Chapter 1

Groups and Topologies

During the twentieth century enormous progress was made abstracting the mathematical structures critical to nineteenth-century analysis and using them to attack classical problems in new ways. The simple algebraic operations of addition and multiplication of real numbers became part of the more general algebraic structures of groups, rings, fields, and algebras that included many common binary operations in mathematics. Distance functions of various kinds that were frequently used in analytic arguments lead to the concept of a metric, the study of metric spaces, and eventually general topology. These new areas of algebra and topology thrived as independent mathematical subjects. More importantly for us, their cross fertilization proved to be very productive ground for mathematical research, particularly, in what may be called modern analysis. Topological groups provide an ideal area for an introductory exploration of the interaction between algebraic and topological ideas.

In simple terms, a topological group is a group with a continuous multiplication function and a continuous inverse function. This book is a concrete self-contained introduction to topological groups. Since continuity is a topological concept, algebraic and topological ideas must be developed together. The primary topological context will be metric spaces, and the groups studied in Chapters 2, 3 and 4 are intrinsically metric groups. They will reappear many times in the last four chapters.

This chapter begins with introductions to group theory (Section 1.1) and metric space theory (Section 1.2), and then brings them together to begin the study of topological groups (Section 1.3). From that point on, algebraic and topological ideas will be constantly reinforcing each other. New concepts and results about groups and topological spaces will be introduced as needed to build a theory of topological groups.

Coset decompositions of a group and quotient groups are fundamental in all aspects of group theory. The metric-space side of these fundamental group constructions is developed in Section 1.4. The final section (1.5) in the chapter is devoted to the key concepts of compact and locally compact metric groups.

The exercises are an important part of the text. A variety of calculations, elementary results, and supplementary facts have been placed in the exercises and referenced in the text. The intent here is to help readers work through the material at very different speeds and levels depending on their mathematical background. A reference to a specific exercise always includes the page number for that exercise. There is also an index of special symbols on page 362.

1.1 Groups

Let G be a set, finite or infinite, and consider the set $G \times G$ of ordered pairs from G, in other words,

$$G \times G = \{(x, y) : x \in G \text{ and } y \in G\}.$$

(The notation $x \in G$ means that x is an element of the set G, and the colon is read as "such that".) A function $\mu : G \times G \to G$ is called a *binary operation* because it amalgamates two elements of G into one. Binary operations can also be thought of as a kind of abstract multiplication (or addition) of elements of G with $\mu(x, y)$ being the product of two elements x and y in G. Without requiring the function μ to satisfy some algebraic conditions from the familiar world of numbers, it is unlikely that μ would be of interest from an algebraic perspective.

A natural algebraic condition to impose on the function μ is the associative property. Specifically, the function $\mu : G \times G \to G$ is called an *associative binary operation* on G provided that

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z)$$

for all x, y, and z in G. At this point it should be obvious that it would be a lot more convenient to write $\mu(x, y)$ like ordinary multiplication or addition as xy or $x + y$. Then, the associative property is simply

$$x(yz) = (xy)z$$

or

$$x + (y + z) = (x + y) + z$$

for all x, y, and z in G.

A set G with an associative binary operation $\mu : G \times G \to G$ is called a *semigroup*. Semigroups are fairly common, but can also be trivial because the constant function $\mu(x, y) = a \in G$ for all x and y in G defines a semigroup structure on G.

To construct a few more illustrative examples, let $\mathbb{R}$ denote the real numbers and let a be a positive real number. If $a \geq 1$, then $G_a = \{x \in \mathbb{R} : x \geq a\}$ is a semigroup with respect to ordinary multiplication of real numbers. There is a significant difference, however, between the properties of G_1 and G_2.

An element e of a semigroup G is called an *identity element* for G provided that $ex = xe = x$ for all $x \in G$. The semigroup G_1 has an identity element, namely, the number 1, and G_2 does not. The set of all positive real numbers,

$\mathbb{R}^+$ is also a semigroup under the multiplication of real numbers and has an even richer algebraic structure than G_1.

Let G be a semigroup with an identity element e, and let x be an element of G. An *inverse* of x, written x^{-1}, is an element of G such that $xx^{-1} = x^{-1}x = e$. Every real number x in the semigroup $\mathbb{R}^+$ has an inverse, namely, $1/x$, but only 1 has an inverse in G_1.

When all three properties (associativity, an identity element, and inverses) occur together, the structure is called a group. The semigroup $\mathbb{R}^+$ is an example of a group. Specifically, a *group* is a set G with a binary operation $\mu : G \times G \to G$, written $\mu(x, y) = xy$, that satisfies the following conditions:

(a) For all x, y, and z in G,

$$x(yz) = (xy)z.$$

(b) There exists an element e in G such that

$$ex = x = xe$$

for all x in G.

(c) For each x in G there exists x^{-1} such that

$$xx^{-1} = e = x^{-1}x.$$

Although there are algebraic structures like rings that are more complex than groups and ones that are simpler, like semigroups, the concept of a group captures a large middle ground that includes many familiar mathematical environments and serves as a fundamental algebraic building block.

The first example of a group was the positive real numbers, $\mathbb{R}^+$, under multiplication of real numbers. The nonzero real numbers are also a group under multiplication of real numbers. The real numbers, $\mathbb{R}$, are a group under addition of real numbers. Under multiplication $\mathbb{R}$ is a semigroup, but it is not a group because 0 does not have an inverse. Similarly, the complex numbers, $\mathbb{C}$, are a group under complex addition and the nonzero complex numbers are a group under complex multiplication. The integers, $\mathbb{Z}$, are a group under addition, but the nonzero integers are only a semigroup under multiplication.

On a set consisting of exactly one element, say a, there is a unique binary operation, namely, $\mu(a, a) = a$, and with this operation $\{a\}$ is a group called the *trivial group*. A *nontrivial group* is a group with more than one element. As we progress in the study of topological groups, more and more interesting groups will appear.

A basic dichotomy in the study of groups is the distinction between finite and infinite groups. A simple example of a finite group is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, where n is a positive integer. Addition is defined by $i + j = r$ if and only if r is the remainder of the usual sum $i + j$ divided by n. Clearly, 0 is the identity element and $i + (n - i) = 0$. Checking the associative property is an exercise. The group $\mathbb{Z}_n$ is a called the *integers mod n*.

Another basic distinction is between the Abelian groups and all the others. A group G is *Abelian* or *commutative* provided that

$$xy = yx$$

or

$$x + y = y + x$$

for all x and y in G. The additive groups $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{R}$ are Abelian. In fact, when the binary operation of a group is written additively, the group is understood to be an Abelian group, but the converse is not true.. There are plenty of Abelian groups, like $\mathbb{R}^+$, for which the binary operation is written as xy.

For x in a group G written multiplicatively, x^k can be defined for all k in $\mathbb{Z}$. When $k > 0$, set

$$x^k = \underbrace{xx\dots x}_{k-\text{times}}.$$

Set $x^0 = e$. When $k < 0$, set $x^k = (x^{-1})^{-k}$. In this context, the usual rules for exponents hold. Specifically, $x^m x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$ for all m, n in $\mathbb{Z}$. Similarly, one defines kx when G is written additively and obtains $mx + nx = (m+n)x$ and $m(nx) = (mn)x$ for all m, n in $\mathbb{Z}$.

A group G is a *cyclic group* provided that there exists an element a, called a *generator*, in G such that $G = \{a^k : k \in \mathbb{Z}\}$ or $G = \{ka : k \in \mathbb{Z}\}$ in additive notation. The groups $\mathbb{Z}$ and $\mathbb{Z}_m$ are examples of infinite and finite cyclic groups, respectively. Cyclic groups are clearly Abelian.

Two elements x and y in a group G are said to *commute*, if $xy = yx$. Even when a group is not Abelian, it will contain elements that commute. For example, powers of x always commute because $x^m x^n = x^{m+n} = x^{n+m} = x^n x^m$. *Proposition 1.1.1* below provides some less obvious examples.

Functions are another important source of semigroups and groups. Given a set X, there is a natural associative binary operation on the set of functions from X to X, namely , the composition of functions. Specifically, if $f, g : X \to X$ are functions from X to X, then $f \circ g(x) = f(g(x))$ defines another function from X to X. Clearly,

$$f \circ (g \circ h)(x) = f\left(g\big(h(x)\big)\right) = (f \circ g) \circ h(x)$$

for all x in X and

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Thus the set F_X of all functions from X to X with the binary operation of composition of functions denoted by $\circ$ is a semigroup. The map $\iota : X \to X$ defined by $\iota(x) = x$ is an identity element for F_X. To obtain groups of functions additional restrictions must be placed on the set of functions.

We begin with some basic definitions. Let X and Y be sets. A function $f : X \to Y$ is *onto* provided that for every y in Y there exists x in X such that $f(x) = y$. And f is *one-to-one* provided that $f(x) = f(x')$ implies that $x = x'$. If f is both one-to-one and onto, then given y in Y there exists a unique x in X such that $f(x) = y$. Therefore, a function $f^{-1} : Y \to X$ can be defined unambiguously by setting $f^{-1}(y) = x$ if and only if $f(x) = y$. Clearly, $f^{-1}(f(x)) = x$ for all x in X and $f(f^{-1}(y)) = y$ for all y in Y. It follows that f^{-1} is also one-to-one and onto, when f is one-to-one and onto.

Given a set X, let

$$S_X = \{f : X \to X : f \text{ is one-to-one and onto}\} \subset F_X.$$

(The notation $A \subset B$ means that the set A is contained in the set B, or in other words every element of A is in B. It does not exclude the possibility that $A = B$.) If f is in S_X, then $f \circ f^{-1} = \iota = f^{-1} \circ f$. Conversely, if there exists g in S_X such that $f \circ g = \iota = g \circ f$, then $g = f^{-1}$. Therefore, S_X is the group of units of F_X by *Exercise* 3, p. 11.

Given α in S_X, the α-*orbit* of $x \in X$ is defined by

$$\mathcal{O}(x) = \{\alpha^k(x) : k \in \mathbb{Z}\}.$$

A subset E of X is an α-*invariant set* if $\alpha(E) = E$. The α-orbits are the smallest α-invariant sets, and α-invariant sets are simply unions of orbits. Notice that $\mathcal{O}(x) = \mathcal{O}(\alpha^m(x))$ for m in $\mathbb{Z}$ because $\alpha^k(x) = \alpha^{k-m}(\alpha^m(x))$. If z is in $\mathcal{O}(x) \cap \mathcal{O}(y)$, then $\alpha^m(x) = z = \alpha^n(y)$ for some m and n, and it follows that $\mathcal{O}(x) = \mathcal{O}(\alpha^m(x)) = \mathcal{O}(\alpha^n(y)) = \mathcal{O}(y)$. Consequently, given x and y in X, either $\mathcal{O}(x) = \mathcal{O}(y)$ or $\mathcal{O}(x) \cap \mathcal{O}(y) = \phi$. (The symbol ϕ denotes the set that contains no elements and is called the *empty set*.) Thus the α-orbits decompose X into disjoint α-invariant sets.

Now, consider the finite set $X = \{1, 2, 3, \ldots, n\}$, and set $S_n = S_X$. The elements of S_n are called *permutations of n symbols*, and the group S_n is called the *symmetric group on n symbols.* (For convenience, the symbols used here are the integers $1, 2, \ldots, n$, but they could just as well be letters or other distinct symbols.) The group S_n will play an important role in later chapters and will be the primary group studied in this section.

One way to express a permutation in S_n is to write the symbols on a line and the image of each symbol directly underneath it with parentheses as delimiters. For example, the following specifies a particular permutation in S_9:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 6 & 7 & 8 & 3 & 4 & 5 & 2 & 1 \end{pmatrix} \tag{1.1}$$

This permutation will be used to illustrate several basic permutation concepts.

Given a permutation α in S_n, the α-orbits are necessarily finite, and repeats must occur in the sequence $j, \alpha(j), \alpha^2(j), \ldots, \alpha^m(j), \ldots$. The first repeat is particularly useful. Specifically, given a symbol j, $1 \leq j \leq n$, there exists a smallest positive integer m such that $\alpha^m(j) = \alpha^k(j)$ for some k satisfying $0 \leq k \leq m-1$. Applying α^{-k} to both sides of $\alpha^m(j) = \alpha^k(j)$ shows that $\alpha^{m-k}(j) = j$. If $k > 0$, then $0 < m - k < m$ making $\alpha^{m-k}(j) = j$ an earlier repeat. Therefore, $k = 0$ and the α-orbit of j consists precisely of the m distinct points $j, \alpha(j), \ldots, \alpha^{m-1}(j)$. Consequently, the α-orbits and the order of the points in each α-orbit completely describe the permutation α.

For example, the orbits of β defined by equation (1.1) written in the order in which β maps one symbol to another with the last going back to the first are:

$$\{1,9\}, \{2,6,4,8\}, \text{ and } \{3,7,5\}.$$

Because we know these orbits are written in the order in which β maps one symbol to another, it is easy to reconstruct equation (1.1) for β from the above information. Using the wrong order in which β maps one symbol to another will, however, produce a permutation that is not equal to β.

The discussion in the previous paragraphs leads naturally to the concept of a special class of permutations called cycles. The idea of a cycle incorporates both an orbit and the order in which the points occur in that orbit. Let $a_1, \ldots, a_k$ be distinct elements of $\{1, 2, \ldots, n\}$. The *cycle* $(a_1 a_2 \ldots a_k)$ is the permutation that maps a_1 to a_2, a_2 to a_3, $\ldots$, a_k to a_1, and maps j to j for every $j \neq a_i$ for $i = 1, \ldots, k$. In S_9, the cycle (2648) is the same as the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 6 & 3 & 8 & 5 & 4 & 7 & 2 & 9 \end{pmatrix},$$

which agrees with β on the set $\{2,4,6,8\}$. Although the order of a cycle is critical, the first element of a cycle is not. For example, (375) = (753) = (537).

The cycle (a) is the identity map ι. A cycle of the form (ab) is called a *transposition*. If $\alpha = (a_1 a_2 \ldots a_k)$ is a cycle in S_n, then it is easy to see that $\alpha, \alpha^2, \ldots, \alpha^{k-1}$ are distinct permutations and $\alpha^k = \iota$, the identity permutation that fixes every j. In particular, for a transposition $\alpha^2 = \iota$.

Consider an α in S_n. If the α-orbit of the symbol j is

$$\mathcal{O}(j) = \{j, \alpha(j), \ldots, \alpha^{m-1}(j)\},$$

then the cycle $(j\, \alpha(j) \ldots \alpha^{m-1}(j))$ equals α on $\mathcal{O}(j)$. For example, the cycle (2468) agrees with the permutation β given by (1.1) on $\mathcal{O}(2) = \{2,6,4,8\}$. So each α-orbit determines a unique cycle that equals α on that α-orbit.

The following three propositions about cycles will be used to prove an important theorem about S_n at the end of this section.

Proposition 1.1.1 *If $\alpha = (a_1 \ldots a_k)$ and $\beta = (b_1 \ldots b_m)$ are disjoint cycles, that is, $a_i \neq b_j$ for $1 \le i \le k$ and $1 \le j \le m$, then $\alpha \circ \beta = \beta \circ \alpha$.*

Proof. Consider some a_j, $1 \le j \le k$. Since $\beta(a_i) = a_i$ for $1 \le i \le k$ and $\alpha(a_j) = a_q$ for some q, it follows that

$$\alpha(\beta(a_j)) = \alpha(a_j) = a_q = \beta(a_q) = \beta(\alpha(a_j)).$$

Of course, the same argument applies to the points b_j. Obviously, $\alpha(\beta(k)) = k = \beta(\alpha(k))$, when k is neither an a_i nor a b_j. Thus $\alpha \circ \beta = \beta \circ \alpha$. □

Proposition 1.1.2 *If α is a permutation in S_n, then there exist disjoint cycles $\alpha_1, \ldots, \alpha_m$ such that $\alpha = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_m$.*

Proof. Let $\alpha_1, \ldots, \alpha_m$ be the cycles determined by the distinct α-orbits. They are disjoint cycles because distinct orbits are disjoint subsets of $\{1, \ldots, n\}$.

For each i, $\alpha_i = \alpha$ on exactly one of the m distinct α-orbits. Write $\alpha_i = (a_{(i,1)} \ldots a_{(i,k_i)})$ for $1 \leq i \leq m$. Hence, every symbol equals precisely one $a_{(i,j)}$. If $a_{(i,j)}$ is an arbitrary symbol, then

$$\begin{aligned} \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_m(a_{(i,j)}) &= \\ \alpha_i \circ \alpha_1 \circ \ldots \circ \alpha_{i-1} \circ \alpha_{i+1} \circ \ldots \circ \alpha_m(a_{(i,j)}) &= \\ \alpha_i(a_{(i,j)}) = \alpha(a_{(i,j)}) & \end{aligned}$$

for the following reasons: $\alpha_p \circ \alpha_q = \alpha_q \circ \alpha_p$ by *Proposition 1.1.1*, $\alpha_q(a_{(i,j)}) = a_{(i,j)}$ for $q \neq i$, and $\alpha_i(a_{(i,j)}) = \alpha(a_{(i,j)})$. Thus $\alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_m = \alpha$. □

Proposition 1.1.3 *Every permutation α in S_n is the composition of transpositions.*

Proof. By *Proposition 1.1.2*, it suffices to show that a cycle is a composition of transpositions. Applying the transpositions from right to left, it follows that

$$(a_1 \ldots a_k) = (a_1 a_k) \circ (a_1 a_{k-1}) \circ \ldots \circ (a_1 a_3) \circ (a_1 a_2)$$

to complete the proof. □

Returning to the permutation β in S_9 defined by equation (1.1), it follows from the preceding propositions that:

$$\begin{aligned} \beta &= (19) \circ (375) \circ (2648) \\ &= (19) \circ (35) \circ (37) \circ (28) \circ (24) \circ (26). \end{aligned}$$

Let G and G' be arbitrary groups. A *algebraic homomorphism* of G to G' is a function $\varphi : G \to G'$ such that

$$\varphi(xy) = \varphi(x)\varphi(y)$$

for all x and y in G. In other words, you can multiply before or after applying the function, and algebraic properties of G are preserved by φ. In addition, a homomorphism preserves identities and inverses (*Exercise* 11, p. 12). Of course, if the group operation in one or both of the groups is written additively, then the right or left side of the equation would be written as $\varphi(x+y)$ or $\varphi(x)+\varphi(y)$.

The map $\varphi(k) = 2k$ is an example of a algebraic homomorphism of $\mathbb{Z}$ into $\mathbb{Z}$. It is one-to-one, but not onto. To construct an example of a algebraic homomorphism that is not one-to-one define $\psi : \mathbb{Z} \to \mathbb{Z}_n$ by setting $\psi(k) = r$ if and only if q and r are the unique integers such that $k = qn + r$ and $0 \leq r < n$. In other words, r is the *remainder* of k divided by n.

When an algebraic homomorphism, $\varphi : G \to G'$, is both one-to-one and onto, there is a one-to-one onto inverse function $\varphi^{-1} : G' \to G$. If y and y' are elements of G', then, because φ is onto, $y = \varphi(x)$ and $y' = \varphi(x')$ for some x and x' in G. It follows that

$$\varphi^{-1}(yy') = \varphi^{-1}\big(\varphi(x)\varphi(x')\big) = \varphi^{-1}\big(\varphi(xx')\big) = xx' = \varphi^{-1}(y)\varphi^{-1}(y')$$

and φ^{-1} is also an algebraic homomorphism.

A one-to-one onto algebraic homomorphism $\varphi : G \to G'$ is called an *algebraic isomorphism*. As a consequence of the previous paragraph, the inverse of an algebraic isomorphism is an algebraic isomorphism. Two groups are said to be *algebraically isomorphic* if there exists an algebraic isomorphism mapping one group onto the other. Algebraically isomorphic groups have exactly the same algebraic properties. For example, they are either both Abelian or neither is Abelian. So S_3 is not algebraically isomorphic to $\mathbb{Z}_6$, even though they both contain 6 elements, because S_3 is not Abelian by *Exercise* 8, p. 12.

Let n be a positive integer and consider the complex number

$$a = \exp(2\pi i/n) = \cos(2\pi/n) + i\sin(2\pi/n).$$

Then

$$G = \{a^0, a^1, a^2, a^3, \dots, a^{n-1}\},$$

the set of solutions of $z^n = 1$ in $\mathbb{C}$, is an example of a finite cyclic group. The function $\varphi : \mathbb{Z}_n \to G$ defined by $\varphi(r) = a^r$ is an algebraic isomorphism between $\mathbb{Z}_n$ and G. In fact, any two cyclic groups containing n elements are algebraically isomorphic, see *Exercise* 16, p. 12. Thus from a group-theoretic point of view there is one infinite cyclic group and for each positive integer n exactly one cyclic group containing n elements.

Suppose H is a nonempty subset of a group G. Then xy is defined for x and y in H. What conditions will guarantee that H is a group for this multiplication? Certainly, xy must be in H when x and y are in H so that $(x, y) \to xy$ defines a binary operation on H. The associative property holds for any three elements of G, so it automatically holds for any three elements of H. Suppose x^{-1} is also in H whenever x is in H. Since H is assumed to be nonempty, there exists $x \in H$, and hence $e = xx^{-1}$ is in H. It follows that H has an identity, and for every x in H there exists x^{-1} in H such that $xx^{-1} = x^{-1}x = e$. To summarize, if H is a nonempty subset of a group G such that x and y in H implies that both xy and x^{-1} are also in H, then H is also a group under the multiplication of G. This leads naturally to a definition. A nonempty subset H of a group G is a *subgroup of* G provided that for all x and y in H:

(a) $xy \in H$

(b) $x^{-1} \in H$.

For example, $\mathbb{Z}$ is a subgroup of the additive group $\mathbb{R}$, but the positive integers and the odd integers are not subgroups of $\mathbb{R}$. A more interesting example can be found in the complex numbers $\mathbb{C}$. The *conjugate of a complex number* $z = x + iy$ is defined by $\overline{z} = x - iy$. Note that $z\overline{z} = x^2 + y^2$, which is a non-negative real number. The *absolute value of a complex number* is defined by $|z| = \sqrt{z\overline{z}}$. Consider

$$\mathbb{K} = \{z \in \mathbb{C} : |z| = 1\} = \{z = x + iy : x^2 + y^2 = 1\}.$$

Obviously, $\mathbb{K} \subset \mathbb{C} \setminus \{0\}$. If z and w are in $\mathbb{K}$, then an easy calculation shows that $|zw| = 1$ and zw is in $\mathbb{K}$. Similarly, $1/z = \overline{z}$ when $|z| = 1$. So $1/z$ is in $\mathbb{K}$ when z is in $\mathbb{K}$. Thus, $\mathbb{K}$, the familiar unit circle in the complex plane, is a subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$, and the set of solutions of $z^n = 1$ in $\mathbb{C}$ is a subgroup of both $\mathbb{K}$ and of $\mathbb{C}$.

There are two simple remarks to be made about the subgroups of any group G with identity e. First, there are two extreme subgroups of G, namely, G itself and the trivial group $\{e\}$. All other subgroups will be referred to as *proper subgroups*. Second, if a is in G, then $\{a^k : k \in \mathbb{Z}\}$ is an Abelian subgroup of G called the *cyclic subgroup generated by a*.

If $\varphi : G \to G'$ is a algebraic homomorphism, then $K = \{x \in G : \varphi(x) = e'\}$ is a subgroup of G called the *kernel of* φ. Furthermore, $K = \{e\}$ if and only if φ is one-to-one (*Exercise* 12, p. 12). This is the initial observation in an important connection between subgroups and algebraic homomorphisms that will be explored more fully in Section 4.

The roots of $z^2 = 1$ in $\mathbb{C}$ are $\{1, -1\}$, which is a cyclic group containing 2 elements generated by -1. The final goal of this section is to prove the existence of an onto algebraic homomorphism $\operatorname{sgn} : S_n \to \{1, -1\}$. In other words, a sign can be assigned to permutations that is multiplicative when permutations are composed. This sign function will play a significant role in the construction and properties of another homomorphism, the determinant function (Section 2.5).

First, we define a function $N : S_n \to \mathbb{Z}$. Let $|\mathcal{O}(j)|$ denote the cardinality of the orbit $\mathcal{O}(j)$ for a permutation α in S_n, and set

$$N(\alpha) = \sum_{j=1}^{n} \frac{|\mathcal{O}(j)| - 1}{|\mathcal{O}(j)|}.$$

Notice that when $\alpha(j) = j$, the orbit of j is just $\{j\}$ and j contributes nothing to the above sum. It follows that $N(\iota) = 0$. If $\alpha = (a_1 a_2 \ldots a_k)$ is a cycle, then the only terms contributing to the sum are $a_1, \ldots, a_k$ and $\mathcal{O}(a_i)$ consists of the k points $\{a_1, \ldots, a_k\}$ for each i. Hence,

$$N(\alpha) = \sum_{i=1}^{k} \frac{k-1}{k} = k - 1.$$

More generally, if $\mathcal{O}_1 = \{a_1, \ldots, a_k\}$ is an orbit of α, then $\mathcal{O}(a_j) = \mathcal{O}_1$ for $j = 1, \ldots, k$ and

$$\sum_{j \in \mathcal{O}_1} \frac{|\mathcal{O}(j)| - 1}{|\mathcal{O}(j)|} = \sum_{i=1}^{k} \frac{k-1}{k} = k - 1 = |\mathcal{O}_1| - 1,$$

and the orbit $\mathcal{O}_1$ of α contributes $|\mathcal{O}_1| - 1$ to $N(\alpha)$. If the distinct orbits of α are $\mathcal{O}_1, \ldots, \mathcal{O}_p$, then

$$N(\alpha) = \sum_{j=1}^{p} \left(|\mathcal{O}_j| - 1\right).$$

The function sgn : $S_n \to \{1, -1\}$ is defined by setting

$$\operatorname{sgn}(\alpha) = (-1)^{N(\alpha)}. \tag{1.2}$$

The permutation α is said to be an *even permutation* when $\operatorname{sgn}(\alpha) = 1$, and an *odd permutation* when $\operatorname{sgn}(\alpha) = -1$.

Returning to our standard example β defined by equation (1.1), recall that the orbits of β are $\{1,9\}, \{2,6,4,8\}$, and $\{3,7,5\}$. Hence, $N(\beta) = (2-1) + (4-1) + (3-1) = 6$ and $\operatorname{sgn}(\beta) = 1$ making β an even permutation. The cycle (2648) is clearly odd. By using *Propositions 1.1.1 and 1.1.2*, it is easy to show that $\beta \circ (2648) = (19) \circ (24) \circ (68) \circ (375)$. Consequently, $N(\beta \circ (2648)) = 5$ and $\operatorname{sgn}(\beta \circ (2648)) = -1 = \operatorname{sgn}(\beta)\operatorname{sgn}((2648))$. The next task is to prove that this is a general phenomenon.

Understanding when $N(\alpha)$ is odd and even will be critical in showing that sgn is a algebraic homomorphism. The next lemma provides the key.

Lemma 1.1.4 *If $\alpha = (a_1b_1) \circ \ldots \circ (a_mb_m)$ is the composition of m transpositions, then $N(\alpha)$ is odd or even according as m is odd or even.*

Proof. Observe that the statement is certainly true when $m = 1$ because $N((ab)) = 1$ and then proceed by induction. Assuming it holds for m, consider $\alpha = (a_1b_1) \circ \ldots \circ (a_mb_m) \circ (a_{m+1}b_{m+1})$ and set $\beta = (a_2b_2) \circ \ldots \circ (a_mb_m) \circ (a_{m+1}b_{m+1})$.

By *Proposition 1.1.2*, there exist disjoint cycles $\beta_1, \ldots, \beta_k$ such that $\beta = \beta_1 \circ \beta_2 \circ \ldots \circ \beta_k$. Because these cycles are disjoint, a_1 and b_1 each occur in exactly one of the cycles $\beta_1, \ldots, \beta_k$. There are two possibilities: either they occur in the same cycle or in different ones. By *Proposition 1.1.1*, we can assume that the cycle containing a_1 is β_1 and b_1 is in either β_1 or β_2. So either

$$\alpha = (a_1b_1) \circ (a_1c_1 \ldots c_p\, b_1d_1 \ldots d_q) \circ \beta_2 \circ \ldots \circ \beta_k$$

or

$$\alpha = (a_1b_1) \circ (a_1c_1 \ldots c_p) \circ (b_1d_1 \ldots d_q) \circ \beta_3 \circ \ldots \circ \beta_k$$

where p and q are allowed to be zero.

In the first case, it is easy to check that

$$(a_1b_1) \circ (a_1c_1 \ldots c_p\, b_1d_1 \ldots d_q) = (a_1c_1 \ldots c_p) \circ (b_1d_1 \ldots d_q). \tag{1.3}$$

Thus one orbit of β splits into two orbits in α and the other orbits remain unchanged. The orbit $\{a_1, c_1, \ldots, c_p, b_1, d_1, \ldots, d_q\}$ contributes $p + q + 1$ to

$N(\beta)$ and the orbits $\{a_1, c_1, \ldots, c_p\}$ and $\{b_1, d_1, \ldots, d_q\}$ contribute $p+q$ to $N(\alpha)$. Since the other orbits are unchanged, $N(\alpha) = N(\beta) - 1$. It follows that $N(\alpha)$ is odd if and only if $N(\beta)$ is even. Since by induction $N(\beta)$ is even if and only if m is even, it follows that $N(\alpha)$ is odd if and only if $m+1$ is odd. Similarly, $N(\alpha)$ is even if and only if $m+1$ is even, and the first case is done.

Since $(a_1b_1) \circ (a_1b_1) = \iota$, it follows from equation (1.3) that

$$(a_1b_1) \circ (a_1c_1 \ldots c_p) \circ (b_1d_1 \ldots d_q) = (a_1c_1 \ldots c_p\, b_1d_1 \ldots d_q),$$

and two orbits of β combine to form one orbit of α. Thus $N(\alpha) = N(\beta) + 1$ in the second case, and as in the first case, $N(\alpha)$ is odd or even according as $m+1$ is odd or even. □

There are, of course, many ways that a permutation can be written as a composition of cycles. What the lemma tells us is that no matter how one writes a given permutation as a composition of cycles, the number of cycles used will either always be even or always be odd.

Theorem 1.1.5 *The function* $\operatorname{sgn} : S_n \to \{1, -1\}$ *defined by equation (1.2) is an algebraic homomorphism that is onto for* $n \geq 2$.

Proof. Let α and β be elements of S_n. By *Proposition 1.1.3*, α and β are compositions of p and q transpositions respectively. Hence, $\alpha \circ \beta$ is the composition of $p+q$ transpositions. It then follows from the lemma that $\operatorname{sgn}(\alpha) = (-1)^p$, $\operatorname{sgn}(\beta) = (-1)^q$, and $\operatorname{sgn}(\alpha \circ \beta) = (-1)^{p+q}$. Therefore, $\operatorname{sgn}(\alpha \circ \beta) = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)$.

When $n \geq 2$, There is a transposition (ab) in S_n and sgn is onto because $N(\iota) = 0$ and $N((ab)) = 1$. □

The permutations that can be written as the composition of an even number of transpositions are the kernel of the sgn, and hence form a subgroup of S_n. This subgroup of S_n is called the *alternating group* and is denoted by A_n.

EXERCISES

1. Let G be a group. Show that G has only one identity element and each element of G has exactly one inverse.

2. Let x be an element of a group G. Show that $x^m x^n = x^{m+n}$ for all integers m and n.

3. Let G be a semigroup with identity e. An element x in G is a *unit*, if there exists y in G such that $xy = e = yx$. Show that the set of units of the semigroup G is a group. This group is called the *group of units*.

4. Let G be a group and let a be an element of G. Show that the functions $x \mapsto ax$ and $x \mapsto xa$ are one-to-one functions of G onto G. (The arrow $\mapsto$ is read "maps to" and is used to define a function.)

5. For X a finite set, show that a function $f : X \to X$ is one-to-one if and only if it is onto.

6. Show that S_n contains $n! = n(n-1)(n-2)\dots 1$ elements. How many functions are in F_n, the set of functions from $\{1, \dots, n\}$ to itself?

7. Show that S_n contains subgroups algebraically isomorphic to S_k and to $\mathbb{Z}_k$ for $k = 1, 2, 3, \dots, n$.

8. Show that S_n is Abelian if and only if $n \le 2$.

9. Show that every element of the group S_n can be written as a composition of transpositions of the form $(k\,k+1)$.

10. Let H be a nonempty subset of a group G. Show that H is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

11. Let G and G' be groups with identity elements e and e', respectively, and let $\varphi : G \to G'$ be an algebraic homomorphism. Show that $\varphi(e) = e'$ and $\varphi(x^{-1}) = \varphi(x)^{-1}$.

12. Let G and G' be groups with identity elements e and e', respectively, and let $\varphi : G \to G'$ be an algebraic homomorphism. Show that $K = \{x \in G : \varphi(x) = e'\}$ is a subgroup of G. Show that φ is one-to-one if and only if $K = \{e\}$. Also show that $\varphi(G)$ is a subgroup of G'.

13. Let G be a group and let a be an element of G. Show that $\varphi_a(x) = axa^{-1}$ defines an algebraic isomorphism of G onto G. Also show that $\{a : \varphi_a = \iota\}$ is a subgroup of G.

14. Show that a group G is algebraically isomorphic to a subgroup of S_G.

15. Show that if $f : X \to Y$ is one-to-one and onto then S_X and S_Y are algebraically isomorphic groups.

16. Show that if G is a cyclic group, then G is algebraically isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$ for some integer $n \ge 1$.

17. Show that the cyclic subgroup of $\mathbb{K}$ generated by $a = \exp(2\pi i\alpha)$ is finite if and only if α is a rational number.

18. Prove that a group G has no proper subgroups if and only if G is algebraically isomorphic to $\mathbb{Z}_n$ and n is a prime.

19. Calculate the inverse of a cycle $(a_1 \dots a_k)$ in S_n.

1.2 Metric and Topological Spaces

The starting point is again a set, but now distance replaces the concept of a binary operation as the central idea. Let X be a set. A function that measures the distance between two points of X is called a metric. Metrics should reflect enough of the basic properties of measuring distance in the world around us to make them worth studying, but not to the extent that metrics are uncommon.

The real numbers, denoted by $\mathbb{R}$, will play a prominent role in this chapter and subsequent ones. We are assuming that the reader is familiar with all the basic properties of the real numbers including: Every nonempty subset E of $\mathbb{R}$ that is bounded above has a *supremum*, written $\sup E$. To be more specific, a nonempty subset E is *bounded above* provided there exists a real number b such that $x \leq b$ for all x in E, and b is called an *upper bound* of E. If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then there exists a real number $\sup E$ with the following properties:

(a) If x is in E, then $x \leq \sup E$.

(b) If $x \leq a$ for all x in E, then $\sup E \leq a$.

In other words, $\sup E$ is a smallest or *least upper bound* of E. The fact that every nonempty subset E of $\mathbb{R}$ that is bounded above has a supremum is one of the fundamental properties of the real numbers that distinguishes them from the rational numbers.

A *metric* on a set X is a real-valued function $d : X \times X \to \mathbb{R}$ that has the following properties for all x, y, and z in X:

(a) $d(x, y) = d(y, x)$

(b) $d(x, y) \geq 0$

(c) $d(x, y) = 0$ if and only if $x = y$

(d) $d(x, y) \leq d(x, z) + d(z, y)$.

The last property is called the *triangle inequality* and generalizes the fact that in Euclidean geometry the length of one side of a triangle is always less than the sum of the other two sides. Obviously, $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$.

A *metric space* is simply a set X with a specified metric d. Because there is always more than one metric defined on a set X containing more than one point, it is critical at times to make sure there is no ambiguity about the metric being used. The question of when two metrics are or are not fundamentally different must also be addressed.

Among the most basic of all metric spaces are the *n-dimensional Euclidean spaces*, denoted by $\mathbb{R}^n$. The set $\mathbb{R}^n$ is the set of all n-tuples of real numbers and is defined by

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n} = \big\{(x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } 1 \leq j \leq n\big\}.$$

It is convenient to use boldface type to denote points in $\mathbb{R}^n$, that is, $\mathbf{x} = (x_1, \ldots, x_n)$. Note that $\mathbb{R}^n$ is an Abelian group under the binary operation

$$\mathbf{x} + \mathbf{y} = (x_1 + y_2, \ldots, x_n + y_n)$$

with identity $\mathbf{0} = (0, \ldots, 0)$.

The definition of the familiar *dot product* on $\mathbb{R}^3$ can be extended to $\mathbb{R}^n$ by setting

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_n y_n = \sum_{j=1}^{n} x_j y_j.$$

As usual for c, a real number, $c\mathbf{x} = (cx_1, \ldots, cx_n)$. Observe that

(a) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

(b) $(\mathbf{x} + \mathbf{x}') \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x}' \cdot \mathbf{y}$

(c) $c(\mathbf{x} \cdot \mathbf{y}) = (c\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (c\mathbf{y})$

hold for all c in $\mathbb{R}$ and for all $\mathbf{x}$, $\mathbf{x}'$, and $\mathbf{y}$ in $\mathbb{R}^n$.

Set

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} \tag{1.4}$$

and note that $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Then,

$$\|\mathbf{x} - \mathbf{y}\| = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{1/2} \tag{1.5}$$

is the usual Euclidean distance function on $\mathbb{R}^n$, and we will show that it is a metric on $\mathbb{R}^n$. Obviously, $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$ and $\|\mathbf{x} - \mathbf{y}\| \geq 0$. Also $\|\mathbf{x} - \mathbf{y}\| = 0$ if and only if $\mathbf{x} = \mathbf{y}$ is self evident. Proving the triangle property requires a fundamental inequality that will prove to be very useful in general.

Proposition 1.2.1 (Cauchy-Schwarz Inequality) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are in* $\mathbb{R}^n$, *then*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|. \tag{1.6}$$

If $\mathbf{y} \neq \mathbf{0}$, *then equality holds if and only if*

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \, \mathbf{y}.$$

Proof. When $\mathbf{y} = \mathbf{0}$, both sides of the inequality are 0 and there is nothing to prove. So for the rest of the proof $\mathbf{y} \neq \mathbf{0}$.

Suppose $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = 1$ and set $\mathbf{u} = \mathbf{x} - (\mathbf{x}\cdot\mathbf{v})\mathbf{v}$. Then,

$$\begin{aligned} 0 &\le \mathbf{u}\cdot\mathbf{u} \\ &= \mathbf{x}\cdot\mathbf{x} - 2(\mathbf{x}\cdot\mathbf{v})^2 + (\mathbf{x}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{v}) \\ &= \mathbf{x}\cdot\mathbf{x} - (\mathbf{x}\cdot\mathbf{v})^2, \end{aligned}$$

and $0 = \mathbf{u}\cdot\mathbf{u}$ if and only if $\mathbf{u} = \mathbf{0}$ or equivalently $\mathbf{x} = (\mathbf{x}\cdot\mathbf{v})\mathbf{v}$.

For $\mathbf{y} \neq \mathbf{0}$, the above inequality applies to

$$\mathbf{v} = \frac{1}{\|\mathbf{y}\|}\,\mathbf{y}$$

and

$$0 \le \mathbf{x}\cdot\mathbf{x} - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2}.$$

The latter can be rewritten as

$$(\mathbf{x}\cdot\mathbf{y})^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2$$

or by taking the positive square root as

$$|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

It also follows from the previous paragraph that holds if and only if

$$\mathbf{x} = \left(\mathbf{x}\cdot\frac{1}{\|\mathbf{y}\|}\,\mathbf{y}\right)\frac{1}{\|\mathbf{y}\|}\,\mathbf{y} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\,\mathbf{y} = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\,\mathbf{y}$$

to complete the proof. (Note that $\mathbf{x}$ occurs on both sides of this equation. And it holds for $c\mathbf{x}$ when it holds for $\mathbf{x}$.) □

Corollary 1.2.2 *For all* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *in* $\mathbb{R}^n$,

$$\|\mathbf{x}-\mathbf{y}\| \le \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{y}\|,$$

and $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ *is a metric on* $\mathbb{R}^n$.

Proof. To prove the triangle inequality observe that it suffices to prove

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|, \tag{1.7}$$

and then set $\mathbf{u} = \mathbf{x} - \mathbf{z}$ and $\mathbf{v} = \mathbf{z} - \mathbf{y}$. Next, an easy calculation shows that

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2.$$

The Cauchy-Schwarz Inequality can be applied here because $\mathbf{u}\cdot\mathbf{v} \le |\mathbf{u}\cdot\mathbf{v}|$. Specifically,

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{u}\|^2 \\ &= \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2 \end{aligned}$$

which implies that $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Hence, the triangle inequality holds and equation (1.5) defines a metric on $\mathbb{R}^n$. □

The "metric space $\mathbb{R}^n$" will always mean $\mathbb{R}^n$ with this standard metric. When $n = 1$, the metric given by (1.5) equals the metric $d(x, y) = |x - y|$ on $\mathbb{R}$ and, unless stated otherwise, the metric on $\mathbb{R}$.

The *open ball* of radius $r > 0$ with center y in a metric space X is defined by

$$B_r(y) = \{x \in X : d(x, y) < r\}.$$

Of course, the open ball $B_r(y)$ also depends on the space X and the metric d, but making the specific X and d clear from the context is preferable to using more cumbersome notation like $B_r(y, X, d)$.

Suppose x and y are distinct points in a metric space X with metric d. If $0 < r < d(x, y)/2$, then it is easy to see that $B_r(x) \cap B_r(y) = \phi$ because $z \in B_r(x) \cap B_r(y)$ would imply that $d(x, y) \leq d(x, z) + d(z, y) < r + r < d(x, y)$, which is impossible. The existence of disjoint open balls centered at any two distinct points of a metric space is called the *Hausdorff property* of metric spaces.

When X is a metric space with metric d and Y is a nonempty subset of X, the restriction of d to $Y \times Y$ defines a metric on Y. For y in Y, the open ball of radius $r > 0$ with center y in Y is just $B_r(y) \cap Y$. So every nonempty subset of a metric space is a metric space in its own right.

It is easy to extend the definitions of convergent sequences and continuous functions from $\mathbb{R}$ to metric spaces. A sequence x_k, $k = 1, 2, 3, \ldots$, in a metric space X with metric d *converges* to y in X, if given $\varepsilon > 0$, there exists $K > 0$ such that $d(x_k, y) < \varepsilon$, whenever $k \geq K$. The Hausdorff property guarantees that the limit of a convergent sequence in a metric space is unique (*Exercise* 1, p. 27). We will usually say simply "Let x_k be a sequence in ...." with it understood that for each k in $\mathbb{Z}^+$, the positive integers, there is an x_k in a prescribed set.

Now, let X and Y be metric spaces with metrics d_X and d_Y, respectively, and let $f : X \to Y$ be a function from X to Y. The function f is *continuous* at z in X, if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(z)) < \varepsilon$, whenever $d_X(x, z) < \delta$. Convergent sequences can be used to characterize continuity as follows:

Proposition 1.2.3 *Let X and Y be metric spaces with metrics d_X and d_Y and let $f : X \to Y$ be a function from X to Y. The following are equivalent:*

(a) The function f is continuous at z in X.

(b) Whenever x_k is a sequence in X converging to z, the sequence $f(x_k)$ converges to $f(z)$ in Y.

Proof. First, suppose f is continuous at z, and let x_k be a sequence in X converging to z. To show that $f(x_k)$ converges to $f(z)$, consider any $\varepsilon > 0$. Since f is continuous at z, there exists $\delta > 0$ such that $d_Y(f(x), f(z)) < \varepsilon$

whenever $d_X(x,z) < \delta$. Because x_k converges to z, there exists $K > 0$ such that $d_X(x_k, z) < \delta$ whenever $k \geq K$. It follows that $d_Y(f(x_k), f(z)) < \varepsilon$, whenever $k \geq K$, and $f(x_k)$ converges to $f(z)$.

For the second half of the proof, suppose that whenever x_k is a sequence in X converging to z, the sequence $f(x_k)$ converges to $f(z)$ in Y. The proof will proceed by contradiction, that is, it will be shown that the assumption that the conclusion is false leads to a contradiction. Therefore, the conclusion must be true.

Let $\varepsilon > 0$. Suppose there does not exist $\delta > 0$ such that $d_Y(f(x), f(z)) < \varepsilon$ whenever $d_X(z, x) < \delta$. Then for every positive integer k there exists an $x_k \in X$ such that $d_X(x_k, z) < 1/k$ and $d_Y(f(x_k), f(z)) \geq \varepsilon$. Since $1/k$ converges to 0 as k goes to infinity, it follows that x_k converges to z. It is obvious that $f(x_k)$ does not converge to $f(z)$ because $d_Y(f(x_k), f(z)) \geq \varepsilon$ for all k, contradicting the starting assumption that whenever x_k is a sequence in X converging to z, the sequence $f(x_k)$ converges to $f(z)$ in Y. Therefore, there must exist δ such that $d_Y(f(x), f(z)) < \varepsilon$ whenever $d_X(x, z) < \delta$, and f is continuous z. □

A slightly different, but equivalent, way of stating the definition of $f : X \to Y$ being continuous at z is that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f\big(B_\delta(z)\big) \subset B_\varepsilon\big(f(z)\big).$$

For a subset E of Y, set $f^{-1}(E) = \big\{x \in X : f(x) \in E\big\}$. Using this notation, the above can be restated as the function $f : X \to Y$ is continuous at z if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$B_\delta(z) \subset f^{-1}\left[B_\varepsilon(f(z))\right]. \tag{1.8}$$

This point of view is helpful when working with functions that are continuous at every point of X and is linked to the definition of an open set.

Let X be a metric space with metric d. A subset U of X is an *open set* in X provided that for every x in U there exists $r > 0$ such that $B_r(x) \subset U$. The next proposition uses open sets to characterize functions that are continuous at every point in a metric space, such functions will be called simply *continuous functions.*

Proposition 1.2.4 *Let X and Y be metric spaces and let $f : X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(U)$ is an open set in X, whenever U is an open set in Y.*

Proof. Suppose f is continuous at every x in X, and let U be an open set in Y. It must be shown that for any x in $f^{-1}(U)$ there exists $r > 0$ such that $B_r(x) \subset f^{-1}(U)$. Since U is open and $f(x)$ is in U, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset U$. Now using the continuity of f at x, there exists $\delta > 0$ such that (1.8) holds. Clearly, $f^{-1}\left[B_\varepsilon(f(x))\right] \subset f^{-1}(U)$, and hence

$$B_\delta(x) \subset f^{-1}\left[B_\varepsilon(f(x))\right] \subset f^{-1}(U).$$

Therefore, $f^{-1}(U)$ is an open set in X.

Next, suppose that whenever U is an open set in Y, then $f^{-1}(U)$ is an open set in X. Let x be an arbitrary point in X and let $\varepsilon > 0$. Because $B_\varepsilon(f(x))$ is an open subset of Y by *Exercise* 2, p. 27, the set $f^{-1}(B_\varepsilon(f(x))$ is an open subset of X by hypothesis, and it contains x. Hence, there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))$. Thus (1.8) holds and f is continuous at x. □

Corollary 1.2.5 *Let X, Y and Z be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is a continuous function.*

Proof. $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ □

Since *Proposition 1.2.4* shows that the open sets completely determine the continuous functions from one metric space into another without specific reference to the metric, what role does the metric really play? To address this question, we first need to examine the union and intersection properties of open sets. To do this we work with collections of sets or sets whose elements are sets. Let $\mathcal{S}$ be a collection of subsets of a set X. The *union* and *intersection* of the sets in $\mathcal{S}$ are subsets of X defined by

$$\bigcup_{E\in\mathcal{S}} E = \{x : x \in E \text{ for some } E \in \mathcal{S}\}$$

and

$$\bigcap_{E\in\mathcal{S}} E = \{x : x \in E \text{ for all } E \in \mathcal{S}\},$$

respectively. Sometimes it is more convenient to define a finite collection of sets in X by indexing them as in $E_1, \ldots, E_m$.

Proposition 1.2.6 *Let X be a metric space with metric d. Then the open sets have the following properties:*

(a) Both X and the empty set ϕ are open sets.

(b) If $\mathcal{S}$ is any collection of open sets in X, then

$$\bigcup_{U\in\mathcal{S}} U = \{x : x \in U \text{ for some } U \in \mathcal{S}\}$$

is an open set in X.

(c) If $U_1, \ldots, U_m$ is any finite collection of open sets in X, then

$$\bigcap_{j=1}^{m} U_j = \{x : x \in U_j \text{ for } j = 1, \ldots, m\}$$

is an open set in X.

Proof. Obviously, X is an open set. The empty set, ϕ, is open because it is impossible to find x in ϕ such that $B_r(x)$ is not contained in ϕ for all $r > 0$. In other words, the empty set satisfies the definition of an open set vacuously.

For part *(b)*, let x be any point in

$$\bigcup_{U \in \mathcal{S}} U.$$

Then x is in at least one particular V in $\mathcal{S}$. Since V is open, there exists $r > 0$ such that $B_r(x) \subset V$. It follows that

$$B_r(x) \subset \bigcup_{U \in \mathcal{S}} U,$$

and the union of the sets in $\mathcal{S}$ is open because x was an arbitrary point in it.

For the third part, let x be an arbitrary point in

$$\bigcap_{j=1}^{m} U_j.$$

Consequently, x is in every U_j for $j = 1, \ldots, m$. Since U_j is open, for each j, there exists $r_j > 0$ such that $B_{r_i}(x) \subset U_j$. Set $r = \min\{r_j : j = 1, \ldots, m\}$, which is positive. It follows that $B_r(x) \subset U_j$ for $j = 1, \ldots, m$ and

$$B_r(x) \subset \bigcap_{j=1}^{m} U_j.$$

Therefore, the intersection of a finite collection of open sets is open. □

Let X be a nonempty set. A collection $\mathcal{T}$ of subsets of X such that (a) both X and ϕ are in $\mathcal{T}$; (b) if $\mathcal{S}$ is any collection of sets in $\mathcal{T}$, that is, $\mathcal{S} \subset \mathcal{T}$, then $\bigcup_{U \in \mathcal{S}}$ is in $\mathcal{T}$; (c) if $U_1, \ldots, U_m$ is any finite collection of sets in $\mathcal{T}$, then $\bigcap_{j=1}^{m} U_j$ is in $\mathcal{T}$. is called a *topology* on X, and the sets in $\mathcal{T}$ are called *open sets.* A *topological space* is simply a nonempty set with a specified topology. Thus the content of *Proposition 1.2.6* is that a metric on X determines a topology on X, making X is a topological space.

A topology on a set X has the *Hausdorff property* provided that given distinct points x and y in X there exist disjoint open sets U and V in the topology such that $x \in U$ and $y \in V$. When the topology on X has the Hausdorff property, X is called a *Hausdorff topological space.* It has already been shown that every metric topology has the Hausdorff property. Not every topological space has the Hausdorff property, and hence not every topological space is a metric space. For example, $\mathcal{T} = \{X, \phi\}$ is the smallest topology for any nonempty set X. If X contains more than one point, this topology does not have the Hausdorff property and does not come from a metric.

For any nonempty set X, the collection of all subsets of X is obviously a topology on X and the largest topology possible on X. This topology is called the *discrete topology.* The discrete topology on any nonempty set is always a metric topology. For example, on any nonempty set X

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}, \tag{1.9}$$

is a metric that will produce the discrete topology on X because $B_{1/2}(x) = \{x\}$ for all $x \in X$. So on every nonempty set there is at least one metric.

Let X be a topological space with topology $\mathcal{T}_X$, and let Y be a nonempty subset of X. It is easy to verify that

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}_X\} \tag{1.10}$$

is a topology on Y. The topology $\mathcal{T}_Y$ is called the *relative topology* and is the default topology on any nonempty subset of a topological space. Note that if Z is a nonempty subset of Y, then the relative topology on Z from Y is the same as the relative topology on Z from X because $(U \cap Y) \cap Z = U \cap Z$.

A nonempty subset Y of a topological space X is *discrete* provided that for every $y \in Y$ there exists an open set U of X such that $U \cap Y = \{y\}$. Thus for every y in Y, the set $\{y\}$ is an open subset of Y with the relative topology. It follows that every subset of Y is an open subset of Y with the relative topology because the union of open sets is an open set. So the relative topology of a nonempty discrete subset of a topological space is the discrete topology.

If X is a metric space with metric d, then $\mathcal{T}(d)$ will denote the topology on X determined by d, that is, the collection of open sets determined by d. Two metrics d_1 and d_2 on X will be called *equivalent metrics* provided $\mathcal{T}(d_1) = \mathcal{T}(d_2)$. When working with open sets, equivalent metrics are interchangeable.

Theorem 1.2.7 *If d is a metric on X, then*

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is a metric on X that is equivalent to d and satisfies $d'(x,y) < 1$ for all x and y in X.

Proof. It is easy to see that $d'(x,y) < 1$ and satisfies the first three conditions for being a metric.

To prove the triangle inequality, first observe that the real-valued function

$$f(t) = \frac{t}{1+t} = \frac{1}{1 + \frac{1}{t}}$$

is increasing for $t \geq 0$. Hence $d(x,y) \leq d(x,z) + d(z,y)$ implies that

$$\frac{d(x,y)}{1 + d(x,y)} \leq \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)},$$

and it follows that

$$\begin{aligned} d'(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &\leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\ &= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= d'(x,z)+d'(z,y). \end{aligned}$$

Thus $d'(x,y) \leq d'(x,z) + d'(z,y)$, and $d'(x,y)$ is a metric on X.

It remains to be shown that $\mathcal{T}(d) = \mathcal{T}(d')$. Actually, more is true; the two metrics have the same open balls, just the radii change. Let $B'_r(x)$ denote an open ball of radius r in the d' metric. It is easy to verify that for $r > 0$

$$B_r(x) = B'_{\frac{r}{1+r}}(x)$$

and that for $0 < r < 1$

$$B'_r(x) = B_{\frac{r}{1-r}}(x).$$

It now follows from the definition of an open set that $\mathcal{T}(d) = \mathcal{T}(d')$. □

The complementary behavior of open sets is also an important topological matter. If E is a subset of X, then the *complement* of E is by definition $X \setminus E = \{x \in X : x \notin E\}$. A subset E of a topological space X is a *closed set* provided that $X \setminus E$ is an open set. For metric spaces, we have the following useful characterization of closed sets:

Proposition 1.2.8 *Let X be a metric space. A subset F of X is closed if and only if the limit of every convergent sequence x_k in F is also in F.*

Proof. Suppose F is closed and x_k is a sequence in F converging to x. Since F is closed, $X \setminus F$ is open. If x is not in F, then x is in the open set $X \setminus F$ and there exists $r > 0$ such that $B_r(x) \subset X \setminus F$. Now there exists K such that $x_k \in B_r(x)$ for $k \geq K$, contradicting the fact that x_k is in F for all k.

For the second half of the proof, suppose that the limit of every convergent sequence x_k in F is also in F. It suffices to show that $X \setminus F$ is open. Let x be a point in $X \setminus F$. If $B_{1/k}(x) \subset X \setminus F$ does not hold for any k, then for every k there exists $x_k \in F \cap B_{1/k}(x)$. Clearly, x_k is a sequence in F converging to $x \in X \setminus F$, contradicting the hypothesis that the limit of every convergent sequence x_k in F is also in F. Therefore, $B_{1/k}(x) \subset X \setminus F$ for some k, and $X \setminus F$ is open. □

Proposition 1.2.9 *Let X be a topological space. Then the closed sets of X have the following properties:*

(a) Both X and the ϕ are closed sets.

(b) If $\mathcal{S}$ is any collection of closed sets in X, then

$$\bigcap_{F\in\mathcal{S}} F = \{x : x \in F \text{ for all } F \in \mathcal{S}\}$$

is a closed set in X.

(c) If $F_1, \ldots F_m$ is any finite collection of closed sets in X, then

$$\bigcup_{j=1}^{m} F_j = \{x : x \in F_j \text{ for some } j\}$$

is a closed set in X.

Proof. Since $X \setminus \phi = X$ and $X \setminus X = \phi$, both X and ϕ are closed because ϕ and X are open sets.

For *(b)* and *(c)*, it suffices to show that

$$X \setminus \bigcap_{F\in\mathcal{S}} F \quad\text{and}\quad X \setminus \bigcup_{j=1}^{m} F_j$$

are open sets. Using de Morgan's formulas from *Exercise* 6, p. 27, these sets can be expressed as follows:

$$X \setminus \bigcap_{F\in\mathcal{S}} F = \bigcup_{F\in\mathcal{S}} (X \setminus F)$$

and

$$X \setminus \bigcup_{j=1}^{m} F_j = \bigcap_{j=1}^{m} (X \setminus F_j).$$

It follows from the definition of a topology that the right-hand sides are open sets because the sets $X \setminus F$ for $F \in \mathcal{S}$ and $X \setminus F_j$ for $j = 1, \ldots, m$ are open sets. Therefore,

$$\bigcap_{F\in\mathcal{S}} F \quad\text{and}\quad \bigcup_{j=1}^{m} F_j$$

are closed sets. □

Let E be a subset of a topological space X. Define the *closure* of E by

$$E^{-} = \big\{x : E \cap U \neq \phi \text{ for all open } U \text{ containing } x\big\}. \tag{1.11}$$

Note that x is not in E^{-} if and only if there exists an open set V containing x such that $E \cap V = \phi$. In particular, if V is an open set such that $E \cap V = \phi$, then $V \subset X \setminus E^{-}$. The closure construction has a number of useful properties that are worth stating as a proposition.

Proposition 1.2.10 *Let E be a subset of a topological space X. Then the following hold:*

(a) The set E^- is a closed set containing E.

(b) The set E is closed if and only if $E = E^-$.

(c) If F is a closed set such that $E \subset F$, then $E^- \subset F$.

(d) $E^- = \bigcap\{F : E \subset F = F^-\}$.

Proof. Obviously, $E \subset E^-$. If x is not in E^-, then there exists an open set V containing x such that $E \cap V = \phi$. It follows that $V \subset X \setminus E^-$ and $X \setminus E^-$ is open by *Exercise* 8, p. 28, a basic fact that will be used frequently. Thus E^- is a closed set and *(a)* holds.

If E is closed, then $X \setminus E$ is open, and a point in $X \setminus E$ cannot be in E^- because $E \cap (X \setminus E) = \phi$. Hence, $E^- \subset E \subset E^-$ and $E = E^-$. The converse is obvious.

For part *(c)*, observe that $X \setminus F$ is an open set such that $E \cap (X \setminus F) = \phi$ and hence $E^- \cap (X \setminus F) = \phi$.

Finally, part *(d)* follows from parts *(a)*, *(b)*, and *(c)*. □

Given sets $X_1, X_2, \ldots, X_m$, their *Cartesian product* is defined by

$$X_1 \times \ldots \times X_m = \big\{(x_1, \ldots, x_m) : x_j \in X_j \text{ for } j = 1, \ldots, m\big\},$$

and for $1 \le j \le m$ the projections

$$p_j : X_1 \times \ldots \times X_m \to X_j$$

are defined by $p_j(x_1, \ldots, x_m) = x_j$.

Suppose $X_1, \ldots, X_m$ are also metric spaces with metrics $d_1, \ldots, d_m$. Then

$$d\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) = \sum_{j=1}^{m} d_j(x_j, y_j) \tag{1.12}$$

defines a metric on $X_1 \times \ldots \times X_m$ called the *product metric*. An important property of the product metric is that its open sets can be characterized in terms of the open sets of $X_1, \ldots, X_m$.

Theorem 1.2.11 *Let $X_1, \ldots, X_m$ be metric spaces with metrics $d_1, \ldots, d_m$, and let d be the metric on $X_1 \times \ldots \times X_m$ defined by (1.12). The following are equivalent for a subset U of $X_1 \times \ldots \times X_m$:*

(a) U is an open subset for the metric d.

(b) For every point $(x_1, \ldots, x_m) \in U$ there exist open sets U_j of X_j for $j = 1, \ldots, m$ such that

$$(x_1, \ldots, x_m) \in U_1 \times \ldots \times U_m \subset U.$$

In particular, $U_1 \times \ldots \times U_m$ is an open set for the metric d, when U_j is an open set in X_j for $j = 1, \ldots, m$.

Proof. To begin the proof, first observe that for $r > 0$

$$B_r(x_1) \times \ldots \times B_r(x_m) \subset B_{mr}\big((x_1, \ldots, x_m)\big) \tag{1.13}$$

and

$$B_r\big((x_1, \ldots, x_m)\big) \subset B_r(x_1) \times \ldots \times B_r(x_m). \tag{1.14}$$

If U is an open set in $X_1 \times \ldots \times X_m$ and $(x_1, \ldots, x_m)$ is in U, then there exists $r > 0$ such that

$$B_{mr}\big((x_1, \ldots, x_m)\big) \subset U,$$

and *(b)* follows from (1.13) because open balls are open sets.

If condition *(b)* holds, then, given $(x_1, \ldots, x_m) \in U$, for each j there exists $r_j > 0$ such that $B_{r_j}(x_j) \subset U_j$. Set $r = \min\{r_j : j = 1, \ldots, m\}$, and note that

$$B_r(x_1) \times \ldots \times B_r(x_m) \subset U_1 \times \ldots \times U_m \subset U.$$

Now(1.14) implies that

$$B_r\big((x_1, \ldots, x_m)\big) \subset U,$$

and U is an open set for the metric d. □

Corollary 1.2.12 *Let $X_1, \ldots, X_m$ be metric spaces. If d_j and d'_j are equivalent metrics on X_j for $j = 1, \ldots, m$, then $d\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) = \sum_{j=1}^{m} d_j(x_j, y_j)$ and $d'\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) = \sum_{j=1}^{m} d'_j(x_j, y_j)$ are equivalent metrics on $X_1 \times \ldots \times X_m$.*

Condition *(b)* in *Theorem 1.2.11* now leads us to a topology on any product of a finite number of topological spaces $X_1, \ldots, X_m$. Let $\mathcal{T}$ be the collection of subsets U of $X_1 \times \ldots \times X_m$ such that for every point $(x_1, \ldots, x_m) \in U$ there exist open sets U_j of X_j satisfying

$$(x_1, \ldots, x_m) \in U_1 \times \ldots \times U_m \subset U.$$

We need to show that $\mathcal{T}$ is a topology. Obviously, $X_1 \times \ldots \times X_m$ is in $\mathcal{T}$ because each X_j is an open set in X_j. The empty set satisfies the required condition vacuously and is in $\mathcal{T}$. It is also clear that the union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

To show that the finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$, it suffices to show that $U \cap V$ is in $\mathcal{T}$ when U and V are in $\mathcal{T}$. If $(x_1, \ldots, x_m)$ is in $U \cap V$, then there exist open sets U_j and V_j of X_j for $j = 1, \ldots, m$ such that $(x_1, \ldots, x_m) \in U_1 \times \ldots \times U_m \subset U$ and $(x_1, \ldots, x_m) \in V_1 \times \ldots \times V_m \subset V$. It follows that

$$(x_1, \ldots, x_m) \in (U_1 \times \ldots \times U_m) \cap (V_1 \times \ldots \times V_m) \subset U \cap V.$$

Since

$$(U_1 \times \ldots \times U_m) \cap (V_1 \times \ldots \times V_m) = (U_1 \cap V_1) \times \ldots \times (U_m \cap V_m),$$

$U \cap V$ is in $\mathcal{T}$ when U and V are in $\mathcal{T}$. Thus $\mathcal{T}$ is a topology on $X_1 \times \ldots \times X_m$, and it is called the *product topology*. From *Theorem 1.2.11*, we know that the product topology is a metric topology when $X_1, \ldots, X_m$ are metric spaces

In the remainder of this section, we will establish a few useful properties of functions from one topological space to another. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* at $z \in X$ provided that for every open set U of Y containing $f(z)$ there exists an open set V of X such that $z \in V \subset f^{-1}(U)$. It is easy to verify that for metric spaces this definition is equivalent to the definition of continuity at a point z found on page 16. As before, a *continuous function* means a function that is continuous at every point of its domain. We also have the following topological versions of *Proposition 1.2.4* and its corollary:

Proposition 1.2.13 *Let X and Y be topological spaces and let $f : X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(U)$ is an open subset of X, whenever U is an open subset of Y.*

Proof. Suppose f is continuous at every x in X, and let U be an open subset of Y. Consider an arbitrary x in $f^{-1}(U)$. Then, $f(x)$ is in U, and by the definition of continuity there exists an open subset V of X such that $x \in V \subset f^{-1}(U)$. It now follows from *Exercise* 8, p. 28 that $f^{-1}(U)$ is open. The other direction is trivial. $\square$

Corollary 1.2.14 *Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is a continuous function.*

Corollary 1.2.15 *Let X and Y be topological spaces and let $f : X \to Y$ be a function. If the topology on X is the discrete topology, then f is continuous.*

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be open if $f(U)$ is an open subset of Y, whenever U is an open subset of X.

Proposition 1.2.16 *If $X_1 \times \ldots \times X_m$ is a product of topological spaces with the product topology, then the projections are continuous open functions.*

Proof. If V is an open set in X_j, then

$$p_j^{-1}(V) = X_1 \times \ldots \times X_{j-1} \times V \times X_{j+1} \times \ldots \times X_m$$

which is an open set in the product topology. So the projections are continuous.

Given an open set U of $X_1 \times \ldots \times X_m$, it must be shown that $p_j(U)$ is open in X_j. It suffices by *Exercise* 8, p. 28, to show that for each y in $p_j(U)$ there exist an open set V in X_j such that $y \in V \subset p_j(U)$. If y is in $p_j(U)$, then there exists $(x_1, \ldots, x_m)$ in U such that $x_j = y$. So there exist open sets U_i of X_i for $i = 1, \ldots, m$ such that $(x_1, \ldots, x_m) \in U_1 \times \ldots \times U_m \subset U$. It follows that $y = x_j \in U_j = p_j(U_1 \times \ldots \times U_m) \subset p_j(U)$. Thus U_j is the required open set V of X_j and the projection p_j is an open function. $\square$

Proposition 1.2.17 *Let $X_1 \times \ldots \times X_m$ be a product of topological spaces with the product topology, and let Y be a topological space. A function $f : Y \to X_1 \times \ldots \times X_m$ is continuous if and only if the functions $f_j = p_j \circ f$ are continuous for $j = 1, \ldots, m$.*

Proof. If f is continuous, then the functions $f_j = p_j \circ f$ are all continuous because the projections are continuous and the composition of continuous functions is continuous by *Corollary 1.2.14*.

Suppose the functions $f_j = p_j \circ f$ are all continuous. These functions are called *coordinate functions* because $f(y) = (f_1(y), \ldots, f_m(y))$. Let U be an open set in $X_1 \times \ldots \times X_m$. To show that $f^{-1}(U)$ is open it suffices to show that for every point y in $f^{-1}(U)$, there exists an open set V of Y such that $y \in V \subset f^{-1}(U)$.

By *Theorem 1.2.11*, there exist open sets U_j of X_j such that $f(y) \in U_1 \times \ldots \times U_m \subset U$, and so

$$y \in f^{-1}(U_1 \times \ldots \times U_m) \subset f^{-1}(U).$$

Observe that

$$\bigcap_{j=1}^{m} (p_j \circ f)^{-1}(U_j) = f^{-1}(U_1 \times \ldots \times U_m).$$

Therefore, $V = f^{-1}(U_1 \times \ldots \times U_m)$ is the required open set because the functions $p_j \circ f$ are continuous. □

Let X and Y be topological spaces. A function $f : X \to Y$ is a *homeomorphism* provided that f is a one-to-one and onto function such that both f and f^{-1} are continuous functions. Suppose $f : X \to Y$ is a one-to-one and onto function. Since $(f^{-1})^{-1} = f$, the function f^{-1} is continuous if and only if f is an open function. Therefore, a one-to-one and onto function is a homeomorphism if and only if it is both continuous and open. It follows that for a homeomorphism f, a subset U of X is open if and only if $f(U)$ is an open set in Y. Since $f(X \setminus U) = Y \setminus f(U)$ because f is one-to-one and onto, a subset E of X is closed if and only if $f(E)$ is an closed set in Y. Consequently, two topological spaces are regarded as equal when they are homeomorphic.

A property of topological spaces is a *topological invariant* if given two homeomorphic topological spaces either they both have the property or neither one has it. A topological space X with topology $\mathcal{T}$ is said to be *metrizable* provided that there exists a metric d for X such that $\mathcal{T}(d) = \mathcal{T}$. Being metrizable is a topological invariant (*Exercise* 20, p. 29).

We close this section with a word of caution about the differences between metric spaces and topological spaces. *Propositions 1.2.3 and 1.2.8* are not true for all topological spaces. Convergence is a more delicate issue in general topological spaces and requires using *nets* instead of sequences. Many results like *Propositions 1.2.3 and 1.2.8* can be generalized to topological spaces by replacing sequences with nets, but this requires a substantial detour into the theory of nets. The interested reader can find a comprehensive treatment of nets in

Kelley [7], Chapter 2 or Willard [16], Chapter 4. These two books both with the same title, *General Topology*, will be our standard general topology references. Although not recent publications, they have stood the test of time and are still readily available.

EXERCISES

1. Show that a sequence in a metric space cannot converge to two distinct points.

2. Prove that an open ball of a metric space X is an open set in X.

3. Let X and Y be sets, let $f : X \to Y$ be a function, and let $\mathcal{S}$ be a collection of subsets of Y. Prove that

$$f^{-1}\left(\bigcup_{E\in\mathcal{S}} E\right) = \bigcup_{E\in\mathcal{S}} f^{-1}(E),$$

$$f^{-1}\left(\bigcap_{E\in\mathcal{S}} E\right) = \bigcap_{E\in\mathcal{S}} f^{-1}(E),$$

and

$$f^{-1}(Y \setminus E) = X \setminus f^{-1}(E).$$

(The usual way to prove two sets are equal is to show that each point in the set on the left is in the set on the right and visa versa.)

4. Let X be a metric space with metric d, and let y be a point in X. Show that the function $d_y(x) = d(x, y)$ is continuous on X. Given $\varepsilon > \delta > 0$, show that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that the function $f = g \circ d_y$ has the following properties:

   (a) f is continuous

   (b) $0 \le f(x) \le 1$ for all $x \in X$

   (c) $f(x) = 1$ if and only if $d(x, y) \le \delta$,

   (d) $f(x) = 0$ if and only if $d(x, y) \ge \varepsilon$.

5. Show that $d_1(x, y) = |x^3 - y^3|$ is a metric on $\mathbb{R}$ equivalent to $d(x, y) = |x - y|$.

6. Let $\mathcal{S}$ be a collection of subsets of a set X. Prove *de Morgan's formulas*:

$$X \setminus \bigcap_{E\in\mathcal{S}} E = \bigcup_{E\in\mathcal{S}} (X \setminus E)$$

and

$$X \setminus \bigcup_{E\in\mathcal{S}} E = \bigcap_{E\in\mathcal{S}} (X \setminus E).$$

7. Prove that in a metric space X with metric d, the closed balls defined by $\{x \in X : d(x,y) \le r\}$ are closed sets and that

$$B_r(y)^- \subset \{x \in X : d(x,y) \le r\}.$$

Prove that in $\mathbb{R}^n$ the closure of an open Euclidean ball $B_r(\mathbf{y})$ is the closed ball $\{\mathbf{x} : d(\mathbf{x},\mathbf{y}) \le r\}$, and show by example that this is not true in all metric spaces.

8. Let X be a topological space. Prove that a subset U of X is an open set if and only if for every x in U there exists an open set V such that $x \in V \subset U$.

9. Let E be a nonempty subset of a metric space X. A point $x \in X$ is in E^- if and only if there exists a sequence x_k in E converging to x.

10. Let U be an open set in a metric space X, and let x be a point in U. Prove that there exists an open set V such that $x \in V \subset V^- \subset U$. A topological space with this property is said to be *regular* .

11. Let X be a Hausdorff topological space. Show that if x is in X, then $\{x\}$ is a closed set in X.

12. Let F be a closed subset of a metric space X with metric d. Define the *distance from the point x in X to the closed set F* by $d(x,F) = \inf\{d(x,y) : y \in F\}$ for any x in X. (If E is a set of real numbers that is bounded below, then the *infinum* of E, written $\inf E$, is the greatest lower bound and $\inf E = -\sup(-E)$.) For $r > 0$, show that set $\{x \in X : d(x,F) \ge r\}$ is a closed set or equivalently that the set $\{x \in X : d(x,F) < r\}$ is open.

13. Show that if E is a nonempty subset of $\mathbb{R}$, then $\sup E$ or $\inf E$ is in E^- according as E is bounded above or below.

14. Let x_k and y_k be sequences in the metric spaces X and Y. Show that the sequence (x_k, y_k) in $X \times Y$ with the product topology converges to (x,y) if and only if x_k converges to x in X and y_k converges to y in Y.

15. Show that the metrics $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ and $d'(\mathbf{x},\mathbf{y}) = \sum_{j=1}^n |x_1 - y_2|$ on $\mathbb{R}^n$ are equivalent.

16. Let X be a metric space with metric d and let Y be a nonempty subset of X. Show that the relative topology on Y is the same the metric topology from restricting d to Y.

17. Show that the product topology on $\mathbb{R}^m \times \mathbb{R}^n$ coincides with the usual topology on $\mathbb{R}^{n+m}$. (One approach is to make use of *Exercise 15* above.)

18. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous function. Show that $f(A^-) \subset f(A)^-$. Then show that $f(A)^- = f(A^-)$, when f is a homeomorphism of X onto Y.

19. Let X and Y be metric spaces. An onto function $f : X \to Y$ such that $d_Y(f(u), f(v)) = d_X(u, v)$ for all u and v in X is called an *isometry*. Show that an isometry is a homeomorphism. Prove that the isometries of a metric space onto itself are a group under composition of functions. Give an example of a distance preserving function of a metric space to itself that is not onto.

20. Let X and Y be homeomorphic topological spaces. Prove that X is metrizable if and only if Y is metrizable.

1.3 Continuous Group Operations

Let G be both a group and a topological space. So we are assuming there is a specified binary operation on G that satisfies the group axioms on page 3 and a collection of open sets $\mathcal{T}$ satisfying the axioms of a topology on page 19. No assumption, however, is being made of any connection between these two structures on G. In general, there is no reason to believe that multiplying two elements in G or taking the inverse of an element in G are continuous functions. When these functions are continuous, however, algebraic and analytic ideas come together in a rich theory.

A *topological group* is a group that is also a topological space such that the function $(x, y) \to xy^{-1}$ is a continuous function from $G \times G$ with the product topology to G. When the topology on G comes from a metric, G will be called a *metric group*.

Proposition 1.3.1 *Let G be both a group and a topological space. Then G is a topological group if and only if both the function $x \to x^{-1}$ from G to G and the function $(x, y) \to xy$ from $G \times G$ with the product metric to G are continuous.*

Proof. Suppose G is a topological group and e is the identity element of G. The function $y \to (e, y)$ from G into $G \times G$ is continuous by *Proposition 1.2.17*. Since the composition of continuous maps is continuous (*Corollary 1.2.14*), the function $y \to (e, y) \to ey^{-1} = y^{-1}$ is continuous.

To prove that the function $(x, y) \to xy$ is continuous, consider the function from $G \times G$ with the product topology to itself defined by $f(x, y) = (x, y^{-1})$. Then $p_1 \circ f = p_1$ is obviously continuous, and $p_2 \circ f(x, y) = (p_2(x, y))^{-1}$ is continuous because p_2 is continuous and $y \to y^{-1}$ was just shown to be continuous. Thus f is also continuous by *Proposition 1.2.17*. Consequently, the composition of f followed by $(x, y) \to xy^{-1}$ is continuous, that is, $(x, y) \to (x, y^{-1}) \to x(y^{-1})^{-1} = xy$ is continuous.

For the second half of the proof, suppose the functions $x \to x^{-1}$ and $(x, y) \to xy$ are continuous. Then as above $f(x, y) = (x, y^{-1})$ is continuous because $y \to y^{-1}$ is continuous. It follows that the composition $(x, y) \to f(x, y) = (x, y^{-1}) \to xy^{-1}$ is continuous. □

Of course, if G is a topological group with the group operation written additively, then the function $(x, y) \to x - y$ is continuous by definition and equivalent to the functions $x \to -x$ and $(x, y) \to x + y$ being continuous.

Until Chapter 5, all of our examples of topological groups will be metric groups. The most obvious example of a metric group is the additive real numbers $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. The product metric on $\mathbb{R} \times \mathbb{R}$ is $d'((x, y), (u, v)) = |x - u| + |y - v|$. Since

$$|(x + y) - (a + b)| \leq |x - a| + |y - b| = d'\big((x, y), (a, b)\big),$$

addition is a continuous function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ at every (a, b) in $\mathbb{R}^2$. The inverse functions, $x \to -x$, is an isometry of $\mathbb{R}$ onto itself and obviously continuous. So $\mathbb{R}$ is a metric group by *Proposition 1.3.1.* In fact, the same argument with $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ replacing $d(x, y) = |x - y|$ shows that $\mathbb{R}^n$ under addition is also a metric group for $n > 1$.

With more careful estimates, it is not difficult to show that the groups $\mathbb{R}^+$ and $\mathbb{R} \setminus \{0\}$ are also metric groups with the metric $d(x, y) = |x - y|$ (*Exercise* 1, p. 38). Similarly, one shows that $\mathbb{C} \setminus \{0\}$ and $\mathbb{K}$ are metric groups using the metric $d(z, w) = |z - w|$, where $|z - w|$ is the complex absolute value (*Exercise* 2, p. 38). *In the future,* $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^+$, $\mathbb{R} \setminus \{0\}$, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ *and* $\mathbb{K}$ *will always refer to these specific metric groups.*

Any group can be made into a metric group by using the discrete metric defined by (1.9) because the product topology of two spaces with discrete topologies is also a discrete topology and all functions from one space with the discrete topology to another metric space are continuous. Occasionally it will be useful to consider $\mathbb{R}_d$, the real numbers with the discrete topology. It will be understood that $\mathbb{Z}$ is always the metric group with the discrete topology on the additive group $\mathbb{Z}$, which is the same as its relative topology as a subset of $\mathbb{R}$. It is also convenient to assume that every finite group is a discrete metric space and hence a metric group (see *Exercise* 3, p. 38).

Proposition 1.3.2 *Let G be a topological group and let a and b be elements of G. Then the following functions are all homeomorphisms of G onto G:*

(a) $x \to xb$

(b) $x \to ax$

(c) $x \to axb$

(d) $x \to axa^{-1}$,

(e) $x \to x^{-1}$

Proof. The function $x \to xb$ is one-to-one and onto by *Exercise* 4, p. 11. It is continuous because it is the composition of the continuous functions $x \to (x, b)$ and $(x, y) \to xy$. The function $x \to xb^{-1}$ is thus also continuous, and clearly the inverse of $x \to xb$, making $x \to xb$ a homeomorphism. Similar arguments work in the remaining parts. ☐

Let A and B be subsets of a group G. The following notations will be convenient:

$$A^{-1} = \{a^{-1} : a \in A\},$$

and

$$AB = \{ab : a \in A \text{ and } b \in B\}.$$

Notice that if the identity element e is in A, then $B \subset AB$.

It will also be helpful to use the concept of a neighborhood. Let X be a topological space and let x be in X. A *neighborhood* of x is a subset U of X such that there exist and open set V satisfying $x \in V \subset U$. A neighborhood need not be an open set. A set U is open if and only if it is a neighborhood of each of its points. Also, a function $f : X \to Y$ is continuous at z if and only if $f^{-1}(U)$ is a neighborhood of z whenever U is a neighborhood of $f(z)$.

Lemma 1.3.3 *Let G be a topological group. If U is an open neighborhood of the identity e in G, then there exists an open neighborhood V of the identity such that $VV \subset U$.*

Proof. Because multiplication is continuous, the set $V' = \{(x, y) : xy \in U\}$ is an open set in $G \times G$ with the product topology. Obviously, (e, e) is in V'. It follows from the definition of the product topology on page 24 that there exist open neighborhoods V_1 and V_2 of e such that $V_1 \times V_2 \subset V'$. Hence, $V_1V_2 \subset U$. Set $V = V_1 \cap V_2$ to get $VV \subset U$. ☐

For a topological space X, a *countable neighborhood base at x* is a sequence of neighborhoods U_m of x such that for every neighborhood U of x there exists m such that $U_m \subset U$. (The concept of a set or collection of sets being countable will be examined in more detail in Section 3.4 and its exercises. For now, countable is just an adjective that occurs in the terminology.)

Every metric space X has a countable neighborhood base at any point x; just let $U_m = B_{1/m}(x)$. When there is a countable neighborhood base at every point of topological space, the space is said to be *first countable.* Metric spaces are first countable, but not all topological spaces are first countable.

If G is a topological group and U_m is a countable neighborhood base at the identity e, then *Proposition 1.3.2* implies that xU_m and U_mx are countable neighborhood bases at x for any x in G. Thus a topological group is first countable, if it has a countable neighborhood base at the identity.

For metric groups, there are countable neighborhood bases at e that reflect the algebraic structure, as well as the metric structure.

Proposition 1.3.4 *Let G be a metric group. There exists a countable neighborhood base U_m at the identity element e with the following properties:*

(a) U_m is open for all m.

(b) $U_m U_m U_m \subset U_{m-1}$ *for all* $m \geq 2$.

(c) $U_m^{-1} = U_m$ *for all* m.

(d) $\bigcap_{m=1}^{\infty} U_m = \{e\}$.

Proof. If V is an open set containing e, then so is V^{-1} because $x \to x^{-1}$ is a homeomorphism. So $U = V \cap V^{-1}$ is an open neighborhood of e such that $U^{-1} = U$.

Let d be a metric for G. The construction of the neighborhood base will proceed by induction. To start the process, let $V_1 = B_1(e)$ and set $U_1 = V_1 \cap V_1^{-1}$, making U_1 is an open neighborhood of e such that $U_1^{-1} = U_1 \subset B_{1/1}(e)$.

Suppose $U_1, \ldots, U_m$ have been constructed satisfying conditions *(a)*, *(b)*, *(c)*, and $U_m \subset B_{1/m}(e)$. Note that $U_m \cap B_{1/(m+1)}(e)$ is an open neighborhood of e. By applying *Lemma 1.3.3* twice, there exists an an open neighborhood V_{m+1} of e such that

$$V_{m+1}V_{m+1}V_{m+1}V_{m+1} \subset U_m \cap B_{1/(m+1)}(e).$$

Since e is in V_{m+1}, it clearly follows that $V_{m+1}V_{m+1}V_{m+1} \subset U_m$ and $V_{m+1} \subset B_{1/(m+1)}(e)$. Set $U_{m+1} = V_{m+1} \cap V_{m+1}^{-1}$, so U_{m+1} satisfies condition *(a)* and *(c)*. Since $U_{m+1} \subset V_{m+1}$, it follows that U_{m+1} satisfies conditions *(b)* and $U_{m+1} \subset B_{1/(m+1)}(e)$.

Therefore by induction there exists a sequence of open neighborhoods U_m of e satisfying conditions *(a)*, *(b)*, *(c)*, and $U_m \subset B_{1/m}(e)$. The last inclusion guarantees that the sequence of sets U_m is a neighborhood basis at e and

$$\{e\} \subset \bigcap_{m=1}^{\infty} U_m \subset \bigcap_{m=1}^{\infty} B_{1/m}(e) = \{e\}.$$

It follows that *(d)* holds. □

The usual metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ on $\mathbb{R}^n$ has the additional invariant property that $d(\mathbf{u} + \mathbf{x}, \mathbf{u} + \mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. In fact, every metric group has such a metric. These invariant metrics will be very useful, but the proof of their existence is delicate.

Theorem 1.3.5 *Let* G *be a metric group with metric* d. *There exists a metric* ρ *on* G *that is equivalent to* d *and satisfies*

$$\rho(ax, ay) = \rho(x, y)$$

for all x, y, *and* a *in* G.

Proof. Let U_m be a countable neighborhood base at e given by *Proposition 1.3.4*. The proof will depend only on the properties of this sequence of sets. For technical reasons, set $U_0 = G$. Define a function $f : G \times G \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1/2^m & \text{if } x^{-1}y \in U_m \setminus U_{m+1} \\ 0 & \text{if } x^{-1}y \in U_m \text{ for all } m \geq 0. \end{cases}$$

The function f has the following properties:

(a) $0 \leq f(x,y) \leq 1$

(b) $f(x,y) = f(y,x)$ because $y^{-1}x = (x^{-1}y)^{-1}$ and $U_m^{-1} = U_m$

(c) $f(ax,ay) = f(x,y)$ because $(ax)^{-1}ay = x^{-1}y$

(d) $f(x,y) > 0$ if $x \neq y$ because $\bigcap_{m=1}^{\infty} U_m = \{e\}$

(e) $f(x,y) \leq 1/2^m$ implies that $x^{-1}y \in U_m$ because $U_{k+1} \subset U_k$ for all k.

For each positive integer n define

$$F_n : \underbrace{G \times \ldots \times G}_{n+1 \text{ copies}} \to \mathbb{R}$$

by

$$F_n(x_0, x_1, \ldots, x_n) = \sum_{j=1}^{n} f(x_{j-1}, x_j).$$

In particular, $F_1(x,y) = f(x,y)$.

Set

$$\rho(x,y) = \inf\{F_n(x_0, \ldots, x_n) : x_0 = x, x_n = y, n \geq 1\}$$

The function ρ has the following elementary properties:

(a) $0 \leq \rho(x,y) \leq f(x,y) \leq 1$

(b) $\rho(x,y) = \rho(y,x)$ because $f(x,y) = f(y,x)$ and consequently

$$F_n(x_0, x_1, \ldots, x_{n-1}, x_n) = F_n(x_n, x_{n-1}, \ldots, x_1, x_0) \tag{1.15}$$

(c) $\rho(ax,ay) = \rho(x,y)$ because

$$F_n(ax, x_1, \ldots, x_{n-1}, ay) = F_n(x, a^{-1}x_1, \ldots, a^{-1}x_{n-1}, y).$$

The next step is to show that ρ satisfies the triangle inequality. Given x, y, and z in G, it suffices to show that $\rho(x,z) + \rho(z,y) + 2\varepsilon \geq \rho(x,y)$ for every $\varepsilon > 0$. From the definition of ρ, there exist $x_1, \ldots, x_{n-1}$ such that

$$\rho(x,z) + \varepsilon \geq F_n(x, x_1 \ldots, x_{n-1}, z),$$

and there exist $y_1, \ldots, y_{m-1}$ such that

$$\rho(z,y) + \varepsilon \geq F_m(z, y_1 \ldots, y_{m-1}y).$$

Consequently,

$$\begin{aligned}
\rho(x,z) + \rho(z,y) + 2\varepsilon &\geq F_n(x, x_1, \ldots, x_{n-1}, z) + F_m(z, y_1, \ldots, y_{m-1}, y) \\
&= F_{n+m}(x, x_1 \ldots, x_{n-1}, z, y_1, , \ldots, y_{m-1}, y) \\
&\geq \rho(x,y).
\end{aligned}$$

To show that ρ is a metric, it remains only to show that $\rho(x,y) > 0$ when $x \neq y$. It has already been shown that ρ has the required invariance property $\rho(ax, ay) = \rho(x, y)$ for all x, y, and a in G. Completing the proof that ρ is a metric will not, however, complete the proof of the theorem. It must also be shown that d and ρ produce the same topology on G, that is, $\mathcal{T}(d) = \mathcal{T}(\rho)$. The following lemma will be used to complete these last two steps of the proof:

Lemma 1.3.6 *For all $x, x_1, \ldots, x_{n-1}, y$ in G, and $n \geq 1$*

$$f(x,y) \leq 2F_n(x, x_1, \ldots, x_{n-1}, y).$$

Proof. Without loss of generality we can assume that consecutive terms in the finite sequence $x, x_1, \ldots, x_{n-1}, y$ are distinct. (For example, if $x_1 = x_2$, then $F_n(x, x_1, \ldots, x_{n-1}, y) = F_{n-1}(x, x_2, \ldots, x_{n-1}, y)$, and it suffices to prove that $f(x,y) \leq 2F_{n-1}(x, x_2, \ldots, x_{n-1}, y)$.)

The $n = 1$ case is trivial. For $n = 2$, let m be the smallest positive integer such that $1/2^m \leq F_2(x, x_1, y) = f(x, x_1) + f(x_1, y)$. If $xx_1^{-1} \notin U_m$, then the definition of f implies that $1/2^{m-1} \leq f(x, x_1) \leq F_2(x, x_1, y)$, contradicting the choice of m. Thus $xx_1^{-1} \in U_m$ and likewise $x_1y^{-1} \in U_m$. It follows that $xy^{-1} \in U_mU_m \subset U_{m-1}$, and then $f(x,y) \leq 1/2^{m-1} = 2/2^m \leq 2F_1(x, x_1, y)$. Similarly, $f(x,y) \leq 2F_3(x, x_1, x_2, y)$, using $U_mU_mU_m \subset U_{m-1}$.

Proceeding by induction for $n \geq 4$, assume that the conclusion of the lemma holds for all integers from 1 to n. To show that it is also true for $n+1$, consider $F_{n+1}(x, x_1, \ldots, x_n, y)$. Using $f(x,y) = f(y,x)$ and equation (1.15), it suffices to prove either $f(x,y) \leq 2F_{n+1}(x, x_1, \ldots, x_n, y)$ or $f(y,x) \leq 2F_{n+1}(y, x_n, \ldots, x_1, x)$. We can assume without loss of generality that

$$f(x, x_1) = F_1(x, x_1) \leq \frac{1}{2}F_{n+1}(x, \ldots, y).$$

Now there exists k, $1 \leq k \leq n$, such that the following inequalities hold:

$$\begin{aligned} F_k(x, \ldots, x_k) &\leq \frac{1}{2}F_{n+1}(x, \ldots, y) \\ F_{k+1}(x, \ldots, x_{k+1}) &\geq \frac{1}{2}F_{n+1}(x, \ldots, y). \end{aligned}$$

We will assume that $1 \leq k < n$ and leave the case $k = n$ to the reader. It follows that

$$F_{n-k}(x_{k+1}, \ldots, y) \leq \frac{1}{2}F_{n+1}(x, \ldots, y).$$

The induction assumption now implies that

$$f(x, x_k) \leq 2F_k(x, \ldots, x_k) \leq F_{n+1}(x, \ldots, y)$$

and

$$f(x_{k+1}, y) \leq 2F_{n-k}(x_{k+1}, \ldots, y) \leq F_{n+1}(x, \ldots, y)$$

It is also clear that

$$f(x_k, x_{k+1}) \leq F_{n+1}(x, \ldots, y)$$

These last three inequalities are the crucial ingredients for the finale of the proof.

Choose the smallest positive integer m such that

$$\frac{1}{2^m} \leq F_{n+1}(x, \ldots, y).$$

If xx_k^{-1} is not in U_m, then the definition of f implies that $1/2^{m-1} \leq f(x, x_k)$ and the first of our three crucial inequalities implies that

$$\frac{1}{2^{m-1}} \leq f(x, x_k) \leq F_{n+1}(x, \ldots, x_n, y),$$

contradicting the choice of m. Therefore, xx_k^{-1} must lie in U_m.

The same reasoning shows that $x_{k+1}y^{-1} \in U_m$ and $x_k x_{k+1}^{-1} \in U_m$. It follows that

$$xy^{-1} = (xx_k^{-1})(x_k x_{k+1}^{-1})(x_{k+1}y^{-1}) \in U_m U_m U_m,$$

and then part *(b)* of *Proposition 1.3.4* implies that $xy^{-1} \in U_{m-1}$. (In the $k = n$ case, $x_n y^{-1}$ is in U_m and $U_m U_m \subset U_{m-1}$ suffices.) Therefore,

$$f(x, y) \leq \frac{1}{2^{m-1}} = \frac{2}{2^m} \leq 2F_{n+1}(x, \ldots, y),$$

and the proof of the lemma is complete. □

As a consequence of the lemma

$$\rho(x, y) = \inf\left\{F_n(x_0, \ldots, x_n) : x_0 = x, x_n = y, n \geq 1\right\} \geq \frac{1}{2} f(x, y)$$

or simply

$$\rho(x, y) \geq \frac{1}{2} f(x, y)$$

In particular, $\rho(x, y) > 0$ when $x \neq y$ because $f(x, y) > 0$ when $x \neq y$. Hence, ρ is a metric. In fact, it is a bounded metric because $\rho(x, y) \leq 1$. Furthermore, setting $B'_r(y) = \{x : \rho(x, y) < r\}$, it is easy to see that $B'_r(y) = yB'_r(e)$ because $\rho(yx, y) = \rho(x, e)$.

Finally it must be shown that d and ρ generate the same open sets in G, that is $\mathcal{T}(d) = \mathcal{T}(\rho)$. For each y in G the open sets yU_m are a countable neighborhood base at y for $\mathcal{T}(d)$. For $\mathcal{T}(\rho)$, the open sets $B'_{1/2^m}(y)$ are a countable neighborhood base at y. So it suffices (using the definition of a countable base and *Exercise* 8, p. 28 again.) to show that

$$B'_{1/2^m}(y) \subset yU_m \subset B'_{1/2^{m-1}}(y)$$

for all $m \geq 1$. Since $B'_r(y) = yB'_r(e)$ for all y and $r > 0$, the above is further reduced to showing that

$$B'_{1/2^m}(e) \subset U_m \subset B'_{1/2^{m-1}}(e).$$

If $x \in B'_{1/2^m}(e)$, then

$$\frac{1}{2^m} > \rho(x,e) \geq \frac{1}{2} f(x,e)$$

and $1/2^{m-1} > f(x,e)$. Since the positive values of $f(x,y)$ are always of the form $1/2^k$, it follows that $1/2^m \geq f(x,e) = f(e,x)$. Thus $x = e^{-1}x$ must be in U_m by property (e) of $f(x,y)$. Therefore, and $B'_{1/2^m}(e) \subset U_m$.

Next, consider $y \in U_m$. Then, $f(e,y) \leq 1/2^m < 1/2^{m-1}$ and $\rho(e,y) \leq f(e,y) < 1/2^{m-1}$, proving that $U_m \subset B'_{1/2^{m-1}}(e)$. □

A metric ρ for a topological group G such that $\rho(ax, ay) = \rho(x,y)$ for all x, y, and a in G is called a *left-invariant metric.* For a left invariant metric the functions $\psi_a : G \to G$ defined by $\psi_a(x) = ax$ for $a \in G$ are not just homeomorphisms of G onto G but they are also isometries because they preserve the distance between points and are onto (see *Exercise* 19, p. 29).

A left-invariant metric need not be right-invariant, but the proof of the existence of left-invariant metrics can obviously be modified to prove the existence of right-invariant metrics. *As needed, we can assume the metric on a metric group is left- or right-invariant without changing the topology on the group.*

In Section 1.1 (page 7), we defined an algebraic homomorphism from one group to another. For metric groups, we want to consider algebraic homomorphisms that are also continuous. Since topological groups are the central topic, the adjective "algebraic" will be dropped and the assumption of continuity added for topological groups.

Let G and G' be topological groups. A function $\varphi : G \to G'$ is a *homomorphism* of G to G' provided that φ is continuous on G and that

$$\varphi(xy) = \varphi(x)\varphi(x) \tag{1.16}$$

for all x and y in G. *It is important to stress that a homomorphism will always be a continuous function unless the adjective "algebraic" precedes it.*

A homeomorphism φ of a topological group G onto a topological group G' that satisfies (1.16) for all x and y in G is a called an *isomorphism* and G and G' are *isomorphic topological groups.* The inverse of an isomorphism between two topological groups is an isomorphism of topological groups because it has already been shown (page 8) that the inverse of an algebraic isomorphism is an algebraic isomorphism and the inverse of a homeomorphism is a homeomorphism. Finally, an *automorphism* is an isomorphism of a topological group onto itself.

Here are some familiar examples of homomorphisms and isomorphisms. The function $f(x) = \exp(x) = e^x$ is an isomorphism from $\mathbb{R}$ onto $\mathbb{R}^+$ with $f^{-1}(x) = \ln(x)$. Similarly, $\varphi(x) = \exp(2\pi i x) = \cos(2\pi x) + i\sin(2\pi x)$ is a homomorphism of $\mathbb{R}$ onto $\mathbb{K}$, but it is not one-to-one. The functions $f(x) = ax$, where a is a nonzero real number are all automorphisms of $\mathbb{R}$ (see *Exercises 6 and 7*, p. 39). The simple complex polynomial $f(z) = z^n$, n a positive integer, is a homomorphism of $\mathbb{C} \setminus \{0\}$ onto itself that is not an isomorphism for $n > 1$.

Let $G_1, \ldots, G_m$ be groups. There is a natural group structure on $G_1 \times \ldots \times G_m$ called the *direct product.* Specifically,

$$(x_1, \ldots, x_m)(y_1, \ldots, y_m) = (x_1 y_1, \ldots, x_m y_m),$$

$$e = (e_1, \ldots, e_m),$$

and

$$(x_1, \ldots, x_m)^{-1} = (x_1^{-1}, \ldots, x_m^{-1}).$$

Proposition 1.3.7 *If $G_1, \ldots, G_m$ are topological groups, then the direct product $G_1 \times \ldots \times G_m$ with the product topology is a topological group.*

Proof. It must be shown that the function

$$f\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) = (x_1 y_1^{-1}, \ldots, x_m y_m^{-1})$$

from $(G_1 \times \ldots \times G_m) \times (G_1 \times \ldots \times G_m)$ to $G_1 \times \ldots \times G_m$ is continuous. By *Proposition 1.2.17,* it suffices to show that the functions

$$p_j \circ f\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) = x_j y_j^{-1}$$

from $(G_1 \times \ldots \times G_m) \times (G_1 \times \ldots \times G_m)$ to G_j are continuous at an arbitrary point $\big((a_1, \ldots, a_m), (b_1, \ldots, b_m)\big)$. For convenience, we will assume that $j = 1$; the proof is the same for all j but the notation is easier with $j = 1$.

Let U be an open neighborhood of $a_1 b_1^{-1}$ in G_1. Because G_1 is a topological group, $\{(x_1, y_1) : x_1 y_1^{-1} \in U\}$ is an open neighborhood of (a_1, b_1) in $G_1 \times G_1$ and there exist open neighborhoods V_1 and V_2 of a_1 and b_1, respectively, such that $V_1 \times V_2 \subset \{(x_1, y_1) : x_1 y_1^{-1} \in U\}$ and hence $V_1 V_2^{-1} \subset U$. Clearly,

$$(V_1 \times G_2 \ldots \times G_m) \times (V_2 \times G_2 \times \ldots \times G_m) \subset (p_1 \circ f)^{-1}(U)$$

and

$$(V_1 \times G_2 \ldots \times G_m) \times (V_2 \times G_2 \times \ldots \times G_m)$$

is an open neighborhood of $\big((a_1, \ldots, a_m), (b_1, \ldots, b_m)\big)$ by *Theorem 1.2.11.* □

The list of topological groups has just grown significantly and now includes $\mathbb{K}^n$ for all positive integers and metric groups like $\mathbb{R}^m \times \mathbb{K}^n \times \mathbb{Z}_p$. There are also some new connections worth mentioning. The function $\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C} \setminus \{0\}$ defined by $\varphi(r, \theta) = re^{i\theta}$ is a homomorphism of the product metric group $\mathbb{R}^+ \times \mathbb{R}$ onto $\mathbb{C} \setminus \{0\}$, and there is a homomorphism of metric groups behind the polar form of a nonzero complex number. Similarly, The function $\psi(r, z) = e^r z$ defines an isomorphism from the product metric group $\mathbb{R} \times \mathbb{K}$ onto $\mathbb{C} \setminus \{0\}$, and $\mathbb{C} \setminus \{0\}$ can be thought of as a product group that looks like an infinite cylinder.

We close this section with a topological characterization of when a topological group is metrizable. It provides a different viewpoint of what a metric group is and is not.

Theorem 1.3.8 *A topological group G is metrizable if and only if G is first countable and the points of G are closed sets.*

Proof. If G is a metric group, then it is first countable because every metric space is first countable and points are closed sets by *Exercise* 11, p. 28, because metric spaces have the Hausdorff property.

Conversely, suppose G is first countable and the points of G are closed sets. Let W_m be a countable neighborhood base at e the identity of G. Without loss of generality we can assume that $W_{m+1} \subset W_m$ by replacing W_m with $\bigcap_{k=1}^{m} W_k$.

If $x \neq e$, then $G \setminus \{x\}$ is an open neighborhood of e because $\{x\}$ is a closed set by hypothesis. Hence, $W_m \subset G \setminus \{x\}$ for some m, and for every $x \neq e$ in G, there exists some W_m such that x is not in W_m. Therefore, $\bigcap_{m=1}^{\infty} W_m = \{e\}$.

The proof of *Proposition 1.3.4* can be applied here by replacing $B_{1/m}(e)$ with W_m to obtain a countable neighborhood base U_m at the identity element e with the following properties:

(a) U_m is open for all m,

(b) $U_m U_m U_m \subset U_{m-1}$ for all $m \geq 2$

(c) $U_m^{-1} = U_m$ for all m

(d) $\bigcap_{m=1}^{\infty} U_m = \{e\}$.

Now the proof of *Theorem 1.3.5* can be reused word for word to prove that there exists a left-invariant metric ρ such that $\mathcal{T}(\rho)$ equals the original topology on G. Thus G is metrizable by definition. □

Metric groups are a rich class of topological groups that cut a wide swath across the theory of topological groups. They are certainly ample for a solid introduction to topological groups and have the advantage of requiring a lot less mathematical machinery. Although we will frequently restrict our attention to metric groups, we do not want to loose sight of how they fit into the larger picture. To help provide the reader with a broader perspective than metric groups, a variety of basic ideas and introductory results will be presented in the general context of topological groups and used to discuss results beyond the scope of this book.

EXERCISES

1. Show that the groups $\mathbb{R}^+$ and $\mathbb{R} \setminus \{0\}$ are metric groups with the metric $d(x, y) = |x - y|$.

2. Prove that the groups $\mathbb{C} \setminus \{0\}$ and $\mathbb{K}$ are metric groups.

3. Show that if G is a finite metric group, then the topology on G is the discrete topology.

4. Let G be a group and a topological space such that the function $(x, y) \to xy$ is a continuous function from $G \times G$ with the product topology to G. Prove that if the inverse function $x \to x^{-1}$ from G to G is continuous at the identity e, then it is continuous on G and G is a topological group.

5. Let G and G' be topological groups. Show that if $\varphi : G \to G'$ is an algebraic homomorphism that is continuous at e, then φ is a homomorphism.

6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a homomorphism of $\mathbb{R}$ to itself. Prove the following:

   (a) $(1/q)\varphi(x) = \varphi(x/q)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Z} \setminus \{0\}$.

   (b) $\varphi(rx) = r\varphi(x)$ for all $x \in \mathbb{R}$ $r \in \mathbb{Q}$, the rational numbers.

   (c) $\varphi(sx) = s\varphi(x)$ for all s and x in $\mathbb{R}$.

   (d) There exists $a \in \mathbb{R}$ such that $\varphi(x) = ax$.

7. Show that the group of automorphisms of the metric group $\mathbb{R}$ is algebraically isomorphic to the group $\mathbb{R} \setminus \{0\}$.

8. Let G and G' be metric groups with left-invariant metrics d and d', respectively. If $\varphi : G \to G'$ is a homomorphism, then given $\varepsilon > 0$, there exists $\delta > 0$ such that $d'(\varphi(x), \varphi(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

9. Let G be a metric group. Given a neighborhood U of the identity of G and a positive integer n, show that there exists a neighborhood V of the identity such that $V^n \subset U$.

10. Let G be a topological group. Prove the following:

    (a) If V is a neighborhood of the identity and $x \in V^-$, then $xV \cap V \neq \phi$.

    (b) If U and V are neighborhoods of the identity such that $VV^{-1} \subset U$, then $V^- \subset U$.

    (c) G is a *regular* topological space, that is, given an open set U and $y \in U$, there exists an open set V such that $y \in V \subset V^- \subset U$.

11. Let G be a metric group. Prove that there exists a two-sided invariant metric if and only if there exist a neighborhood base of open sets at e such that $xU_m x^{-1} = U_m$ for all $x \in G$ and $m \geq 1$.

12. Let F be a closed subset of a topological group G. Prove that if x is in $G \setminus F$, then there exists an open neighborhood of e such that $xV \cap FV = \phi$.

1.4 Subgroups and Their Quotient Spaces

Let G be a topological group, and let H be a subgroup of G as defined in Section 1.1 on page 8. Then H is a group under the same binary operation that defines the group structure on G. If H has the relative topology from G, then the product topology on $H \times H$ is the same as the relative topology from the product topology on $G \times G$ because

$$(U \cap H) \times (V \cap H) = (U \times V) \cap (H \times H).$$

Hence, the function $(x, y) \to xy^{-1}$ on $H \times H$ with the product topology is a continuous function to H. *Therefore, with the relative topology a subgroup of a topological group is a topological group. It will be understood henceforth that a subgroup of a topological group is the topological group obtained by using the relative topology on the subgroup.*

Unless H is a closed subset of G, however, the topological group H can be a strange beast. Consequently, the focus will be on subgroups that are closed subsets of G or simply *closed subgroups.*

Let G be a topological group. When the topology on G has the Hausdorff property defined on page 19, G is called a Hausdorff topological group. They are the most common topological groups that one encounters in the literature. A set consisting of a single point is always a closed set in a Hausdorff topological group by *Exercise* 11, p. 28. When a subgroup H of G is also a discrete subset of G (page 20), H is said to be a *discrete subgroup* of G. For example, $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$. Recall that the relative topology of a discrete subset of a topological space is the discrete topology.

Proposition 1.4.1 *Let H be a subgroup of a Hausdorff topological group G. The following hold:*

(a) If there exists an open set in G such that $U \cap H = \{e\}$, then H is a discrete subgroup of G.

(b) If H is a discrete subgroup of G, then H is a closed subgroup of G.

Proof. Given an open set U such that $U \cap H = \{e\}$, let a be an element of H. Then Ua is an open set containing a because $x \to xa$ is a homeomorphism of G onto itself by *Proposition 1.3.2.* Suppose x is in $Ua \cap H$. Then $x = ua$ for some $u \in U$. Since x is in H, it follows that $xa^{-1} = u \in U \cap H$, and consequently $xa^{-1} = e$. Therefore, $x = a$, $Ua \cap H = \{a\}$, and H is discrete.

Let H be a discrete subgroup of G, and consider $x \in G \setminus H$. There exists an open set containing e such that $U \cap H = \{e\}$, and another open set V containing e such that $VV^{-1} \subset U$ because $(x, y) \to xy^{-1}$ is continuous. Suppose there are two points a and b in $Vx \cap H$. Then $a = v_1 x$ and $b = v_2 x$ with v_1 and v_2 in V. So $ab^{-1} = (v_1 x)(v_2 x)^{-1} = v_1 v_2^{-1} \in VV^{-1} \subset U$. Hence $ab^{-1} \in U \cap H = \{e\}$ and $a = b$. Thus either $Vx \cap H$ is empty or contains exactly one point. If $Vx \cap H = \{a\}$, then $a \neq x$ because $x \notin H$, and there exists an open set W containing x such that $a \notin W$ because G is Hausdorff. It follows that

$(Vx \cap W) \cap H = \phi$. Thus every x in $G \setminus H$ has an open neighborhood that is contained in $G \setminus H$. Therefore, and $G \setminus H$ is open, and H is closed. □

Proposition 1.4.2 *If H is a subgroup of a topological group G, then H^- is also a subgroup of G.*

Proof. Let a and b be arbitrary points in H^-. It suffices by *Exercise* 10, p. 12, to show that ab^{-1} is in H^-. Let U be an open neighborhood of ab^{-1}. There exist open neighborhoods V and W of a and b, respectively, such that $VW^{-1} \subset U$. Because a and b are in H^- there exist $x \in V \cap H$ and $y \in W \cap H$. It follows that xy^{-1} is in $U \cap H$. As U is an arbitrary neighborhood of ab^{-1}, this means that ab^{-1} is in H^-. □

Proposition 1.4.3 *If H is an Abelian subgroup of a Hausdorff topological group G, then H^- is also an Abelian subgroup of G.*

Proof. Suppose a and b are elements of H^- and $ab \neq ba$. Then there exist disjoint open neighborhoods U_1 and U_2 of ab and ba, respectively, because G is Hausdorff. Since multiplication is continuous, there exist open neighborhoods V_1, V_2 of a and W_1, W_2 of b such that $V_1W_1 \subset U_1$ and $W_2V_2 \subset U_2$. Set $V = V_1 \cap V_2$ and $W = W_1 \cap W_2$. Because a and b are in H^-, there exists $x \in V \cap H$ and $y \in W \cap H$. It follows that xy is in U_1 and yx is in U_2. Therefore, $xy \neq yx$ and H is not Abelian, a contradiction. □

Quotient spaces are constructed by decomposing a space into disjoint sets and then letting the sets of the decomposition be the points of the quotient space. To understand better how a subgroup naturally decomposes a group into disjoint sets, we return to the idea of the orbits from Section 1.1 (page 5).

Let X be a nonempty set and let L be a subgroup of the group S_X. Then there is a natural *L-orbit* or simply *orbit*, when unambiguous, for each a in X defined by $\mathcal{O}(a) = \{\psi(a) : \psi \in L\}$. This definition extends the idea of an orbit used in Section 1.1 by replacing the cyclic group generated by an element α of S_n with any subgroup L of S_X. Like the permutation case, orbits decompose a set X into disjoint pieces.

Proposition 1.4.4 *Let L be a subgroup of S_X and let a and b be points in X. Then either $\mathcal{O}(a) = \mathcal{O}(b)$ or $\mathcal{O}(a) \cap \mathcal{O}(b) = \phi$.*

Proof. It suffices to show that $\mathcal{O}(a) \cap \mathcal{O}(b) \neq \phi$ implies $\mathcal{O}(a) = \mathcal{O}(b)$. If $\mathcal{O}(a) \cap \mathcal{O}(b) \neq \phi$, then there exist ψ_1 and ψ_2 in L such that $\psi_1(a) = \psi_2(b)$ or equivalently $a = \psi_1^{-1} \circ \psi_2(b)$. For any ψ in L, it follows that $\psi \circ \psi_1^{-1} \circ \psi_2 \in L$ because L is a subgroup of S_X, and hence $\psi(a) = \psi \circ \psi_1^{-1} \circ \psi_2(b)$ is in $\mathcal{O}(b)$. Thus $\mathcal{O}(a) \subset \mathcal{O}(b)$. Similarly, $\mathcal{O}(b) \subset \mathcal{O}(a)$ and $\mathcal{O}(a) = \mathcal{O}(b)$. □

Now, let H be a subgroup of a group G, and as before, let $\psi_y : G \to G$ be defined by $\psi_y(x) = yx$. Each ψ_y is one-to-one, onto, and an element of S_G. The set $L = \{\psi_y : y \in H\}$ is a subgroup of S_G because $\psi_y \circ \psi_{y'} = \psi_{yy'}$ and

$\psi_y^{-1} = \psi_{y^{-1}}$. (In fact, $y \to \psi_y$ is algebraic isomorphism of H onto L. With $H = G$, this is the solution to *Exercise* 14, p. 12.) The L-orbit of a in G is

$$\begin{aligned} \mathcal{O}(a) &= \{\psi_y(a) : y \in H\} \\ &= \{ya : y \in H\} \\ &= Ha \end{aligned}$$

and is called a *left coset*. By the previous proposition, left cosets are either disjoint or equal. Each $a \in G$ is in the left coset Ha because e is in H. The subgroup itself is always a coset because $He = H$. Thus the left cosets decompose G into disjoint subsets, one of which is H itself. There is an important criterion for determining when two cosets are equal.

Proposition 1.4.5 *Let H be a subgroup of a group G, and let a and b be in G. Then $Ha = Hb$ if and only if ab^{-1} is in H.*

Proof. If $Ha = Hb$, then $a = yb$ for some $y \in H$ and $ab^{-1} = y \in H$. Conversely, $ab^{-1} = y \in H$ implies that $Ha = Hab^{-1}b = Hyb = Hb$ because $Hy = H$ for any $y \in H$. □

For topological groups, left cosets have additional topological properties.

Proposition 1.4.6 *Let G be a topological group.*

(a) If H is a closed subgroup, then every coset Ha is a closed set in G.

(b) If H is an open subgroup of G, then every coset Ha is an open set in G.

(c) If H is an open subgroup of G, then H is also a closed subgroup of G.

Proof. Since $x \to xa$ is a homeomorphism of G onto G, the left coset Ha is closed if and only if H is closed, and Ha is open if and only if H is open, proving parts *(a)* and *(b)*. (See the discussion following the definition of homeomorphism on page 26.)

If H is an open subgroup, then the cosets are all open sets and

$$\bigcup_{a \notin H} Ha$$

is an open subset of G. The fact that $Ha = H$ if and only if $a \in H$ implies that

$$H = G \setminus \bigcup_{a \notin H} Ha.$$

Thus H is the complement of an open set and closed. □

As an example consider the multiplicative group $\mathbb{R} \setminus \{0\}$. Clearly, $\mathbb{R}^+$ is a subgroup of $\mathbb{R} \setminus \{0\}$ and there are two cosets, $\mathbb{R}^+$ and $\mathbb{R}^- = \mathbb{R}^+(-1)$. Note that as in part *(c)*, $\mathbb{R}^+$ is both an open and a closed set in $\mathbb{R} \setminus \{0\}$. Open-and-closed subgroups will play an important role in Chapter 5.

Let G/H denote the collection of left cosets. So G/H is a set whose elements are the orbits of the points of G under the subgroup $L = \{\psi_y : y \in H\}$ of S_G. In other words,

$$G/H = \{E \subset G : E = Ha \text{ for some } a \in G\}.$$

There is a natural function π from G onto G/H defined by $\pi(a) = Ha$. We will restrict our study of quotient spaces to closed subgroups of metric groups. The choice of left cosets is arbitrary, and every result has an analogue for right cosets.

Theorem 1.4.7 *Let G be a metric group, and let d be a left-invariant metric for G. If H is a closed subgroup of G, then*

$$\rho(Ha, Hb) = \inf\{d(xa, yb) : x, y \in H\} \tag{1.17}$$

is a metric on G/H. Furthermore, a subset U of G/H is an open subset of G/H if and only if $\pi^{-1}(U)$ is an open subset of G.

Proof. Obviously, $\rho(Ha, Hb) = \rho(Hb, Ha)$, and $\rho(Ha, Hb) \geq 0$ for all Ha and Hb in G/H. Since d is a left-invariant metric, $d(xa, yb) = d(a, x^{-1}yb)$ and

$$\rho(Ha, Hb) = \inf\{d(a, wb) : w \in H\} = \inf\{d(wa, b) : w \in H\} \tag{1.18}$$

are alternative formulas for the function ρ.

Clearly, $Ha = Hb$ implies $\rho(Ha, Hb) = 0$. If $\rho(Ha, Hb) = 0$, then equation (1.18) implies that there exists a sequence x_k in H such that $d(a, x_k b)$ converges to zero, which is equivalent to saying that $x_k b$ converges to a. Since H is closed, its cosets are closed sets by *Proposition 1.4.6*. Hence a is in Hb and $Ha = Hb$.

To prove the triangle inequality, consider three cosets Ha, Hb, and Hc. Let $\varepsilon > 0$ be arbitrary. By using equation (1.18), there exist x and y in H such that

$$\begin{aligned} d(xa, c) &\leq \rho(Ha, Hc) + \varepsilon \\ d(c, yb) &\leq \rho(Hc, Hb) + \varepsilon. \end{aligned}$$

Equation (1.17) and these inequalities imply that

$$\begin{aligned} \rho(Ha, Hb) &\leq d(xa, yb) \\ &\leq d(xa, c) + d(c, yb) \\ &\leq \rho(Ha, Hc) + \rho(Hc, Hb) + 2\varepsilon. \end{aligned}$$

Since the inequality $\rho(Ha, Hb) \leq \rho(Ha, Hc) + \rho(Hc, Hb) + 2\varepsilon$ holds for all $\varepsilon > 0$, it must be true that $\rho(Ha, Hb) \leq \rho(Ha, Hc) + \rho(Hc, Hb)$. This completes the proof that ρ defines a metric on G/H.

Let $B'_r(Ha)$ denote the open ball around Ha in G/H determined by the metric ρ. In preparation for proving that U, a subset of G/H, is an open set in G/H if and only if $\pi^{-1}(U)$ is an open set in G, it will be shown that

$$\pi^{-1}\big(B'_r(Ha)\big) = HB_r(a). \tag{1.19}$$

Observe that x is in $\pi^{-1}(B'_r(Ha))$ if and only if $\rho(Hx, Ha) < r$ if and only if $d(x, ya) < r$ for some $y \in H$ if and only if $x \in B_r(ya) = yB_r(a)$ for some $y \in H$ if and only if $x \in HB_r(a)$. This proves the validity of equation (1.19).

Suppose that U is an open set in G/H and let a be in $\pi^{-1}(U)$. Then $B'_r(Ha) \subset U$ for some $r > 0$ and $B_r(a) \subset HB_r(a) = \pi^{-1}(B'_r(Ha)) \subset \pi^{-1}(U)$ by (1.19). So $\pi^{-1}(U)$ is open.

Suppose that $\pi^{-1}(U)$ is open and Ha is in U. Then a is in $\pi^{-1}(U)$ and $B_r(a) \subset \pi^{-1}(U)$ for some $r > 0$. Since $\pi^{-1}(U)$ is a union of cosets, $HB_r(a) \subset \pi^{-1}(U)$. Using (1.19) again, it follows that $B'_r(Ha) \subset U$ and U is open. □

Once again we are in the enviable position of having a formula for a metric and a characterization of the open sets that is independent of the metric. Given a left-invariant metric d for G and a closed subgroup H of G, the metric given by equation (1.17) or (1.18) on G/H is called a *quotient metric*. The topology determined by a quotient metric is a *quotient topology* on G/H and does not depend on the choice of the invariant metric d. With the quotient topology G/H is called a *quotient space*.

Corollary 1.4.8 *If H is a closed subgroup of the metric group G, then the function $\pi : G \to G/H$ from G to the quotient space G/H defined by $\pi(g) = Hg$ is continuous and open.*

Proof. The function π is obviously continuous because $\pi^{-1}(U)$ is an open set if and only if U is an open set in G by the theorem.

To prove that π is an open function, first observe that for any subset V of G

$$\begin{aligned} \pi^{-1}(\pi(V)) &= HV \\ &= \bigcup_{y \in H} yV \\ &= \bigcup_{y \in H} \psi_y(V). \end{aligned}$$

If V is open, then $\psi_h(V)$ is open because ψ_h is a homeomorphism of G onto itself, and then $\pi^{-1}(\pi(V))$ is open because it is a union of open sets. It follows from the theorem that $\pi(V)$ is open, when V is open □

Corollary 1.4.9 *Let H be a closed subgroup of a metric group G. If H' is a subgroup of G containing H, then H'/H is a subset of G/H and the relative topology on H'/H is the same as its quotient topology.*

Proof. Clearly H is a closed subgroup of H' and H'/H is a subset of G/H. Let d be a left-invariant metric for G, and let ρ be the metric on G/H given by equation (1.17). Let d' be the restriction of d to H'. Because d' is a left-invariant metric on H', it can be used to construct a quotient metric ρ' on H'/H by setting $\rho'(Ha, Hb) = \inf\{d'(xa, yb) : x, y \in H\}$ for a and b in H'. It is now obvious that ρ' is just the restriction of ρ to H'/H. Hence the relative topology on H'/H is the same as its quotient topology by *Exercise* 16, p. 28. □

Corollary 1.4.10 *Let H be a closed subgroup of a metric group. The quotient topology on G/H is the discrete topology if and only if H is an open subgroup of G.*

Proof. If the quotient topology is discrete, then the points of G/H are open sets, and, in particular, $\{H\}$ is an open set in G/H. It follows from the theorem that $\pi^{-1}(\{H\}) = H$ is an open set in G.

Conversely, if H is an open set in G, then every coset Ha is an open subset of G, and each point of G/H is an open set by *Corollary 1.4.8.* So the quotient topology is discrete. □

The next proposition provides a criterion for the convergence of a sequence in G/H in terms of the convergence of sequences in the metric group G. It will be used in the proofs of several results in the remainder of this section.

Proposition 1.4.11 *Let H be a closed subgroup of a metric group G. A sequence Hx_k in G/H converges in the quotient topology to Hx if and only if there exists a sequence y_k in H such that the sequence $y_k x_k$ converges to x in the group G.*

Proof. Let d be a left-invariant metric on G, and let $\rho(Ha, Hb)$ be the metric on G/H defined by equation (1.18). Suppose Hx_k converges to Hx in G/H, so the sequence of numbers $\rho(Hx_k, Hx)$ converges to 0. It follows from equation (1.18) that for each k there exists y_k in H such that $d(y_k x_k, x) < \rho(Hx_k, Hx) + 1/k$. Hence $d(y_k x_k, x)$ converges to 0, and $y_k x_k$ converges to x in G.

Conversely, if there exists a sequence y_k in H such that $y_k x_k$ converges to x in G, then $Hx_k = Hy_k x_k = \pi(y_k x_k)$ converges to $\pi(x) = Hx$ in G/H because π is continuous. □

For a group G, defining a group structure on G/H by $(Ha, Hb) \mapsto HaHb$ does not work in general because $HaHb$ need not equal any Hc. An additional hypothesis is required to produce a natural group structure on G/H. A subgroup H of G is a *normal subgroup* of G provided $aHa^{-1} = H$ for all $a \in G$ or equivalently $aH = Ha$ for all $a \in G$.

Recall that the function $\varphi_a(x) = axa^{-1}$ is an algebraic automorphism of a group G. These automorphisms are called the *inner automorphisms.* Alternatively, a subgroup H of G is normal if and only if every inner automorphism maps H onto itself. In particular, the subgroups G and $\{e\}$ of a group G are normal subgroups, and every subgroup of an Abelian group is normal.

Theorem 1.4.12 *If H is a closed normal subgroup of a metric group G, then G/H with the quotient topology and the binary operation $(Ha, Hb) \mapsto HaHb = Hab$ is a metric group. Furthermore, $\pi(a) = Ha$ is an open homomorphism of G onto G/H.*

Proof. Routine calculations show that $(Ha)(Hb) = H(aHa^{-1})ab = HHab = Hab$ is an associative binary operation on G/H, the coset H is an identity element, and $(Ha)^{-1} = Ha^{-1}$, giving G/H a group structure. Clearly, $\pi(a)\pi(b) = (Ha)(Hb) = Hab = \pi(ab)$ and π is an onto algebraic homomorphism.

We already know that when H is closed, G/H is a metric space and π is an open continuous function. To complete the proof, it remains to show that $(Ha, Hb) \to Hab^{-1}$ is a continuous function from $G/H \times G/H$ to G/H .

Let d be a left-invariant metric on G, and let ρ be the metric on G/H given by (1.17). By *Proposition 1.2.3*, it suffices to prove that the sequence $Ha_k b_k^{-1}$ converges to Hab^{-1}, whenever the sequence (Ha_k, Hb_k) converges to (Ha, Hb) in $G/H \times G/H$. It follows from *Exercise* 14, p. 28, that (Ha_k, Hb_k) converges to (Ha, Hb) if and only if Ha_k and Hb_k converge to Ha and Hb, respectively, in G/H.

By *Proposition 1.4.11*, there exist sequences x_k and y_k in H such that $x_k a_k$ and $y_k b_k$ converge to a and b respectively in G. Because G is a metric group, $x_k a_k (y_k b_k)^{-1}$ converges to ab^{-1}. Note that $x_k a_k (y_k b_k)^{-1} = x_k a_k b_k^{-1} y_k^{-1} = x_k w_k a_k b_k^{-1}$ for some w_k in H because $a_k b_k^{-1} H = H a_k b_k^{-1}$ for all k by normality. Using *Proposition 1.4.11* again, it follows that $Ha_k b_k^{-1}$ converges to Hab^{-1} because $x_k w_k$ is in H for all k and $x_k w_k a_k b_k^{-1}$ converges to ab^{-1}. □

When G is a metric group and H is a closed normal subgroup of G, the open homomorphism $\pi : G \to G/H$ defined by $\pi(x) = Hx$ will be called the *canonical homomorphism.*

Proposition 1.4.13 *Let G and G' be metric groups. If $\varphi : G \to G'$ is homomorphism of G to G', then the kernel $K = \{g \in G : \varphi(g) = e'\} = \varphi^{-1}(e')$ is a closed normal subgroup of G.*

Proof. By *Exercise* 12, p. 12, the kernel $K = \varphi^{-1}(e')$ is a subgroup, and it is closed by *Exercise* 3, p. 50, because φ is continuous and $\{e'\}$ is a closed subset of G'.

If x is in K, then

$$\begin{aligned} \varphi(axa^{-1}) &= \varphi(a)\varphi(x)\varphi(a^{-1}) \\ &= \varphi(a)e'\varphi(a)^{-1} \\ &= e'. \end{aligned}$$

Hence $aKa^{-1} \subset K$ for all a in G, and K is normal by *Exercise* 5, p. 50. □

Theorem 1.4.14 (First Isomorphism Theorem) *Let G and G' be metric groups. If $\varphi : G \to G'$ is homomorphism of G onto G' with kernel K, then $\tilde{\varphi}(Ka) = \varphi(a)$ defines a one-to-one homomorphism of G/K onto G' satisfying $\tilde{\varphi} \circ \pi = \varphi$. If φ an open function, then $\tilde{\varphi}$ is an isomorphism of G/K onto G'.*

Proof. If $Ka = Kb$, then ab^{-1} is in K and $\varphi(ab^{-1}) = e'$ or $\varphi(a) = \varphi(b)$. Thus setting $\tilde{\varphi}(Ka) = \varphi(a)$ unambiguously defines a function from G/K to G' which is onto because φ is onto. Clearly, $\tilde{\varphi} \circ \pi = \varphi$. Moreover,

$$\begin{aligned} \tilde{\varphi}(KaKb) &= \tilde{\varphi}(Kab) \\ &= \varphi(ab) \\ &= \varphi(a)\varphi(b) \\ &= \tilde{\varphi}(Ka)\tilde{\varphi}(Kb) \end{aligned}$$

and φ is an algebraic homomorphism.

To show that $\tilde{\varphi}$ is continuous, it suffices to show that $\pi(\varphi^{-1}(U)) = \tilde{\varphi}^{-1}(U)$ because φ is continuous and π is open. An element of $\pi(\varphi^{-1}(U))$ has the form Ka with $a \in \varphi^{-1}(U)$. Hence, $\tilde{\varphi}(Ka) = \varphi(a) \in U$ and $\pi(\varphi^{-1}(U)) \subset \tilde{\varphi}^{-1}(U)$. If Ka is in $\tilde{\varphi}^{-1}(U)$, then $Ka = \pi(a)$ and $\varphi(a) = \tilde{\varphi}(Ka) \in U$, proving that $\tilde{\varphi}^{-1}(U) \subset \pi(\varphi^{-1}(U))$. Thus $\pi(\varphi^{-1}(U)) = \tilde{\varphi}^{-1}(U)$, and φ is continuous.

If $\tilde{\varphi}(Ka) = \varphi(a) = e'$, then a is in K and $Ka = K$, the identity element of G/K. So the kernel of $\tilde{\varphi}$ is the trivial subgroup of G/K, and $\tilde{\varphi}$ is one-to-one.

Although it has now been shown that $\tilde{\varphi}$ is an algebraic isomorphism of G/K onto G', it does not follow that it is an isomorphism of metric groups just because $\tilde{\varphi}$ is continuous. The difficulty is that $\tilde{\varphi}^{-1}$ need not be continuous (see *Exercise* 12, p. 51).

The function $\tilde{\varphi}^{-1}$ is continuous if and only if $\tilde{\varphi}$ is an open function because

$$\left(\tilde{\varphi}^{-1}\right)^{-1}(U) = \tilde{\varphi}(U).$$

If the function φ is an open function and U is an open set in G/K, then $\pi^{-1}(U)$ is an open subset of G, and $\varphi(\pi^{-1}(U))$ is an open subset of G'. Since

$$\varphi\big(\pi^{-1}(U)\big) = \tilde{\varphi} \circ \pi\big(\pi^{-1}(U)\big) = \tilde{\varphi}(U),$$

the homomorphism $\tilde{\varphi}$ is an open function and an isomorphism of the metric group G/K onto G'. $\square$

Theorem 1.4.15 (Second Isomorphism Theorem) *Let G and G' be metric groups and let φ be an open homomorphism of G onto G' with kernel K. If H' is a closed normal subgroup of G' and $H = \varphi^{-1}(H')$, then H is a closed normal subgroup of G containing K, and H/K is a closed normal subgroup of G/K. Furthermore, the three metric groups G/H, G'/H' and $(G/K)/(H/K)$ are all isomorphic.*

Proof. It follows from *Exercises 7 and 8*, p. 50, that H is a closed normal subgroup of G. Since K is a normal subgroup of G, it is obviously a normal subgroup of any subgroup of G that contains K. Consequently, G'/H', G/H, G/K, and H/K are all well-defined metric groups.

Clearly, H/K is a subgroup of G/K. If $a \in G$, then $KaKxKa^{-1} = Kaxa^{-1}$ lies in H/K for all $x \in H$ because H normal. It follows that H/K is a normal subgroup of G/K. From *Corollary 1.4.9*, we know that the relative topology on H/K is the same as its quotient topology. (See *Exercises 9 and 10*, p. 51, for a full description of subgroups of quotient groups.)

To show that H/K is a closed set in G/K, it suffices to show that $(H/K)^{-} \subset H/K$. Consider Ka in $(H/K)^{-}$. Let U be any open neighborhood of a in G. Then $\pi(U)$ is an open neighborhood of Ka, and there exists x in H such that Kx is in $\pi(U) \cap H/K$ because Ka is in $(H/K)^{-}$. There exists y in U such that $Ky = \pi(y) = Kx$. It follows that y is in H because $K \subset H$ and $x \in H$. Thus every open neighborhood of a contains point y in H, proving that a is in $H^{-} = H$. Therefore, Ha is in H/K.

Let $\pi' : G' \to G'/H'$ be the canonical open homomorphism. By *Theorem 1.4.14*, there exists an isomorphism $\tilde{\varphi}$ of G/K onto G'. Consider the composition $\pi' \circ \tilde{\varphi} : G/K \to G'/H'$. It is an open homomorphism because both $\tilde{\varphi}$ and π' are open homomorphisms. Note that $\pi' \circ \tilde{\varphi}(Ka) = H'\varphi(a) = H'$ if and only if $\varphi(a)$ is in H' if and only if a is in H. Therefore, the kernel of $\pi' \circ \tilde{\varphi}$ is H/K and $(G/K)/(H/K)$ is isomorphic to G'/H' by *Theorem 1.4.14*.

Lastly, consider the composition $\pi' \circ \varphi : G \to G'/H'$, which is an open homomorphism because both φ and π' are open homomorphisms. Its kernel is clearly H. Therefore, G/H and G'/H' are isomorphic by *Theorem 1.4.14*. □

Corollary 1.4.16 *Let H and K be closed normal subgroups of the metric group G. If $K \subset H$, then the metric groups $(G/K)/(H/K)$ and G/H are isomorphic.*

Proof. Set $G' = G/K$ and $H' = H/K$. Then as in the proof of the theorem, H' is a closed subgroup of G'. Now the theorem can be applied with $\varphi = \pi$, the canonical homomorphism of G onto G', and $G'/H' = (G/K)/(H/K)$. □

If H is a subgroup of a Hausdorff topological group G, then *Theorem 1.4.7* provides the definition of a *quotient topology* on G/H by letting $\mathcal{T} = \{U \subset G/H : \pi^{-1}(U) \text{ is open}\}$ and applying *Exercise* 3, p. 27, to show that $\mathcal{T}$ is a topology. This is a Hausdorff topology when H is closed, and G/H is a topological group when H is normal by *Exercises 18, 19, and 20* beginning on page 193, which can be worked without further prerequisites. Then the proofs of the First and Second Isomorphism Theorems for Hausdorff topological groups are essentially the same as the proofs just presented for metric groups.

To illustrate the use of the First and Second Isomorphism Theorems (*Theorems 1.4.14 and 1.4.15*), we will determine all the closed subgroups of $\mathbb{R}$, their quotients, and the quotients of the quotients. The first proof requires the *greatest integer function* denoted by $[x]$ for $x \in \mathbb{R}$ and defined by $[x]$ equals the largest integer less than or equal to x. Note that $0 \le x - [x] < 1$, and that x is an integer if and only if $x - [x] = 0$.

Theorem 1.4.17 *If H is a closed subgroup of $\mathbb{R}$, then exactly one of the following holds:*

(a) $H = \mathbb{R}$

(b) $H = \{0\}$

(c) there exists a positive number a such that $H = \{ma : m \in \mathbb{Z}\}$, making H a discrete subgroup of $\mathbb{R}$ isomorphic to $\mathbb{Z}$.

Proof. The first step is to show that either $H = \mathbb{R}$ or H is discrete. If H is not discrete, then by *Proposition 1.4.1* for every positive integer k there exists $x_k \in H$ with $0 < |x_k| < 1/k$. Observe that every interval of length $1/k$ contains an element of H of the form mx_k with $m \in \mathbb{Z}$. Given any real number x, for each positive integer k choose m_k such that $|m_k x_k - x| < 1/k$. Clearly, $m_k x_k$

converges to x, and x is in the closed subgroup H. Since x was an arbitrary real number, $H = \mathbb{R}$.

If H is discrete, then either $H = \{0\}$ or H contains a nonzero real number. Suppose H contains a nonzero real number, and hence its inverse. Thus H contains a positive number. Set $a = \inf\{x \in H : x > 0\}$. Then *Exercise* 13, p. 28, implies that a must lie in H because H is closed. Moreover, a is positive because H is discrete. Let x be an arbitrary element of H, and set $m = [x/a]$. Then $x - ma$ is in H. It follows from $0 \leq x/a - [x/a] < 1$ that $0 \leq x - ma < a$, contradicting the choice of a unless $0 = x - ma$. Therefore, $x = ma$ and $H = \{ma : m \in \mathbb{Z}\}$. Obviously, $m \mapsto ma$ defines an isomorphism from $\mathbb{Z}$ onto H to complete the proof. □

Corollary 1.4.18 *If H is a proper closed subgroup of $\mathbb{R}$, then $\mathbb{R}/H$ is isomorphic to $\mathbb{K}$.*

Proof. The function $\varphi(x) = \exp(2\pi i x/a)$ is a homomorphism of $\mathbb{R}$ onto $\mathbb{K}$ and its kernel is precisely $H = \{ma : m \in \mathbb{Z}\}$. Since it maps open intervals of $\mathbb{R}$ to open circular arcs, it is an open function. (In Section 1.5, we will prove a more general result, *Theorem 1.5.18*, that also implies that φ is open.) An application of the First Isomorphism Theorem completes the proof. □

The next corollary is usually proved by purely algebraic means, but we will take advantage of the theorem.

Corollary 1.4.19 *If H is a subgroup of $\mathbb{Z}$, then exactly one of the following holds:*

(a) $H = \mathbb{Z}$

(b) $H = \{0\}$

(c) there exists a positive integer $a > 1$ such that $H = \{ma : m \in \mathbb{Z}\}$.

Proof. Every subgroup of $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$, and hence a closed subgroup of $\mathbb{R}$ by *Proposition 1.4.1.* □

Theorem 1.4.20 *If H' is a closed subgroup of $\mathbb{K}$ other than $\mathbb{K}$, then*

(a) there exists a positive integer n such that $H' = \{z \in \mathbb{K} : z^n = 1\}$

(b) H' is a finite cyclic group

(c) H' is a discrete subgroup of $\mathbb{K}$

(d) $\mathbb{K}/H'$ is isomorphic to $\mathbb{K}$.

Proof. Consider the homomorphism $\varphi(x) = \exp(2\pi i x)$ of $\mathbb{R}$ onto $\mathbb{K}$. Then, as noted in the proof of *Corollary 1.4.18*, φ is an open homomorphism with kernel $\mathbb{Z}$. Set $H = \varphi^{-1}(H')$. It follows that H cannot be $\{0\}$ because H contains $\mathbb{Z}$, and H cannot be $\mathbb{R}$ because $H' \neq \mathbb{K}$. Thus $H = \{ma : m \in \mathbb{Z}\}$ for some $a > 0$.

Since $\mathbb{Z} \subset H$, there must exist a positive integer n such that $1 = na$ or $a = 1/n$. Therefore,

$$\begin{aligned} H' &= \{\exp(2\pi ima) : m \in \mathbb{Z}\} \\ &= \{\exp(2\pi im/n) : m \in \mathbb{Z}\} \\ &= \{\exp(2\pi i/n)^m : m \in \mathbb{Z}\} \\ &= \{\exp(2\pi i/n)^m : 1 \leq m \leq n\} \end{aligned}$$

and completes the proof of parts *(a)* and *(b)*. Since a finite subgroup of a metric group is always a discrete subgroup, *(c)* holds.

For the last part, The Second Isomorphism Theorem implies that $\mathbb{R}/H$ and $\mathbb{K}/H'$ are isomorphic. It follows form *Corollary 1.4.18* that $\mathbb{R}/H$ is isomorphic to $\mathbb{K}$. □

EXERCISES

1. Prove that a finite subgroup of a Hausdorff topological groups is a discrete subgroup.
2. Let H be a subgroup of a metric group G with identity e. Show that if H is a neighborhood of the identity, then H is an open-and-closed subgroup of G.
3. Let X and Y be topological spaces. Prove that $f : X \to Y$ is a continuous function if and only if $f^{-1}(E)$ is a closed set in X when E is a closed set in Y.
4. Show that the quotient space for right cosets is homeomorphic to the quotient space for left cosets. (One approach is to use the function $gH \to (gH)^{-1} = Hg^{-1}$.)
5. Let H be a subgroup of a group G. Prove that H is a normal subgroup of G if and only if $aHa^{-1} \subset H$ for all a in G.
6. Prove that if H is a normal subgroup of a topological group G, then H^- is a normal subgroup of G.
7. Let G and G' be groups, and let $\varphi : G \to G'$ be an algebraic homomorphism. Prove the following:
   (a) If H' is a subgroup of G', then $\varphi^{-1}(H')$ is a subgroup of G containing the kernel of φ.
   (b) If H' is a normal subgroup of G', then $\varphi^{-1}(H')$ is a normal subgroup of G.
   (c) If H is a subgroup of G, then $\varphi(H)$ is a subgroup of G'.
   (d) If H is a normal subgroup of G and φ is onto, then $\varphi(H)$ is a normal subgroup of G'.

8. Let G and G' be topological groups, and let $\varphi : G \to G'$ be a homomorphism. Prove the following:

   (a) If H' is a closed subgroup of G', then $\varphi^{-1}(H')$ is a closed subgroup of G.

   (b) If φ is an open onto homomorphism and $\varphi^{-1}(H')$ is a closed subgroup of G, then H' is a closed subgroup of G'.

9. Let K be a normal subgroup of the group G, and let H' be a subset of G/K. Show that H' is a subgroup of G/K if and only if there exists a subgroup H of G such that $K \subset H$ and $H' = H/K$. In addition, show that H' is a normal subgroup of G/K if and only if there exists a normal subgroup H of G such that $K \subset H$ and $H' = H/K$.

10. Let G be a metric group, and let K be a closed normal subgroup of G. Show that H' is a closed subgroup of G/K if and only if there exists a closed subgroup H of G containing K such that $H' = H/K$ and that H' is normal if and only if H is normal.

11. Let H be a closed normal subgroup of a metric group G with left-invariant metric d. Show that the metric ρ defined by equation (1.17) is a left-invariant metric on G/H, and that if d is a left-invariant and right-invariant metric for G, then ρ is a left- and right-invariant metric.

12. Show that $\mathbb{K}$ is the continuous image of $\mathbb{R}_d$, the real numbers with the discrete topology, with kernel $\mathbb{Z}$, but the metric groups $\mathbb{K}$ and $\mathbb{R}_d/\mathbb{Z}$ are not isomorphic.

13. Let α be a real number and let H be the cyclic subgroup of the circle generated by $a = \exp(2\pi i\alpha)$. Show that $H^{-} = \mathbb{K}$ if and only if α is irrational.

14. Let $\varphi : G \to G'$ be a homomorphism of the metric group G onto the metric group G'. Prove that φ is open if and only if $\varphi(V)$ is a neighborhood of e' whenever V is a neighborhood of e.

15. Let G and G' be metric groups, let $\varphi : G \to G'$ be a homomorphism, and let H be a closed normal subgroup of G. Prove that there exists a homomorphism $\tilde{\varphi} : G/H \to G'$ such that $\varphi = \tilde{\varphi} \circ \pi$ if and only if H is contained in the kernel K of φ.

1.5 Compactness and Metric Groups

A particularly important class of subsets of topological spaces are the compact subsets. Although the definition may not seem to be an intuitively natural one, it turns out to be a useful property in the analytical study of topological spaces.

A subset C of a topological space X is a *compact set* provided that whenever $\mathcal{S}$ is a collection of open sets such that

$$C \subset \bigcup_{U \in \mathcal{S}} U, \tag{1.20}$$

there exists a finite collection $U_1, \ldots, U_k$ of sets in $\mathcal{S}$ such that

$$C \subset \bigcup_{i=1}^{k} U_i. \tag{1.21}$$

Obviously, any finite subset of a topological space is compact. When X itself is compact, X is called a *compact topological space.*

A collection of open sets $\mathcal{S}$ of X that satisfies condition (1.20) is called an *open cover of* C, and $U_1, \ldots, U_k$ such that (1.21) holds is called a *finite subcover.* An alternative way of stating the definition of a compact set is that C is compact provided that every open cover of C contains a finite subcover of C.

Proposition 1.5.1 *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous function. If C is a compact subset of X, then $f(C)$ is a compact subset of Y.*

Proof. Let $\mathcal{S}$ be an open covering of $f(C)$. Then $\mathcal{S}' = \{f^{-1}(U) : U \in \mathcal{S}\}$ is an open covering of C because f is continuous. Hence, there exists a finite subcover $f^{-1}(U_1), \ldots, f^{-1}(U_k)$ of C. It follows that $U_1, \ldots, U_k$ is a finite subcover of $f(C)$. □

Corollary 1.5.2 *Let $f : X \to Y$ be a homeomorphism of the topological space X onto the topological space Y and let C be a subset of X. Then C is a compact subset of X if and only if $f(C)$ is a compact subset of Y.*

Proposition 1.5.3 *A closed set contained in a compact set of a topological space is compact.*

Proof. Let F be a closed set contained in the compact set C, and let $\mathcal{S}$ be an open cover of F. Note that $X \setminus F$ is an open set and

$$C \subset \left(\bigcup_{U \in \mathcal{S}} U \right) \cup (X \setminus F).$$

So $\mathcal{S}' = \{U : U \in \mathcal{S} \text{ or } U = X \setminus F\}$ is an open cover for C. Because C is compact, there exist a finite subcover $U_1, \ldots, U_k$ of C from $\mathcal{S}'$. Without loss of generality we can assume that $U_k = X \setminus F$ and U_j is in $\mathcal{S}$ for $j = 1, \ldots, k-1$. It follows that

$$F \subset \bigcup_{i=1}^{k-1} U_i,$$

and F is compact. □

Proposition 1.5.4 *A compact set in a Hausdorff topological space is a closed set.*

Proof. Let C be a compact subset of a Hausdorff topological space. It suffices to show that $X \setminus C$ is open. Let y be in $X \setminus C$. For each $x \in C$, there exist disjoint open neighborhoods U_x and V_y of x and y, respectively, because X is Hausdorff. Obviously, $\{U_x : x \in X\}$ is an open cover of C. Since C is compact, there exist a finite set of points $x_1, \ldots, x_m$ in C such that $C \subset \bigcup_{k=1}^m U_{x_k} = U$. Then $V = \bigcap_{k=1}^m V_{x_k}$ is an open neighborhood of y such that $V \cap U = \phi$. Since $C \subset U$, it follows that $V \cap C = \phi$ and $y \in V \subset X \setminus C$. Thus $X \setminus C$ is open.

Corollary 1.5.5 *A compact subset of a metric space is closed.*

Proof. Metric spaces are Hausdorff.

A subset E of $\mathbb{R}$ is *bounded* provided that it is both bounded above and below or equivalently there exists $R > 0$ such that E is contained in the closed interval $[-R, R] = \{x : -R \leq x \leq R\}$.

Theorem 1.5.6 (Heine-Borel) *A subset of* $\mathbb{R}$ *is compact if and only if it is both closed and bounded.*

Proof. First, suppose that C is a compact subset of $\mathbb{R}$. Then C is closed by *Corollary 1.5.5*. If C is not bounded, then the open intervals $(-n, n)$, $n = 1, 2, \ldots$ are an open cover of C that does not have a finite subcover.

To prove that a closed and bounded subset of $\mathbb{R}$ is compact it suffices by *Proposition 1.5.3* to show that the closed interval $[-R, R]$ is compact for positive R.

Let $\mathcal{S}$ be an open cover of $[-R, R]$. Clearly $-R$ is in U for some $U \in \mathcal{S}$, and because U is open, there exists $r > 0$ such that $(-R - r, -R + r) \subset U$. So there exists a (very simple) finite subcover of $[-R, -R + r/2]$. The idea of the proof is to push the property that there is finite subcover for $[-R, x]$ for as large an x as possible using the same ideas.

Set

$$A = \left\{ x > -R : [-R, x] \subset \bigcup_{i=1}^{k} U_i \textit{ for some } U_1, \ldots, U_k \in \mathcal{S} \right\}.$$

From the previous paragraph, we know A is not the empty set. Moreover, if $-R < y < x$ and x is in A, then y is in A.

It suffices to prove that R is in A. Suppose R is not in A. Then, A must be contained in the open interval $(-R, R)$. Hence, $-R < c = \sup A \leq R$, and c is in U' for some U' in $\mathcal{S}$. Because U' is open, there exist $\varepsilon > 0$ such that the open interval $(c - \varepsilon, c + \varepsilon)$ is contained in U'. By the construction of c, there exists $a \in A$ such that $c - \varepsilon < a \leq c$, and there exist $U_1, \ldots, U_k$ such that $[-R, a] \subset \bigcup_{j=1}^k U_j$. It follows that

$$[-R, c + \varepsilon/2] \subset [-R, a] \cup (c - \varepsilon, c + \varepsilon) \subset \bigcup_{j=1}^{k} U_j \cup U'$$

and $c + \varepsilon/2$ is in A, which is impossible because c is an upper bound for A. Therefore R must be in A and $[-R, R]$ is compact. ☐

Corollary 1.5.7 *The metric group* $\mathbb{K}$ *is compact.*

Proof. The function $f(t) = e^{2\pi it} = \cos(2\pi t) + i\sin(2\pi t)$ maps $\mathbb{R}$ continuously onto $\mathbb{K}$ such that $f([0,1]) = \mathbb{K}$. Since $[0,1]$ is compact by the theorem, $\mathbb{K}$ is compact by *Proposition 1.5.1.* ☐

Proposition 1.5.8 *Let X be a topological space, and let $f : X \to \mathbb{R}$ be a continuous function. If C is a compact subset of X, then $f(C)$ is a closed and bounded subset of $\mathbb{R}$ and there exist x_m and x_M in C such that*

$$f(x_m) \leq f(x) \leq f(x_M)$$

for all x in C.

Proof. By *Proposition 1.5.1*, $f(C)$ is a compact subset of $\mathbb{R}$, and hence closed and bounded by *Theorem 1.5.6.* It follows that $a = \inf\{t \in \mathbb{R} : t \in f(C)\}$ and $b = \sup\{t \in \mathbb{R} : t \in f(C)\}$ are finite. Because $f(C)$ is closed, a and b are in $f(C)$ by *Exercise* 13, p. 28. So there exist x_m and x_M such that $f(x_m) = a$ and $f(x_m) = b$. ☐

The remainder of this section examines the basic properties of compact subsets of metric spaces with an emphasis on compact and locally compact metric groups.

Proposition 1.5.9 *If x_k is a sequence in a compact subset C of a metric space X, then x_k has a convergent subsequence.*

Proof. If for some integer k, the set $\{j : x_j = x_k\}$ is infinite, then x_k has a constant subsequence that obviously converges. Thus it can be assumed without loss of generality that for all k the set $\{j : x_j = x_k\}$ is finite. It follows that the set $E = \{y \in C : y = x_j \text{ for some } j\}$ is infinite.

Suppose x_k does not have a convergent subsequence. Then for each y in C there exists an integer $n(y)$ depending on y such that $\{k : x_k \in B_{1/n(y)}(y)\}$ is the empty set when $y \neq x_k$ for all k and a nonempty finite set when $y = x_k$ for some k. Clearly,

$$\mathcal{S} = \{B_{1/n(y)}(y) : y \in C\}$$

is an open cover of C. Since C is compact, there exist $y_1, \ldots, y_m$ such that

$$C \subset \bigcup_{i=1}^{m} B_{1/n(y_i)}(y_i).$$

Because E is infinite, it follows that for some y_i the set $\{k : x_k \in B_{1/n(y_i)}(y_i)\}$ is infinite, a contradiction. Therefore, x_k has a convergent subsequence. ☐

Corollary 1.5.10 *Every bounded sequence of real numbers contains a convergent subsequence.*

Proof. Apply the proposition to a closed and bounded interval containing the sequence. □

The product of a collection of compact sets is compact in a very general setting. The next result, *Theorem 1.5.11*, is a finite version of this major theorem for metric spaces that will suffice for the next three chapters. We will return to this subject in Chapter 5.

Theorem 1.5.11 *Let $X_1, \ldots, X_m$ be metric spaces. If $C_1, \ldots, C_m$ are compact subsets of $X_1, \ldots, X_m$, respectively, then $C_1 \times \ldots \times C_m$ is a compact subset of $X_1 \times \ldots \times X_m$ with the product topology.*

Proof. The proof proceeds by induction with the theorem being trivial when $m = 1$. Assume the theorem is true for $m-1$ metric spaces with $m \geq 2$ and consider compact subsets $C_1, \ldots, C_m$ of the metric spaces $X_1, \ldots, X_m$, respectively. Given x in X_1, the function $(x_2, x_3, \ldots, x_m) \to (x, x_2, x_3, \ldots, x_m)$ is a continuous function from $X_2 \times \ldots \times X_m$ to $X_1 \times \ldots \times X_m$. By induction, $C_2 \times \ldots \times C_m$ is compact and the set $C_x = \{x\} \times C_2 \times \ldots \times C_m$ is compact set because it is the continuous image of a compact set.

Let $\mathcal{S}$ be an open covering of $C_1 \times \ldots \times C_m$. For each x in C_1, the collection of open sets $\{U \in \mathcal{S} : x \in p_1(U)\}$ from $\mathcal{S}$ is an open cover of the compact set C_x. Therefore, for each x in C_1, there exists a finite collection of open sets, $\mathcal{S}_x$ from $\mathcal{S}$ such that $x \in p_1(U)$ for each $U \in \mathcal{S}_x$ and

$$C_x \subset \bigcup_{U \in \mathcal{S}_x} U.$$

Although this shows that $\mathcal{S}$ contains a finite subcover $\mathcal{S}_x$ of C_x for each x in C_1, there are possibly infinitely many points in C_1. To complete the proof, the compactness of C_1 must be used to prove that there exists a finite number of points $x_1, \ldots, x_k$ in C_1 such that together the collections of open sets $\mathcal{S}_{x_1}, \ldots, \mathcal{S}_{x_k}$ cover $C_1 \times \ldots \times C_m$.

The next step is to show that for $x \in C_1$ there exists an open ball $B_{1/k}(x)$ in X_1 such that

$$B_{1/k}(x) \times C_2 \times \ldots \times C_m \subset \bigcup_{U \in \mathcal{S}_x} U.$$

Suppose no such $B_{1/k}(x)$ exists. Then for each positive integer k there exists $x_k \in B_{1/k}(x) \subset X_1$ and $\mathbf{y}_k \in C_2 \times \ldots \times C_m$ such that

$$(x_k, \mathbf{y}_k) \notin \bigcup_{U \in \mathcal{S}_x} U.$$

(The boldface notation $\mathbf{y}$ is being used for points in $X_2 \times \ldots \times X_m$ to remind us that they are points in a product space with coordinates.) By applying

Proposition 1.5.9, there exists a subsequence $\mathbf{y}_{k_i}$ converging to $\mathbf{y}$ in $C_2 \times \ldots \times C_m$. It follows that $(x_{k_i}, \mathbf{y}_{k_i})$ converges to $(x, \mathbf{y})$ in C_x. Thus $(x, \mathbf{y})$ must be in some U in $\mathcal{S}_x$. Consequently, $(x_{k_i}, \mathbf{y}_{k_i})$ is in the same U for large i, contradicting the construction of $(x_k, \mathbf{y}_k)$. Therefore, the required $B_{1/k}(x)$ exists. Let $k(x)$ be the smallest positive integer k such that $B_{1/k}(x) \times C_2 \times \ldots \times C_m \subset \bigcup_{U \in \mathcal{S}_x} U$, and set $V_x = B_{1/k(x)}(x)$.

Clearly, $\{V_x : x \in C_1\}$ is an open cover of C_1, and there exist $y_1, \ldots, y_k$ in X_1 such that

$$C_1 \subset \bigcup_{j=1}^{k} V_{y_j}.$$

To complete the proof it suffices to show that

$$\mathcal{S}' = \bigcup_{j=1}^{k} \mathcal{S}_{y_j}$$

is a finite subcover of $C_1 \times \ldots \times C_m$. By construction, $\mathcal{S}'$ is a finite collection of open sets in the original cover $\mathcal{S}$.

Let $(w_1, \ldots, w_m)$ be an arbitrary point in $C_1 \times \ldots \times C_m$. Then w_1 is in some V_{y_j}. It follows that

$$(w_1, w_2, \ldots, w_m) \in V_{y_j} \times C_2 \times \ldots \times C_m \subset \bigcup_{U \in \mathcal{S}_{y_j}} U \subset \bigcup_{U \in \mathcal{S}'} U$$

and the proof is complete. □

Corollary 1.5.12 *The metric groups $\mathbb{K}^n$ are compact.*

A subset of $\mathbb{R}^n$ is bounded provided all the coordinates of all the points in it is a bounded set of real numbers. In other words, a subset E of $\mathbb{R}^n$ is *bounded* provided there exists $R > 0$ such that

$$E \subset [-R, R]^n.$$

Theorem 1.5.13 *A subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Proof. Suppose E is a closed and bounded subset of $\mathbb{R}^n$. Then $E \subset [-R, R]^n$ for some $R > 0$. By *Theorems 1.5.6 and 1.5.11* the set $[-R, R]^n$ is a compact subset of $\mathbb{R}^n$. Hence, E is a closed set contained in a compact set and is compact by *Proposition 1.5.3.*

Suppose E is a compact subset of $\mathbb{R}^n$. Then it is closed by *Proposition 1.5.4.* Because the projections $p_j : \mathbb{R}^m \to \mathbb{R}$ are continuous, the sets $p_j(E)$ are compact by *Proposition 1.5.1.* The Heine-Borel Theorem now implies that $p_j(E)$ is bounded for $j = 1, \ldots, m$, and hence E is bounded. □

Small compact neighborhoods can be as important as large compact sets. When every point of a topological space has a compact neighborhood, the space is said to be *locally compact*. Every compact topological space is, of course, locally compact, and $\mathbb{R}^n$ is an example of a locally compact space that is not compact. Discrete metric spaces are locally compact because points are both compact and open sets. Not all topological spaces, however, are locally compact (see *Exercise* 14, p. 61). Local compactness is a critical property in the study of Hausdorff topological groups.

If C is a compact neighborhood of x in a metric space X, then there exists $r > 0$ such that $B_\varepsilon(x) \subset C$ for $0 < \varepsilon \leq r$. Because C is closed by *Corollary 1.5.5*, it follows that $B_\varepsilon(x)^- \subset C$ and $B_\varepsilon(x)^-$ is compact by *Proposition 1.5.3*. Since $B_\varepsilon(x)^- \subset \{y : d(x,y) \leq \varepsilon\}$ by *Exercise* 7, p. 28, the sequence of sets $B_{r/k}(x)^-$ is a countable neighborhood base of compact sets at x. The point of this discussion can be summarized as follows:

Proposition 1.5.14 *A metric space X is locally compact if and only if for every $x \in X$ there exists a countable neighborhood base of compact sets.*

Proposition 1.5.15 *Let G be a metric group. Then G is locally compact if and only if there is a compact neighborhood of the identity.*

Proof. Let C be a compact neighborhood of the identity e of G. Since the map $\psi_a(x) = ax$ is a homeomorphism of G onto itself mapping e to a, the set $\psi_a(C)$ is a compact neighborhood of a, proving that G is locally compact. The other direction is trivial. □

A subset E of a metric space X is *dense* in X, provided $E^- = X$. Equivalently, E is dense if $E \cap B_r(x) \neq \phi$ for all $x \in X$ and $r > 0$. Both the rational numbers and the irrational numbers are dense subsets of $\mathbb{R}$. When α is an irrational number, $\{e^{2\pi i k\alpha} : k \in \mathbb{Z}\}$ is a dense subgroup of $\mathbb{K}$ (*Exercise 13*, p. 51).

Theorem 1.5.16 (Baire) *Let X be a locally compact metric space. If D_k is a sequence of open dense subsets in X, then*

$$\bigcap_{k=1}^{\infty} D_k$$

is a dense subset of X.

Proof. Set $D = \bigcap_{k=1}^{\infty} D_k$ and let x be an arbitrary point in X. It suffices to show that $B_{r'}(x) \cap D \neq \phi$ for all $r' > 0$. Because X is a locally compact metric space, for every $r' > 0$ there exists $r > 0$ such that $B_r(x)^- \subset B_{r'}(x)$ and $B_r(x)^-$ is compact. This further reduces the proof to showing that $B_r(x)^- \cap D \neq \phi$ when $B_r(x)^-$ is compact and $r > 0$.

Since D_1 is both open and dense, there exists $r_1 > 0$ and x_1 such that $B_{r_1}(x_1)^- \subset B_r(x) \cap D_1$. In fact, we can define, r_k and x_k inductively so that

$$B_{r_k}(x_k)^- \subset B_{r_{k-1}}(x_{k-1}) \cap \bigcap_{j=1}^{k} D_j \subset B_r(x)$$

because $\bigcap_{j=1}^{k} D_j$ is also both open and dense (*Exercise* 8, p. 61). It follows that

$$B_{r_k}(x_k)^- \subset B_{r_{k-1}}(x_{k-1})^- \subset B_r(x)^-$$

for $k > 1$ and

$$\bigcap_{k=1}^{\infty} B_{r_k}(x_k)^- \subset \bigcap_{k=1}^{\infty} D_k = D.$$

Thus,

$$\bigcap_{k=1}^{\infty} B_{r_k}(x_k)^- \subset B_r(x)^- \cap D,$$

and it suffices to show that

$$\bigcap_{k=1}^{\infty} B_{r_k}(x_k)^- \neq \phi.$$

Suppose it is empty. Then by de Morgan's formulas (*Exercise* 6, p. 27),

$$\bigcup_{k=1}^{\infty} \left(X \setminus B_{r_k}(x_k)^-\right) = X$$

and the open sets

$$U_k = X \setminus B_{r_k}(x_k)^-$$

form an open cover of X and of $B_r(x)^-$ in particular. So there exists a finite subcover of $B_r(x)^-$ because it is compact. Observe that $B_{r_k}(x_k)^- \subset B_{r_{k-1}}(x_{k-1})^-$ implies that $U_{k-1} \subset U_k$ for all $k > 1$. Thus any finite subcover reduces to $B_r(x)^- \subset U_k$ for some k. It follows that

$$X \setminus B_r(x)^- \supset X \setminus U_k = B_{r_k}(x_k)^-,$$

which contradicts the property that

$$B_{r_k}(x_k)^- \subset B_r(x)^-$$

holds for all $k \geq 1$. Therefore, D is a dense subset of X. □

Theorem 1.5.16 is usually referred to as the Baire Category Theorem. (For a purely topological version of this theorem see *Exercise* 20, p. 62.) Because of the complementary relationship between open and closed sets, The Baire Category Theorem has an equivalent version stated in terms of closed sets The complement of an open dense subset is a closed set that is not the neighborhood of any point in the metric space. Such a set is called a *closed nowhere dense set.* The Baire Category Theorem can also be stated using closed nowhere dense sets instead of open dense sets.

Corollary 1.5.17 *Let X be a locally compact metric space. If F_k is a sequence of closed sets such that*

$$\bigcup_{k=1}^{\infty} F_k = X,$$

then some F_k contains a nonempty open set. In particular, a locally compact metric space cannot be the union of a sequence of closed nowhere dense subsets.

Proof. Suppose none of the sets F_k contains an open set. Set $D_k = X \setminus F_k$, which is open for all k. Since no open set U is contained in F_k, the set $U \cap D_k \neq \phi$ and D_k is dense.

The theorem implies that $\bigcap_{k=1}^{\infty} D_k$ is dense, and hence not empty. By de Morgan's formulas

$$\phi = X \setminus \bigcup_{k=1}^{\infty} F_k = \bigcap_{k=1}^{\infty} (X \setminus F_k) = \bigcap_{k=1}^{\infty} D_k \neq \phi.$$

This contradiction completes the proof. □

There is also a version of the Baire Category Theorem based on a purely metric hypothesis instead of a topological one. A sequence x_k in a metric space X with metric d is a *Cauchy sequence*, provided that given $\varepsilon > 0$ there exists a positive integer K such that $d(x_n, x_m) < \varepsilon$ when both $m \geq K$ and $n \geq K$. A metric space X with metric d is said to be *complete* if every Cauchy sequence in X converges to a point in X. The hypothesis that X is a "locally compact metric space" can be replaced by "complete metric space" and the theorem is still valid. This result (*Exercise* 19, p. 62) and several other basic facts about complete metric spaces are contained in the exercises.

A metric space X is *σ-compact or sigma-compact* provided that there exists a sequence of compact sets C_k in X such that

$$X = \bigcup_{k=1}^{\infty} C_k.$$

An example of a σ-compact metric space is $\mathbb{R}^n$. If H is a closed subgroup of a σ-compact metric group G, then clearly H and G/H are σ-compact.

Theorem 1.5.18 (Open Homomorphism Criterion) *Let G and G' be locally compact metric groups, and let $\varphi : G \to G'$ be an onto homomorphism. If G is σ-compact, then φ is an open homomorphism.*

Proof. It suffices to show that φ maps neighborhoods of e to neighborhoods of e' by *Exercise* 14, p. 51. Let U be a neighborhood of e. Because G is locally compact, there exists a compact neighborhood V of e such that $VV^{-1} \subset U$ and an open set W such that $e \in W \subset V$.

The open sets Wx, $x \in G$, are an open cover of G. Since G is σ-compact, there exists a sequence x_k such that $G = \bigcup_{k=1}^{\infty} Wx_k$ by *Exercise* 9, p. 61. It follows that $G = \bigcup_{k=1}^{\infty} Vx_k$ and

$$G' = \bigcup_{k=1}^{\infty} \varphi(V)\varphi(x_k).$$

Because V is compact, $\varphi(V)$ is compact and hence closed by *Corollary 1.5.5.* Now *Proposition 1.3.2* implies that the sets $\varphi(V)\varphi(x_k)$ are closed for all k. By *Corollary 1.5.17*, there exists k such that $\varphi(V)\varphi(x_k)$ contains an nonempty open set, and then $\varphi(V)$ must contain an open set V'.

Let y be a point in V' and pick x in V such that $\varphi(x) = y$. Then $V'y^{-1}$ is an open set in G' such that

$$e' \in V'y^{-1} \subset \varphi(V)\varphi(x^{-1}) = \varphi(Vx^{-1}) \subset \varphi(VV^{-1}) \subset \varphi(U).$$

Thus φ maps neighborhoods of e to neighborhoods of e'. □

Corollary 1.5.19 *Let G and G' be locally compact metric groups and let $\varphi : G \to G'$ be an onto homomorphism. If G is σ-compact, then G' is isomorphic to G/K where K is the kernel of φ.*

Proof. Since φ is open by the theorem, *Theorem 1.4.14* applies. □

Theorem 1.5.18 and its corollary provide another proof that $x \to e^{2\pi i x}$ is an open homomorphism of $\mathbb{R}$ onto $\mathbb{K}$, and the metric groups $\mathbb{R}/\mathbb{Z}$ and $\mathbb{K}$ are isomorphic. Actually, we can say more. Suppose G' is a locally compact group and $\varphi : \mathbb{R} \to G'$ is an onto homomorphism with kernel K. *Corollary 1.5.19* implies that $\mathbb{R}/K$ is isomorphic to G'. If K equals $\mathbb{R}$ or $\{0\}$, then G' is isomorphic to the trivial group or to $\mathbb{R}$. What happens when $K = \{ka : k \in \mathbb{Z}\}$ for some $a > 0$? Note that the function $x \to e^{2\pi i x/a}$ is a homomorphism of $\mathbb{R}$ onto $\mathbb{K}$ with kernel K. It follows that $\mathbb{K}$ is isomorphic to $\mathbb{R}/K$ and hence to G'. Thus we have determined all the locally compact metric groups that are homomorphic images of $\mathbb{R}$.

Since the Baire Category Theorem holds for locally compact regular topological spaces (*Exercise* 20, p. 62) and every topological group is regular (*Exercise* 10, p. 39), the Baire Category Theorem can be applied to sequences of open dense sets or sequences of closed nowhere dense sets in a locally compact topological group. For example, the proof of the Open Homomorphism Criterion can be used to prove a more general version of this theorem, with one caveat. The one step in the proof of *Theorem 1.5.18* that uses the metric hypothesis is the inference that $\varphi(V)$ is closed because it is compact. If the "locally compact metric groups" hypothesis is replaced with the hypothesis that the groups are locally compact Hausdorff topological groups, then *Proposition 1.5.4* can be applied to show that $\varphi(V)$ is closed because it is compact. Consequently, the Open Homomorphism Criterion holds with the words "metric groups" replaced by the words "Hausdorff topological groups".

EXERCISES

1. Let X be a compact metric space with metric d. Show that there exists a constant M such that $d(x, y) \leq M$ for all x and y in X.

2. Let X and Y be Hausdorff topological spaces and let $f : X \to Y$ is a continuous one-to-one onto function. Prove that if X is compact, then f^{-1} is continuous and f is a homeomorphism. (One approach is to use *Exercise* 3, p. 50.)

3. Let G be a metric group. Prove that if C is a compact subset of G and F is a closed subset of G such that $C \cap F = \phi$, then there exists an open neighborhood V of the identity such that $(CV) \cap (FV) = \phi$. (One approach is to use *Exercise* 12, p. 39.)

4. Let C_1 and C_2 be disjoint compact subsets of a metric space X with metric d. Prove that $0 < \inf \{d(x, y) : x \in C_1 \text{ and } y \in C_2\}$.

5. Prove that with the relative topology, open and closed subsets of locally compact metric spaces are locally compact metric spaces.

6. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous onto open function. Prove that Y is locally compact, if X is locally compact.

7. Let $X_1, \ldots, X_n$ be metric spaces. Prove that $X_1, \ldots, X_n$ are locally compact if and only if $X_1 \times \ldots \times X_n$ is locally compact.

8. Prove that in a metric space the intersection of a finite number of open dense sets is open and dense.

9. Let $\mathcal{S}$ be an open cover of a topological space X. Show that if X is σ-compact, then there exists a sequence of open sets U_k in $\mathcal{S}$ such that $X = \bigcup_{k=1}^{\infty} U_k$.

10. Show that a closed subset of a σ-compact metric space is σ-compact.

11. Show that for a σ-compact space there exists a sequence of compact subsets C_k such that $C_k \subset C_{k+1}$ for all $k \geq 1$ and $X = \bigcup_{k=1}^{\infty} C_k$.

12. Prove that a finite product of σ-compact spaces is σ-compact.

13. Let U be an open subset of a σ-compact metric space X. Show that U with the relative topology is also a σ-compact metric space. (One approach is to use *Exercise* 12, p. 28, with $F = X \setminus U$.)

14. Show that $\mathbb{Q}$, the rational numbers, with the relative topology from $\mathbb{R}$, is not locally compact.

15. Use *Theorem 1.5.16* to prove that there does not exist a one-to one function from $\mathbb{Z}^+ = \{k \in \mathbb{Z} : k \geq 1\}$ onto $\mathbb{R}$.

16. Let $\mathbb{R}_d$ denote the reals with the discrete topology. Show that $\mathbb{R}_d$ is not σ-compact. (One approach is to use *Exercise 15* above.)

17. Let X be a metric space with metric d. Prove the following:

    (a) Every convergent sequence in X is a Cauchy sequence.

    (b) A Cauchy sequence converges if and only if it has a convergent subsequence.

    (c) If X is compact, then it is a complete metric space.

18. Prove that $\mathbb{R}^n$ is a complete metric space for the metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

19. Prove the Baire category theorem for complete metric spaces. (One approach is to modify the proof in the text by choosing $r_k \leq 1/2^k$ and showing that x_k is Cauchy.)

20. Prove the Baire Category Theorem for regular (see *Exercise* 10, p. 39 for the definition of regular) locally compact topological spaces by showing that the sets $B_{r_k}(x_k)$ in the proof of *Theorem 1.5.16* can be replaced with open sets V_k such that V_k^- is compact.