

1 Generalized Quantifiers in Natural Language Semantics*

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1. Introduction

Generalized quantifiers have been standard tools in natural language semantics since at least the mid-1980s. It is worth briefly recalling how this came about. The starting point was Richard Montague's compositional approach to meaning (Montague, 1974). Frege and Russell had shown how to translate sentences with quantified subjects or objects in first-order logic, but the translation was not compositional. Indeed, Russell made a point of this, concluding that the subject-predicate form of, say, English was *misleading*, since there are no subjects in the logical form. No constituents of the translations

- (1) a. $\exists x(\text{professor}(x) \wedge \text{smoke}(x))$
 b. $\exists x(\forall y(\text{king-of-}F(y) \leftrightarrow y = x) \wedge \text{bald}(x))$

correspond to the subjects “some professors” or “the king of France” in

- (2) a. Some professors smoke
 b. The king of France is bald

respectively. Montague in effect laid this sort of reasoning to rest. He showed that there are compositional translations into *simple type theory*,

- (3) a. $((\lambda X \lambda Y \exists x(X(x) \wedge Y(x)))(\text{professor}))(\text{smoke})$
 b. $((\lambda X \lambda Y \exists x(\forall y(X(y) \leftrightarrow y = x) \wedge Y(x)))(\text{king-of-}F))(\text{bald})$

that, moreover, β -reduce precisely to (1a) and (1b). (Montague used an intensional type theory; only the extensional part is relevant here.) The constituent $(\lambda X \lambda Y \exists x(X(x) \wedge Y(x)))(\text{professor})$ of (3a), of type $\langle\langle e, t \rangle, t\rangle$, directly translates the DP “some professors,” and similarly $(\lambda X \lambda Y \exists x(\forall y(X(y) \leftrightarrow y = x) \wedge Y(x)))(\text{king-of-}F)$ translates “the king of France.” Moreover, these English DPs have the form [Det N'], and their determiners are translated by $\lambda X \lambda Y \exists x(X(x) \wedge Y(x))$ and $\lambda X \lambda Y \exists x(\forall y(X(y) \leftrightarrow y = x) \wedge Y(x))$, of type $\langle\langle e, t \rangle, \langle\langle e, t \rangle, t\rangle\rangle$. Both types of formal expressions denote generalized quantifiers.

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Generalized quantifiers had been introduced in logic, for purposes completely unrelated to natural language semantics, by Mostowski (1957) and, in full generality, Lindström (1966). Montague did not appeal to generalized quantifiers, but around 1980 semanticists began to realize that objects of type $\langle\langle e, t \rangle, t\rangle$ and $\langle\langle e, t \rangle, \langle\langle e, t \rangle, t\rangle\rangle$ could interpret arbitrary DPs and Dets, and that logical GQ theory had something to offer; the seminal papers were Barwise and Cooper (1981); Higginbotham and May (1981); Keenan and Stavi (1986). In particular, many common Dets, such as “*most, more than half, an even number of,*” are not definable in first-order logic (FO), in contrast with Montague’s “*some, every, the.*” But generalized quantifiers are first-order in another sense: they all quantify over individuals. In effect, these authors focused attention on objects of level at most 2 in the type hierarchy. Even when higher types are ignored, a surprising number of linguistic phenomena turn out to be amenable to this setting.

A further step towards classical model theory was taken in van Benthem (1984). Quantifiers of the above-mentioned types are (on each universe) functions from (characteristic functions of) sets to truth values (for DPs), or functions from sets to such functions (for Dets). Van Benthem showed that it was fruitful to construe them as *relations* (unary or binary) *between sets*, and he developed powerful tools for the model-theoretic study of Det denotations. The relational approach ignores the compositional structure that had been the motive to introduce generalized quantifiers into semantics in the first place. But on the other hand it exhibits many of their properties more conspicuously, and makes the applicability of methods from model theory more direct. Besides, for most purposes the functional and the relational approach to generalized quantifiers are essentially notational variants.

In this chapter I will present some highlights of the use of generalized quantifiers in semantics, from the beginning up to the present day. Although many things cannot be covered here, my hope is that the reader will get an impression of the power of these model-theoretic tools in the study of real languages. There are several surveys available where more details concerning particular applications can be found; I will point to them when called for. The reader should *not* leave with the impression, however, that all linguistically interesting issues concerning DPs or determiners (or corresponding means of quantification) can be treated with these tools. Generalized quantifiers are *extensional* objects, and there are subtleties about the meaning of DPs and determiners that they are insensitive to; I will note a few as we go along.¹ This indicates that the tools of GQ theory need to be complemented with other devices, not that they must in the end be abandoned. Indeed, my aim in this chapter is to show that there is a level of semantic analysis for which these tools are just right.

2. Definitions

Quantifiers (from now on I will usually drop “generalized”) have a syntactic and a semantic aspect. Syntactically, one constructs a formal language where quantifier symbols are variable-binding operators, like $\forall$ and $\exists$. Unlike $\forall$ and $\exists$, these operators may need to bind the same variable in distinct formulas. For example, a Det interpretation Q concerns two formulas φ and ψ , corresponding to the N' and the VP in a sentence $[[\text{Det } N'] \text{ VP}]$, and the operator binds the same variable in each. The resulting formula can be written

$$(4) \quad Qx(\varphi, \psi)$$

as in standard first-order logic with generalized quantifiers, or

$$(5) \quad Q(\hat{x}[\varphi])(\hat{x}[\psi])$$

as in Barwise and Cooper (1981), or

$$(6) \quad [Qx: \varphi]\psi$$

as in Higginbotham and May (1981).² The latter two reflect the constituent structure $[[\text{Det } N'] \text{ VP}]$, whereas (4)—the notation I will use here—fits the relational view of quantifiers. Once a logical language L for quantifiers is fixed, a formal semantics for a corresponding fragment of English can be given via compositional rules *translating* (analyzed) English phrases into L .

However, for this translation to have anything to do with meaning, we need a *semantics* for L . Following a main tradition, this will be a model-theoretic semantics, that is, a specification of a notion of model and a “truth definition”; more accurately, a *satisfaction* relation holding between models, certain L -expressions, and suitable assignments to the variables of corresponding objects in the model. But because our quantifiers are first-order (in the sense explained above), models are just ordinary first-order models, variables range over individuals in universes of such models, and we can help ourselves to the familiar format of the inductive truth definition in first-order logic, with an extra clause for each quantifier besides $\forall$ and $\exists$. To formulate these clauses, we need a precise notion of quantifiers as model-theoretic (not syntactic) objects.

Here it is important to note that quantifiers are *global*: on each non empty set M , a quantifier Q is a relation Q_M between relations over M (i.e. a second-order relation on M), but Q itself is what assigns Q_M to M , that is, it is a function from non empty sets to second-order relations on those sets. (This means that Q is not itself a set but a proper class, a fact without practical consequences in the present context.) The type of Q specifies the number of arguments and the arity of each argument; we use Lindström’s simple typing: $\langle n_1, \dots, n_k \rangle$, where k and each n_i is a positive natural number, stands for a k -ary second-order relation where the i :th argument has arity n_i . So the quantifier in (4) has type $\langle 1, 1 \rangle$ and DP denotations have type $\langle 1 \rangle$; in general, quantifiers of type $\langle 1, \dots, 1 \rangle$ (relations between *sets*) are called *monadic*, and the others *polyadic*.

Why is it important that quantifiers are global? A reasonable answer is that the meaning of “every” or “at least four” is independent not only of the nature of the objects quantified over but also the size of the universe (of discourse). “At least four” has the *same* meaning in “at least four cars,” “at least four thoughts,” and “at least four real numbers.” These properties are not built into the most general notion of a quantifier. The “topic neutrality” of, for example, “at least four” is a familiar model-theoretic property, shared by many (but not all) Det interpretations, but something more is at stake here. A quantifier that meant *at least four* on universes of size less than 100, and *at most ten* on all larger universes would still be “topic-neutral,” but it would not mean “the same” on every universe, and presumably no natural language determiner behaves in this way.

We will discuss these properties presently. For now the point is just that the meaning of determiners is such that the universe of discourse is a parameter, not something fixed. This is what makes quantifiers in the model-theoretic sense eminently suitable to interpret them. Indeed, Lindström (1966) defined a quantifier of type τ as a class of models of that type. This is a notational variant of the relational version: for example, for $\tau = \langle 1, 1 \rangle$, writing $(M, A, B) \in Q$ or $Q_M(A, B)$ makes no real difference. But the relational perspective brings out issues that otherwise would be less easily visible, so this is the format we use.

In full generality, then, a (global) *quantifier of type* $\langle n_1, \dots, n_k \rangle$ is a function Q assigning to each non-empty set M a second-order relation Q_M (if you wish, a local quantifier) on M of that type. Corresponding to Q is a variable-binding operator, also written Q ,³ and $FO(Q)$ is the *logic* obtained from first-order logic FO by adding formulas of the form

$$(7) \quad Qx_{i1} \dots x_{i n_1}; \dots; x_{k1} \dots x_{k n_k} (\psi_1, \dots, \psi_k)$$

whenever $\psi_1, \dots, \psi_k$ are formulas. Here all free occurrences of $x_{i1} \dots x_{i n_i}$ (taken to be distinct) are bound in ψ_i by Q . Let $\bar{x}_i$ abbreviate $x_{i1} \dots x_{i n_i}$ and let $\bar{y} = y_1 \dots y_m$ be the remaining free variables

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in any of $\psi_1, \dots, \psi_k$. Then the clause corresponding to Q in the truth (satisfaction) definition for $FO(Q)$ is

$$\mathcal{M} \models Q\bar{x}_1; \dots; \bar{x}_k(\psi_1, \dots, \psi_k)[\bar{b}] \Leftrightarrow Q_M(R_1, \dots, R_k)$$

where $\mathcal{M}$ is a model with universe M , $\bar{b} = b_1, \dots, b_m$ is an assignment to $\bar{y}$, and R_i is the set of n_i -tuples $\bar{a}_i = a_{i1}, \dots, a_{in_i}$ such that $\mathcal{M} \models \psi_i[\bar{a}_i, \bar{b}]$. As noted, for monadic Q we can simplify and just use one variable:

$$Qx(\psi_1, \dots, \psi_k)$$

Then, relative to x , and an assignment to the other free variables (if any) in $\psi_1, \dots, \psi_k$, each ψ_i defines a subset of M .

We will mostly deal with the quantifiers themselves rather than the logical languages obtained by adding them to FO . The logical language is, however, useful for displaying scope ambiguities in sentences with nested DPs. And it is indispensable for proving negative expressibility results: To show that Q is *not* definable from certain other quantifiers, you need a precise language for these quantifiers, telling you exactly which the possible defining sentences are.

As noted, a main role for GQ theory in semantics will be played by a certain class of type $\langle 1, 1 \rangle$ quantifiers: those interpreting determiners.⁴ Here are some examples.

- (8) $every_M(A, B) \Leftrightarrow A \subseteq B$
 $some_M(A, B) \Leftrightarrow A \cap B \neq \emptyset$
 $no_M(A, B) \Leftrightarrow A \cap B = \emptyset$
 $some\ but\ not\ all_M(A, B) \Leftrightarrow A \cap B \neq \emptyset\ and\ A - B \neq \emptyset$
 $at\ least\ four_M(A, B) \Leftrightarrow |A \cap B| \geq 4\ \ (|X| \text{ is the cardinality of } X)$
 $between\ six\ and\ nine_M(A, B) \Leftrightarrow 6 \leq |A \cap B| \leq 9$
 $most_M(A, B) \Leftrightarrow |A \cap B| > |A - B|$
 $more\ than\ a\ third\ of\ the_M(A, B) \Leftrightarrow |A \cap B| > 1/3 \cdot |A|$
 $infinitely\ many_M(A, B) \Leftrightarrow A \cap B \text{ is infinite}$
 $an\ even\ number\ of_M(A, B) \Leftrightarrow |A \cap B| \text{ is even}$
 $(the_{sg})_M(A, B) \Leftrightarrow |A| = 1 \text{ and } A \subseteq B$
 $(the_{pl})_M(A, B) \Leftrightarrow |A| > 1 \text{ and } A \subseteq B$
 $the\ ten_M(A, B) \Leftrightarrow |A| = 10 \text{ and } A \subseteq B$
 $Mary's_M(A, B) \Leftrightarrow \emptyset \neq A \cap \{b: has(m, b)\} \subseteq B$
 $some\ professors'_M(A, B) \Leftrightarrow professor \cap \{a: A \cap \{b: has(a, b)\} \subseteq B\} \neq \emptyset$
 $no...except\ Sue_M(A, B) \Leftrightarrow A \cap B = \{s\}$

The first three are classical Aristotelian quantifiers, except that Aristotle seems to have preferred the universal quantifier with *existential import* (or else he just restricted attention to properties with non-empty extensions):

- (9) $(all_{ei})_M(A, B) \Leftrightarrow \emptyset \neq A \subseteq B$

The next three are numerical quantifiers: let us say that Q is *numerical* if it is a Boolean combination of quantifiers of the form *at least n*, for some $n \geq 0$. Note that this makes *every*, *some*, and *no* numerical, as well as the two *trivial* quantifiers **0** and **1**:

- (10) $1_M(A, B) \Leftrightarrow |A \cap B| \geq 0$, i.e. $1_M(A, B)$ holds for all M and $A, B \subseteq M$
 $0 = \neg 1$, i.e. $0_M(A, B)$ holds for no M, A, B

(This is for type $\langle 1, 1 \rangle$; similarly for other types.) Then come two proportional quantifiers: Q is *proportional* if the truth value of $Q_M(A, B)$ depends only on the proportion of B s among the A s:

- (11) For $A, A' \neq \emptyset$, if $|A \cap B|/|A| = |A' \cap B|/|A'|$ then $Q_M(A, B) \Leftrightarrow Q_M(A', B)$.⁵

When proportional quantifiers are discussed, we assume that only *finite* universes are considered; this restriction will be written FIN.

Infinitely many and *an even number of* are more mathematical examples (though they interpret perfectly fine Dets), not falling under any of the categories mentioned so far. Then come three *definite* quantifiers; the first two can be taken to interpret the singular and plural definite article, respectively. The issue of whether definiteness can be captured as a property of quantifiers is interesting and we come back to it in section 9. The list (8) ends with two *possessive* and one *exceptive* quantifier.

A linguist might object that not all the names of quantifiers in (8) are English determiners. For example, “*more than a third of the cats*” should perhaps be analyzed as “*more than a third*” (which in turn could be analyzed further) plus “*of*” plus “*the cats*.” But I am not insisting on syntactic categories at this point, only that the labels in (8) *could* be construed as specifiers that need to combine with a nominal. This is why the truth conditions of sentences with these phrases as subjects can be expressed in terms of type $\langle 1, 1 \rangle$ quantifiers, and to a large extent these truth conditions seem correct, even though a few details may be disputed.

Type $\langle 1 \rangle$ quantifiers play an equally fundamental role for semantics, at least for languages allowing the formation of DPs; this is exemplified in the next section. We will also see that the semantics of some linguistic constructions appears to require polyadic quantifiers.

3. Determiner Phrases (DPs) and Quantifiers

I said that one of Montague’s insights was that, in principle, *all* DPs can be interpreted as type $\langle 1 \rangle$ quantifiers. In the following list, the first two are the familiar $\forall$ and $\exists$ from first-order logic.

- (12) $everything_M(B) \Leftrightarrow B = M \Leftrightarrow \forall_M(B)$
 $something_M(B) \Leftrightarrow B \neq \emptyset \Leftrightarrow \exists_M(B)$
 $nothing_M(B) \Leftrightarrow B = \emptyset$
 $at\ least\ four\ things_M(B) \Leftrightarrow |B| \geq 4$
 $most\ things_M(B) \Leftrightarrow (Q^R)_M(B) \Leftrightarrow |B| > |M - B|$ (the Rescher quantifier)
 $(Q_{even})_M(B) \Leftrightarrow |B| \text{ is even}$

Other examples are proper names and bare plurals. Montague proposed that a proper name like “*Mary*” should be interpreted as the type $\langle 1 \rangle$ quantifier consisting of all sets containing Mary. In general, for any individual a , the *Montagovian individual* I_a is defined, for all M and all $B \subseteq M$, by

- (13) $(I_a)_M(B) \Leftrightarrow a \in B$

Bare plurals come in universal and an existential versions; cf.

- (14) a. Firemen wear black helmets.
 b. Firemen are standing outside your house.

In general, for any set C we can define:⁶

- (15) a. $(C^{pl,u})_M(B) \Leftrightarrow \emptyset \neq C \subseteq B$
 b. $(C^{pl,e})_M(B) \Leftrightarrow C \cap B \neq \emptyset$

Next, a large class of English DPs are *compound*, of the form [Det N’]. The meaning of these is obtained by restricting or *freezing* the first argument of the determiner denotation to the extension of the nominal. Define, for any type $\langle 1, 1 \rangle$ quantifier Q and any set A , the type $\langle 1 \rangle$ quantifier Q^A as follows.

$$(16) \quad (Q^A)_M(B) \Leftrightarrow Q_{A \cup M}(A, B)$$

The universe is extended on the right-hand side to take care of the case that A is not a subset of M (as we must if Q^A is to be defined on *every* universe). One could instead build the requirement $A \subseteq M$ into the definition:

$$(17) \quad (Q^A)_M(B) \Leftrightarrow A \subseteq M \text{ and } Q_M(A, B)$$

The two definitions coincide when $A \subseteq M$ in fact holds. There is a reason to prefer (16), however, as we will see in the next section (and in section 9).

4. Meaning the Same on Every Universe

We noted in the introduction that there do not seem to exist any determiners whose meaning differs radically on different universes, such as a Det meaning *at least four* on universes of size less than 100, and *at most ten* on all larger universes. The property of *extension*, introduced in van Benthem (1984), captures the gist of this idea. It can be formulated for quantifiers of any type; we use type $\langle 1, 1 \rangle$ as an example:

$$(EXT) \quad \text{If } A, B \subseteq M \subseteq M', \text{ then } Q_M(A, B) \Leftrightarrow Q_{M'}(A, B).$$

In other words, the part of the universe that lies outside $A \cup B$ is irrelevant to the truth value of $Q_M(A, B)$. For quantifiers satisfying EXT we can dispense with the subscript M and write simply

$$Q(A, B)$$

This is a practice I will follow from now on whenever feasible.

It appears that all Det interpretations satisfy EXT (and similarly for the polyadic quantifiers we will encounter). As for type $\langle 1 \rangle$ quantifiers, Montagovian individuals and (both versions of) bare plurals are EXT, and so are all quantifiers of the form Q^A , *provided* they are defined as in (16). They would *not* be EXT if (17) were used; this is one argument in favor of (16).

The only exceptions to EXT so far are some of the quantifiers in (12): *everything*, *most things* (but not *something*, *nothing*, or *at least four things*). Obviously, the reason is the presence of the words like “*thing*,” which must denote the universe M , and M may enter in the truth conditions in a way that violates EXT.

One might say that it is still the case that, for example, “*everything*” means the same on every universe. This reflects perhaps an imprecision in “mean the same.” It is not clear how one could define sameness of meaning in a way that allowed also the non-EXT quantifiers in (12) to mean the same on all universes. In any case, it seems that with the sole exception of some type $\langle 1 \rangle$ quantifiers interpreting phrases that contain words like “*thing*,” all quantifiers needed for natural language semantics are EXT.

5. Domain Restriction

Determiners denote type $\langle 1, 1 \rangle$ quantifiers but, due to the syntactic position of the corresponding expressions, the two arguments are not on equal footing. The first argument is the extension of the noun belonging to the determiner phrase while the second comes from a verb phrase. The semantic correlate of this syntactic fact is that quantification is *restricted* to the first argument.

There are two equivalent ways to explain how this restriction works. The first is in terms of the property of *conservativity*: for all M and all $A, B \subseteq M$,

$$(CONSERV) \quad Q_M(A, B) \Leftrightarrow Q_M(A \cap B, B)$$

That all (interpretations of) determiners satisfy CONSERV is reflected in the fact that the following pairs are not only equivalent but a clear redundancy is felt in the b versions:

- (18) a. Several boys like Sue.
b. Several boys are boys who like Sue.
- (19) a. All but one student passed.
b. All but one student is a student who passed.

In other words, the truth value of $Q_M(A, B)$ doesn't depend on the elements of $B - A$. However, conservativity in itself is not sufficient for domain restriction. If $Q_M(A, B)$ depended on elements outside A and B , we could hardly say that quantification was restricted to A . (For example, if $Q_M = \text{every}_M$ if $17 \in M$, and $= \text{some}_M$ otherwise.) To avoid this, EXT is also required. In other words,

$$\text{domain restriction} = \text{CONSERV} + \text{EXT}$$

The other way to express domain restriction is in terms of the model-theoretic notion of *relativization*. A quantifier Q of any type τ can be relativized, that is, there is a quantifier Q^{rel} of type $\langle 1 \rangle \cap \tau$, which describes the behavior of Q restricted to the first argument. It suffices here to consider the case $\tau = \langle 1 \rangle$. Then Q^{rel} is the type $\langle 1, 1 \rangle$ quantifier defined as follows:

$$(20) \quad (Q^{rel})_M(A, B) \Leftrightarrow Q_A(A \cap B)$$

So Q^{rel} indeed has its quantification restricted to the first argument. And the two ways to cash out the idea of domain restriction are equivalent:

Fact 1. A type $\langle 1, 1 \rangle$ quantifier is CONSERV and EXT iff it is the relativization of a type $\langle 1 \rangle$ quantifier.

Proof. It is readily checked that Q^{rel} is CONSERV and EXT. In the other direction, if the type $\langle 1, 1 \rangle$ Q is CONSERV and EXT, define a type $\langle 1 \rangle$ Q' by

$$Q'_M(B) \Leftrightarrow Q_M(M, B)$$

Using CONSERV and EXT, one readily verifies that $Q = (Q')^{rel}$. □

6. Boolean Operations on Quantifiers

A main reason for Montague to treat proper names as denoting type $\langle 1 \rangle$ quantifiers was facts about coordination. Proper names can be freely conjoined with quantified DPs:

- (21) a. Two students and a few professors left the party.
b. Mary and a few professors left the party.

For familiar reasons, $[DP_1 \text{ and } DP_2 \text{ VP}]$ cannot in general be analyzed as $[DP_1 \text{ VP and } DP_2 \text{ VP}]$; it has to use a coordinate structure. (“*Some boy sings and dances*” is not equivalent to “*Some boy sings and some boy dances*.”) So we need $[DP_1 \text{ and } DP_2]$ to denote a type $\langle 1 \rangle$ quantifier. In (21), “and” is just intersection. Another relevant fact is that individuals cannot be conjoined but names can:

- (22) Henry and Sue work at NYU.

A correct interpretation of “*Henry and Sue*” in (22) is $I_h \cap I_s$.⁷

Boolean operations apply to Dets too: *some but not all* is the intersection of *some* and *not all*, *between three and five* is the intersection of *at least three* and *at most five*. Likewise, “*either exactly three or more than five*” is a perfectly fine complex Det.

Negation usually occurs as VP negation in English, although sentence-initial position is possible with some Dets, like “*every, more than five*.” Accordingly, there are two ways to negate a type $\langle 1, 1 \rangle$ quantifier, often called *outer* and *inner* negation. So we have the following Boolean operations, restricting attention here to CONSERV and EXT type $\langle 1, 1 \rangle$ quantifiers:

- (23) a. $(Q \wedge Q')(A, B) \Leftrightarrow Q(A, B) \text{ and } Q'(A, B)$
 b. $(Q \vee Q')(A, B) \Leftrightarrow Q(A, B) \text{ or } Q'(A, B)$
 c. $\neg Q(A, B) \Leftrightarrow \text{not } Q(A, B)$ (outer negation)
 d. $Q\neg(A, B) \Leftrightarrow Q(A, A - B)$ (inner negation)

In addition, there is the *dual* of Q :

$$(24) \quad Q^d = \neg(Q\neg) \quad [= (\neg Q)\neg]$$

(Corresponding Boolean operations are defined, *mutatis mutandis*, for type $\langle 1 \rangle$ quantifiers; in particular, we then have $(Q\neg)_M(B) \Leftrightarrow Q_M(M - B)$.) The negations and the dual all satisfy *cancellation*: $\neg\neg Q = Q\neg\neg = (Q^d)^d = Q$. Using this, one checks that each CONSERV and EXT type $\langle 1, 1 \rangle$ quantifier spans a *square of opposition*,

$$\text{square}(Q) = \{Q, Q\neg, \neg Q, Q^d\}$$

which is unique in the sense that if $Q' \in \text{square}(Q)$, then $\text{square}(Q') = \text{square}(Q)$. For example, $\{\text{all, no, not all, some}\}$ is (a modern version of) the classical Aristotelian square; another example with numerical quantifiers is $\{\text{all but at most four, at most four, all but at least five, at least five}\}$.⁸

Negations and duals are well represented among English Dets: $(\text{every})^d = \text{some}$, $(\text{at most five})\neg = \text{all but at most five}$, $(\text{the six})\neg = \text{none of the six}$, $(\text{at most two-thirds of the})^d = \text{fewer than one-third of the}$, $(\text{all - except Henry})\neg = \text{no - except Henry}$, $(\text{Mary's})\neg = \text{none of Mary's}$, $(\text{exactly half the})\neg = \text{exactly half the}$. The distribution and properties of outer and inner negation and duals in English have been studied in particular by Keenan; for example Keenan and Stavi (1986) and Keenan (2005, 2008).

7. Quantifiers in the Number Triangle

Van Benthem (1984) introduced a very useful tool for the study of DP and Det denotations: the number triangle. It has turned out to be invaluable for (i) the discovery of various properties of quantifiers and connections between them, (ii) identifying counterexamples to suggested generalizations, and (iii) studying the expressive power of quantifiers. We will see examples of all three. But first we need to present a property of many but not all Det denotations, a property that justifies the label “quantifier.”

The property is *isomorphism closure* (or *isomorphism invariance*); I will call it ISOM. It is usually presupposed in logical GQ theory. Indeed, recalling Lindström’s definition of a quantifier as a class of relational structures of the same type, ISOM is precisely the requirement that this class be closed under isomorphic structures: if $\mathcal{M} \in Q$ and $\mathcal{M} \cong \mathcal{M}'$, then $\mathcal{M}' \in Q$. For monadic quantifiers there is an equivalent and more useful formulation. If Q is of type $\langle 1, \dots, 1 \rangle$, with k arguments, a structure $\mathcal{M} = (M, A_1 \dots, A_k)$ partitions M into 2^k parts (some of which may be empty), and it is easy to see that ISOM amounts to the requirement that whenever the corresponding parts in $\mathcal{M}$ and $\mathcal{M}' = (M', A'_1 \dots, A'_k)$ have the same cardinality, $Q_{\mathcal{M}}(A_1 \dots, A_k)$ iff $Q_{\mathcal{M}'}(A'_1 \dots, A'_k)$.

Now let Q be a CONSERV and EXT type $\langle 1, 1 \rangle$ quantifier. We have seen that in this case, given (M, A, B) , the parts $B - A$ and $M - (A \cup B)$ do not matter. So ISOM boils down to this:

$$(25) \quad \text{If } |A \cap B| = |A' \cap B'| \text{ and } |A - B| = |A' - B'|, \text{ then } Q(A, B) \Leftrightarrow Q(A', B').$$

This in effect means that Q can be identified with a binary relation between (cardinal) numbers, which I will also denote Q . In other words, the following is well defined for any cardinal numbers k, m :

$$(26) \quad Q(k, m) \text{ iff there are } M \text{ and } A, B \subseteq M \text{ s.t. } |A - B| = k, |A \cap B| = m, \text{ and } Q_M(A, B).$$

Now, looking at list (8) in section 2, we see that all except the last three are ISOM:

- (27) *every* $(k, m) \Leftrightarrow k = 0$
some $(k, m) \Leftrightarrow m \neq 0$
no $(k, m) \Leftrightarrow m = 0$
some but not all $(k, m) \Leftrightarrow m \neq 0 \text{ and } k \neq 0$
at least four $(k, m) \Leftrightarrow m \geq 4$
between six and nine $(k, m) \Leftrightarrow 6 \leq m \leq 9$
most $(k, m) \Leftrightarrow m > k$
more than a third of the $(k, m) \Leftrightarrow m > 1/3 \cdot (k + m)$
infinitely many $(k, m) \Leftrightarrow m \text{ is infinite}$
an even number of $(A, B) \Leftrightarrow m \text{ is even}$
the_{sg} $(k, m) \Leftrightarrow k = 0 \text{ and } m = 1$
the_{pl} $(k, m) \Leftrightarrow k = 0 \text{ and } m > 1$
the ten $(k, m) \Leftrightarrow k = 0 \text{ and } m = 10$

But *Mary's*, *some professor's*, *no ... except Sue* all involve particular individuals or properties and hence are not ISOM. One could make them ISOM by adding extra arguments, but then we would lose their natural correspondence with determiners.

ISOM type $\langle 1 \rangle$ quantifiers are binary relations between numbers too:

$$(28) \quad Q(k, m) \text{ iff there are } M \text{ and } B \subseteq M \text{ s.t. } |M - B| = k, |B| = m, \text{ and } Q_M(B).$$

Indeed, each CONSERV, EXT, and ISOM type $\langle 1, 1 \rangle$ quantifier is of the form Q^{rel} for some ISOM type $\langle 1 \rangle$ Q , and it is easy to check that Q and Q^{rel} define the *same* binary relation between numbers.

Now assume FIN. Then these quantifiers are all subsets of N^2 , where $N = \{0, 1, 2, \dots\}$. Turn N^2 clockwise 45 degrees. This is the *number triangle* (Figure 1.1).

A point (m, k) in the number triangle belongs to the *row* $(m, 0), (m, 1), \dots$ and the *column* $(0, k), (1, k), \dots$. Its *level* is the diagonal (horizontal) line $(m + k, 0), (m + k - 1, 1), \dots, (0, m + k)$. A

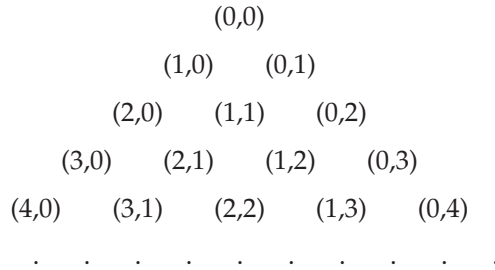


Figure 1.1. The number triangle.

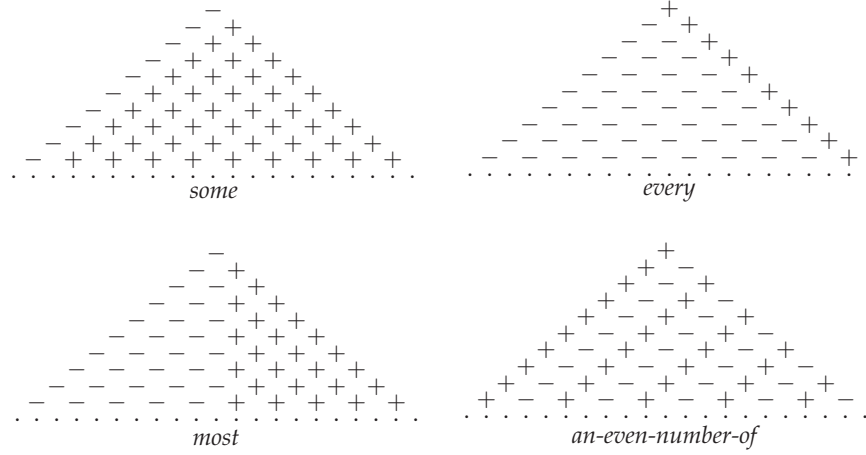


Figure 1.2. Some quantifiers in the number triangle.

type $\langle 1, 1 \rangle$ quantifier Q constitutes an area in the triangle; we can mark the points in Q with $+$ and the others with $-$. Given A, B , the corresponding point is $(|A - B|, |A \cap B|)$ and the level is $|A|$. In the type $\langle 1 \rangle$ case, given $B \subseteq M$, the point is $(|M - B|, |B|)$ and the level is $|M|$. So in this case, the local quantifier Q_M is fully represented at level $|M|$. The patterns in Figure 1.2 represent *some*, *every*, *most*, *an even number of*, and, equally, $\exists$, $\forall$, Q^R , Q_{even} .

8. Basic Properties

Here are some basic properties of CONSERV and EXT type $\langle 1, 1 \rangle$ quantifiers, together with their representations (under ISOM and FIN) in the number triangle.

8.1 Symmetry

Q is *symmetric* if

$$(\text{SYMM}) \quad Q(A, B) \Rightarrow Q(B, A)$$

Under CONSERV, this is easily seen to be equivalent to what Keenan called *intersectivity*: the truth value of $Q(A, B)$ depends only on $A \cap B$:

$$(\text{INT}) \quad \text{If } A \cap B = A' \cap B', \text{ then } Q(A, B) \Leftrightarrow Q(A', B').$$

So under ISOM, the truth value depends only on $|A \cap B|$, which is to say that whenever a point (m, k) is in Q , so are all points on the column $(0, k), (1, k), \dots$; we illustrate this in Figure 1.3.

Directly from the number triangle, *some* and *an even number of* are symmetric, *every* and *most* are not. *Every* is *co-symmetric*: $Q(A, B)$ depends only on $A - B$ (as in Figure 1.3 but for rows instead), but *most* is neither.

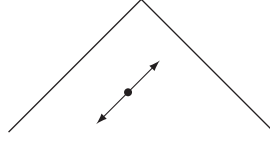


Figure 1.3. Symmetry.

8.2 Negations

The pattern for $\neg Q$ is obtained from that for Q by switching $+$ and $-$. $Q\neg$ is the *converse* of Q :

$$Q\neg(k, m) \Leftrightarrow Q(m, k)$$

And Q^d results from switching $+$ and $-$ in $Q\neg$. From this we see that $Q \neq \neg Q$ (obviously), $Q \neq Q^d$ (consider the point $(0,0)$)⁹, but numerous quantifiers are such that $Q = Q\neg$, namely, all those whose corresponding relation between numbers is symmetric. They are called *midpoint quantifiers* in Westerstahl (2012a) (using a term from Keenan (2008) in a slightly different sense). To see that there are in fact uncountably many midpoints (even under FIN), draw a vertical line in the triangle through $(0,0)$, $(1,1)$, $(2,2)$, Any set of points on the left side of that line yields a midpoint quantifier by mirroring that set to the right of the line; indeed, Q is a midpoint iff it can be obtained in this way. As elaborated in Keenan (2008), here we find curious natural language examples, illustrated by equivalent pairs like the following:

- (29) a. Exactly three of the six boys passed the exam.
b. Exactly three of the six boys didn't pass the exam.
- (30) a. Between 40 and 60% of the professors left.
b. Between 40 and 60% of the professors didn't leave.
- (31) a. Either exactly five or else all but five students came to the party.
b. Either exactly five or else all but five students didn't come to the party.

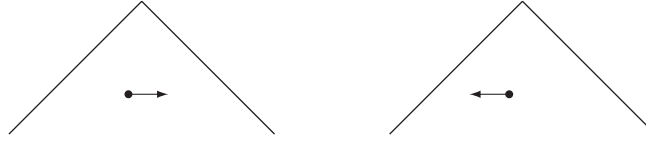
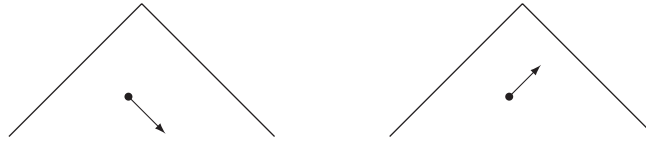
8.3 Monotonicity

Most natural language quantifiers exhibit some form of monotonicity behavior, and all forms are easily representable in the number triangle. To begin, if Q has two arguments, it can be increasing or decreasing in the right or left argument. We write

- (32) a. $\text{MON}\uparrow$: $Q(A, B) \ \& \ B \subseteq B' \Rightarrow Q(A, B')$
b. $\uparrow\text{MON}$: $Q(A, B) \ \& \ A \subseteq A' \Rightarrow Q(A', B)$

and similarly for $\text{MON}\downarrow$ and $\downarrow\text{MON}$, as well as combinations like $\downarrow\text{MON}\uparrow$. In the number triangle, this becomes as illustrated in Figures 1.4 and 1.5.

Combining with negation, we can see how the monotonicity behavior of Q completely determines that of the other quantifiers in $\text{square}(Q)$. For example, if Q is $\downarrow\text{MON}$, then so is $Q\neg$, whereas $\neg Q$ and Q^d are $\uparrow\text{MON}$. Combining with symmetry, we see that if Q is both $\text{MON}\uparrow$ and SYMM , it has to be of the form *at least n*, for some n (or else the trivial 0). We also see that there are four

Figure 1.4. $\text{MON}\uparrow$ and $\text{MON}\downarrow$ Figure 1.5. $\uparrow\text{MON}$ and $\downarrow\text{MON}$ Figure 1.6. $\uparrow_{\text{SE}}\text{MON}$ and $\downarrow_{\text{NE}}\text{MON}$

more “monotonicity directions” in the triangle: up and down along the axes. They can be named using compass directions, as in Figure 1.6. These correspond to *left*, but restricted, monotonicity properties:

- (33) a. $\uparrow_{\text{SE}}\text{MON}$: $Q(A, B) \ \& \ A \subseteq A' \ \& \ A - B = A' - B' \Rightarrow Q(A', B)$
 b. $\downarrow_{\text{NE}}\text{MON}$: $Q(A, B) \ \& \ A' \subseteq A \ \& \ A \cap B = A' \cap B' \Rightarrow Q(A', B)$

Similarly for the other two directions. The combination $\uparrow_{\text{SE}}\text{MON} + \downarrow_{\text{NE}}\text{MON}$, called *smoothness* (see Figure 1.7) is particularly interesting, in that most $\text{MON}\uparrow$ Det denotations actually have the stronger property of smoothness, such as *at least n* and the proportional *more than m/n:ths of the, at least m/n:ths of the* (and correspondingly for right downward monotonicity and co-smoothness). And, of course, $\uparrow\text{MON}$ is $\uparrow_{\text{SE}}\text{MON} + \uparrow_{\text{SW}}\text{MON}$, etc.

Almost all Det denotations have some combination of these monotonicity properties. Even a seemingly non monotone quantifier like *an odd number of*—which has neither of the standard left and right monotonicity properties, nor is it a Boolean combination of quantifiers with these properties—satisfies such a combination, as it is *symmetric*, and we see directly in the triangle that $\text{SYMM} = \downarrow_{\text{NE}}\text{MON} + \uparrow_{\text{SW}}\text{MON}$.

Monotonicity offers several illustrations of how the number triangle helps thinking about quantifiers. Let us look at a few examples.

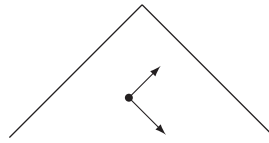


Figure 1.7. Smoothness.

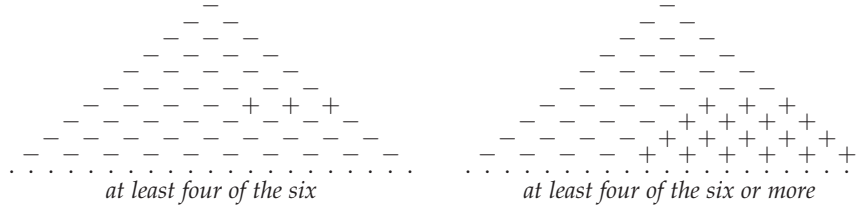
8.3.1 Counterexamples Among several *monotonicity universals* in Barwise and Cooper (1981), one was:

(U1) If a Det denotation Q is left increasing ($\uparrow\text{MON}$; Barwise and Cooper called this *persistence*), it is also right increasing ($\text{MON}\uparrow$).

This holds for a large number of English Dets. However (van Benthem, 1984), the number triangle immediately shows that *some but not all* is a counterexample (see Figure 1.8). Going right on the same level you will hit a $-$, violating $\text{MON}\uparrow$, but the downward triangle from any point in Q remains in Q . A similar conjecture was made in Väänänen and Westerståhl (2002), also backed by a large number of examples:

(U2) If a Det denotation Q is $\text{MON}\uparrow$, it is in fact smooth.

But there are simple patterns in the number triangle violating this, and some of them can be taken to interpret English Dets. For example:



These patterns are immediately seen to be $\text{MON}\uparrow$ but not smooth. For both (U1) and (U2), the number triangle was instrumental in finding the counterexamples.

8.3.2 Generalizations Facts discovered in the number triangle hold under the rather restrictive assumption of **CONSERV**, **EXT**, **ISOM**, and **FIN**. But some Det denotations presuppose infinite models (e.g. *finitely many*) and some are not **ISOM** (e.g. *Mary's* and *every ... except John*). However, it often happens that facts from the number triangle generalize to arbitrary **CONSERV** quantifiers. For example, it is immediate in the triangle that smoothness implies $\text{MON}\uparrow$. And indeed, we have

Fact 2. Any **CONSERV** quantifier satisfying (33a) and (33b) is $\text{MON}\uparrow$.

Proof. Suppose that $Q_M(A, B)$ and $B \subseteq B' \subseteq M$. Let $A' = A - (B' - B)$. It follows that $A' \subseteq A$ and $A' \cap B = A \cap B = A' \cap B'$. By (33b), $Q_M(A', B)$, so, using **CONSERV** twice, $Q_M(A', B')$. But we also have $A' \subseteq A$ and $A' - B = A - B'$. Thus, by (33a), $Q_M(A, B')$. $\square$

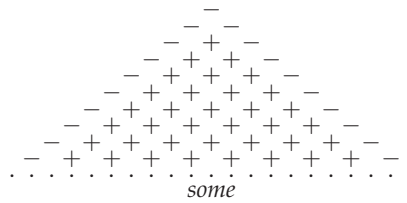
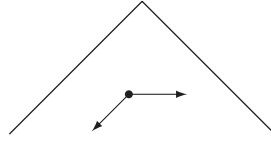


Figure 1.8. *Some but not all.*

Figure 1.9. $\uparrow\text{MON}\uparrow$.

Another example is the characterization of symmetry just mentioned: $\text{SYMM} = \downarrow_{\text{NE}}\text{MON} + \uparrow_{\text{SW}}\text{MON}$. It is not difficult to show that for any CONSERV quantifier, symmetry is equivalent to the conjunction of (33b) and the general property corresponding to $\uparrow_{\text{SW}}\text{MON}$. But it would have been hard to even come up with the two relevant properties without the use of the number triangle.

8.3.3 Expressive power Questions of expressive power take various forms. One is: given certain properties of quantifiers, exactly which quantifiers have them? Several early results concerned *relational* properties like reflexivity and transitivity of Q as a binary relation. As to monotonicity, one may ask, for example, exactly which quantifiers are $\uparrow\text{MON}\uparrow$? The answer comes from looking in the number triangle (but here the presuppositions of that representation are necessary): each point $(n - k, k)$ in Q determines a downward trapezoid, a quantifier $Q_{n,k}$, whose right edge aligns with the right axis of the triangle, and whose left edge is parallel to the left axis (see Figure 1.9). But you can only take a finite number of steps from any point before hitting the left axis. Hence Q must be a finite disjunction of quantifiers of the form $Q_{n,k}$. Expressing the latter in English, and including the trivial 0 , we have proved:

Fact 3 (CONSERV, EXT, ISOM, FIN). Q is $\uparrow\text{MON}\uparrow$ iff it is a finite disjunction of quantifiers of the form at least k of the $n - 1$ or more ($k < n$).

Here is a similar example (without proof). A strengthening of $\downarrow\text{MON}$ sometimes turns up in linguistic contexts: *left anti additivity*:

$$\text{LAA } Q_M(A, C) \ \& \ Q_M(B, C) \Leftrightarrow Q_M(A \cup B, C)$$

One such context is when “and” seems to mean “or,” as in

(34) Every boy and girl was invited to the party.

Here $\text{boy} \cap \text{girl} = \emptyset$, but that is not a necessary condition for this reading. So a natural question is which quantifiers are LAA. The answer, which can be obtained just by reasoning in the number triangle (see Peters and Westerståhl (2006), section 5.3) is:

Fact 4 (CONSERV, EXT, ISOM, FIN). The only LAA quantifiers, besides 0 and 1 , are every, no, and the quantifier $Q(A, B) \Leftrightarrow A = \emptyset$.

The number triangle can also be used for standard logical notions like first-order definability. For example,

Fact 5 (CONSERV, EXT, ISOM, FIN). All Boolean combinations of $\uparrow\text{MON}$ quantifiers are first-order definable.

Proof. It suffices to show that all $\uparrow\text{MON}$ quantifiers are so definable. But each point in such a quantifier determines a downward triangle, included in the quantifier, and one can only take finitely many steps toward the edges, so the quantifier must be a finite disjunction of such triangles, each of which is obviously first-order definable. $\square$

It follows that, under these constraints, there are only countably many left monotone quantifiers (since there are only countably many defining first-order sentences), whereas it is easy to see that there are uncountably many right monotone—and even smooth—quantifiers.

The converse to Fact 5 also holds—all first-order definable quantifiers (satisfying the constraints) are Boolean combinations of $\uparrow$ MON quantifiers—but proving this requires logical techniques we have not yet discussed.

Monotonicity is ubiquitous in natural language semantics, and not only because almost all (all?) Det denotations have some such property, and many have the basic right or left properties. For one thing, downward monotonicity, and also stronger properties like LAA, have been instrumental in explaining the distribution of so-called *polarity items*; see, for example, Ladusaw (1996) or Peters and Westerståhl (2006), section 5.9, for surveys. For another, monotonicity plays a crucial role in *reasoning*. A lot of everyday reasoning can be analyzed with the help of “one-step monotonicity inferences,” such as the following.

All students smoke.

Hence: All philosophy students smoke or drink.

Such inferences have been taken to be part of a *natural logic*; van Benthem (2008) and Moss Chapter 18 of this volume give overviews. Moreover, Aristotelian *sylogistics* is really all about monotonicity. For example, the syllogism *Cesare*,

no B is C	
all A are B	
no A is C	

simply says that the quantifier *no* is $\downarrow$ MON. This was an early observation, but recently a systematic logical study of syllogistic style reasoning, extended with other forms in order to increase expressive power while still keeping the systems *decidable*, has been initiated by Larry Moss; see Chapter 18 of this volume.

Finally, monotonicity has been used in connection with measuring the *processing effort* required by various determiners. An early idea from Barwise and Cooper (1981) was that monotone quantifiers are easier to process than non-monotone ones, and upward monotonicity is easier than the downward variant. The idea was developed into a notion of “minimal count complexity” by van Benthem, who showed that the quantifiers with this property are exactly the smooth ones.¹⁰ He went on to study the complexity of processing quantifiers with methods from automata theory. The area of quantification and complexity has received much attention by logicians, and recently also by semanticists; we give some hints in section 11.4.

9. Definiteness

Definiteness (and correspondingly, indefiniteness) is a fundamental, but still much debated linguistic property of DPs (and Dets). (There are several overviews where further references can be found; let me mention here Abbott (2004, 2010).) It “starts out” as a syntactic property, with typical English instances containing the definite article such as “*the student*,” “*the ten boys*,” but is extended to proper names, pronouns, and, for example, “*Mary’s students*.” No general morphosyntactic definition seems to be agreed on, but there are various *tests* for definiteness, such as inability to appear in existential there sentences:

- (35) a. There were three students at the meeting.
 b. *There were the students/Mary’s students at the meeting.

At the same time, definiteness seems to have a lot to do with *meaning*: What is its semantic definition? This seems easier: already Russell identified *unique reference* as the characteristic property of *singular* quantified DPs. He didn't assign meaning directly to a phrase like "*the present king of Sweden*," but his truth conditions for sentences with this DP as subject required the existence of a unique present king of Sweden.¹¹ We only need to extend this to cover plural definites, such as "*the books*" or "*the 18th century kings of Sweden*," and an obvious suggestion is to let such DPs denote the *set* of (salient) objects satisfying the description (where a singleton is identified with its element in the singular case). But far from every DP can be taken to denote a set, so the task is to single out the ones that can. This is where GQ theory should be useful.

Since the advent of *dynamic* or *discourse* semantics, semantic criteria for definiteness are often couched in terms of *familiarity* or *uniqueness*. Familiarity is a thoroughly discourse-related notion, requiring the object(s) satisfying the description to have been introduced (in some sense) in earlier parts of the discourse:

- (36) a. A tall man wearing a hat entered the room. The man sat down on a bench.
b. John and Henry entered the room. The two men sat down on a bench.

However, there are many instances in which a definite description can be used without any previous introduction, simply to convey information about the individual described, and familiarity nowadays seems to have lost its appeal as a defining property of definiteness, while uniqueness remains. Thus, it ought to be rather uncontroversial (but see below) that unique reference, or simply *referentiality* (extended to plural DPs), is what semantically characterizes definites.

We need to be careful, of course, to distinguish a DP occurrence in a sentence actually referring to something—or, if you wish, the utterer of the sentence *using* it to refer—from the semantic property of the DP that it *can* be so used. A particular use of a definite DP may not refer at all, even though other uses do. But with this distinction in mind, which DPs are semantically definite?

The answer was given in Barwise and Cooper (1981). They defined definiteness for Dets, but the definition extends readily to DPs. To begin, let M be a fixed universe. A type $\langle 1 \rangle$ quantifier Q_M , which we can think of as a set of subsets of M , is *definite* if it is either empty or generated by some non empty set X , in the sense that for all $B \subseteq M$,

$$Q_M(B) \Leftrightarrow X \subseteq B$$

Call X , when it exists, the *generator* of Q_M . Then X is the intersection of the sets in Q_M : $X = \cap Q_M$. Next, a DP (as well as the Det if the DP has the form [Det N']) is (semantically) *definite* if its denotation is definite in this sense. Finally, if a DP is definite *and* has a non-trivial denotation (neither 0_M nor 1_M), let us say that the DP *refers* to the generator X .

For example,

- (37) a. *Henry* refers to h (i.e. Henry), since I_h is generated by $\{h\}$ (provided $h \in M$).
b. *the five girls* refers to the the salient set A of girls in M , since *the five*^A is generated by A (provided $|A| = 5$).
c. *John's books* refers to the set of books (A) that John has, more generally, to $A \cap R_j$, since *John's*^A is generated by $A \cap R_j$ (provided $A \cap R_j \neq \emptyset$; where R is some possessive relation and $R_j = \{b : R(j, b)\}$ is the set of things possessed by j).

But if the provisos in (37) are *not* satisfied, the quantifiers have no generators, and these DPs, although semantically definite, do not refer at all. This seems correct.¹²

An important technical point (not discussed by Barwise and Cooper) needs to be addressed here: the generator, as we defined it, depends on M , so we should really write X_M instead. However, if the reference of a DP would change when the universe is extended (without changing facts about salience), it would hardly be a satisfactory account of reference. Fortunately, this situation

does not arise. More precisely, it is proved in Peters and Westerståhl (2006), ch. 4.6, that if Q_1 is CONSERV, EXT, and definite in the above sense, then Q_1^A is generated by the *same* set whenever $(Q_1^A)_M$ is nontrivial. (By the way, this proof essentially uses definition (16) in section 3 of freezing, providing a further reason to prefer that definition over (17).)

Thus, this semantic notion of definiteness is robust, and one would think it does a good job of capturing the idea of definiteness as referentiality. However, it has met with at least two kinds of opposition in the linguistic literature. One is that it doesn't conform to facts about existential there sentences. The Det "*every*" is not definite according to Barwise and Cooper's definition—since when $A = \emptyset$, $(every^A)_M$ is generated by $\emptyset$ and hence trivial ($= 1_M$)—but still unacceptable in existential there sentences:

- (38) *There is every student at the party.

But this criticism is misplaced: Barwise and Cooper did not try to explain acceptability in existential there sentences in terms of definiteness. They suggested an explanation in terms of their notion of a (positive or negative) *strong* Det, relying on an idea that reluctance to utter logically true (false) sentences had been grammaticalized in existential there constructions. That idea can be criticized (and it has been, e.g. in Keenan (1987)), but it has little to do with definiteness. Indeed, *every* is not definite but positive strong, according to Barwise and Cooper.

Another influential criticism of Barwise and Cooper's notion of definiteness is based on the fact, that they themselves noted, that "*both*" and "*the two*" are indistinguishable as type $\langle 1, 1 \rangle$ quantifiers, but behave differently in *partitive* constructions. This is often taken to be another characteristic of definites (but see below), and indeed we have

- (39) a. One of the two men is guilty.
b. *One of both men is guilty.

Ladusaw (1982) proposed that a DP appearing after "*of*" in a partitive actually refers to a *group*, the one made up of the individuals in the set which Barwise and Cooper say a definite DP refers to.

The standard model-theoretic GQ framework cannot be used to explain the facts in (39). On the other hand, this seems to be just about the only case where a distinction in terms of referentiality is required. So instead of abandoning the standard setting in favor of (much more complicated) lattice-like structures where individuals as well as groups of individuals are elements of the universe, we may simply accept that GQ theory fails in the case of "*both*" and "*the two*." Except for this one case, replacing "set" by "group" would change little as regards facts about definiteness. In particular, both approaches rely on the same idea of definiteness as referentiality.

More generally, this is an instance of the fact, mentioned in the introduction, that the usefulness of GQ theory in semantics hinges on its simplicity and its familiarity from logic. But since it is an extensional and first-order framework, we should not expect it to be sensitive to all aspects of the meanings of Dets and DPs. As long as it yields robust descriptions, and sometimes explanations, of semantic features of these phrases that are correct in a large number of cases, we ought to be satisfied. And it seems to me that Barwise and Cooper's notion of semantic definiteness has proved its mettle in this respect. As a precise model-theoretic account of definiteness as referentiality in the GQ framework, it has no rivals.

Finally, a few more words about the role of definites in partitives. The commonly assumed *Partitive Constraint* says that the DP in a partitive phrase of the form

- (40) Det of DP

must be (plural and) definite. This is supported by examples like the following.

- (41) a. three of the dogs
b. *three of some dogs

- c. most of those flights
- d. *most of all flights
- e. some of the six cars
- f. *some of few cars

Jackendoff (1977), where the partitive constraint was introduced, observed that *possessive* DPs are also fine in these phrases,

- (42) a. many of his friends
b. two of Mary's books

but nevertheless concluded that the characteristic property of these DPs is semantic, namely, that "it designates a set out of which certain individuals (or a certain subset) is selected" (pp. 108–9). This is essentially referentiality in our sense, and indeed it is often claimed that possessive DPs are definite.

However, it seems in fact that *all* (plural) possessive DPs are fine in (40), and many of these are *in no sense referential*:

- (43) a. at least two of most students' term papers
b. one of each girl's parents

For example,

- (44) At least two of most students' term papers got an A.

doesn't say that among a set of term papers (written by members of some sufficiently large set of students), at least two got an A. It says that most students x are such that at least two of x 's term papers got an A. This is an instance of the uniform truth conditions of sentences with possessive DPs, studied in detail in Peters and Westerstål (2013). The conclusion there is that although both definite and possessive DPs are allowed in (40), the semantic rules are different in the two cases. For some simple DPs, like those in (42), the partitive and the possessive rules result in the same truth conditions, but in more complex cases they come apart. This is illustrated by

- (45) Two of the ten boys' books are missing.

Sentence (45) is *three* ways ambiguous. One interpretation, in which *two* quantifies over boys, says about two of the ten boys that each one's books are missing. In addition, there are two readings where *two* instead quantifies over books. To see this, consider the number of missing books. If the DP "*two of the ten boys' books*" is analyzed as a partitive, which requires treating "*the ten boys' books*" as a definite DP referring to the set of books belonging to (one or more of) the ten boys, (45) says that *two* books in this set are missing. If "*two of the ten boys' books*" is instead analyzed as a possessive, (45) says that each of the ten boys is such that two of his books are missing, so *twenty* books in all could be missing.

In conclusion, the partitive constraint applies to one analysis of phrases of the form (40), but definiteness is not in general a requirement on the DP in this form.

10. Decomposition

The interpretation of a DP of the form [Det N'] is $Q = Q_1^A$, where Q_1 interprets the Det and A the N'. In the build-up of sentence meaning, this step entails *loss of information*, unless Q_1 and A can be

recovered from Q , which in general is not the case. Could this be a problem for compositionality? Let us first take a quick general look at the possibility of *decomposing* type $\langle 1 \rangle$ quantifiers, and then come back to potential repercussions for compositionality.

Call Q *decomposable* if there is a CONSERV and EXT type $\langle 1, 1 \rangle$ quantifier Q_1 and a set A such that $Q = Q_1^A$. If Q_1 is also ISOM we say that Q is *ISOM decomposable*. CONSERV and EXT are natural requirements, since all Det interpretations satisfy them (and indeed one can show that without them, every type $\langle 1 \rangle$ quantifier would be trivially decomposable). ISOM is a further guarantee that Q_1 is well behaved, though there are cases when ISOM cannot be had.

Since Q_1^A is EXT if Q_1 is, EXT is a necessary requirement for decomposability. Are there any EXT but not decomposable quantifiers? To answer this it is useful to note:

- (46) A decomposable quantifier Q *lives on* some set C , in the sense that *for all* M and all $B \subseteq M$, $Q_M(B) \Leftrightarrow Q_M(C \cap B)$.

This is immediate: Q_1^A lives on A since Q_1 is CONSERV. Westerståhl (2008) shows that living on some set is also *sufficient* for decomposability; here we just need the easy direction. It appears that non-decomposable quantifiers are of two kinds:

Fact 6. (a) *Non-trivial ISOM type $\langle 1 \rangle$ quantifiers are not decomposable.*

- (b) *Let, for some $D \neq \emptyset$, $(\text{only } D)_M(B) \Leftrightarrow \emptyset \neq B \subseteq D$. Then only D is EXT, but not decomposable.*

Proof. (a) Suppose Q non-trivial and ISOM. We may also suppose it is EXT. Assume first that $\neg Q(\emptyset)$. By non-triviality, there are M and $B \subseteq M$ such that $Q_M(B)$ and hence, by EXT, $Q_B(B)$. Suppose Q lives on C . Then take any set B' such that $|B'| = |B|$ and $C \cap B = \emptyset$. By ISOM we have $Q_{B'}(B')$. By the live-on property, $Q'_B(C \cap B')$, which contradicts our assumption. So Q lives on no set, and therefore cannot be decomposable. If instead $Q(\emptyset)$ holds, the same argument shows that $\neg Q$ lives on no set, which also implies that Q is not decomposable.

(b) Clearly, *only D* is EXT. Suppose it lives on some set C . Take $b \notin C \cup D$. Then *only D*(D), hence *only D*($C \cap D$). Since $C \cap D = C \cap (D \cup \{b\})$, we have *only D*($C \cap (D \cup \{b\})$), and therefore, again by the live-on property, *only D*($D \cup \{b\}$), which contradicts the definition of *only D*. $\square$

Note that *only D* is a perfectly good DP denotation; cf.

- (47) a. Only Mary was absent. ($D = \{m\}$)
b. Only pilots can fly.

So Fact 6(b) is an(other) indication that “*only*” is not a Det (other indications being that its interpretation would not be CONSERV, and that it doesn’t combine with an N' in (47) but with a proper noun and a bare plural).

We will come back to the significance of Fact 6(a) presently, but let us first note that usually non-quantified DPs do denote decomposable quantifiers; many of which are ISOM decomposable. We have (cf. section 3):

- (48) a. $I_a = \text{every}^{\{a\}} = \text{all}_{ei}^{\{a\}} = \text{the}_{sg}^{\{a\}} = \text{some}^{\{a\}}$
b. $C^{pl,u} = \text{all}_{ei}^C$ and $C^{pl,e} = \text{some}^C$

Furthermore, using the characterization of decomposability mentioned above, one can show:

Fact 7. *The class of decomposable quantifiers is closed under Boolean operations, including inner negation (and hence dual).*

(The crucial observation is that if Q lives on C and Q' lives on C' , then $Q \wedge Q'$ and $Q \vee Q'$ live on $C \cup C'$.)

However, ISOM decomposability is not in general preserved under Boolean operations: although simple DPs are ISOM decomposable, we have, for example:

Fact 8. *The quantifier John, or Mary and Sue, i.e. $I_j \vee (I_m \wedge I_s)$, is not ISOM decomposable. More generally, the quantifier $Q = \text{some}^D \vee \text{every}^E$ is (decomposable but) not ISOM decomposable, provided $D \neq \emptyset$, $|E| > 1$, and $D \cap E = \emptyset$.*

Proof. The first claim is a special case of the second (with $D = \{j\}$ and $E = \{m, s\}$). That Q is decomposable follows from Fact 7. By the definition of Q we have $Q(\{a\})$ for all $a \in D$, but $\neg Q(\{b\})$ for $b \in E$. Now suppose $Q = Q_1^A$. If $a \in D$, we thus have $Q_1(A, \{a\})$, and hence $Q_1(A, A \cap \{a\})$, so $a \in A$ since $\neg Q_1(A, \emptyset)$. Also $\text{every}^E(E)$, so $Q(E)$, i.e. $Q_1(A, E)$, and hence $Q_1(A, A \cap E)$. It follows that $A \cap E \neq \emptyset$. Take $b \in A \cap E$. We then have $Q_1(A, \{a\})$ but $\neg Q_1(A, \{b\})$ for some $a, b \in A$. This shows that Q_1 is not ISOM. $\square$

Indeed, the quantifier Q_1 used to decompose $I_j \vee (I_m \wedge I_s)$ is quite artificial from a natural language point of view: essentially it has to have the form

$$(Q_1)_M(A, B) \Leftrightarrow A = C \ \& \ (I_j \vee (I_m \wedge I_s))(A \cap B)$$

for some *fixed* set C such that $j, m, s \in C$. This Q_1 is CONSERV and EXT but hardly the interpretation of any English determiner.

By contrast, CONSERV and EXT type $\langle 1, 1 \rangle$ quantifiers where the first argument is anchored, not to a fixed set, but to the universe M , are frequent in natural language. Such a quantifier in effect has type $\langle 1 \rangle$, and we saw several examples in section 3: *everything, something, nothing, most things*, etc. These DPs have the form [Det N'], but the N' is a word like “*thing*,” which denotes the universe. One might try to say that they are decomposable in a different sense, namely, that there is a CONSERV and EXT Q_1 such that $Q_M(B) \Leftrightarrow (Q_1)_M(M, B)$. But then *every* type $\langle 1 \rangle$ quantifier is decomposable in this way, since we saw from (the proof of) Fact 1 in section 5 that $Q_M(B) \Leftrightarrow Q_M^{rel}(M, B)$.

So we have the somewhat peculiar situation that although the DPs just mentioned involving “*thing*” are “syntactically decomposed,” the most natural corresponding semantic notion of decomposability is trivial, whereas for other DPs, “syntactically decomposed” or not, the semantic notion makes good sense.

On the theme of “natural” decomposition, consider also English prenominal possessive DPs. These have the form [Det N'], where the Det in turn is possessive: [DP $'s$]; for example “*Mary's dogs*,” “*some students' books*.” As possessive Dets are not ISOM, this is not ISOM decomposition. Is there also an ISOM decomposition? The answer is again: only in simple cases. Recall that, when “*Mary's*” means *all of Mary's*,¹³

$$(49) \text{ Mary's}(A, B),$$

as in

$$(50) \text{ Mary's dogs are well behaved.}$$

means

$$(51) \emptyset \neq A \cap R_m \subseteq B$$

where R is a possessive relation and $R_m = \{b : R(m, b)\}$ is the set of things “possessed” by m . Since (50) can also be written

$$(52) \text{ all}_{ei}^{A \cap R_m}(B)$$

Mary's^A is ISOM decomposable. But with an argument similar to the one for Fact 8 one can show that, for example, *exactly two students' books*, as in

(53) Exactly two students' books are missing

is not ISOM decomposable. Note that (50) can also be expressed as follows (provided Mary has at least one dog):

(54) All/the dogs belonging to Mary are well behaved.

where "*dogs belonging to Mary*" is an N' restricting "*Mary's*", whereas (53) does not mean

(55) All/the books belonging to exactly two students are missing.¹⁴

The notion of decomposability concerns the *existence* of decompositions. What about *uniqueness*? As (48a) illustrates, Q_1 is *never* recoverable from $Q = Q_1^A$. But Westerståhl (2008) shows that A is uniquely recoverable when the decomposition is ISOM (also illustrated by (48a)): if $Q_1^A = Q_2^B$, where Q_1, Q_2 are both ISOM, then $A = B$.

This leads us back to the issue of compositionality. If there are cases where, in order to arrive at a correct meaning of a sentence containing a DP of the form [Det N'], the semantic composition function applied immediately above that DP needs access not just to [DP] but to [Det] or [N'] or both, there might be a problem. Are there such cases? I will mention two examples.¹⁵

Consider the *reciprocal* sentence

(56) Three of the pirates stared at each other in surprise.

A familiar analysis (see Dalrymple *et al.* (1998)) has "*each other*" denote a type $\langle 1, 2 \rangle$ quantifier; in this case presumably (if the event is instantaneous and you cannot stare at more than one person at a time)

$$EO(A, R) \Leftrightarrow \forall a \in A \exists b \in A (a \neq b \ \& \ R(a, b))$$

Now if (56) has the structure

(57) [_S [_{DP} Three of the pirates] [_{VP} stared at each other in surprise]]

it is reasonable to let [three of the pirates] have the form Q_1^A and let [stared at each other in surprise] = $\lambda X EO(X, R)$ (where R is 'stare in surprise'). Then the meaning of (56) is obtained using a so-called *Ramsey quantifier*:

(58) $\text{Ram}(Q_1)(A, R) \Leftrightarrow \exists X \subseteq A (Q_1^A(X) \ \& \ EO(X, R))$

Thus, we see that, on this analysis, the semantic composition function needs access not only to Q_1^A but also to A . And if Q_1 is ISOM, by the result just mentioned, A can indeed be recovered from Q_1^A . In other words, what seems like a potential problem for compositionality is resolved by a technical result from GQ theory.

The next case, however, is more problematic. It again concerns possessive DPs, and the fact that these usually exhibit what Barker (1995) called *narrowing*, as illustrated by the following sentence:

(59) Most people's grandchildren love them.

Sentence (59) does not quantify over all people, only over people *with grandchildren*. Indeed, most people don't have grandchildren (they are too young for that), but (59) is not trivially true for that reason, nor should its truth entail that most people have grandchildren. In general, narrowing is the property of a quantified possessive DP that its quantification is only over "possessors," that

is over individuals who “possess” something of the relevant kind. (In (59) the possessive relation comes from a relational noun.)

This means that a typical sentence with a quantified possessive subject DP has the following truth conditions:

$$(60) \quad Q_1 C\text{'s } A \text{ are } B \Leftrightarrow Q_1(C \cap \text{dom}_A(R), \{a: A \cap R_a \subseteq B\})$$

where $\text{dom}_A(R) = \{a: A \cap R_a \neq \emptyset\}$ is the set of things “possessing” something in A .¹⁶ Thus, C is narrowed to $C \cap \text{dom}_A(R)$. Moreover, applying (60) to sentences like (59) (with the obvious constituent structure), evaluation of the possessive DP requires access not only to the restricting set but also to the quantifier Q_1 itself. But Q_1 *cannot* be recovered from Q_1^X . One can always find a different quantifier Q_2 such that $Q_1^X = Q_2^X$, but using Q_2 and $X = C \cap \text{dom}_A(R)$ in the right-hand side of (60) gives a different (and unintended) truth condition. This can be used to find direct counterexamples to the compositionality of this analysis.

I will not dwell on the options here (the situation is discussed at length in Peters and Westerståhl (2013)). The point has just been to show that issues of decomposition can be important for natural language semantics.

11. Questions of Expressive Power

Quantifiers are (extensional versions of) properties of, or relations between, properties and relations, so expressivity questions in this context take the form: “Is property P expressible in terms of $Q_1, Q_2, \dots$?” But the answers, and their interest, depend very much on how “expressible in terms of” is spelled out.

For example, “Is the property of having exactly three elements expressible in terms of quantifiers of the form *at least* n ?” has an obvious positive answer, since *exactly three* = *at least three* but *not at least four*. Here “expressible in terms of” can be taken as “is a Boolean combination of.” Similarly, the question whether *most* is thus expressible has a negative answer, which is almost equally obvious. Questions like these have some logical interest, though usually one would allow definitions that make use of first-order logic FO , rather than just propositional logic. We described logics of the form $FO(Q_1, Q_2, \dots)$ in section 2, and now the definability question “Is Q expressible in terms of $Q_1, Q_2, \dots$?” has a completely precise meaning; illustrated here with Q of type $\langle 1, 1, 2 \rangle$:

- (61) Q is *definable* in a logic $L = FO(Q_1, Q_2, \dots)$ if there is an L -sentence ψ , whose non logical vocabulary consists of two 1-place predicate symbols and one 2-place predicate symbol, such that for all M , all $A, B \subseteq M$, and all $R \subseteq M^2$,

$$Q_M(A, B, R) \Leftrightarrow (M, A, B, R) \models \psi$$

And indeed one can show that *most* is not definable in $L = FO(\text{at least } n: n = 1, 2, \dots)$. But each *at least* n is of course definable already in FO , so FO and L are *equivalent*, and the result says that *most* is not first-order definable.

In general, for logics L, L' , we write $L \leq L'$ (L' is at least as strong as L) iff every L -sentence is logically equivalent to—true in the same models as—some L' -sentence, and $L \equiv L'$ iff $L \leq L'$ and $L' \leq L$. For these logics, expressive power is closely tied to definability; it is easy to show:

Fact 9. Q is *definable* in L iff $FO(Q) \leq L$.

So although the logics FO and $L = FO(\text{at least } n: n = 1, 2, \dots)$ use different variable-binding operators and hence rather different logical syntax, they have the same expressive power: $FO \equiv L$, whereas $FO(\text{most}) \not\leq FO$, i.e. $FO < FO(\text{most})$.

Is there a linguistic interest to this type of question? Apart from the fact that they concern general issues of *meaning* that should interest semanticists, they say something about the logical machinery needed for semantics. For example, the hypothesis that *FO* would suffice is refuted by the fact about *most* just mentioned. Also, one must distinguish positive and negative definability results. A positive result may use a logical defining sentence without any “natural” English translation, rendering its linguistic interest dubious. By contrast, some logical definitions are close enough to linguistic forms that their existence (or non existence) may be relevant to, for example, questions of compositionality. A negative result, on the other hand, is more interesting—and harder to prove—the more general the allowed forms of definition are: if *all* defining forms are ruled out, so are the linguistically “natural” ones.

In the rest of this section I give examples illustrating these and other points concerning expressive power. I start with some familiar Det interpretations.

11.1 *Most vs. more_than*

I mentioned that *most* is not *FO*-definable, but much more can be said. Although it is the relativization of the Rescher quantifier $Q^R = \text{most things}$, $\text{most} = (\text{most things})^{\text{rel}}$, it is not definable from it, even over finite models, as Barwise and Cooper (1981) showed. *Some*, *all*, *at least three* are all definable from the type $\langle 1 \rangle \exists, \forall, \exists_{\geq 3}$, respectively—of which they are the relativizations—but not so for *most*. This shows emphatically that type $\langle 1 \rangle$ quantifiers are not enough for semantics: we need the type $\langle 1, 1 \rangle$ Det interpretations. In fact, an even stronger result holds about *most*:

Theorem 10 (Kolaitis and Väänänen (1995)). *Most is not definable from any finite number of type $\langle 1 \rangle$ quantifiers.*

Most undefinability results for generalized quantifiers in logic rely on so-called Ehrenfeucht–Fraïssé games, plus some finite combinatorics; Peters and Westerståhl (2006), Chapter 13–15, gives an overview.

So what does it take to define *most*? Obviously, to be able to compare cardinal numbers. But the quantifier

$$MO_M(A, B) \Leftrightarrow |A| > |B|$$

is not CONSERV, and hence does not interpret any English Det. However, there is a natural way to express its truth conditions in English: “*There are more As than Bs.*” Paraphrasing once more:

(62) More As than Bs exist.

And this does use an English Det, albeit one that takes *two* noun arguments instead of one:

- (63) a. More women than men smoke.
b. Fewer Swedes than Danes cross the Öresund bridge every day.

Thus, with the type $\langle 1, 1, 1 \rangle$ quantifier

$$\text{more_than}_M(A, B, C) \Leftrightarrow |A \cap C| > |B \cap C|$$

we have

- (64) a. $\text{most}(A, B) \Leftrightarrow |A \cap B| > |A - B| \Leftrightarrow \text{more_than}(A \cap B, A - B, M)$
b. $\text{more_than } x(Ax \wedge Bx, Ax \wedge \neg Bx, x = x)$
c. More As that are B than As that are not B exist.

where (64a) is the set-theoretic definition, (64b) is the defining sentence in $FO(more_than)$, and (64c) is its English rendering. Either way, we see that $FO(most) \leq FO(more_than)$.

What about the other direction? Restricting attention to *finite* universes, we note:

$$|A| > |B| \Leftrightarrow |A - B| + |A \cap B| > |B - A| + |A \cap B| \Leftrightarrow |A - B| > |B - A|$$

Using this with the definitions of *more_than* and *most*, and simplifying somewhat, we eventually obtain:

$$(65) \quad more_than(A, B, C) \Leftrightarrow most(((A \cap C) - B) \cup ((B \cap C) - A), A)$$

This time an English version of the right-hand side would be more cumbersome. Still, it follows that on finite universes, the Det interpretations *most* and *more_than* have the same expressive power.

In fact, all (65) requires is that $A \cap B \cap C$ is finite. Furthermore, when $A \cap B \cap C$ is infinite one can show that another definition of *more_than* in terms of *most* works instead:

$$(66) \quad more_than(A, B, C) \Leftrightarrow most(((A \cap C) - B) \cup ((B \cap C) - A), A) \\ \& \; most(A \cap C, \bar{B})$$

This entails that over all universes

$$(67) \quad FO(most, infinitely_many) \equiv FO(more_than)$$

For one can use the quantifier *infinitely many* to distinguish the two cases just described, and moreover this quantifier is definable in terms of *more_than*: a set is infinite iff removing one element doesn't decrease its cardinality:

$$infinitely_many(A, B) \Leftrightarrow A \cap B \text{ is infinite} \\ \Leftrightarrow \exists a \in A \cap B \text{ s.t. } |(A - \{a\}) \cap B| \geq |A \cap B| \\ \Leftrightarrow \exists a \in A \cap B \text{ s.t. } \neg more_than(A, A - \{a\}, B)$$

The defining sentence in $FO(more_than)$, i.e.

$$\exists x(Ax \wedge Bx \wedge \neg more_than \ y(Ay, Ay \wedge y \neq x, By))$$

seems rather hard to express in “natural” English. But these are the logical facts. Restricted to finite universes, the type $\langle 1, 1 \rangle$ Det denotation *most* and the type $\langle 1, 1, 1 \rangle$ Det denotation *more_than* are equally expressive. But over arbitrary universes, we need to be able to distinguish finite from infinite sets in order to define *more_than* from *most*. That is, one can show that *more_than* is *not* definable over arbitrary universes in $FO(most)$.

As these examples illustrate, we have a number of hard facts about the relative expressive power of various Det interpretations. Sometimes the available definitions are readily expressed in English, other times not. The results are logically significant and often non-trivial. I leave it to the reader to estimate the interest of this kind of results for semantics.

11.2 Definability from monotone quantifiers

Assuming *ISOM* and *FIN*, as we also do in the present subsection, we found a definability result for monotonicity in the number triangle (section 8.3): left *MON* Det interpretations first-order definable. Now consider right monotonicity. Given the ubiquity of right monotone—and usually smooth—Dets, it is of some interest to see how basic these are in terms of expressive power. Again, the number triangle turns out to be instrumental.

The first thing to note is that definability in terms of type $\langle 1 \rangle$ quantifiers is different from definability from their type $\langle 1, 1 \rangle$ relativizations, even though they amount to the same relation between numbers. That is, it differs when Q^{rel} is not itself definable in terms of Q .¹⁷ One can show that a type $\langle 1 \rangle$ quantifier Q is EXT iff Q^{rel} is symmetric (this holds without any assumption on Q), and in this case we have $Q_M^{rel}(A, B) \Leftrightarrow Q_A(A \cap B) \Leftrightarrow Q_M(A \cap B)$, so Q^{rel} is definable from Q . For example, *infinitely many* is definable from Q_0 (see (12), section 3), and *an even number of* is definable from Q_{even} . But, as we saw in the previous subsection, the non-symmetric *most* = $(most\ things)^{rel}$ is not definable from *most things*.

This affects definability from monotone quantifiers. Väänänen (1997) has a perspicuous characterization of definability from type $\langle 1 \rangle$ monotone quantifiers using the number triangle. Say that Q has *bounded oscillation* if, in the number triangle, there is a finite bound to the number of sign switches (from $+$ to $-$ and from $-$ to $+$) on the levels of the triangle.

Theorem 11 (Väänänen (1997)). *A quantifier is definable from monotone type $\langle 1 \rangle$ quantifiers iff it has bounded oscillation.*

For example, *an even number of* clearly has unbounded oscillation, and hence is not definable from any monotone type $\langle 1 \rangle$ quantifiers. But a surprising fact is that it is definable from relativizations of such quantifiers.

To see this, observe first that a monotone increasing type $\langle 1 \rangle$ quantifier can (under ISOM and FIN) be characterized by a function f that yields, for each level n , the point $(n - f(n), f(n))$ where the $+$'s start. That is, for any function f from N to N such that $f(n) \leq n + 1$, define the quantifier Q_f by

$$(Q_f)_M(B) \Leftrightarrow |B| \geq f(|M|)$$

Then Q is monotone increasing iff Q is of the form Q_f for some f . (The case $f(n) = n + 1$ is when $Q_M(B)$ is false for all $B \subseteq M$, $|M| = n$.) Also,

$$(68) \quad (Q_f)^{rel}(A, B) \Leftrightarrow |A \cap B| \geq f(|A|)$$

Now (68) can be utilized to define Q_{even} (and hence *an even number of*) from just one quantifier of the form $(Q_f)^{rel}$. Let

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

Then we can define Q_{even} with the following trick:

$$\begin{aligned} (Q_{even})_M(A) &\Leftrightarrow |A| \text{ is even} \\ &\Leftrightarrow 1 \geq f(|A|) \\ &\Leftrightarrow A = \emptyset \text{ or } \exists a \in A (|\{a\}| \geq f(|A|)) \\ &\Leftrightarrow A = \emptyset \text{ or } \exists a \in A (Q_f)_M^{rel}(A, \{a\}) \\ &\Leftrightarrow (M, A) \models \neg \exists x A(x) \vee \exists x (A(x) \wedge (Q_f)^{rel} y(A(y), y = x)) \end{aligned}$$

In fact, a similar construction works for any EXT type $\langle 1 \rangle$ quantifier (equivalently, any symmetric CONSERV and EXT type $\langle 1, 1 \rangle$ quantifier). Again, however, the defining sentence seems far removed from anything easily expressed in English.

As to definability in general from CONSERV and EXT MON $\uparrow$ quantifiers, Väänänen and Westerstål (2002) generalize Theorem 11 to this case, using a rather more involved notion of

bounded oscillation. However, finding a quantifier *not* definable in this way turns out to be quite complicated, and certainly no natural language examples are known. It seems a fairly safe bet that all Det interpretations are definable from right monotone quantifiers, but it is unclear whether this fact has any linguistic significance. What about definability from (CONSERV and EXT) *smooth* quantifiers? Väänänen and Westerståhl (2002) show that if Q is definable from smooth quantifiers it has bounded oscillation in the original sense so, for example, Q_{even} is not definable in this way. A characterization of definability from smooth quantifiers might be illuminating, but is still open.

11.3 Polyadic quantifiers and reducibility

We may now seem to have strayed somewhat from natural language quantification, so let us turn to a concrete issue related to a particular form of definition, deriving from sentences of the common form

(69) $DP_1 [V DP_2]$

such as

- (70) a. John and Mary read at least one book.
 b. Three critics reviewed one film.
 c. Most students know at least two professors.

Since all involve a transitive verb, their truth conditions can be expressed with a *polyadic* quantifier, which would be of type $\langle 1, 1, 2 \rangle$ for (70b) and (70c), and of type $\langle 1, 2 \rangle$ for (70a).¹⁸ But clearly, the truth conditions can be stated, in a uniform way, using the usual DP and Det interpretations. To formulate this, let the *iteration* $Q \cdot Q'$ of two type $\langle 1 \rangle$ quantifiers Q and Q' be defined, for any M and any $R \subseteq M^2$, by

$$(71) (Q \cdot Q')_M(R) \Leftrightarrow Q_M(\{a : Q'_M(R_a)\})$$

where again $R_a = \{b : R(a, b)\}$. Then we call a quantifier Q of type $\langle 1, 1, 2 \rangle$ *reducible* (Keenan) if there are CONSERV and EXT type $\langle 1, 1 \rangle$ quantifiers Q_1 and Q_2 such that

$$(72) Q(A, B, R) \Leftrightarrow Q_1^A \cdot Q_2^B(R)$$

Similarly for a type $\langle 1, 2 \rangle$ or type $\langle 2 \rangle$ quantifier.¹⁹ The point is that the apparent polyadic quantification is reduced to monadic quantification, and, moreover, in a way that reflects a straightforward compositional analysis of sentences of the form (69).

The polyadic quantifiers in (70) are reducible, since their truth conditions can be written:

- (73) a. $(I_j \wedge I_m) \cdot \text{some}^{\text{book}}(\text{read})$
 b. $\text{three}^{\text{critic}} \cdot \text{one}^{\text{film}}(\text{reviewed})$
 c. $\text{most}^{\text{student}} \cdot \text{at least two}^{\text{professor}}(\text{know})$

That is, these are the default or narrow scope readings of (70). The wide scope readings can be written in as *inverse iterations*, permuting the order of the type $\langle 1 \rangle$ quantifiers and the arguments of R .²⁰

$$Q' \cdot Q(R^{-1})$$

However, it is an interesting fact that many sentences apparently of the form (69) express truth conditions that are *not* reducible. This phenomenon was discussed in detail in van Benthem (1989) and Keenan (1992). The latter paper presents a multitude of examples; let us look at three kinds:

- (74) a. John and Mary read the same books.
 b. 32 critics reviewed 60 films (at the festival).
 c. Exactly one man loves exactly one woman.

These sentences can be taken to use the following quantifiers:

- (75) a. $A \cap R_j = A \cap R_m \neq \emptyset$
 b. $32^A \cdot \text{some}^B(R) \wedge 60^B \cdot \text{some}^A(R^{-1})$ (32 = *exactly thirty-two*)
 c. $\exists X \subseteq A \exists Y \subseteq B (|X| = |Y| = 1 \ \& \ R = X \times Y)$

Here (74b) is given a so-called *cumulative* reading, and (74c) a *branching* reading.²¹ We have:

Fact 12. *None of the quantifiers in (74) are reducible.*

This can be proved in various ways; I outline Keenan's proof for (a sentence similar to) (74a) as an example. The crucial observation is Keenan's *Product Theorem*, which says that if two iterations (i.e. two quantifiers defined as in (71)) are equal when the relation argument is a cross-product, they are equal on all relation arguments. Now compare

- (76) a. John and Mary read the same books.
 b. John and Mary read some books.

(76b) (with the default scope) means $(I_j \wedge I_m) \cdot \text{some}^{\text{book}}(\text{read})$. But it is easy to see that if *read* is a cross-product, (76a) is true iff (76b) is true. That is,

$$\text{if } R = B \times C, \text{ then } A \cap R_j = A \cap R_m \neq \emptyset \Leftrightarrow A \cap R_j \neq \emptyset \ \& \ A \cap R_m \neq \emptyset.$$

So if (a) were reducible, (a) and (b) would be synonymous, by the Product Theorem. But they clearly aren't: (b) doesn't require any book to have been read by both John and Mary.²²

We may conclude that no straightforward compositional analysis of sentences (74) exists. This doesn't mean that none is available, but at least one has to work harder to find one. Barker (2007) presents an analysis of "*same*" as a scope-taking adjective, and uses it in a compositional account of sentences like (74a). And branching or partially ordered quantification was the inspiration for Hintikka's *Independence-Friendly Logic* (see Hintikka and Sandu (1997)), which was given a compositional semantics in Hodges (1997), which in turn led to *Dependence Logic* (see Väänänen (2007)), where dependencies between variables are treated with special atomic formulas. In general, the search for (reasonable) compositional semantics in the face of apparent counterexamples has been quite fruitful, both in logic and linguistics.

11.4 Resumption, polyadicity, and processing

Let us end by looking at one more type of polyadic quantification, so-called *resumption*, and use it illustrate two aspects of complexity: one logical and one computational. We continue to assume ISOM and FIN. The resumption of a monadic quantifier Q simply applies Q to pairs (more generally, k -tuples) of individuals. If Q has type $\langle 1 \rangle$ and $R \subseteq M^2$,

$$(77) \text{Res}^2(Q)_M(R) \Leftrightarrow Q_{M^2}(R)$$

In semantics, resumptions of Det denotations such as *most* have been proposed for the analysis of so-called adverbs of quantification (Lewis, 1975), as in

- (78) Men are usually taller than women.

Once you have Q , it seems fairly straightforward to apply it to pairs; after all, Q assigns a local quantifier to *every* universe, including those of the form M^2 . Nevertheless, from a logical point of view, $\text{Res}^2(Q)$ is much more complicated than Q .

Since $\text{Res}^2(Q)$ only cares about how Q behaves on universes whose number of elements is a square, Q is not in general definable from $\text{Res}^2(Q)$. However, one can show that if Q is Ext (behaves “the same” on all universes), Q can be defined from $\text{Res}^2(Q)$ (Peters and Westerståhl, 2006, section 15.5.1). When is $\text{Res}^2(Q)$ definable from Q ?

This is a good place to emphasize something that has only been implicit so far: We are talking about *uniform* definability, where the same defining sentence works for all universes. There is also a weaker notion of *local* definability over a fixed universe M . If M is finite, the defining sentence may then contain, for example, conjunctions and disjunctions indexed by the elements of M , or the subsets of M . Thus, Keenan and Stavi (1986) prove that every CONSERV type $\langle 1, 1 \rangle$ quantifier on M can be written as a Boolean combination of Montaguevian individuals (one of their “effability theorems”), a result which has no uniform counterpart. And, returning to the present case, van Benthem (1989) shows that each resumption $\text{Res}^2(Q)_M$ is a Boolean combination of iterations and inverse iterations (section 11.3), also called a *unary complex*, where again the defining sentence depends on the size of M . In this case, however, since $\text{Res}^2(Q)$ seems so closely related to Q , one might expect a uniform result. But this expectation fails.

In fact, one can show fairly easily with Ehrenfeucht–Fraïssé methods that $\text{Res}^2(\text{most})$ is not definable in $\text{FO}(\text{most})$. This leaves open that other monadic quantifiers might work, perhaps even with a unary complex. But a celebrated result (celebrated for the complex combinatorics used in its proof) by Kerkko Luosto shows that this is impossible. Let **Mon** be the class of *all* monadic quantifiers, and recall the Rescher quantifier $Q^R = \text{most things}$ from section 3.

Theorem 13 (Luosto (2000)). $\text{Res}^2(Q^R)$ is not definable in $\text{FO}(\text{Mon})$.

It easily follows that $\text{Res}^2(\text{most})$ is not definable in $\text{FO}(\text{Mon})$ either.

This shows that, as regards logical definability, there is an unbridgeable gap between polyadic (in fact type $\langle 2 \rangle$) and monadic quantification, even for the resumption of common Det interpretations.

But logical expressive power is not the only way to gauge the complexity of quantification. Specifically, is there some other notion of complexity that supports the intuition that the resumption of Q is not that much more complicated than Q itself?

On finite universes, sets and relations, i.e. finite structures, can be coded as *words* (finite strings of symbols), and quantifiers as sets of words. A computing device computes Q if it *accepts* exactly the words in Q . In van Benthem (1986), quantifiers are classified according to what kind of *finite automata* is required to compute them. For example, it is shown that first-order definable monadic quantifiers are precisely those computed by acyclic and permutation-closed automata, and that proportional quantifiers like *most* can be computed by push-down automata.

Another approach is to measure the *computational complexity* of computing various quantifiers. Using the most general computing devices, Turing machines, one estimates how many steps it takes (*time complexity*; alternatively how much space is needed; *space complexity*), for the machine corresponding to Q , to accept or reject an input of length n . If the least number of steps required is bounded by a polynomial in n , Q is said to be in PTIME (or P). If the same holds when non deterministic Turing machines are also allowed, the quantifier is in NP. There are numerous results relating computational complexity to logical expressive power. A famous early example is *Fagin’s Theorem*, stating that the NP properties of finite structures are precisely those expressible in *existential second-order logic*. On the other hand, Lauri Hella proved that no logic of the form $\text{FO}(Q_1, \dots, Q_n)$ can capture precisely the PTIME properties. The interest in PTIME is due to the common assumption that problems in PTIME are “tractable,” amenable to a reasonably efficient algorithm, whereas problems beyond PTIME are not.

Recently, computational complexity has been brought to bear on quantifiers occurring in natural language, in particular polyadic quantifiers. All of these can be seen as *lifts* of monadic quantifiers: they result from applying certain operations on monadic quantifiers: iteration, cumulation, branching, “Ramseyfication” (cf. (58) in section 10), resumption. A natural question is then if these operations preserve the property of being tractable, i.e. PTIME. Szymanik (2010) presents a number of results in this area. Specifically, branching (this is due to Sevenster) and “Ramseyfication” can lead to intractable quantifiers, but iteration, cumulation, and resumption preserve tractability. The facts about iteration and cumulation are expected, but it is noteworthy that resumption also preserves PTIME. This lends some support to the view that in terms of (human) computation, quantifying over pairs rather than individuals is a fairly simple addition.

Szymanik argues that these facts are indeed significant for human processing, and he and other theorists have begun carrying out various psychological experiments that seem to vindicate some of these claims. In any event, theoretical as well as experimental research on human processing and quantification is a live and developing field today, complementing the initially purely model-theoretic approach to generalized quantifiers in natural language.

NOTES

- 1 Some theorists consider Dets like “many,” “few,” “more than enough” as intensional, and either exclude them from a GQ treatment (Keenan and Stavi, 1986) or use possible worlds semantics for their interpretation (Fernando and Kamp, 1996). I do not discuss intensional interpretations here. The alternative, if one wants to admit Dets like these, is to impose a fixed context assumption.
- 2 In (5) we could have written $Q(\tilde{x}[\varphi])(\tilde{y}[\psi])$ instead; similarly it is not essential that the same variable is bound in the two formulas in (4), as long as one fixed variable is bound in each. So instead of (4) we could also use $Qx, y(\varphi, \psi)$ with the understanding that (each free occurrence of) x is bound in φ , and y in ψ ; see (7) below.
- 3 Similar abuses of notation will occur frequently in this chapter, to avoid clutter. For example, the extension of a word like “dog” will often be written simply *dog*, rather than, say, $\llbracket \text{dog} \rrbracket$, or dog' , or **dog**. Similarly, the quantifier interpreting an English determiner like “most” is written *most*, as is the corresponding variable-binding operator in $FO(\text{most})$. No confusion should result from this.
- 4 In English and many other languages, determiners form a rich and productive class of expressions denoting $\langle 1, 1 \rangle$ quantifiers. Other languages mainly employ so-called A quantification (see Bach *et al.* (1995) for examples), with, for instance, adverbs or auxiliaries used for the same purpose.
- 5 Some proportional Dets like “most” are context-dependent in that the actual proportion may vary with context. So “most” could sometimes mean “more than half” and sometimes “at least 75 %”. Interestingly, the definition of proportionality here (which is due to Ed Keenan) works also when such context-dependence is allowed.
- 6 This is what we can say about bare plurals within GQ theory. From a linguistic point of view there is much more to their semantics; in particular the relation between their universal reading (but not the existential one) and *generics*; see, for example, Carlson and Pelletier (1995).
- 7 There are semantic frameworks, different from the present one, where individuals can be conjoined, in order to deal with plurals, as in “Henry and Sue met outside the NYU Linguistics Department”; for example Link (1983, 1987). But no plural individual is involved in (22).
- 8 Westerståhl (2012b) studies how this modern version of the Aristotelian square compares to the classical one, and how it applies to various Det denotations, in particular possessive Dets. It is not always the case that all four corners of the square are occupied by interpretations of well-formed English Dets; for example “all but at least five” above seems slightly deviant.
- 9 More generally, no CONSERV Q is *self-dual* in this sense, since that would require $Q_M(A, B) \Leftrightarrow \neg Q_M(A, A - B)$, which is impossible for $A = B = \emptyset$. On the other hand, a *local* type $\langle 1 \rangle$ quantifier Q_M can be self-dual; a typical example is $(I_a)_M$, provided $a \in M$.
- 10 Van Benthem identified the smoothness property (which he called *continuity*) precisely in this connection; see van Benthem (1986).

- 11 It is another matter, not directly relevant to our present concerns, that according to Russell unique existence is part of *what is said* by such a sentence, whereas most linguists nowadays would regard it as a *presupposition*.
- 12 Barwise and Cooper treated the quantifiers as *undefined* in these cases, as a way of imposing the existence of a generator as a presupposition. Here we let $Q_M = \emptyset$ instead, in order to avoid making the logic of quantifiers partial. For the purpose of discussing definiteness, the difference is negligible.
- 13 And not *some of Mary's*, as it does in
- (i) When Mary's dogs escape, her neighbors usually catch them.
- However, for any Q_2 , the quantifier Q_2 of Mary's A , as in
- (ii) Most of Mary's dogs are well behaved.
- is in fact ISOM decomposable.
- 14 It does mean the wide scope reading of (55), but that is not a reading where "*belonging to exactly two students*" semantically restricts "*all books*".
- 15 Of course there are trivial ways to achieve compositionality in such cases; for example, let the meaning of $[\text{Det } N']$ be the pair $([\text{Det}], [N'])$. But this semantics has no interest. My presupposition here is that DPs mean type $\langle 1 \rangle$ quantifiers and Dets mean type $\langle 1, 1 \rangle$ quantifiers.
- 16 This is the special case where quantification over 'possessions' is universal. For discussion, see Peters and Westerståhl (2013).
- 17 Q is always definable from Q^{rel} : $Q_M(B) \Leftrightarrow Q_M^{rel}(M, B)$.
- 18 More naturally perhaps, (a) would use a type $\langle 0, 0, 1, 2 \rangle$ quantifier, allowing individuals too as arguments. I stick to the standard notion of quantifier here.
- 19 We can omit the universe in (72) since all quantifiers involved are EXT.
- 20 To semantically represent these readings is easy. To derive them compositionally is another matter, requiring some extra interpretation mechanism: quantifying in (Montague), Quantifier Raising (May), Cooper storage, ...
- 21 In addition to its two reducible readings, i.e. that exactly one man is such that he loves just one woman (whereas other men may love several or no women), and the corresponding wide scope version. It is branching in the sense that the two DPs are independent of each other: neither is in the scope of the other, as the symmetric form of (75c) shows. If you doubt that the branching reading of (74c) exists, consider this example (due to Irene Heim):

John has published exactly one article in exactly one journal.

- 22 For more about when various polyadic quantifiers are equivalent to iterations, see Westerståhl (1994).

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