micro/nanochannels requires a driving force (e.g., in the form of a pressure drop). Contrast necessary "pumps" providing such a pressure differential for: (a) conventional (i.e., macroscale) systems and (b) micro/nanoscale systems.

1.4 Considering an ideal gas, derive expressions for pressure p, temperature T, and dynamic viscosity μ on the molecular level and explain.

1.5 Discuss the *continuum mechanics assumption* in light of "fluid flow" borderline cases, such as grain flow in a silo or jelly movement in food processing.

1.6 Is "open-channel flow" an example of *internal flow* or *external flow*?

1.7 Derive Eq. (1.10) and plot the case $\vec{v} = (0.5 + 0.8x)\hat{i} + (1.5 - 0.8y)\hat{j}$ in $[\mathrm{m/s}]$.

1.8 Employing *scale analysis,* derive the following dimensionless groups: Peclet number, Euler number, Nusselt number, Weber number, and Sherwood number; also provide sample applications.

1.9 Considering Figure 1.3, set up a general energy balance (i.e., a first-order difference equation) for a transient open system, and show that any closed system is just a special case. What are the advantages/disadvantages of such a lumped-parameter approach vs. a differential approach with spatial resolution?

1.10 Recast Eqs. (1.15) to (1.17) so that the material properties all appear as (momentum, thermal, and mass) *diffusivities*, say, all in $[\mathrm{m^2/s}]$.

1.11 Design an illustrative figure depicting the four "derivation approaches" mentioned in Sect. 1.3.1. Specifically, list math and physics aspects as well as possible approach interactions and applications.

1.12 Considering the extended cases of the RTT discussed in Sect. 1.3.2, expand Eq. (1.19) and derive two more general RTTs, i.e., one for a translating, accelerating, and deforming control volume and another one for a rotating control volume.

1.13 Concerning Example 1.3: (a) what are the limits on vertical coordinate z and (b) derive Eq. (E1.3-4), i.e., Torricelli's law.

1.14 Looking at Sect. 1.3.2.2, expand the D/Dt operator in the LHS of Eq. (1.28a) and express properly the RHS to arrive *directly* at the Navier-Stokes equation for linear momentum transfer (see Eq. (1.80)).

1.15 Provide some additional math and physics details to the "derivation" of Eq. (1.33), starting with the energy conservation law (1.31) in Sect. 1.3.2.3.

1.16 Starting with Eq. (1.37), derive a useful form of Eq. (1.40). Contrast that to the derivation approach when using the energy RTT.

1.17 What are the mathematical expressions for the three heat fluxes making up the total heat flux in Eq. (1.40)?

1.18 List and comment on the advantages of the conservation laws in *differential* form (see Sect. 1.3.3) over the integral form.

1.19 What is the mathematical reason and give a physical explanation for attaching the unit tensor to the pressure p in Eq. (1.52) and in other similar cases?

1.20 Explain the need and meaning of the transpose of the "velocity field gradient" in Eq. (1.54); also comment on the advantages/disadvantages of equations written/manipulated in "vector notation" versus "index notation."

1.21 Illustrate and contrast with an application each the following: the stress vector versus the stress tensor versus the total stress tensor.

1.22 Why is the (total) stress tensor symmetric?

1.23 Considering shear flow, illustrate Eq. (1.62) to explain Eqs. (1.63) and (1.64).

1.24 Prove Eq. (1.65) and illustrate/explain the "collapse" of the vorticity tensor to the vorticity vector

1.25 Equation (1.69) implies a host of boundary conditions. Discuss three basic cases.

1.26 Most solved book examples assume steady laminar fully developed flow, known as "Poiseuille-type" flow. Discuss the mathematical and physical implications and show three applications for which that flow field simplification would be *incorrect*.

1.27 In Sect. 1.3.3.3, four assumptions are listed for the transformation of the Cauchy equation to the Navier-Stokes equation: (a) justify these four assumptions; (b) derive Eq. (1.81) from Eq. (1.52); (c) give two examples when using the N-S equation would be inadequate.

1.28 Prandtl's boundary layer equations plus Euler's equation are still quite popular in aerodynamics and hardly used everywhere else. Why? In contrast, Stokes' equation (see Sect. 1.3.3.3) is often applicable in micro/nanofluidics. Why?

1.29 On a macroscale, enthalpy encapsulates internal energy plus flow energy and hence the essential contributions to "total energy" for many flow systems, devices, and processes. Hence, replacing e_{total} with h in Eq. (1.92), derive Eq. (1.93) and Eq. (1.95).

1.30 Starting with the "species mass" RTT, i.e., $B_S = m/\forall = c$, and assuming Fourier's law plus constant properties, derive Eq. (1.96). What are examples of S_c?

1.31 Show that, in general, hydrodynamic and thermal entrance lengths differ, i.e., provide math/physics/graphical explanations. Under what condition are they the same, if any?

1.32 Looking at Table 1.3, why is Nu ($q_{\text{wall}} = \text{const.}$) always greater than Nu ($T_{\text{wall}} = \text{const.}$)?

1.33 Derive Eq. (1.98) from the inequality stated in (1.97).

1.34 Explain/illustrate the two proportionalities for S_{gen} in Eq. (1.99a).

1.35 Derive Eqs. (1.104) and (1.105) and show that Eq. (1.106) is correct.

1.4.2 Text Problems

1.36 *Tailored continuity equations in differential form:* Set up the continuity equation for:

(a) steady 1-D compressible flow in a tube;

(b) compressible isothermal gas flow in terms of pressure p;

(c) an incompressible fluid flowing just radially into: (i) a line sink or (ii) a point sink.

1.37 *Velocity components of incompressible flows:* Determine and plot:

(a) $v(x,y)$ for planar flow with $u(x,y) = 10 + \dfrac{5x}{x^2 + y^2}$ if $v(x,0) = 0$

(b) $v_r(r,\theta)$ for cylindrical flow with $v_\theta = -\left(10 + 4/r^2\right)\cos\theta$ if $v_r(2,\theta) = 0$

(c) $v(x,y)$ for a near-surface 2-D flow with $u(x,y) = 10\left[2\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2\right]$, assuming $\delta(x) = Cx^{4/5}$, $v(x,0) = 0$, and $w = 0$

1.38 *Reduced system of Navier-Stokes equations:* Consider a large horizontal plate beneath a liquid to oscillate with $u_{\text{wall}} = u_0 \sin\omega t$. Propose the conservation laws for: (i) constant viscosity and (ii) temperature-dependent viscosity.

1.39 *Transient fully developed pipe flow:* Based on measurements, the following 1-D velocity profile has been constructed:

$$u(r,t) = u_m\left[1-\left(\frac{r}{r_0}\right)^2\right]\left(1-e^{t/\tau}\right)$$

where $u_m = 2$ m/s, $r_0 = 0.02$ m, and $\tau = 10$ s. Find the fluid element velocity and acceleration for $t = 1$ s at $r = 0$ and $r = 0.25r_0$. At what times t do the maximum values appear at these locations? Plot the parabolic profiles at different time levels during $0 \le t \le \infty$.

1.40 P*eriodically changed temperature field:* In a steady flow field described by $u = u_m\left(1-\hat{y}^2\right)$, the temperature varies as

$$T\left(\hat{y},t\right) = T_m\left(1-\hat{y}^2\right)\cos\left(\pi\frac{t}{\tau}\right) \text{ in } ^\circ\text{C}$$

where $\hat{y} \equiv y/d$, $u_m = 2$ m/s, $T_m = 20^\circ$C, and $\tau = 100$ s. Find $T\left(\hat{y}=0, t=20 \text{ s}\right)$, plot $T\left(\hat{y},t\right)$, and comment!

1.41 *Is the pipe entrance length important?* Consider a horizontal 2 mm tube 3 m long connected to a constant-head reservoir. If 9 L/h of water at 15°C is collected, determine the ratio of $L_{\text{entrance}}/L_{\text{tube}}$ and comment.

1.42 *Show qualitatively that* $\left(\Delta p/\Delta x\right)_{\text{entrance}} > \left(\Delta p/\Delta x\right)_{\text{fully developed}}$. Now, consider an $RE\forall$ (radius R and thickness Δx) for steady laminar pipe flow, and obtain $\Delta p/\Delta x$ from a 1-D force balance. Based on physical insight conclude the stated results.

1.43 *Poiseuille flow in a slanted pipe:* Its radius is R and elevation change is h, with fluid properties μ and $\gamma = \rho g$. First show that the average velocity

$$v_{\text{avg}} = -\frac{R^2}{8\mu}\frac{d}{dx}\left(p+\gamma h\right)$$

Then compute the Reynolds number and wall shear stress for 20°C water flow in a pipe ($D = 4$ mm and $L = 10$ m) slanted upward by 10° and with a pressure rise of 6 kPa over the 10 m.

1.44 *Poiseuille flow in a horizontal pipe or in an annulus:*

(a) Find the radial ratio r_x/R where $v\left(r_x\right) = v_{\text{avg.}}$.

(b) Find the radial ratio r_x/R where $\tau\left(r_x\right) = 0.5\tau_{\text{wall}}$.

(c) Find $Q_{\text{pipe}}/Q_{\text{annulus}}$ where the pipe radius is R and the annulus is formed by $R_{\text{inner}} = R/2$ and $R_{\text{out}} = R$.

1.45 *Poiseuille flow in an annulus of radii* R_1 *and* R_2, $R_2 > R_1$:

(a) Show that $Q = \frac{\pi}{8\mu}\left(\frac{dp}{dx}\right)\left[R_2^4 - R_1^4 - \frac{\left(R_2^2 - R_1^2\right)}{\ln\left(R_2/R_1\right)}\right]$.

(b) Determine the special case of $R_1 \to 0$ (pipe flow) and $R_1 \to R_2$ where $R_2 >> 1$ (i.e., parallel plate flow).

1.46 *Run-off from a parking lot:* Consider a film of height $h = 10$ mm of water at 20°C flowing down a 100 m-wide plane which is sloped at 0.00015. Check the Reynolds number and determine the volumetric flow rate. Plot $Q(h)$ and comment.

1.47 *Torque on a shaft:* A cylinder of $D = 40$ cm rotates with 30 rad/s in a housing ($L = 80$ cm) and is lubricated with oil at 20°C, forming an 800-μm gap. Find the necessary torque, assuming first a linear velocity profile and then using the actual velocity distribution. Determine the induced error made. Plot the torque as a function of lubrication gap.

1.48 *Rotating cone:* A 90° cone of 10 cm side length rotates with 50 rad/s in a housing with a 2 mm lubrication film ($\mu = 0.01\ \mathrm{N \cdot s/m^2}$). Find the necessary torque when assuming: (a) a linear velocity profile and (b) a more realistic velocity distribution. Comment on the error invoked.

1.49 In the process of tape coating, the tape (width w and thickness d) is pulled at constant velocity U through a housing of height H and length $L(H << L)$, containing the coating liquid of viscosity μ. Develop an expression for $U_{\max}$ not quite reaching the maximum tensile force, F_{pull}, the tape can withstand. Plot $U_{\max}(\mu)$ and comment.

1.50 Pulling vertically a rigid sheet out of an oil-bath will leave a film on the surface. Given a constant U_{pull}, the sheet dimensions $(a \times b)$, and the oil properties, develop a dimensionless equation for the velocity profile and estimate the oil thickness h and the necessary pulling force.