1

Russell and Moore

The question which Kant put at the beginning of his philosophy, namely ''How is pure mathematics possible?'' is an interesting and difficult one, to which every philosophy which is not purely sceptical must find some answer. (Russell 1959a/1912, p. 84)

Empiricism, Mathematics, and Symbolic Logic

Bertrand Russell – aristocrat (3rd Earl Russell), anti-war activist, prolific writer, and brilliant philosopher and mathematician – is the father of Anglo-American analytic philosophy. Russell did the hard work of expounding and promulgating the new symbolic logic that was to revolutionize the method of philosophy. Equally important for analytic philosophy, he introduced others to the works of Gottlob Frege and Ludwig Wittgenstein, who might otherwise have languished unappreciated. Russell proposed and energetically pursued philosophical issues that were keenly examined by philosophers throughout the twentieth century. Without Bertrand Russell's work, especially the work he produced early in his career in logic and the philosophy of language, there would have been no Anglo-American analytic philosophy.

Russell says that Frege was the pioneer and no doubt this is true. ''Many matters which, when I was young, baffled me by the vagueness of all that had been said about them, are now amenable to an exact technique, which makes possible the kind of progress that is customary in science. . . . [T]he pioneer was Frege, but he remained solitary until his old age''(Russell 1963/1944, p. 20).[1] Russell's optimism about

[1] Citations here and throughout are indicated as (Name date/original publication date if significantly different, page)

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philosophical progress may seem overstated, but not his judgment of Frege. Frege did revolutionary work on the foundations of mathematics and was the first to clarify and investigate issues in the philosophy of language that were central to twentieth-century philosophy and are still central today. Indeed Gottlob Frege *was* the pioneer of the techniques that gave life to analytic philosophy, but he would not have had an impact without Russell's influence. Frege would have remained solitary. Russell brought Frege to the attention of other philosophers and mathematicians, especially in the English-speaking world, and developed and improved Frege's pioneering ideas.

Russell's greatest contribution to logic, philosophy, and mathematics was his publication of *Principia Mathematica* with Alfred North Whitehead (published in three volumes, 1910–13). Based on ideas originally articulated by Frege in the late nineteenth century, Russell developed and founded the field of symbolic logic. Symbolic logic today is central not only to philosophy but to many other areas including mathematics and computer science.[2] In addition to *Principia Mathematica* (often referred to simply as *PM*), Russell expounded the ideas and methods of the new symbolic logic energetically in his *Principles of Mathematics* and many other influential publications early in the twentieth century. The influence, importance, and central role of *PM* cannot be overemphasized. For example, Kurt Gödel titled his historic paper ''On formally undecidable propositions of *Principia Mathematica* and related systems.'' (More on this influential mathematical paper below (see pp. 162–4).)

The methodology that gives analytic philosophy its strength and structure is the logic and philosophy of language generated by the original work of Frege, Russell, and Whitehead.

Their results in logic and the philosophy of language have also had major impacts in other areas of philosophy. The revolution in logic in the early years of the twentieth century gave analytic philosophers the tools to articulate and defend a sophisticated form of empiricism. [*Background 1.1* – Epistemology: empiricism versus rationalism (The background snippets are found at the end of the chapter.)] With the new tools in logic and philosophy of language, philosophers were able to repair the flaws and gaps in thinking of the classical British empiricists. The major gap was the lack of an explanation of how pure mathematics is possible. Modern logic as developed by Frege, Russell, and Whitehead yielded definite results in the foundations of mathematics and the philosophy of language that, though technical

[2] *Principia Mathematica* was voted number 23 of the 100 most important nonfiction books of the twentieth century – the highest rated philosophy book. (http://www.infoplease.com/ipea/A0777310.html)

and expounded in daunting detail, went to the heart of epistemological issues. Empiricists could claim to have solved the outstanding problems plaguing their theory – namely our knowledge of mathematics – by using the techniques of mathematical logic. (This is explained in the next section.)

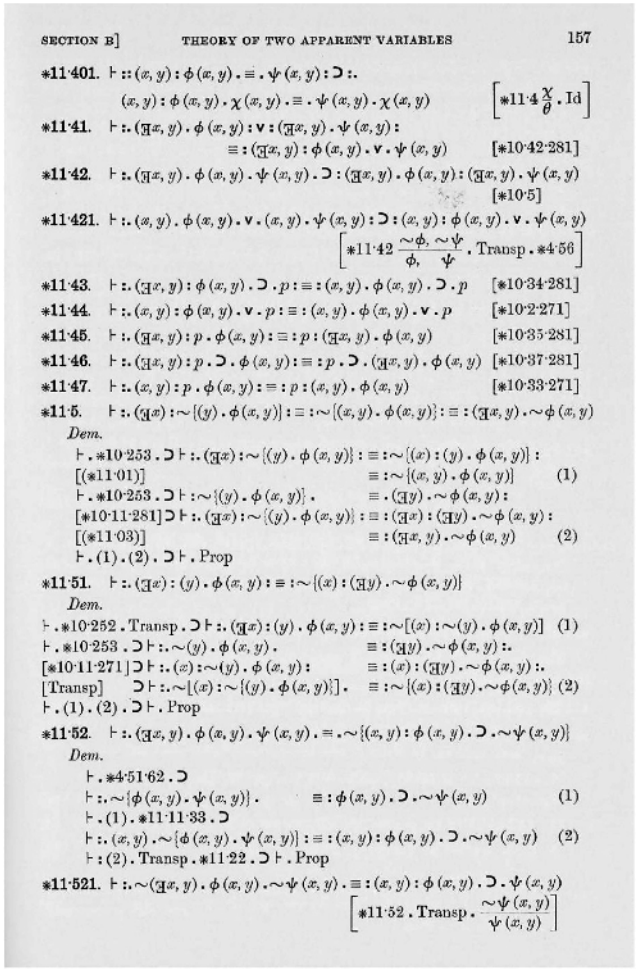
Although Russell was uneasy with empiricism, his sympathy was with the classical British empiricists. Virtually all analytic philosophers have shared this sympathy while at the same time becoming increasingly uneasy with the details and presuppositions of classical empiricism. Russell could not accept ''pure empiricism'' – the view that all knowledge is derived from immediate sensory experience – but sought to move only as far from it as was absolutely necessary. Speaking of his very early views Russell says: ''it seemed to me that pure empiricism (which I was disposed to accept) must lead to skepticism . . . ''(Russell 1959b/1924, p. 31). Even worse than skepticism, Russell came to believe that pure empiricism led to solipsism and could not account for our knowledge of scientific laws or our beliefs about the future. Still, Russell always seemed to feel that these were problems for empiricism, not reasons to discard it outright.

Despite his sympathy with empiricism, in places Russell sounds like an unabashed rationalist: ''It is, then, possible to make assertions, not only about cases which we have been able to observe, but about all actual or possible cases. The existence of assertions of this kind and their necessity for almost all pieces of knowledge which are said to be founded on experience shows that traditional empiricism is in error and that there is *a priori* and universal knowledge'' (Russell 1973, p. 292. From a lecture given in 1911). [*Background 1.2* – A priori, analytic, necessary]

Despite his wavering philosophical sympathies, Russell's mathematical logic gave later empiricists the tools to respond to the troubling difficulties with their position that Russell was pointing out. Mathematics is *a priori* and universal, so how can it be empirical? Twentieth-century analytic philosophy got its first shot of energy from a plausible answer to this question – an answer offered by the logical investigations of Frege and Russell.[3]

Frege and Russell were able to use symbolic logic to reconceptualize the very nature of mathematics and our mathematical knowledge (Figure 1.1). I must emphasize that symbolic logic as developed in

[3] Whitehead, also a brilliant philosopher, logician, and mathematician did much of the technical work of developing symbolic logic but did not play the kind of role that Russell did in publishing it, publicizing it, and making it accessible to fellow philosophers, and showing how fruitful and valuable a tool it was.



SECTION B] THEORY OF TWO APPARENT VARIABLES 157

$*11{\cdot}401.\ \vdash :: (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset :.$
$(x, y) : \phi(x, y) . \chi(x, y) . \equiv . \psi(x, y) . \chi(x, y) \qquad \left[*11{\cdot}4\,\frac{\chi}{\theta} . \mathrm{Id}\right]$

$*11{\cdot}41.\ \vdash :. (\exists x, y) . \phi(x, y) : \vee : (\exists x, y) . \psi(x, y) :$
$\equiv : (\exists x, y) : \phi(x, y) . \vee . \psi(x, y) \qquad [*10{\cdot}42{\cdot}281]$

$*11{\cdot}42.\ \vdash :. (\exists x, y) . \phi(x, y) . \psi(x, y) . \supset : (\exists x, y) . \phi(x, y) : (\exists x, y) . \psi(x, y)$
$[*10{\cdot}5]$

$*11{\cdot}421.\ \vdash :. (x, y) . \phi(x, y) . \vee . (x, y) . \psi(x, y) : \supset : (x, y) : \phi(x, y) . \vee . \psi(x, y)$
$\left[*11{\cdot}42\,\frac{\sim\phi,\ \sim\psi}{\phi,\ \psi} . \mathrm{Transp} . *4{\cdot}56\right]$

$*11{\cdot}43.\ \vdash :. (\exists x, y) : \phi(x, y) . \supset . p : \equiv : (x, y) . \phi(x, y) . \supset . p \qquad [*10{\cdot}34{\cdot}281]$

$*11{\cdot}44.\ \vdash :. (x, y) : \phi(x, y) . \vee . p : \equiv : (x, y) . \phi(x, y) . \vee . p \qquad [*10{\cdot}2{\cdot}271]$

$*11{\cdot}45.\ \vdash :. (\exists x, y) : p . \phi(x, y) : \equiv : p : (\exists x, y) . \phi(x, y) \qquad [*10{\cdot}35{\cdot}281]$

$*11{\cdot}46.\ \vdash :. (\exists x, y) : p . \supset . \phi(x, y) : \equiv : p . \supset . (\exists x, y) . \phi(x, y) \qquad [*10{\cdot}37{\cdot}281]$

$*11{\cdot}47.\ \vdash :. (x, y) : p . \phi(x, y) : \equiv : p : (x, y) . \phi(x, y) \qquad [*10{\cdot}33{\cdot}271]$

$*11{\cdot}5.\ \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : \sim\{(x, y) . \phi(x, y)\} : \equiv : (\exists x, y) . \sim\phi(x, y)$

Dem.

$\vdash . *10{\cdot}253 . \supset \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : \sim\{(x) : (y) . \phi(x, y)\} :$
$[(*11{\cdot}01)] \qquad \equiv : \sim\{(x, y) . \phi(x, y)\} \qquad (1)$
$\vdash . *10{\cdot}253 . \supset \vdash : \sim\{(y) . \phi(x, y)\} . \qquad \equiv . (\exists y) . \sim\phi(x, y) :$
$[*10{\cdot}11{\cdot}281] \supset \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : (\exists x) : (\exists y) . \sim\phi(x, y) :$
$[(*11{\cdot}03)] \qquad \equiv : (\exists x, y) . \sim\phi(x, y) \qquad (2)$
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*11{\cdot}51.\ \vdash :. (\exists x) : (y) . \phi(x, y) : \equiv : \sim\{(x) : (\exists y) . \sim\phi(x, y)\}$

Dem.

$\vdash . *10{\cdot}252 . \mathrm{Transp} . \supset \vdash :. (\exists x) : (y) . \phi(x, y) : \equiv : \sim[(x) : \sim(y) . \phi(x, y)] \qquad (1)$
$\vdash . *10{\cdot}253 . \supset \vdash :. \sim(y) . \phi(x, y) . \qquad \equiv : (\exists y) . \sim\phi(x, y) :.$
$[*10{\cdot}11{\cdot}271] \supset \vdash :. (x) : \sim(y) . \phi(x, y) : \qquad \equiv : (x) : (\exists y) . \sim\phi(x, y) :.$
$[\mathrm{Transp}] \qquad \supset \vdash :. \sim[(x) : \sim\{(y) . \phi(x, y)\}] . \qquad \equiv : \sim\{(x) : (\exists y) . \sim\phi(x, y)\} \qquad (2)$
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*11{\cdot}52.\ \vdash :. (\exists x, y) . \phi(x, y) . \psi(x, y) . \equiv . \sim\{(x, y) : \phi(x, y) . \supset . \sim\psi(x, y)\}$

Dem.

$\vdash . *4{\cdot}51{\cdot}62 . \supset$
$\vdash :. \sim\{\phi(x, y) . \psi(x, y)\} . \qquad \equiv : \phi(x, y) . \supset . \sim\psi(x, y) \qquad (1)$
$\vdash . (1) . *11{\cdot}11{\cdot}33 . \supset$
$\vdash :. (x, y) . \sim\{\phi(x, y) . \psi(x, y)\} : \equiv : (x, y) : \phi(x, y) . \supset . \sim\psi(x, y) \qquad (2)$
$\vdash : (2) . \mathrm{Transp} . *11{\cdot}22 . \supset \vdash . \mathrm{Prop}$

$*11{\cdot}521.\ \vdash :. \sim(\exists x, y) . \phi(x, y) . \sim\psi(x, y) . \equiv : (x, y) : \phi(x, y) . \supset . \psi(x, y)$
$\left[*11{\cdot}52 . \mathrm{Transp} . \frac{\sim\psi(x, y)}{\psi(x, y)}\right]$

Figure 1.1 This is a page chosen at random from *Principia Mathematica. PM* is extremely daunting and is today studied only by specialists, although it has an accessible introduction that is a brief overview of symbolic logic.

PM was not just the use of symbols – so that for example we use ''∨'' instead of the word ''or'' and ''(∃x)'' for ''some.'' That would be impressive perhaps and simplifying in some ways, but not revolutionary. The revolution in logic, pioneered by Frege, and expounded by *PM* was based on the concept of treating logic mathematically, and then treating mathematics as a form of logic. This is Frege's and Russell's

logicism.[4] [*Background 1.3* – Mathematical logic of *PM* versus traditional Aristotelian logic and a note on symbolism]

Symbolic logic is not only of technical interest for those concerned about the foundations of mathematics. Virtually every philosophy major in every college and university in the United States and elsewhere is required to pass a course in symbolic logic. Not only philosophy majors, but other students as well – computer science majors, mathematics majors, not to mention English majors – take symbolic logic courses. Symbolic logic has also been central to the development of computers, and it is now a branch of mathematics, and is an indispensable tool for theoretical linguistics and virtually anyone working in technical areas of the study of language.

Symbolic logic has been the central motivating force for much of analytic philosophy. Besides giving philosophers the tools to solve problems that have concerned thinkers since the Greeks, the notion that mathematics is logic points to an answer to the question posed by Kant in the quote that opens this chapter. ''How is pure mathematics possible?'' This is an answer that removes mathematics as an obstacle to empiricism. Mathematics is possible because it is analytic.

Logicism

Whitehead and Russell's *PM* was an elaborate argument for logicism, which in turn was based on earlier work by Frege. The logicist program is succinctly stated by Russell and attributed to Frege: ''Frege showed in detail how arithmetic can be deduced from pure logic, without the need of any fresh ideas or axioms, thus disproving Kant's assertion that '$7 + 5 = 12$' is synthetic'' (Russell 1959b, p. 32).

Logicism was one of several responses to difficulties that emerged in the foundations of mathematics toward the end of the nineteenth century. These difficulties perplexed Russell and many others. We can skip over the technicalities for now, and keep in mind that none of the difficulties that troubled Russell would matter for any practical applications of mathematics or arithmetic. You could still balance your checkbook even if the foundations of mathematics had not been put on a firm footing. Nevertheless, to a philosopher of Russell's uncompromising character these difficulties were intellectually troubling. The results of his investigations have thrilled and baffled philosophers ever

[4] As opposed, e.g., to psychologism, the view that mathematics is derived from human psychology. Frege was deeply opposed to psychologism.

since and tormented and fascinated (at least a few) students taking Symbolic Logic.

The nature of mathematics and arithmetic is a central problem for philosophers, especially in the area of epistemology. In the argument between the empiricists and rationalists, the question of our knowledge of mathematical facts plays a key role. Even an impure (i.e., moderate) empiricist must answer the question how we know that $7 + 5 = 12$, that the interior angles of a triangle equal 180°, that there are infinitely many prime numbers, and so on. ''Of course, we know them because we were told them in school and read them in the textbook.'' This answer, while having an appealing simplicity, would disappoint both the rationalist and the empiricist and is abjectly unphilosophical. We know those mathematical facts because we can figure them out, ''see the truth of them,'' especially when we've been shown the proofs or done the calculations. [*Background 1.4* – Proofs that the sum of the interior angles of a triangle is 180° and that there are infinitely many prime numbers] And the marvelous thing is, not only that we ''see'' the truths, but also understand that they must be so, could not be otherwise, and are necessary and absolute. No experience could impart such certainty. Mathematical knowledge dooms the empiricist claim that all knowledge is based on experience.

Russell's assertions about geometry in the following quote apply to all of mathematics. (When he uses the term ''idealists'' his description applies to rationalists.)

> Geometry, throughout the 17th and 18th centuries, remained, in the war against empiricism, an impregnable fortress of the idealists. Those who held – as was generally held on the Continent – that certain knowledge, independent of experience, was possible about the real world, had only to point to Geometry: none but a madman, they said, would throw doubt on its validity, and none but a fool would deny its objective reference. The English Empiricists, in this matter, had, therefore, a somewhat difficult task; either they had to ignore the problem, or if, like Hume and Mill, they ventured on the assault, they were driven into the apparently paradoxical assertion that Geometry, at bottom, had no certainty of a different kind from that of Mechanics . . . (Russell 1897, p. 1)[5]

The problem that empiricism has with mathematics is worth pondering. Even if ''$7 + 5 = 12$'' and ''the interior angles of a triangle equal

[5] Infamously, John Stuart Mill claimed that mathematical truths were based on experience. Few empiricists have agreed with Mill's view. Russell could not accept it and surely Russell is right.

180°'' are derived in some way from experiences of counting and measuring angles, it is impossible that, e.g., our knowledge of the infinitude of primes comes from experience. Although perhaps the idea could be led back by many steps to experiences with counting and dividing and so on, I do not see how any experience or observation (other than ''seeing'' the proof) could get one to know with certainty that there are infinitely many prime numbers. Using a computer to generate prime numbers wouldn't help. It would just keep calculating primes, but how could we know it would never get to the last one? There is no possible empirical test that would establish that there are infinitely many primes. Yet the proof is so simple and obvious that there can be no doubt. If you are troubled by the indirect nature of the proof, be assured there are direct proofs. In any case, Euclid's proof assures us there is a larger prime given any series of primes.

Empirical evidence and observations even if pervasive and universal cannot explain the certainty and necessity of mathematical propositions. In the case of a mathematical proposition such as ''$7 + 5 = 12$,'' empirical observations are not evidence or support. If a proposition is based on observational evidence, then there must be possible observations that one could describe that would refute the proposition. No possible observations would refute ''$7 + 5 = 12$.'' If every possible observation, test, and experiment is compatible with the truth of the proposition, then observation, test, and experiment is irrelevant to the proposition. This is the case with the true mathematical propositions that I cited. A simple example should suffice: If I put 7 sheep in the pen, and then 5 more and counted all the sheep and kept getting 11, I would assume that one of the sheep was stolen, escaped, or had been kidnapped by aliens. The last thing I would ever judge is that $7 + 5$ does not equal 12. Indeed, I would never judge that unless I had lost all sense of reason. To repeat, if no possible experience or observation would lead us to give up a proposition, then it is not based on experience or observation. In mathematics we have decisive counterexamples to empiricism: propositions that are true, that we know to be true and in fact are absolutely certain but are not based on observation, test, experiment, or experience.

This much was accepted by Russell and empiricists (other than Mill) and has been accepted by most philosophers since. Our mathematical statements and ones like them are necessarily true and are not based on sensory experience in that they are not empirical scientific results established in the lab or field by the scientific method and observation. The only alternative source seems to be pure reason. The victory cheers of the rationalists are ringing through the ages. Here are clear examples

of important, useful, evident items of knowledge based on and derivable from pure reason. There's an old saying that goes something like this: ''If the camel once gets his nose in the tent, his body will soon follow.'' If we once grant that mathematical knowledge is non-empirical and based on pure reason as the rationalist claims, then there will be no stopping the rest of the body getting into the tent: metaphysics, religion, ontology, cosmology, ethics, aesthetics, etc. all will follow. Empiricism will be bankrupt.

The most influential modern version of the rationalist claim is Kant's view that, e.g., ''$7 + 5 = 12$'' is synthetic *a priori*. The empiricist response to this apparently devastating claim, a response based on the logical system of Frege and Russell, forms a main current of analytic philosophy. One central tenet of the logical positivists (see Chapter 2) – often called logical empiricists – is that there are no synthetic *a priori* propositions. Russell (and Whitehead) following Frege made the first key step in the empiricist response. This step is their logicism.[6]

> From Frege's work it followed that arithmetic, and pure mathematics generally, is nothing but a prolongation of deductive logic. This disproved Kant's theory that arithmetical propositions are ''synthetic'' and involve a reference to time. The development of pure mathematics from logic was set forth in detail in *Principia Mathematica* by Whitehead and myself. (Russell 1945, p. 830)

Kant's views on mathematics were outmoded in any case and ready to be replaced by more modern ones. For example, many developments in nineteenth-century mathematics such as non-Euclidean geometry cast suspicion on Kant's method of arguing for his synthetic *a priori*.

Frege attempted to demonstrate that all of mathematics could be derived from pure logic – thus logicism. The derivation of mathematics from logic was a complex and stepwise procedure. Many previous mathematical results were required. By the end of the nineteenth century, mathematicians believed they had shown how all statements of traditional mathematics could be expressed as statements about the natural numbers, so all statements of mathematics could be reformulated as statements about the natural numbers. And furthermore, all theorems about the natural numbers could be derived from Peano's five axioms.[7] So it was believed that all of traditional mathematics could be derived from the Peano postulates. This made the plans of

[6] Neither Russell, Whitehead, nor Frege ever used the term ''logicism.'' The term was coined by Rudolf Carnap, one of the foremost logical empiricists, in the 1930s.

[7] Named after Giuseppe Peano (1858–1932), an Italian mathematician who developed the postulates and influenced Russell's view of mathematics.

Russell, following upon the work of Frege, vastly simpler. Instead of focusing on all of the vastness of mathematics, he could focus just on the Peano postulates. Russell discovered a fatal flaw in Frege's work (described below) and working with Whitehead attempted to overcome this flaw and thus successfully complete the work begun by Frege.

If this logicist program could be neatly carried out, it would solve both the problems with the synthetic *a priori* and the view that mathematical knowledge is based on observation. The point is that logic is analytic, so if the Peano postulates could be shown to be purely logical in nature, this would demonstrate that the postulates are analytic – and thus that anything that followed from them by pure logic would also be analytic. We could see, then, exactly why and how ''$7 + 5 = 12$'' is not synthetic *a priori*. It is analytic, and thus *a priori* at least in the special sense that it could be justified without appeal to observation or experience – justified simply by its form once the meanings of its constituent terms are fully spelled out.

Although Frege and Russell had determined that mathematics is reducible to logic – that mathematics and logic are one – they had not fully realized the extent to which this trivialized mathematics. Russell was reluctantly driven to the view that mathematics is all tautological. ''I thought of mathematics with reverence, and suffered when Wittgenstein led me to regard it as nothing but tautologies'' (Russell 1963/1944, p. 19). (In the next chapter we will see how Wittgenstein developed this idea.) The view that mathematics consists of nothing but tautologies disappointed Russell but energized the empiricist philosophers who followed Frege, Russell, and Wittgenstein. It answers Mill and at once appeals to the empiricists, because it demystifies mathematics – gets the camel's nose out of the tent. Neither the rationalists nor anybody else would be happy to say that the truths of religion, metaphysics, ethics, etc. are tautologies like the truths of mathematics. The empiricists now had a simple, clear explanation of how we know and how we can justify mathematical knowledge without Kant's synthetic *a priori* or any other form of knowledge that would give support to rationalism.

The logicist program consists of several steps. First mathematics is reduced to a relatively simple base such as the Peano postulates. Then the base is shown to be translatable into terms solely from pure logic. Then the axioms are shown to be truths of logic. The final step due to Wittgenstein is the claim that the truths of logic are tautological. This explains how mathematical knowledge can be *a priori*, but the rationalist can no longer use mathematics as a weapon against the empiricist. The rationalist is left holding an empty bag. Granted, mathematical propositions are not based on experience or observation,

but they are not the results of pure rational insight into the ultimate nature of reality either. They are based on a clear process that involves no rationalist mystification and involves only epistemic procedures that are acceptable to the empiricists. We know mathematical propositions in the way that we know that ''All grandmothers are mothers'' or that ''Either all insects have eight legs or it is not the case that all insects have eight legs.''

> The further property needed to make a proposition one of mathematics or logic . . . is the property traditionally expressed by saying that the propositions concerned are 'analytic' or 'logically necessary'. Or we may say that the propositions of logic or mathematics are 'true in virtue of their form'. If I say 'Socrates was wise', I say something substantial, which is known from history and cannot be known otherwise. But if I say 'Socrates was wise or not wise', I say something which requires no knowledge of history; its truth follows from the meanings of the words. (Russell 1973, p. 303)

If we spin out more and more complex versions of these analytic propositions, things can get very hairy and even surprising, but there is nothing mysterious or epistemically wonderful about the process.

Alas, the work of Russell and Whitehead did not result immediately in the smooth resolution of foundational issues that Russell had hoped for.

> In June 1901, this period of honeymoon delight came to an end. Cantor had a proof that there is no greatest cardinal; in applying this proof to the universal class, I was led to the contradiction about classes that are not members of themselves. It soon became clear that this is only one of an infinite class of contradictions. I wrote to Frege, who replied with the utmost gravity that ''*die Arithmetik ist ins Schwanken geraten*.'' At first, I hoped the matter was trivial and could be easily cleared up; but early hopes were succeeded by something very near to despair. (Russell 1963/1944, p. 13)

The problems Russell had found with Frege's program had to do with self-reference and self-containment of sets. The axiom that caused the problems was the comprehension axiom that said that any formula of the form ''x is Φ'' yields a set of x's that satisfies Φ. E.g., ''x is red'' yields the set of things that are red. Since there is always the empty set, how could the comprehension axiom cause problems? If Φ was screwy enough or did not apply to anything, then the set of x's that satisfied Φ would be the empty set. Oddly enough and to Frege's despair and Russell's chagrin there are Φ that do not and cannot yield any set,

not even the empty set, thus the unrestricted comprehension axiom despite its obviousness had to be revised. [*Background 1.5* – Technical explanation of Russell's paradox and other problems and a brief tour through Cantorian set theory]

Russell considered many ways of dealing with the problems created by his eponymous paradox. Ultimately, work on resolving the paradoxes led to the full complexities of *Principia Mathematica*. The problems were solved by Russell and Whitehead by introducing the theory of types. The idea of Russell and Whitehead's theory of types[8] was that propositions were not allowed to apply to themselves and sets were not allowed to contain themselves. The technical details need not detain us, but they never fully resolved the problems in a satisfactory manner. Avoiding problems of self-reference seems simple enough in theory, but becomes very complex in practice when one is trying to generate all of mathematics. How can one be sure that a proposition does not apply to itself? There is no easy way to be certain that self-reference has not crept in somewhere.[9]

To the ordinary chap perhaps such apparently minor problems as self-reference do not seem worthy of the years of agonizing work of Russell and Whitehead. As Desdemona (*Othello* act 2, scene 1) says: ''These are old fond paradoxes to make fools laugh i' th' alehouse.'' Well in a sense they were old paradoxes, because they were based on the liar's paradox well known to the ancients and mentioned in the Bible, but Russell and Whitehead were not fools and they weren't laughing. (I don't know how much time they spent in the alehouse – probably not much given all the work they had to do generating *PM*.) Keep in mind that Russell mentions in his autobiography that at age 11 he was troubled by the lack of proof for Euclid's axioms. Since Frege, Whitehead, and Russell were concerned to establish firm foundations for mathematics because of technical problems that emerged in the nineteenth century, they could hardly ignore technical problems of their own. Russell would be the last person to do that.

No one worried that these paradoxes would cause any difficulties for practicing mathematicians. The problems that Russell was confronting were and are philosophical problems. They would be of concern to someone who craved an account of the human capacity for knowledge of mathematics. Unfortunately, as Russell's logical system became more

[8] Usually just called ''Russell's Theory of Types.''

[9] In Chapter 5 we will see that Gödel's famous proof, which helped to undermine logicism, is based on a form of self-reference.

and more complex in order to avoid the paradoxes it was no longer able to generate all of classical mathematics without the addition of more axioms that did not seem to be purely logical. For example, Russell needed to add the axiom of infinity – that there are infinitely many items available. This does not look to be analytic nor logically necessary. Other such axioms were also required. Russell and Whitehead were able to show that all of classical mathematics was derivable with the addition of these non-logical axioms. Although this in itself was a great achievement, they could no longer plausibly claim that mathematics was reducible to pure logic – that it was all analytic. So much for logicism. Today the consensus is that an astonishing amount of classical mathematics can be derived from pure logic and set theory (suitably complicated to avoid paradoxes). All notions of classical mathematics can be defined using only logical notions such as ''or,'' ''if . . . , then . . . ,'' ''all . . . '' and ''$\in$'' of set theory.

Despite the logicist program's failure, twentieth century set theory, model theory, and proof theory – the core of foundations of mathematics – would not have existed without *PM. PM* is the great source of what comes later, even if it did not succeed in its purpose. And logicism did not die a quiet, peaceful death. The claim that mathematics and geometry are analytic is too appealing to give up without a fight, because it offers a neat explanation of the certainty of mathematics, the necessity of mathematical truths, and why they are *a priori*. It solves one of the philosophical riddles of the ages – how is pure mathematics possible? Such a wonderful thing should not be given up lightly. The logical positivists (see Chapter 2) needed logicism, and were unwilling to abandon it. Logical positivists continued to claim that mathematics was derivable from pure logic and that it was analytic well into the 1950s. The modern empiricists made many attempts to refine and defend logicism, and these attempts still continue today.

Particularly frustrating to Russell and the logical positivists was that the technical program of logicism was so close to being successfully completed. To the outside observer, even to the philosopher not steeped in these issues, the problems that defeated Russell and Whitehead seem like the tiniest little flyspecks. To give up a program that accomplished so much, and that promises the solution to such deep and troubling problems on the basis of a few technical difficulties that could just be ignored seems like extreme intellectual fastidiousness. It is also forthright honesty and commitment to seeking the truth that would have made Socrates proud. This is the essence of philosophy.

Russell on Definite Descriptions

Russell's 1905 article ''On Denoting,'' published in the leading British philosophy journal *Mind*, is considered by philosophers of language to be the most seminal article in twentieth-century philosophy of language. Philosophy of language, along with logic, has been central to the development of analytic philosophy. ''On Denoting'' was a key source of this analytic revolution. The results that Russell propounded in ''On Denoting'' are based on his symbolic logic and his work on the foundations of mathematics.

The question Russell is attempting to answer is how to properly interpret what he calls ''definite descriptions.'' A definite description is a phrase that is meant to pick out one object; for example, ''the first president of the United States,'' or ''the computer on which I composed this text.'' Definite descriptions are distinguished from names such as ''George Washington'' and general descriptive phrases such as ''a computer.'' A definite description picks out its object by the descriptions involved, whereas a name just tags an object without its fitting any specific description. One notoriously troubling problem is how to understand definite descriptions that seem to pick out something non-existent, for example ''the king of the United States in 2008.'' Russell famously concentrated his attention in 1905 on the definite description ''the present king of France.'' The question is how to understand the statement ''The present king of France is bald.'' Is this true or false? Or perhaps it has no truth-value.

Before we consider Russell's answers to these questions, we need to address more general issues about the philosophy of language. Put bluntly: Why is philosophy of language an interesting, valuable, and worthwhile pursuit for otherwise busy and intelligent adults? We all know how to talk, read, and write. While no doubt most of us could use some improvement in these skills, the results in philosophy of language are not going to help with our language skills. In some ways, philosophy of language is like philosophy of mathematics. Mathematicians do not need philosophy of mathematics, and language users do not need philosophy of language. We will get along talking and writing just fine without it.

So then, what is the value of philosophy of language? It is not just satisfying an idle curiosity and a fascination with puzzles, although having an interest in puzzle solving helps add a bit of spice to an admittedly dry subject. As a preliminary point, no one should be deceived that issues in the philosophy of language are simple and pellucid just because it is oh so easy to talk. Just because we can all

use language competently does not mean we understand how it works. We are all able to walk just fine, or at least most of us are who are temporarily abled, but walking is an amazingly complex process that involves difficult and continuous feats of balance. The physics and physiology of walking are dauntingly complex. Likewise, the problems in the philosophy of language, while easy to ask, are anything but easy to answer satisfactorily.

The value and importance of philosophy of language stems from various applications. 1) Much that goes under the heading of philosophy of language is actually logic or informal logic. Logic is the science of reasoning and drawing inferences. Reasoning can be difficult, and always could use careful, even meticulous attention. People are subject to screw-ups when reasoning. 2) More directly relevant to our topics here, results in the philosophy of language are relevant to epistemology, metaphysics, and ethics and these areas do have practical and theoretical importance. People claim to know things that they do not know, and vice versa. The entire question of how humans should best pursue knowledge is difficult and conflicted, and of obvious practical importance. While the practical value of metaphysics is not as obvious as epistemology and ethics, one's view of the cosmos and one's place in it affects one's religious views and one's attitude to life's activities. Twentieth-century philosophers have found that apparently small and technical issues in the philosophy of language have had deep implications in these other more exciting realms. In fact, the study of linguistic issues has proven so rich in philosophical insights that twentieth-century philosophers have tended to start with philosophy of language as a way into other areas of philosophy. Methods and tools developed in the philosophy of language throughout the twentieth century have been applied fruitfully to these other areas.

Before examining Russell's ''On Denoting,'' I should emphasize that my writing of ''results'' in the philosophy of language is a bit disingenuous. To the extent that issues in philosophy of language are actually part of logic we can speak of results, but there are no, or few, results in philosophy outside of logic. Russell's theory of definite descriptions while widely admired by philosophers is not universally accepted as a result in the sense in which there are results in mathematics or natural science. There are alternative views, as we shall see (Chapter 4), and the issue of how to interpret definite descriptions is still being energetically examined by philosophers. Nevertheless, Russell's theory of definite descriptions is considered by analytic philosophers to be a paradigm of philosophy and to represent real progress in the philosophy of language and linguistics.

The question is ''How do definite descriptions work?'' Here's a simple answer: The description mentions certain features and whichever is the unique object that has the features is the object denoted (i.e., picked out) by the description. So ''the first president of the United States of America'' denotes George Washington, because he uniquely fits that description. Alas, what seems obvious, easy, and commonsensical, so often won't work when we get to puzzle cases. Certainly George Washington fits that description, and uniquely, but the claim that this explains why the description denotes him is uncomfortable. If we just considered George Washington our discomfort might be minimal, but we want a uniform treatment of definite descriptions. Our mechanism for explaining the denoting of ''the first president of the United States of America'' does not comfortably work across the board. There are too many puzzle cases left. Even with George Washington there is a bit of a puzzle, since he himself is no longer around to be denoted. Note that this problem will also arise with proper names. How can we use a name to talk about a man who does not now exist? Let's leave names and this puzzle aside for the moment. (We will see in Chapter 7 that Kripke and Putnam have an answer.)

Puzzles arise about definite descriptions when there is no single object that fits the description or when nothing fits the description. How does one decide if an assertion that contains such a description is true or false? If I assert ''The only prime number between 10 and 15 is larger than 2×6,'' is what I've said true, false, or neither? The trouble is we do not know what ''the only prime number between 10 and 15'' denotes if anything. Does it denote 11, or 13, both, or neither? Our simple theory that handled ''the first president of the United States of America'' will not handle this example. Even more troubling difficulties arise with definite descriptions which nothing fits – Russell's famous example ''the present king of France'' or for another example ''The president of Canada.'' How are we to respond if someone said ''The president of Canada is a woman''? We would naturally respond ''Canada doesn't have a president.'' So is the statement true, false, or neither? There are good reasons in favor of each of these answers and eminent philosophers have argued for each of them. For example, since the *head of state* of Canada is a woman, we might suppose that ''the president of Canada'' must denote Elizabeth II and so the statement is true. But maybe the speaker wasn't thinking of Elizabeth II, and was just simply mistaken about Canada's political system. What are we to say then? Annoyingly, reasons in favor of one answer are also reasons against the other two and there are plenty of reasons pro and con to go around.

One answer that Russell particularly disliked, but that was popular in the period before his article appeared in 1905, was that there are objects that have being but do not exist. This view has come to be called Meinongianism after Alexius Meinong who propounded it in the late nineteenth century. Meinongianism is a result of taking our simple common sense theory of definite descriptions very seriously. So according to this view there is a present king of France and even a president of Canada (not head of state, president). They are not real physical people; they would be counted in no census. These abstract beings are the entities that are denoted by the phrases ''the present king of France'' and ''the president of Canada.'' If the proposition ''The president of Canada is a woman'' is meaningful as it appears to be, then there must be an object denoted by ''the president of Canada,'' and that object either is or is not a woman. Since no real person is the president of Canada, the object denoted must be a nonexistent being. Now other problems begin to swarm. No such nonexistent objects are observable or even members of our physical universe. Worse still, there are too many of them – one for each possible description thought of or not. This is a felonious violation of Occam's Rule – ''Don't multiply entities when you don't need to'' – in other words don't make things more complex than they absolutely have to be, ''Keep it simple!'' The simple, common sense theory of definite descriptions is simple, but in order to make it work we need this swarm of nonexistent entities. These entities are not subject to empirical scientific investigation and they aren't well behaved like mathematical objects. Unlike numbers and sets, there are no clear criteria of identity or individuation for unreal entities. Is the president of Canada the same or different from the present king of Canada? Is the nonexistent president of Canada a woman? Can a nonexistent woman be a king? Why not? Do these questions even make sense? None of these problems is insurmountable. There are Meinongians today in philosophy, fortunately only a very few, and they are by no means considered to be kooks.

Nevertheless, the problems with Meinongianism were severe enough in Russell's view to demand an alternative. Russell's theory of definite descriptions offers that alternative. Russell was engaged in reducing, that is eliminating, the need for postulating mathematical entities, so he would hardly abide other ethereal objects. Of course, if there is no alternative, then no matter how complex and no matter how much we dislike a theory, we must accept it. If there is an alternative, then we can in good conscience turn away from that bad old overly complex theory. One of the reasons that Russell's theory of definite descriptions was so joyfully received by philosophers is that many could not see

an alternative to Meinongianism. To those of us who are empirically-minded and favor demystification, Russell's theory was like finding a working flashlight when you are stumbling in the dark.

Russell's method of dealing with definite descriptions, which he also applied to many other types of expressions, is to deny they have a denotation in isolation. The key confusion according to Russell is the assumption that ''the present king of France'' or ''the president of Canada'' are denoting expressions. In fact when properly analyzed using his and Frege's methods of logic the expressions disappear entirely. Definite descriptions are not denoting expressions by themselves. So ''the president of Canada'' does not occur in the analysis of the statement ''The president of Canada is a woman.'' According to Russell the statement ''The president of Canada is a woman'' is a somewhat incorrect and shorthand way of saying ''There is one and only one president of Canada and that one is a woman.'' The latter is the real statement, the former is a misleading but natural of way of saying it.

We now have a clear and uniform answer to the question about truth and falsity. It is false that there is one and only one president of Canada and that one is a woman, because this implies that there is a president of Canada, which there is not. In cases where more than one thing fits the description, the statement is false because the statement logically implies that there is only one. When one and only one thing fits the description, such as ''the first president of the United States,'' then the statement is true if the predicate is true of that thing and false otherwise. So the statement ''The first president of the United States was born in Virginia'' is true. There is one and only one first president of the United States and he was born in Virginia. Here it is in contemporary symbolic logic: $\exists x\{[Fx \ \& \ \forall y(Fy \supset x = y)] \ \& \ Vx\}$. The first part says there is at least one thing that is the first president of the United States (there is a first president of the United States), the next part is impossible to put into plain English from the symbols but says in effect that there is at most one, and the last part says that that unique president was born in Virginia. Voila!

Although the symbolic form of the statement is not strictly necessary to understand the analysis, the analysis would not have occurred to Russell or appealed so much to others if it had not been so readily symbolized. Russell's use of logical symbols does not just come from a relish for technical and mathematical symbolization. According to the old way of symbolizing, our statement about the president of Canada would look like this: ''Wa'' where ''W'' stands for ''is a woman'' and ''a'' stands for the non-existent object ''the president of Canada.'' When we

look at the Russellian symbolization we see how profoundly different his analysis is: ''$\exists x\{[Hx \,\&\, \forall y(Hy \supset x = y)] \,\&\, Wx\}$.'' Note that in the fully analyzed expression there is no ''a'' or any other term that denotes an individual. In fact, this is a statement not about an individual but about the universe as a whole – it is a general statement. It says that among things that exist in the universe there is an object that is president of Canada – has the property of being the president of Canada – that there is only one such object and that object is a woman. What it says is, of course, false. Such an analysis has the effect of shedding light on mysteries, solving puzzles, and making progress by giving clear defendable answers.

Philosophers inspired by Russell have used symbolic logic since 1905 to attempt to solve their own philosophical problems, often with less success but no less energy.

Russell developed a metaphysics and philosophy of language based on his symbolic logic. He called it ''logical atomism.'' The name is revealing. The metaphysics of logical atomism can be understood on analogy with physics and chemistry. Larger objects are built up out of smaller atomic ones. Russell held that language when fully analyzed consists of atomic propositions and molecular ones constructed out of them by the logical functions – not, or, and, if . . . , then . . . . In this fashion, all of language and thought could be constructed out of the simplest atomic elements.

Although developments in physics influenced Russell, the prime motivation for his logical atomism came from his work on the foundations of mathematics. Since mathematics could be ''rebuilt'' firmly on the basis of the most simple elements and operations, Russell embraced the idea that every organized area of knowledge could be so treated. First analyze a body of knowledge – say, for example, physics – into the simplest basic elements and then rebuild it step by step from those simple elements. The rebuilt theory or body of knowledge would be vastly improved over the old vague and disordered one, but still contain all its truths. The rebuilding process would exhibit clearly the essential structure of the area of knowledge. Note, however, that the basic elements of physics, according to Russell, would not be atoms. Atoms are already highly derived.

> If your atom is going to serve purposes in physics, as it undoubtedly does, your atom has got to turn out to be a construction, and your atom will in fact turn out to be a series of classes of particulars. The same process which one applies to physics, one will also apply elsewhere. (Russell 1971/1918, p. 274)

The classes of particulars Russell is here talking about are not such things as space/time points or energy packets. They are classes of metaphysical simples discovered by philosophical analysis. Russell here is concerned with the logical structure of an area of knowledge, not the material structure it describes.

Russell analyzes our knowledge of the external world: that is, our common sense knowledge of tables, chairs, other people, and our own bodies along these lines. The world is built of logical atoms.

> Our purpose that has run through all that I have said, has been the justification of analysis, i.e., the justification of the logical atomism, of the view that you can get down in theory, if not in practice, to ultimate simples, out of which the world is built, and that those simples have a kind of reality not belonging to anything else. (Russell 1971/1918, p. 270)

Russell does not explain what the simples are, just that they are metaphysically necessary for the construction of reality. Sometimes they are taken to be particular instantaneous sense impressions, and they can also be qualities or relations.

> The reason that I call my doctrine *logical* atomism is because the atoms I wish to arrive at as the sort of last residue in analysis are logical atoms. Some of them will be what I call 'particulars' – such things as little patches of colour or sounds, momentary things – and some of them will be predicates or relations and so on. The point is that the atom I wish to arrive at is the atom of logical analysis, not the atom of physical analysis. (Russell 1971/1918, p. 178)

Atomic facts are built of simples. Molecular facts are built of atomic facts. Language when suitably reconstructed will consist of atomic propositions representing atomic facts, and molecular propositions. The molecular propositions are built up out of the atomic ones by logical functions: not, or, and, if . . . , then. The common structure of language and the world is represented by symbolic logic. The structure of language is also the structure of thought (when suitably analyzed).

Russell's vision of logical analysis has been amazingly influential given its highly theoretical nature. As we will see in the next chapter Wittgenstein's *Tractatus Logico-Philosophicus*, which influenced the logical positivists and others, was based on Russell's views and is a version of logical atomism. A great deal of analytic philosophy, such as Oxford ordinary language philosophy (see Chapter 4), is a reaction against logical atomism.

G. E. Moore's Philosophy of Common Sense

While Frege influenced Russell's development of symbolic logic and logicism, Russell's wider outlook was shaped by his contact with G. E. Moore – a philosopher only slightly less significant in creating analytic philosophy than Russell himself. As young men at Cambridge University both Russell and Moore embraced versions of Hegelian philosophy filtered through the British Hegelians, F. H. Bradley and J. M. E. McTaggart, philosophers hardly known or read today (although see mention of McTaggart in the Epilogue). Bradley denied that there are any individual things, and argued that reality consists of a single purely spiritual Absolute. McTaggart's most famous book is titled *The Unreality of Time*.

German philosophy after Kant, represented especially by Fichte, Schelling, and Hegel, has come to be called ''German idealism.'' [*Background 1.6* – German idealism and Bradley and McTaggart] In many ways, German idealism and its most ardent and well-known proponent G. W. F. Hegel represents an extreme form of metaphysical speculation. This is the kind of philosophy the logical positivists and other analytic philosophers were reacting against. The logical positivists embraced the term ''positivism'' to emphasize their resistance to German idealism with its emphasis and use and overuse of dialectical negation. Negation in one way or another is at the heart of German idealism. Hegel and the Hegelians are always going on about the ''negation of the negation'' and such things. This sort of metaphysical mystification is anathema to empiricists. The empiricists are ''positivists,'' as opposed to those obscurantist metaphysical ''negationists.''[10] The term ''metaphysical'' came to mean anything even loosely associated with negative Hegelian dialectic and by extension anything other than ''scientific,'' ''clear-headed,'' and ''straightforward.'' To empiricist-minded philosophers ''metaphysical'' meant anything obscure, unscientific, tending to mystification.[11]

[10] This distinction is still alive today. For example, Harold Bloom, not a philosopher but a literary critic associated with Continental philosophy, said in a 1983 interview ''What I think I have in common with the school of deconstruction [a major Continental movement] is the mode of negative thinking or negative awareness, in the technical, philosophical sense of the negative, but which comes to me through negative theology'' (Bloom internet reference).

[11] Abetting this demotion of metaphysics from ''The Queen of Sciences,'' we find that in popular culture today ''metaphysics'' is used to mean ''occult,'' ''supernatural,'' ''mystical.''

The rebellion against Hegelianism and German idealism began with G. E. Moore, and quickly spread to Russell.

> But these motives [dissatisfaction with Hegel's logic and philosophy of mathematics] would have operated more slowly than they did, but for the influence of G. E. Moore. He also had had a Hegelian period, but it was briefer than mine. He took the lead in rebellion, and I followed, with a sense of emancipation. Bradley argued that everything common sense believes in is mere appearance; we reverted to the opposite extreme, and thought that *everything* is real that common sense, uninfluenced by philosophy or theology, supposes real. With a sense of escaping from prison, we allowed ourselves to think that grass is green, that the sun and stars would exist if no one was aware of them, and also that there is a pluralistic timeless world of Platonic ideas. The world, which had been thin and logical, suddenly became rich and varied and solid. Mathematics could be *quite* true, and not merely a stage in dialectic.[12] (Russell 1963/1944, p. 12)

Moore had a profound influence on Russell, but apparently the influence did not go the other way. Moore never seemed to use symbolic logic or very much technical vocabulary at all, nor was he interested in science, mathematics, or anything but pure philosophical reasoning. He has a unique style that is at once daunting in its detail and disarming in its simplicity. Russell in his autobiography writes of Moore:

> In my third year [as a student at Cambridge], however, I met G. E. Moore, who was then a freshman, and for some years he fulfilled my ideal of genius. He was in those days beautiful and slim, with a look almost of inspiration, and with an intellect as deeply passionate as Spinoza's. He had a kind of exquisite purity. I have never but once succeeded in making him tell a lie, and that was by a subterfuge. ''Moore,'' I said, ''do you *always* speak the truth?'' ''No,'' he replied. I believe this to be the only lie he had ever told. (Russell 1968, p. 77)

(Russell's love of paradox comes out even in this brief passage.) Moore must have been very personally compelling, because his writing and style of argumentation is convoluted, humorless, and so often

[12] Although freed of Hegelianism, we can see that Russell was still struggling to emerge from metaphysical intoxication. The ''Platonic ideas'' that Russell mentions do not fit well with an empiricist philosophy. Russell felt that empiricism left out too much. Platonic ideas – eternal, perfect, abstract, and not perceptible by the senses – do not sit well with a ''grass is green'' common sense philosophy. Hard-core empiricists would have trouble seeing much difference between Plato's forms and Bradley's Absolute. Once one is intoxicated with the idea of non-physical entities existing outside of time, one might as well go all the way to the timeless Absolute of the British Hegelians.

unconvincing that I doubt his writings alone would have had the kind of influence that he has had on philosophy.

The two most famous expressions of Moore's reaction against idealism and Hegelianism are his classical articles ''The Refutation of Idealism'' (Moore 1959), originally published in *Mind* in 1903 and ''A Defence of Common Sense''(Moore 1993), published in 1925. Today, among his heirs, Moore is best known, remembered, and loved for these two articles despite their obscurity of style.[13] Although Moore is defending common sense and attacking German philosophical mystification, his argumentation is not available to common sense. One must bring a dedicated close attention to minute detail when studying Moore's articles. Moore has other annoying habits. In ''A Defence of Common Sense'' he attributes numerous views to other philosophers, views that he is claiming to demolish, but only once mentions another thinker – Berkeley. Moore cannot be bothered with citations, quotes, and footnotes – the standard forms of scholarly apparatus.

The charm of Moore's common sense appeal is its simplicity, its air of boldly clearing away the crusty adhesions of generations of philosophy without adding new and doubtful complications of its own. From the time of ancient Greek philosophy to the present day, there have been philosophers, such as Hegel and Bradley, who for one reason or another have denied the ultimate reality of the objects of the common sense world of sense perception. This is the target of Moore's common sense thrust. Moore's claim is that he is more certain of specific facts of common sense, such as that he has two hands, that there are other people, and that the world existed before he was born, than he is of any of the steps of the abstruse or involved philosophical arguments offered by Platonists, Kantians, idealists, Hegelians, skeptics, or even by some of his fellow analytic philosophers, such as Russell. So any philosopher who denies or doubts or questions those facts of common sense must be in the grip of a delusion – he is deluded, or perhaps we should say ''enchanted,'' by his apparently clever arguments. In any case, according to Moore, even if we are not at present able to dissect them, those sophisticated philosophers' arguments that deny or question the facts of common sense cannot be correct.

[13] Moore is also well known for his work in ethics and in particular for his discussion of the so-called naturalistic fallacy. Moore introduced the naturalistic fallacy in his influential work *Principia Ethica* (they had to flaunt that Latin they learned at those English ''public'' schools even if they couldn't get away with writing their books in Latin). Unlike Russell who never devoted much attention to ethics as a branch of philosophy, Moore wrote his only full-length treatise on theoretical ethics. The development and history of twentieth-century ethics is the topic of a separate chapter (Chapter 8), and Moore's work in ethics features prominently there.

Unfortunately, Moore's approach fails to satisfy the philosophical yearning for substance. His common sense argument is too thin, and has failed to triumph even though Oxford ordinary language philosophers and Wittgenstein later tried to keep it alive (see Chapter 4). Moore's common sense philosophy brings to mind those who denied the discoveries of Galileo by claiming that they were more certain that the earth stood still than they could be of his theories and observations using lenses trained on the moon and spots of light in the sky. Isn't the earth's standing still an evident fact of ordinary perception and common sense? It certainly was before Copernicus and Galileo. Common sense does not simply trump the skeptics' and idealists' arguments any more than the common sense of the time trumped modern astronomy.

Moore and Russell on Sense Data

What seemed to be obvious and commonsensical to Moore was often not so to other philosophers. Moore did more than any other thinker to focus attention on the idea of sense data. ''Sense data'' was a term coined by Russell, but Moore put the concept to work in his philosophy and gave it heavy burdens to shoulder. A sense datum is a present sensation of one sense that is private to an individual. Thus a present smell and a colored patch of my visual field are sense data. Sense data are not public objects, although at times Moore states that sense data are on the surfaces of objects. As we will see later in Chapter 4, sense data are introduced primarily by the argument from illusion. For example, a straight stick looks bent when partly in water, and there are optical illusions such as the Checker Shadow illusion (Figure 1.2). The idea is that we do see *something* even if we do not see the object as it is. What do we see? We see or have sense data, supposedly.

Moore argued, or rather claimed, that the existence of sense data is as evident as anything could be.

> ''This is a hand,'' ''That is the sun,'' ''This is a dog,'' etc. etc. etc.
>
> Two things only seem to me to be quite certain about the analysis of such propositions (and even with regard to these I am afraid some philosophers would differ from me) namely that whenever I know, or judge, such a proposition to be true, (1) there is always some *sense-datum* about which the proposition in question is a proposition – some sense-datum which is *a* subject (and, in a certain sense, the principal or ultimate subject) of the proposition in question, and (2) that, nevertheless, *what* I am knowing or

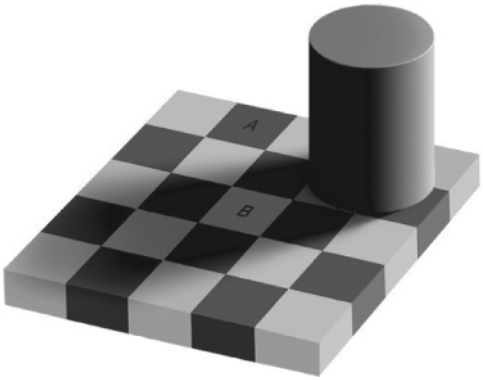


Figure 1.2 Areas A and B are the same shade of gray. This seems incredible but is true. However something is different in our perceptions of A and B. What is it that is different? Copyright Edward H. Adelson.

> judging to be true about this sense-datum is not (in general) that it is *itself* a hand, or a dog, or the sun, etc. etc., as the case may be. . . . But there is no doubt at all that there are sense-data, in the sense in which I am now using that term. I am at present seeing a great number of them, and feeling others. And in order to point out to the reader what sort of things I mean by sense-data, I need only ask him to look at his own right hand. (Moore 1993/1925, p. 128)

Here Moore has ventured far from common sense. Hands, the sun, a dog are all good objects well-known to common sense, but common sense knows no sense data; and note how different the meanings of ''sense'' are in the two uses – common sense and sense data.

Moore's love of common sense was never embraced by later analytic philosophers. The logical positivists who were in sympathy with Moore and Russell in many ways, and were especially influenced by Russell, never adopted common sense as a philosophical source of truth. The logical positivists were more inclined to look to natural science and especially physics for guidance. Later in the twentieth century, Oxford philosophers famously employed ordinary language as a source of argument and insight, but they would not have equated this with common sense. Recently intuition has again been prominent in philosophical research, but this reliance on intuition would not render the sort of results that Moore thought common sense could. Philosophical intuition as it has been used recently is closer to Russell's logical

self-evidence than it is to Moore's common sense. Indeed some of the claims that Moore found to be ''quite certain,'' others would find to be counter-intuitive.

Some members of the logical positivist movement, with their empiricist leanings, focused on the subject of sense data. They shared Moore's view that sense data are fundamental to our empirical knowledge, but unlike Moore and Russell, the logical positivists were troubled by the somewhat unscientific and subjective nature of sense data. Other philosophers, such as the Oxford philosopher J. L. Austin raised questions about the very existence of sense data. Even Wittgenstein and his followers seemed dubious about sense data; if not about their existence and our direct awareness of them, then about whether they can carry any epistemological weight. The nature, function, and even existence of sense data have continued to be issues of intense scrutiny. Sense data remained a central topic in epistemology in analytic circles for most of the twentieth century.

Like Moore, Russell relies on sense data in order to explain the foundations of our knowledge. Russell distinguishes between knowledge by acquaintance and knowledge by description. According to Russell, we are directly acquainted with our sense data.

> All our knowledge, both knowledge of things and knowledge of truths, rests upon acquaintance as its foundation. . . . . Sense data . . . are among the things with which we are acquainted; in fact, they supply the most obvious and striking example of knowledge by acquaintance. (Russell 1959a/1912, p. 48)

Thus, according to Russell, we have direct knowledge by acquaintance of sense data that make up the appearance of the table, but I only indirectly know the physical table by description. According to Russell, we are only acquainted with our sense data, our memories, our present mental states that are available to introspection, and universals such as *whiteness, diversity*, and *brotherhood*. These are the metaphysical atoms of the world that he invoked in his logical atomism. We are not directly acquainted with any physical objects. How far we've strayed from common sense!

An empiricist would gladly follow Russell in agreeing that we are directly acquainted with sense data, and memories, and our present introspectable mental states, but would balk at direct knowledge by acquaintance of metaphysical universals. The classical empiricist John Locke, for example, argued that we create universal ideas as our own mental concepts by abstraction from individual experiences. Russell cannot quite tear himself away from this tradition even though in the

same place he claims to espouse platonism. ''Awareness of universals is called *conceiving*, and a universal of which we are aware is called a *concept*'' (Russell 1959a/1912, p. 52).

Moore's and Russell's Anti-Hegelianism

As we have seen, both Moore and Russell rebelled against the German idealist tradition that culminates with Hegel. Any reader not already entranced by Hegelianism need only spend some time trying to fathom Hegel's writings to appreciate the kind of scorn they provoke. His writings are like a word salad and the reasoning when it apparently can be understood is abstruse. Nevertheless, neither Moore nor Russell ever claimed to be able to refute idealism or Hegelian philosophy. Moore, in particular, is very modest in his claims. Although his article is titled ''The Refutation of Idealism,'' he explicitly denies that anything in the article is a refutation of idealism:

> I say this lest it should be thought that any of the arguments which will be advanced in this paper would be sufficient to disprove, or any refutation of them sufficient to prove, the truly interesting and important proposition that reality is spiritual. For my own part I wish it to be clearly understood that I do not suppose that anything I shall say has the smallest tendency to prove that reality is not spiritual. (Moore 1959/1903, p. 2)

Although Moore does not claim to refute idealism he does argue cogently that no one has any intellectual reason to embrace idealism. In that article Moore also argues, in a way that anticipates Wittgenstein and especially Austin (see Chapter 4), that we directly perceive tables, chairs, and other people. He later repudiated that view.

Moore and Russell may have had other than purely intellectual reasons for rejecting Hegelianism. Here we see Moore and Russell ridiculing Hegel in their famous articles. First Moore from his 1903 ''The Refutation of Idealism'':

> Many philosophers, therefore, when they admit a distinction, yet (following the lead of Hegel) boldly assert their right, in a slightly more obscure form of words, *also* to deny it. The principle of organic unities, like that of combined analysis and synthesis, is mainly used to defend the practice of holding *both* of two contradictory propositions, wherever this may seem convenient. In this, as in other matters, Hegel's main service to philosophy has consisted in giving a name to and erecting into a principle, a type of fallacy to which experience had shown philosophers, along with the rest of mankind, to be addicted. No wonder that he has followers and admirers. (Moore 1959/1903, p. 14)

I note in passing that analytic philosophers have felt that their Continental brothers and sisters were addicted to this fallacy, just as Moore claims. Russell, being Russell, is more sprightly and humorous:

> By the law of excluded middle, either 'A is B' or 'A is not B' must be true. Hence either 'the present King of France is bald' or 'the present King of France is not bald' must be true. Yet if we enumerated the things that are bald, and then the things that are not bald, we should not find the present King of France in either list. Hegelians, who love a synthesis, will probably conclude that he wears a wig. (Russell 1973/1903, p. 110)

The turning away from Hegel was accompanied by a turning toward the British empiricists – John Locke, David Hume, George Berkeley, and John Stuart Mill. Although neither Russell nor Moore were fully committed to the British empiricist tradition in philosophy, they moved now in its spirit, gave it new respect, and were no longer put off by its supposed crudity. They came to view this ''crudity'' as a forthrightness and clarity, a groundedness in reality, that was notably lacking in the ethereal, incomprehensible mystifications of the philosophies of Kant, Hegel, Bradley, and their followers. Analytic philosophers continued to view Continental philosophers as unable and unwilling to shed Hegelianism. And analytic philosophers continued to adopt Russell's and Moore's attitude. Analytic philosophers proudly contrast the clarity, technical proficiency, and respect for natural science of analytic philosophy versus the ultra-sophistication, contrived jargon, and mystification of Continental philosophy. (But as I noted in the Introduction, both the analytic and Continental philosophical traditions have more in common than they may like to recognize in that they are both manifestations of modernism.)

Sociological and cultural reasons also played a role in Moore's and Russell's turning away from Hegelianism and Kantianism and toward British empiricism, and perhaps these sociological and cultural reasons had more influence than either Moore or Russell would like to admit. At the time of the key essays ''On Denoting'' and ''A Refutation of Idealism,'' England was in a period of high tension with Germany and especially Prussia. Both Kant and Hegel were north Germans – Kant spent his life in the Prussian capital Königsberg, and Hegel was at Jena and then Berlin.

When the lines in the above anti-Hegelian quotes from Moore and Russell were being written, Germany was a chief rival and threat to Britain; and in ten years the two nations would be engulfed as opponents in the catastrophe of World War I. In the 1890s and early 1900s Germany was emerging as a naval power to threaten Britain.

According to the Tirpitz Plan, Germany was to force Britain to submit to German hegemony in the international arena. Germany was building a fleet of mighty ships that would cow the Royal Navy. German intentions, which eventually led to World War I, naturally led to a fearful and scornful attitude toward Prussia and the Kaiser among the British. This was occurring when Moore and Russell were turning away from Hegelianism.

Russell (but not Moore), of course, was influenced by and respected Frege, a north German mathematician, and he and Moore were close friends with the Viennese Ludwig Wittgenstein, but Hegelianism represented a different cultural milieu. Hegelianism was something like the official philosophy of Germany, and especially Prussia. Hegel seemed to suggest in his philosophy of history that contemporary Prussia was the culmination of the march of historical change – that there was a destiny to Prussian hegemony. These ideas were perhaps not so baldly stated in Hegel's writings, but his followers saw them there, and this is part of what Hegelianism meant. Later, Hegel was also to be, most unfairly, a source of inspiration to the Nazis as well. Hegel was also a key source of Marx's theories and was closely studied by early communists. The fact that Hegel could be claimed by two such ardently opposed camps as Nazism and Marxism is a tribute to his obscurity.

Anti-German sentiment was ''in the air'' in England in the early 1900s and young men could not help but be influenced by it. This anti-German sentiment would naturally attach to German idealism, if not to specific German friends such as Wittgenstein, or Germans working in very restricted technical fields such as Frege and Cantor.

By no means was Russell simply anti-German and an English chauvinist – Russell was not simply anything. His views and ideas were always conflicted and changing, just like his attitude to empiricism. As World War I was approaching, Russell became a pacifist, or as he later described his position a ''relative pacifist'' – a convenient way to dignify his always conflicted attitude to war and other things. I suppose we could also say that Russell was a relative empiricist,[14] a relative platonist, and so on. In any case, Russell as an opponent of British involvement in World War I courageously refused to follow the multitude to slaughter. As a result of his opposition to the British war effort, he was dismissed from his position at Trinity College, Cambridge, was fined and imprisoned for six months. Russell claimed that his pacifism was the only thing that claimed his true and deep commitment. It replaced mathematical logic. The upheaval of World War I and

[14] We will see in Chapter 3 that Quine uses the term ''relative empiricism'' in *Roots of Reference* in exactly this sense.

its effect on Russell partly explains his moving away from technical philosophy for the rest of his life. About three years after finishing *PM* Russell was still working on mathematical logic . . .

> Then came the war [WW I], and I knew without the faintest shadow of doubt what I had to do. I have never been so whole-hearted or so little troubled with hesitation in any work as in the pacifist work that I did during the war. For the first time I found something to do which involved my whole nature. (Russell 1933, p. 12)

He understandably felt ''a great indignation at the spectacle of the young men of Europe being deceived and butchered in order to gratify the evil passions of their elders'' (Russell 1933, p. 13). Russell's vision was so clear and so courageous, and almost solitary at the time of war fervor, that I marvel in admiration.[15] Alas, he had no effect on the carnage.

Russell came from a distinguished and aristocratic political family. Russell became 3rd Earl Russell on the death of his father – both his parents died when he was quite young. Russell's parents were scandalously liberal for the era, supporting birth control and women's rights. Russell's father was an open atheist and had appointed two ''free thinkers,'' as Russell describes them, to be the guardians of him and his brother. The court overturned his father's will and Russell was brought up almost entirely by his puritanical grandmother at Pembroke Lodge in Richmond Park, London.

Although quite different from his parents and other ancestors, Russell's grandmother had a lasting impact if not on his views, at least on his courage to defend his views, which were often at odds with most of society.

> On my twelfth birthday she gave me a Bible (which I still possess), and wrote her favourite texts on the fly-leaf. One of them was ''Thou shalt not follow a multitude to do evil;'' another, ''Be strong, and of a good courage; be not afraid, neither be Thou dismayed; for the Lord Thy God is with thee whithersoever thou goest.'' These texts have profoundly influenced my life, and still seemed to retain some meaning after I had ceased to believe in God. (Russell 1963/1944, p. 5)

Russell's pacifism, or rather ''relative pacifism,'' guided him throughout his life. He supported the war against the Nazis, but agitated for nuclear disarmament and, toward the end of his long life, against the Vietnam War.

[15] Although I should note that every single member of Trinity College serving in the war effort wrote to support Russell and protest his dismissal.

Figure 1.3 Pembroke Lodge, London. This is now a conference center and restaurant. The Bertrand Russell Suite is one of the fancier venues. Richmond Park is a royal park and the Russells lived here at the pleasure of the Crown. I made a pilgrimage here in 1998. This was before the house was restored, so I could not go inside. It is a sweet place, not too imposing, with sweeping views to the south and west. The park is vast (2360 acres – three times the size of Central Park) with herds of semi-tame deer roaming freely. What a charming place to grow up, and within London! Russell, as a boy, loved roaming and playing in the wilds of Richmond Park. The photo of Pembroke Lodge was made available by the kind permission of Sian Davies. I would also like to acknowledge and thank the photographer, Paul Fairbairn-Tennant.

Much of the writing and publishing that Russell did after his work in technical philosophy was directed more to politics, popular philosophy, and popular histories of philosophy. Russell published many dozens of books and was awarded the Nobel Prize for Literature in 1950. Some of the titles of his books indicate his ideas: *Why I am not a Christian* (1927); *Common Sense and Nuclear Warfare* (1959); *Has Man a Future* (1961); *War Crimes in Vietnam* (1967). Russell was an avid letter writer and wrote over 30,000 letters in his lifetime. I recall, when I was a graduate student at Cornell University, a friend and fellow graduate student wrote to Russell asking a technical question about his early philosophy. Not long after, he received a detailed reply from the great philosopher – a letter which my friend prized and framed and put on his wall over his desk. This was in the early 1960s. Russell is perhaps most famous in the

US as a result of his being denied by a New York court the opportunity to take a professorship at the City University of New York that he was offered in 1940. The court decided that Russell was morally unfit to teach our children.

Unlike Russell, Moore spent his life quietly at Cambridge University. He was the editor of *Mind* for many years and in many other ways continued to be a founder and leader of the analytic tradition in philosophy. He was a friend and supporter of Wittgenstein. Moore also lived a long life but never engaged in politics or disputes outside of philosophy.

Summary

Two initial sources of analytic philosophy are the application of the methods of symbolic logic to philosophical problems and the rejection of German metaphysics. If Russell gets the credit for introducing, popularizing, and applying the revolution in logic begun by Frege, Moore gets the credit for the rebellion against the metaphysical excesses of the German idealists and their English followers. Moore made simplicity, clarity, and careful non-technical analysis respectable in philosophy.

The logical positivists, the subject of the next chapter, were influenced by Russell but not much by Moore. The logical positivists took from Russell, or shared with him, a logicist analysis of mathematics, but rejected any form of platonism. The logical positivists prized Russell's *Principia Mathematica* and the revolution in logic that it represented. Perhaps the greatest cultural affinity that the positivists had with Moore and Russell was the rejection of the excesses of Continental metaphysics, especially in the traditions growing out of Kant and represented most egregiously by Hegel and his followers. Even today, philosophers brought up in the analytic tradition who study the German idealists and Hegel, and there are many who now do, do so with a sense of doing something slightly ''naughty,'' ''exotic,'' ''off-beat,'' and perhaps more political than philosophical.

Final personal note: When I was a graduate student at Cornell in the 1960s, the philosophy department was thoroughly analytic. There were no courses on nineteenth-century philosophy in the philosophy department. We were forbidden to take a course on Nietzsche offered in the German department. There were no questions on Hegel, Kierkegaard, Marx, or Nietzsche on our PhD qualifying exam. The only philosophers that had any regard at all from that period were Frege and Mill, and

they had a very high regard. I am confident that none of my teachers at Cornell would have described themselves as logical positivists. Norman Malcolm and Max Black, both leading analytic philosophers of the era, knew and studied with Moore, Russell, and Wittgenstein, but neither would have been comfortable with logicism or sense data theory. The main shared heritage was a distrust of mystification in philosophy and a scorn for anything that smacked of Hegelianism. Kant was highly regarded, but none of us accepted his synthetic *a priori*. I believe this was a lasting legacy of Moore and Russell.

Background 1.1 – Epistemology: empiricism versus rationalism

Epistemology is the area of philosophy that deals with questions about knowledge and belief. How do we know anything? When are beliefs justified? The central questions of epistemology are the nature of knowledge and skepticism. The philosophical skeptic denies that we have any knowledge of the external world (external to the subjective contents of our own minds). Modern skepticism began with Descartes, the father of modern philosophy. The philosophical project has been to demonstrate that we do have knowledge of the external world and how we have it. There is more extensive discussion of skepticism and contemporary analytic responses to it in the Epilogue.

The classical modern empiricists are John Locke (1632–1704), David Hume (1711–76), and George Berkeley (1685–1753). The classical modern rationalists are René Descartes (1596–1650), Baruch Spinoza (1632–77), and Gottfried Leibniz (1646–1716). Spinoza and Leibniz held that all ideas are innate in the mind, present at birth. Descartes held that most important ideas, like the idea of God and mathematical and geometrical notions, are innate. The rationalists argued that abstract ideas and ideas of God could not be derived from experience. Locke showed by laborious constructions how such ideas could be derived from experience by simple processes. Later rationalists shifted their ground somewhat and claimed that all knowledge is based, like math and geometry, on pure reason. The empiricists claimed that all knowledge is based on experience.

Background 1.2 – A priori, analytic, necessary

An *a priori* proposition is one that can be known independently of any particular experiences. An analytic proposition is one that is true in virtue of the definitions of the terms in the proposition. A necessary proposition is one that must be true, cannot be false. It is true in all possible worlds. All of this is admitted by philosophers to be murky. Empiricists tend to claim that *a priori* = analytic = necessary. Rationalists tend to deny that. Examples are ''$2 + 2 = 4$.'' Most philosophers would agree that this is *a priori* and necessary. Many would also hold that it is analytic. ''All grandmothers are mothers'' is analytic (and *a priori* and necessary).Propositions that are not analytic are synthetic. Kant claimed that $5 + 7 = 12$ is *a priori* and synthetic.

Background 1.3 – Mathematical logic of PM versus traditional Aristotelian logic and note on symbolism

Traditional Aristotelian logic, which was the only logic from about 400 BC until the publication of *PM*, held that every proposition was of the subject/predicate form, or compounds of such propositions. In ''All zebras are animals,'' ''zebras'' is the subject term and ''animals'' is the predicate term. Traditional logic could not recognize relations nor handle singular terms. It treated ''All zebras are animals'' and ''Socrates is an animal'' as logically of the same form. The proposition ''John is married to Betty'' has ''John'' as its subject and attributes the predicate ''married to Betty.'' ''Betty is married to John'' has ''Betty'' as subject term and attributes the predicate ''married to John.'' This treatment misses the fact that ''married to'' is a relation.

Mathematical logic treats logic mathematically. Predicates represent functions from objects to truth-values. The two truth-values are true T and false F. So ''x is an animal'' is a function. ''Socrates is an animal'' is symbolized as ''A*s*'' and it says that Socrates satisfies the function ''animal,'' i.e. A*s* gets the value T. ''All zebras are animals'' says that if any object satisfies the function ''x is a zebra,'' then it satisfies the function ''x is an animal.'' This would be symbolized as ''$\forall x(Zx \supset Ax)$'' which is read as ''For all x, if x is

a zebra, then x is an animal.'' The difference from traditional Aristotelian logic could not be more profound. Relations are just two, three, or more place functions. ''John is married to Betty'' says that the pair John/Betty satisfies the function ''is married to.'' This is symbolized as ''M*jb*''. The mathematical treatment of logic vastly increases the power, simplicity, and usefulness of logic. The original insights of the mathematical treatment of logic are due to Frege.

Note on symbolism: We have two quantifiers: the universal quantifier $\forall$ and the existential quantifier $\exists$. They are used with variables, x, y, z, etc. We also have symbols for truth-functional operators. **or** is symbolized by $\vee$, **and** by &, **not** by $\sim$, **implication** by $\supset$, **equivalence** by $\equiv$. We use upper-case letters as predicate symbols, lower-case letters as individual constants and variables. (See above.) We can make complex formulas by combining these symbols. Parentheses are used as punctuation marks. For example, $\exists x(Fx \,\&\, Gx) \supset \forall y(\sim Gy \supset \sim Fy)$. If something is F and G, then every nonG is nonF.

(See *Background 2.1* – Truth-tables, tautologies, etc. technical introduction to logical ideas in the *Tractatus*, p. 71, for the definition of the truth-functional operators in terms of truth-tables.)

Background 1.4 – Proofs that the sum of the interior angles of a triangle is 180° and that there are infinitely many prime numbers

Proof that the sum of the interior angles of a triangle is 180° (i.e. a straight line): This is an informal proof but is fully convincing. For this to work we need to assume the Euclidean parallel postulate. (At most one line can be drawn through any point not on a given line parallel to the given line in a plane., i.e. a line never gets closer to another line parallel to it.)

In Figure 1.4 Line 1 is parallel to Line 2. It is obvious (but can also be proved) that angle B = angle D, and C = E. D+A+E = B+A+C.

Proof that there are infinitely many prime numbers due to Euclid: Assume that there are finitely many prime numbers, i.e.

assume that there is a largest prime number. Say that l is the largest prime. Take all the finitely many primes and multiply them together, i.e. 2*3*5*. . . *l. The product is a huge composite number. Call it c. Now add 1 to c. This number, call it p, is equal to $c + 1$. p is larger than l, so it must be composite, since l is the largest prime. So p has some prime factors. But if we divide p by any of the prime numbers 2 to l, in each case we get a remainder of 1. So none of those primes is a factor of p. So either p is itself a prime greater than l, or p has a prime factor greater than l. This contradicts the assumption that l is the largest prime. Thus there is no largest prime.

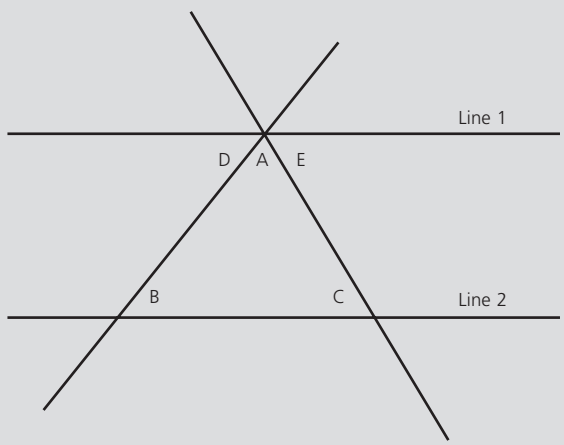


Figure 1.4

Background 1.5 – Technical explanation of Russell's paradox and other problems and a brief tour through Cantorian set theory

Russell's Paradox: Sets can contain themselves, e.g., intuitively, the set of big sets is a big set. Most sets do not contain themselves,

e.g., the empty set does not contain itself. Now consider the set of all sets that do not contain themselves. Call it *S*. Is *S* a member of *S*? If it is, then *S* does not contain itself, since the members of *S* are the sets that do not contain themselves. On the other hand if *S* does not contain itself, then it does. In either case we get a contradiction. Thus there can be no such set as *S*.

Why is this a problem? The Comprehension Axiom was a vital part of the sort of set theory that Russell was using. The Comprehension Axiom says that any specification no matter how nutty specifies a set. This seems intuitively obvious, because if the specification is nutty enough (like, say, the set of round squares), we still get a set, i.e. the empty set. Russell's Paradox demonstrates that the Comprehension Axiom is not correct. At first Russell thought the problem would be easy to fix, but this turned out not to be the case.

Geog Cantor, the inventor of set theory, was able to demonstrate that there are ever larger sets without limit. By using an argument similar to Russell's demonstration of his paradox, but much earlier (by 1884), Cantor was able to establish that the set of subsets of any set has more elements than the set itself. Thus if we consider an infinite set, such as the natural numbers, we now know that the set of subsets of this set has more elements. Thus we get ever-higher levels of infinity. Also paradoxically there is no universal set – there can be no set of everything. It would have more subsets than itself. This is annoying because, e.g., each set should have a complement set – the set of things that are not in it. The complement of the empty set should be the universal set. But no such set exists. Problems mount, and are not easily solved by any intuitively satisfying methods.

Background 1.6 – German idealism and Bradley and McTaggart

German idealism was a loose school that grew out of the philosophy of Immanuel Kant in the nineteenth century. The main proponents were Fichte, Schelling, and Hegel, three thinkers often linked together. The term ''idealism'' in philosophy usually means

a view that is opposed to both dualism and materialism. The idealists hold that everything, in some way or other, is mental or spiritual. The universe ultimately contains no matter or extra-mental energy. These are illusions. Very briefly, Kant held that the phenomena of our experience are the only things we can know about, but that they are generated by an unknown ''thing-in-itself'' which is non-mental, and not part of us. (Materialists hold that everything is matter, or physical; dualists hold that there are two kinds of substances: Mind and matter. Each ontological theory has problems.)

The German idealists get rid of Kant's unknowable thing-in-itself. The only things left are our mental phenomena and these are generated by the mind itself. For example, Fichte claimed that our representations, ideas, or mental images that make up our phenomenal world have their source in our ego, or knowing subject. No external thing-in-itself produces the ideas. Hegel and others expounded other bizarre ideas. For Hegel the world, universe, and history were manifestations of various stages of the development of Absolute Spirit. For the German idealists and their followers individuality was an illusion.

Hegel was much more influential than either Fichte or Schelling and by the end of the nineteenth century Hegelianism of one form or another was the dominant philosophy. Bradley and McTaggart were British Hegelians and idealists. Bradley denied the existence of individual things and, indeed, all of the deliverances of common sense. Instead there was just the one pure Hegelian Absolute. McTaggart famously denied the reality of time. For him, the universe is composed of timeless souls united by love.

Keep in mind that this is the barest snippet. No such snippet can do justice to the philosophers' ideas.

Further Reading

A good place to start for those interested in Frege is *The Frege Reader*, edited by Michael Beany (Wiley-Blackwell 1998). This also has an appendix with suggested further readings on Frege, which are voluminous. Also useful is *Frege: An Introduction to the Founder of Modern Analytic Philosophy* by Anthony Kenny (Wiley-Blackwell 1997).

Russell's autobiography is wonderful reading: *Russell Autobiography* (Routledge 1975).

The Cambridge Companion to Bertrand Russell (Cambridge University Press 2003) is a good place to find scholarly articles on Russell, with special emphasis on his early work and relations to Frege, Moore, and Wittgenstein.

Also *Russell, Idealism, and the Emergence of Analytic Philosophy* by Peter Hylton (Oxford University Press 1990) is a good source on the origins of British analytic philosophy but it is a tough slog.

The Philosophy of G. E. Moore, Vol. IV of *The Library of Living Philosophers*, edited by Paul A. Schilpp (Open Court 1999). This book contains articles by many leading philosophers, written for this volume, on aspects of Moore's philosophy. It includes Moore's replies to each article.

There is also a volume in the series devoted to Russell. *The Philosophy of Bertrand Russell (The Library of Living Philosophers Volume V)*, edited by Paul A. Schilpp (Open Court 1971).

Some Main Problems of Philosophy (Collier 1953) consists of 20 lectures that Moore gave in 1910–11 on many areas of philosophy.

A worthwhile book on Moore is *G. E. Moore* by Thomas Baldwin (Routledge 1990).

Origins of Analytical Philosophy by Michael Dummett (Harvard University Press 1996) supplements, and in some ways diverges from, the account of the subject that I give.

Contemporary Readings in Logical Theory (Macmillan 1967) edited by Irving M. Copi and James A. Gould is an extensive collection of classic articles on formal logic, its history, development, and many other aspects. This anthology contains several of the articles prominently featured in this and other chapters.

Symbolic Logic by Irving Copi (Prentice-Hall 1979) is a good introductory text.