1

Maxwell Equations

> Ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.
>
> —Richard Feynman (American physicist, 1918–1988)

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To master the theory of electromagnetics, we must first understand its history, and find out how the notions of electric charge and field arose and how electromagnetics is related to other branches of physical science. Electricity and magnetism were considered to be two separate branches in the physical sciences until Oersted, Ampère and Faraday established a connection between the two subjects. In 1820, Hans Christian Oersted (1777–1851), a Danish professor of physics at the University of Copenhagen, found that a wire carrying an electric current would change the direction of a nearby compass needle and thus disclosed that electricity can generate a magnetic field. Later the French physicist André Marie Ampère (1775–1836) extended Oersted's work to two parallel current-carrying wires and found that the interaction between the two wires obeys an inverse square law. These experimental results were then formulated by Ampère into a mathematical expression, which is now called Ampère's law. In 1831, the English scientist Michael Faraday (1791–1867) began a series of experiments and discovered that magnetism can also produce electricity, that is, electromagnetic induction. He developed the concept of a magnetic field and was the first to use lines of force to represent a magnetic field. Faraday's experimental results were then extended and reformulated by James Clerk Maxwell (1831–1879), a Scottish mathematician and physicist. Between 1856 and 1873, Maxwell published a series of important papers, such as 'On Faraday's line of force' (1856), 'On physical lines of force' (1861), and 'On a dynamical theory of the electromagnetic field' (1865). In 1873, Maxwell published 'A Treatise on Electricity and Magnetism' on a unified theory of electricity and magnetism and a new formulation of electromagnetic equations since known as Maxwell equations. This is one of the great achievements of nineteenth-century physics. Maxwell predicted the existence of electromagnetic waves traveling at the speed of light and he also proposed that light is an electromagnetic phenomenon. In 1888, the German physicist Heinrich Rudolph Hertz (1857–1894) proved that an electric signal can travel through the air and confirmed the existence of electromagnetic waves, as Maxwell had predicted.

Foundations of Applied Electrodynamics Geyi Wen
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Maxwell's theory is the foundation for many future developments in physics, such as special relativity and general relativity. Today the words 'electromagnetism', 'electromagnetics' and 'electrodynamics' are synonyms and all represent the merging of electricity and magnetism. Electromagnetic theory has greatly developed to reach its present state through the work of many scientists, engineers and mathematicians. This is due to the close interplay of physical concepts, mathematical analysis, experimental investigations and engineering applications. Electromagnetic field theory is now an important branch of physics, and has expanded into many other fields of science and technology.

1.1 Experimental Laws

It is known that nature has four fundamental forces: (1) the strong force, which holds a nucleus together against the enormous forces of repulsion of the protons, and does not obey the inverse square law and has a very short range; (2) the weak force, which changes one flavor of quark into another and regulates radioactivity; (3) gravity, the weakest of the four fundamental forces, which exists between any two masses and obeys the inverse square law and is always attractive; and (4) electromagnetic force, which is the force between two charges. Most of the forces in our daily lives, such as tension forces, friction and pressure forces are of electromagnetic origin.

1.1.1 Coulomb's Law

Charge is a basic property of matter. Experiments indicate that certain objects exert repulsive or attractive forces on each other that are not proportional to the mass, therefore are not gravitational. The source of these forces is defined as the charge of the objects. There are two kinds of charges, called positive and negative charge respectively. Charges are quantitized and come in integer multiples of an **elementary charge**, which is defined as the magnitude of the charge on the electron or proton. An arrangement of one or more charges in space forms a charge distribution. The **volume charge density**, the **surface charge density** and the **line charge density** describe the amount of charge per unit volume, per unit area and per unit length respectively. A net motion of electric charge constitutes an electric current. An electric current may consist of only one sign of charge in motion or it may contain both positive and negative charge. In the latter case, the current is defined as the net charge motion, the algebraic sum of the currents associated with both kinds of charges.

In the late 1700s, the French physicist Charles-Augustin de Coulomb (1736–1806) discovered that the force between two charges acts along the line joining them, with a magnitude proportional to the product of the charges and inversely proportional to the square of the distance between them. Mathematically the force $\mathbf{F}$ that the charge q_1 exerts on q_2 in vacuum is given by **Coulomb's law**

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}\mathbf{u}_R \tag{1.1}$$

where $R = \left|\mathbf{r} - \mathbf{r}'\right|$ is the distance between the two charges with $\mathbf{r}'$ and $\mathbf{r}$ being the position vectors of q_1 and q_2 respectively; $\mathbf{u}_R = (\mathbf{r} - \mathbf{r}')/\left|\mathbf{r} - \mathbf{r}'\right|$ is the unit vector pointing from q_1 to q_2, and $\varepsilon_0 = 8.85 \times 10^{-12}$ is the permittivity of the medium in vacuum. In order that the

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distance between the two charges can be clearly defined, strictly speaking, Coulomb's law applies only to the point charges, the charged objects of zero size. Dividing (1.1) by q_2 gives a force exerting on a unit charge, which is defined as the **electric field intensity E** produced by the charge q_1. Thus the electric field produced by an arbitrary charge q is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 R^2}\mathbf{u}_R = -\nabla\phi(\mathbf{r}) \tag{1.2}$$

where $\phi(\mathbf{r}) = q/4\pi\varepsilon_0 R$ is called the **Coulomb potential**. Here $R = \left|\mathbf{r} - \mathbf{r}'\right|$, $\mathbf{r}'$ is the position vector of the point charge q and $\mathbf{r}$ is the observation point. For a continuous charge distribution in a finite volume V with charge density $\rho(\mathbf{r})$, the electric field produced by the charge distribution is obtained by superposition

$$\mathbf{E}(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0 R^2}\mathbf{u}_R dV(\mathbf{r}') = -\nabla\phi(\mathbf{r}) \tag{1.3}$$

where

$$\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0 R} dV(\mathbf{r}')$$

is the potential. Taking the divergence of (1.3) and making use of $\nabla^2(1/R) = -4\pi\delta(R)$ leads to

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}. \tag{1.4}$$

This is called **Gauss's law**, named after the German scientist Johann Carl Friedrich Gauss (1777–1855). Taking the rotation of (1.3) gives

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0. \tag{1.5}$$

The above results are valid in a vacuum. Consider a dielectric placed in an external electric field. If the dielectric is ideal, there are no free charges inside the dielectric but it does contain bound charges which are caused by slight displacements of the positive and negative charges of the dielectric's atoms or molecules induced by the external electric field. These slight displacements are very small compared to atomic dimensions and form small electric dipoles. The **electric dipole moment** of an induced dipole is defined by $\mathbf{p} = ql\mathbf{u}_l$, where l is the separation of the two charges and $\mathbf{u}_l$ is the unit vector directed from the negative charge to the positive charge (Figure 1.1).

Example 1.1: Consider the dipole shown in Figure 1.1. The distances from the charges to a field point P are denoted by R_+ and R_- respectively, and the distance from the center of the

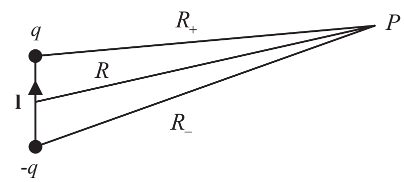

Figure 1.1 Induced dipole

dipole to the field point P is denoted by R. The potential at P is

$$\phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

If $l \ll R$, we have

$$\frac{1}{R_+} = \frac{1}{\sqrt{(l/2)^2 + R^2 - lR\mathbf{u}_l \cdot \mathbf{u}_R}} \approx \frac{1}{R}\left(1 + \frac{1}{2}\frac{l}{R}\mathbf{u}_l \cdot \mathbf{u}_R\right),$$
$$\frac{1}{R_-} = \frac{1}{\sqrt{(l/2)^2 + R^2 + lR\mathbf{u}_l \cdot \mathbf{u}_R}} \approx \frac{1}{R}\left(1 - \frac{1}{2}\frac{l}{R}\mathbf{u}_l \cdot \mathbf{u}_R\right),$$

where $\mathbf{u}_R$ is the unit vector directed from the center of the dipole to the field point P. Thus the potential can be written as

$$\phi \approx \frac{1}{4\pi\varepsilon_0 R^2}\mathbf{p} \cdot \mathbf{u}_R. \tag{1.6}$$

□

The dielectric is said to be polarized when the induced dipoles occur inside the dielectric. To describe the macroscopic effect of the induced dipoles, we define the **polarization vector P** as

$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{1}{\Delta V}\sum_i \mathbf{p}_i \tag{1.7}$$

where ΔV is a small volume and $\sum_i \mathbf{p}_i$ denotes the vector sum of all dipole moments induced inside ΔV. The polarization vector is the volume density of the induced dipole moments. The dipole moment of an infinitesimal volume dV is given by $\mathbf{P}dV$, which produces the potential (see (1.6))

$$d\phi \approx \frac{dV}{4\pi\varepsilon_0 R^2}\mathbf{P} \cdot \mathbf{u}_R.$$

The total potential due to a polarized dielectric in a region V bounded by S may be expressed as

$$\begin{aligned}\phi(\mathbf{r}) &\approx \int\limits_V \frac{\mathbf{P}\cdot\mathbf{u}_R}{4\pi\varepsilon_0 R^2}dV(\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\int\limits_V \mathbf{P}\cdot\nabla'\frac{1}{R}dV(\mathbf{r}') \\ &= \frac{1}{4\pi\varepsilon_0}\int\limits_V \nabla'\cdot\left(\frac{\mathbf{P}}{R}\right)dV(\mathbf{r}') + \frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{-\nabla'\cdot\mathbf{P}}{R}dV(\mathbf{r}') \\ &= \frac{1}{4\pi\varepsilon_0}\int\limits_S \frac{\mathbf{P}\cdot\mathbf{u}_n(\mathbf{r}')}{R}dV(\mathbf{r}') + \frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{-\nabla'\cdot\mathbf{P}}{R}dV(\mathbf{r}')\end{aligned} \tag{1.8}$$

where the divergence theorem has been used. In the above, $\mathbf{u}_n$ is the outward unit normal to the surface. The first term of (1.8) can be considered as the potential produced by a surface charge density $\rho_{ps} = \mathbf{P}\cdot\mathbf{u}_n$, and the second term as the potential produced by a volume charge density $\rho_p = -\nabla\cdot\mathbf{P}$.Both ρ_{ps} and ρ_p are the bound charge densities. The total electric field inside the dielectric is the sum of the fields produced by the free charges and bound charges. Gauss's law (1.4) must be modified to incorporate the effect of dielectric as follows

$$\nabla\cdot\varepsilon_0\mathbf{E} = \rho + \rho_p.$$

This can be written as

$$\nabla\cdot\mathbf{D} = \rho \tag{1.9}$$

where $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ is defined as the **electric induction intensity**. When the dielectric is linear and isotropic, the polarization vector is proportional to the electric field intensity so that $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$, where χ_e is a dimensionless number, called **electric susceptibility**. In this case we have

$$\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E} = \varepsilon\mathbf{E}$$

where $\varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0$ is a dimensionless number, called relative permittivity. Note that (1.5) holds in the dielectric.

1.1.2 Ampère's Law

There is no evidence that magnetic charges or magnetic monopoles exist. The source of the magnetic field is the moving charge or current. **Ampère's law** asserts that the force that a current element $\mathbf{J}_2 dV_2$ exerts on a current element $\mathbf{J}_1 dV_1$ in vacuum is

$$d\mathbf{F}_1 = \frac{\mu_0}{4\pi}\frac{\mathbf{J}_1 dV_1 \times (\mathbf{J}_2 dV_2 \times \mathbf{u}_R)}{R^2} \tag{1.10}$$

where R is the distance between the two current elements, $\mathbf{u}_R$ is the unit vector pointing from current element $\mathbf{J}_2 dV_2$ to current element $\mathbf{J}_1 dV_1$, and $\mu_0 = 4\pi \times 10^{-7}$ is the permeability in

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vacuum. Equation (1.10) can be written as

$$d\mathbf{F}_1 = \mathbf{J}_1 dV_1 \times d\mathbf{B}$$

where $d\mathbf{B}$ is defined as the **magnetic induction intensity** produced by the current element $\mathbf{J}_2 dV_2$

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}_2 dV_2 \times \mathbf{u}_R}{R^2}.$$

By superposition, the magnetic induction intensity generated by an arbitrary current distribution $\mathbf{J}$ is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}') \times \mathbf{u}_R}{R^2} dV(\mathbf{r}'). \tag{1.11}$$

This is called the **Biot-Savart law**, named after the French physicists Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841). Equation (1.11) may be written as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where $\mathbf{A}$ is known as the **vector potential** defined by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{R} dV(\mathbf{r}').$$

Thus

$$\nabla \cdot \mathbf{B} = 0. \tag{1.12}$$

This is called Gauss's law for magnetism, which says that the magnetic flux through any closed surface S is zero

$$\int_S \mathbf{B} \cdot \mathbf{u}_n dS = 0.$$

Taking the rotation of magnetic induction intensity and using $\nabla^2(1/R) = -4\pi\delta(R)$ and $\nabla \cdot \mathbf{J} = 0$ yields

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{r}). \tag{1.13}$$

This is the differential form of Ampère's law.

Example 1.2: Consider a small circular loop of radius a that carries current I. The center of the loop is chosen as the origin of the spherical coordinate system as shown in Figure 1.2. The

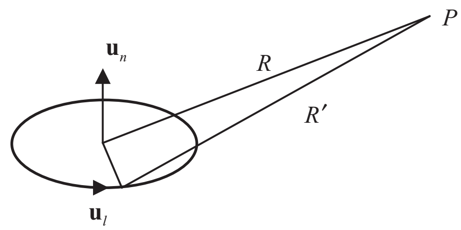

Figure 1.2 Small circular loop

vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_l \frac{1}{R'} \mathbf{u}_l dl(\mathbf{r}')$$

where $\mathbf{u}_l$ is the unit vector along current flow and l stands for the loop. Due to the symmetry, the vector potential is independent of the angle φ of the field point P. Making use of the following identity

$$\int_l \phi \mathbf{u}_l dl = \int_S \mathbf{u}_n \times \nabla \phi dS$$

where S is the area bounded by the loop l, the vector potential can be written as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi} \int_S \mathbf{u}_n \times \nabla' \frac{1}{R'} dS(\mathbf{r}') \\ &= -\frac{\mu_0 I}{4\pi} \int_S \mathbf{u}_n \times \nabla \frac{1}{R'} dS(\mathbf{r}') = \frac{\mu_0 I}{4\pi} \nabla \times \int_S \mathbf{u}_n \frac{1}{R'} dS(\mathbf{r}').\end{aligned}$$

If the loop is very small, we can let $R' \approx R$. Thus

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi} \nabla \times \int_S \mathbf{u}_n \frac{1}{R'} dS(\mathbf{r}') \\ &\approx \frac{\mu_0}{4\pi} \nabla \times \frac{\mathbf{m}}{R} = \frac{\mu_0}{4\pi R^2} \mathbf{m} \times \mathbf{u}_R\end{aligned} \tag{1.14}$$

where $\mathbf{u}_R$ is the unit vector from the center of the loop to the field point P and

$$\mathbf{m} = I \int_S \mathbf{u}_n(\mathbf{r}') dS(\mathbf{r}') = I \mathbf{u}_n \pi a^2$$

is defined as the **magnetic dipole moment** of the loop. □

The above results are valid in a vacuum. All materials consist of atoms. An orbiting electron around the nucleus of an atom is equivalent to a tiny current loop or a magnetic dipole. In the absence of external magnetic field, these tiny magnetic dipoles have random orientations for most materials so that the atoms show no net magnetic moment. The application of an external magnetic field causes all these tiny current loops to be aligned with the applied magnetic field, and the material is said to be magnetized and the **magnetization current** occurs. To describe the macroscopic effect of magnetization, we define a **magnetization vector M** as

$$\mathbf{M} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i \mathbf{m}_i \tag{1.15}$$

where ΔV is a small volume and $\sum_i \mathbf{m}_i$ denotes the vector sum of all magnetic dipole moments induced inside ΔV. The magnetization vector is the volume density of the induced magnetic dipole moments. The magnetic dipole moments of an infinitesimal volume dV is given by $\mathbf{M}dV$, which produces a vector potential (see (1.14))

$$d\mathbf{A} = \frac{\mu_0}{4\pi R^2}\mathbf{M} \times \mathbf{u}_R dV(\mathbf{r}') = \frac{\mu_0}{4\pi}\mathbf{M} \times \nabla' \frac{1}{R} dV(\mathbf{r}').$$

The total vector potential due to a magnetized material in a region V bounded by S is then given by

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0}{4\pi}\int_V \mathbf{M} \times \nabla' \frac{1}{R} dV(\mathbf{r}') \\
&= \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \mathbf{M}}{R} dV(\mathbf{r}') - \frac{\mu_0}{4\pi}\int_V \nabla' \times \frac{\mathbf{M}}{R} dV(\mathbf{r}') \\
&= \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \mathbf{M}}{R} dV(\mathbf{r}') + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{M} \times \mathbf{u}_n(\mathbf{r}')}{R} dS(\mathbf{r}')
\end{aligned} \tag{1.16}$$

where $\mathbf{u}_n$ is the unit outward normal of S. The first term of (1.16) can be considered as the vector potential produced by a volume current density $\mathbf{J}_M = \nabla \times \mathbf{M}$, and the second term as the vector potential produced by a surface current density $\mathbf{J}_{Ms} = \mathbf{M} \times \mathbf{u}_n$. Both $\mathbf{J}_M$ and $\mathbf{J}_{Ms}$ are magnetization current densities. The total magnetic field inside the magnetized material is the sum of the fields produced by the conduction current and the magnetized current and Ampère's law (1.13) must be modified as

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_M).$$

This can be rewritten as

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.17}$$

where $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ is called **magnetic field intensity**. When the material is linear and isotropic, the magnetization vector is proportional to the magnetic field intensity so that $\mathbf{M} = \chi_m \mathbf{H}$, where χ_m is a dimensionless number, called **magnetic susceptibility**. In this case we have

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{E}$$

where $\mu_r = 1 + \chi_m = \mu/\mu_0$ is a dimensionless number, called relative permeability. Notice that (1.12) holds in a magnetized material.

1.1.3 Faraday's Law

Faraday's law asserts that the induced electromotive force in a closed circuit is proportional to the rate of change of magnetic flux through any surface bounded by that circuit. The direction of the induced current is such as to oppose the change giving rise to it. Mathematically, this can be expressed as

$$\int_{\Gamma} \mathbf{E} \cdot \mathbf{u}_t d\Gamma = -\frac{\partial}{\partial t} \int_{S} \mathbf{B} \cdot \mathbf{u}_n dS$$

where Γ is a closed contour and S is the surface spanning the contour as shown in Figure 1.3; $\mathbf{u}_n$ and $\mathbf{u}_t$ are the unit normal to S and unit tangent vector along Γ respectively, and they satisfy the right-hand rule.

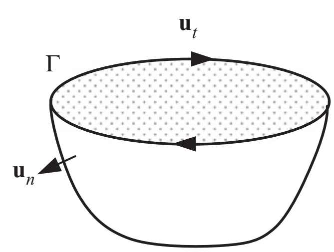

Figure 1.3 A two-sided surface

Loosely speaking, Faraday's law says that a changing magnetic field produces an electric field. The differential form of Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.18}$$

1.1.4 Law of Conservation of Charge

The law of conservation of charge states that the net charge of an isolated system remains constant. Mathematically, the amount of the charge flowing out of the surface S per second is equal to the decrease of the charge per second in the region V bounded by S

$$\int_S \mathbf{J} \cdot \mathbf{u}_n dS = -\frac{\partial}{\partial t} \int_V \rho dV.$$

The law of charge conservation is also known as the **continuity equation**. The differential form of the continuity equation is

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{1.19}$$

1.2 Maxwell Equations, Constitutive Relation, and Dispersion

From (1.18) and (1.17), one can find that a changing magnetic field produces an electric field by magnetic induction, but a changing electric field would not produce a magnetic field. In addition, equation (1.17) implies $\nabla \cdot \mathbf{J} = 0$, which contradicts the continuity equation for a time-dependent field. To solve these problems, Maxwell added an extra term $\mathbf{J}_d$ to Equation (1.17)

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d.$$

It then follows that

$$\nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d = 0.$$

Introducing the continuity equation yields

$$\nabla \cdot \mathbf{J}_d = \frac{\partial \rho}{\partial t}.$$

Substituting Gauss's law (1.4) into the above equation, one may obtain $\mathbf{J}_d = \partial \mathbf{D}/\partial t$. Thus (1.17) must be modified to

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \tag{1.20}$$

The term $\partial \mathbf{D}/\partial t$ is called the **displacement current**. Equation (1.20) implies that a changing electric field generates a magnetic field by electric induction. It is this new electric induction postulate that makes it possible for Maxwell to predict the existence of electromagnetic waves. The mutual electric and magnetic induction produces a self-sustaining electromagnetic vibration moving through space.

1.2.1 Maxwell Equations and Boundary Conditions

It follows from (1.4), (1.12), (1.18) and (1.20) that

$$\begin{aligned}
\nabla \times \mathbf{H}(\mathbf{r}, t) &= \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t),\\
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t},\\
\nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho(\mathbf{r}, t),\\
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0.
\end{aligned} \tag{1.21}$$

The above equations are called **Maxwell equations**, and they describe the behavior of electric and magnetic fields, as well as their interactions with matter. It must be mentioned that the above vectorial form of Maxwell equations is due to the English engineer Oliver Heaviside (1850–1925), and is presented with neatness and clarity compared to the large set of scalar equations proposed by Maxwell. Maxwell equations are the starting point for the investigation of all macroscopic electromagnetic phenomena. In (1.21), $\mathbf{r}$ is the observation point of the fields in meters and t is the time in seconds; $\mathbf{H}$ is the magnetic field intensity measured in amperes per meter (A/m); $\mathbf{B}$ is the magnetic induction intensity measured in tesla (N/A·m); $\mathbf{E}$ is electric field intensity measured in volts per meter (V/m); $\mathbf{D}$ is the electric induction intensity measured in coulombs per square meter (C/m^2); $\mathbf{J}$ is electric current density measured in amperes per square meter (A/m^2); ρ is the electric charge density measured in coulombs per cubic meter (C/m^3). The first equation is Ampère's law, and it describes how the electric field changes according to the current density and magnetic field. The second equation is Faraday's law, and it characterizes how the magnetic field varies according to the electric field. The minus sign is required by Lenz's law, that is, when an electromotive force is generated by a change of magnetic flux, the polarity of the induced electromotive force is such that it produces a current whose magnetic field opposes the change, which produces it. The third equation is Coulomb's law, and it says that the electric field depends on the charge distribution and obeys the inverse square law. The final equation shows that there are no free magnetic monopoles and that the magnetic field also obeys the inverse square law. It should be understood that none of the experiments had anything to do with waves at the time when Maxwell derived his equations. Maxwell equations imply more than the experimental facts. The continuity equation can be derived from (1.21) as

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial \rho(\mathbf{r}, t)}{\partial t}. \tag{1.22}$$

Remark 1.1: The charge density ρ and the current density $\mathbf{J}$ in Maxwell equations are free charge density and currents and they exclude charges and currents forming part of the structure of atoms and molecules. The bound charges and currents are regarded as material, which are not included in ρ and $\mathbf{J}$. The current density normally consists of two parts: $\mathbf{J} = \mathbf{J}_{con} + \mathbf{J}_{imp}$. Here $\mathbf{J}_{imp}$ is referred to as external or impressed current source, which is independent of the field and delivers energy to electric charges in a system. The impressed current source can be of electric and magnetic type as well as of non-electric or non-magnetic origin. $\mathbf{J}_{con} = \sigma \mathbf{E}$, where σ is the conductivity of the medium in mhos per meter, denotes the conduction current induced by the impressed source $\mathbf{J}_{imp}$. Sometimes it is convenient to introduce an external or impressed electric field $\mathbf{E}_{imp}$ defined by $\mathbf{J}_{imp} = \sigma \mathbf{E}_{imp}$. In a more general situation, one can write $\mathbf{J} = \mathbf{J}_{ind}(\mathbf{E}, \mathbf{B}) + \mathbf{J}_{imp}$, where $\mathbf{J}_{ind}(\mathbf{E}, \mathbf{B})$ is the induced current by the impressed current $\mathbf{J}_{imp}$. □

Remark 1.2 (Duality): Sometimes it is convenient to introduce, magnetic current $\mathbf{J}_m$ and magnetic charges ρ_m, which are related by

$$\nabla \cdot \mathbf{J}_m(\mathbf{r}, t) = -\frac{\partial \rho_m(\mathbf{r}, t)}{\partial t} \tag{1.23}$$

and the Maxwell equations must be modified as

$$\begin{aligned}
\nabla \times \mathbf{H}(\mathbf{r}, t) &= \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t), \\
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} - \mathbf{J}_m(\mathbf{r}, t), \\
\nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho(\mathbf{r}, t), \\
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= \rho_m(\mathbf{r}, t).
\end{aligned} \tag{1.24}$$

The inclusion of $\mathbf{J}_m$ and ρ_m makes Maxwell equations more symmetric. However, there has been no evidence that the magnetic current and charge are physically present. The validity of introducing such concepts in Maxwell equations is justified by the equivalence principle, that is, they are introduced as a mathematical equivalent to electromagnetic fields. Equations (1.24) will be called the **generalized Maxwell equations**.

If all the sources are of magnetic type, Equations (1.24) reduce to

$$\begin{aligned}
\nabla \times \mathbf{H}(\mathbf{r}, t) &= \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}, \\
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} - \mathbf{J}_m(\mathbf{r}, t), \\
\nabla \cdot \mathbf{D}(\mathbf{r}, t) &= 0, \\
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= \rho_m(\mathbf{r}, t).
\end{aligned} \tag{1.25}$$

Mathematically (1.21) and (1.25) are similar. One can obtain one of them by simply interchanging symbols as shown in Table 1.1. This property is called **duality**. The importance of

Table 1.1 Duality.

Electric source	Magnetic source
$\mathbf{E}$	$\mathbf{H}$
$\mathbf{H}$	$-\mathbf{E}$
$\mathbf{J}$	$\mathbf{J}_m$
ρ	ρ_m
μ	ε
ε	μ

duality is that one can obtain the solution of magnetic type from the solution of electric type by interchanging symbols and vice versa. □

Remark 1.3: For the time-harmonic (sinusoidal) fields, Equations (1.21) and (1.22) can be expressed as

$$\begin{aligned}
&\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}),\\
&\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mathbf{B}(\mathbf{r}),\\
&\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}),\\
&\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = 0,\\
&\nabla \cdot \mathbf{J}(\mathbf{r}) = -j\omega\rho(\mathbf{r}),
\end{aligned} \tag{1.26}$$

where the field quantities denote the complex amplitudes (phasors) defined by

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}[\mathbf{E}(\mathbf{r})e^{j\omega t}], \text{ etc.}$$ □

We use the same notations for both time-domain and frequency-domain quantities.

Remark 1.4: Maxwell equations summarized in (1.21) hold for macroscopic fields. For microscopic fields, the assumption that the charges and currents are continuously distributed is no longer valid. Instead, the charge density and current density are represented by

$$\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i), \mathbf{J}(\mathbf{r}) = \sum_i q_i \dot{\mathbf{r}}_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{1.27}$$

where q_i denotes the charge of i th particle and $\dot{\mathbf{r}}_i$ (the dot denotes the time derivative) its velocity. Correspondingly, Maxwell equations become

$$\begin{aligned}
&\nabla \times \mathbf{H}(\mathbf{r}, t) = \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t),\\
&\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t},\\
&\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\varepsilon_0},\\
&\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0.
\end{aligned} \tag{1.28}$$

All charged particles have been included in (1.27). The macroscopic field equations (1.21) can be obtained from the microscopic field equations (1.28) by the method of averaging. □

Remark 1.5: Ampère's law and Coulomb's law can be derived from the continuity equation. If we take electric charge Q as a primitive smoothly distributed over a volume V, we can define a charge density $\rho(\mathbf{r}, t)$ such that $Q = \int\limits_V \rho(\mathbf{r}', t)dV(\mathbf{r}')$. Now the assumption that the electric charges are always conserved may be applied, which implies that if the charges within a region V have changed, the only possibility is that some charges have left or entered the region. Based on this assumption, it can be shown that there exists a vector **J**, called current density, such that the continuity equation (1.22) holds (Duvaut and Lions, 1976; Kovetz, 2000). We can define a vector **D**, called electric induction intensity, so that Coulomb's law holds

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t).$$

Then the continuity equation (1.22) implies that the divergence of vector $\partial \mathbf{D}/\partial t + \mathbf{J}$ is zero. As a result, there exists at least one vector **H**, called the magnetic field intensity, so that Ampère's law holds

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t). \qquad \square$$

Remark 1.6: Maxwell equations might be derived from the laws of electrostatics (Elliott, 1993; Schwinger *et al*., 1998) or from quantum mechanics (Dyson, 1990). □

Remark 1.7: The force acting on a point charge q, moving with a velocity **v** with respect to an observer, by the electromagnetic field is given by

$$\mathbf{F}(\mathbf{r}, t) = q[\mathbf{E}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)] \tag{1.29}$$

where **E** and **B** are the total fields, including the field generated by the moving charge q. Equation (1.29) is referred to as **Lorentz force equation**, named after Dutch physicist Hendrik Antoon Lorentz (1853–1928). It is known that there are two different formalisms in classical physics. One is mechanics that deals with particles, and the other is electromagnetic field theory that deals with radiated waves. The particles and waves are coupled through the Lorentz force equation, which usually appears as an assumption separate from Maxwell equations. The Lorentz force is the only way to detect electromagnetic fields. For a continuous charge distribution, the Lorentz force equation becomes

$$\mathbf{f}(\mathbf{r}, t) = \rho \mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \tag{1.30}$$

where **f** is the force density acting on the charge distribution ρ, that is, the force acting on the charge distribution per unit volume. Maxwell equations, Lorentz force equation and continuity equation constitute the fundamental equations in electrodynamics. To completely determine the interaction between charged particles and electromagnetic fields, we must introduce Newton's second law. An exact solution to the interaction problem is very difficult. Usually the fields are first determined by the known source without considering the influence of the moving charged particles. Then the dynamics of the charged particles can be studied by Newton's second law. The electromagnetic force causes like-charged things to repel and oppositely charged things to attract. Notice that the force that holds the atoms together to form molecules is essentially an electromagnetic force, called residual electromagnetic force. □

Remark 1.8: Maxwell equations (1.21) are differential equations, which apply locally at each point in a continuous medium. At the interfaces of two different media, the charge and current and the corresponding fields are discontinuous and the differential (local) form of Maxwell equations becomes meaningless. Thus we must resort to the integral (global) form of Maxwell equations in this case. Let Γ be a closed contour and S be a regular two-sided surface spanning the contour as shown in Figure 1.3. Applying Stokes's theorem to the two curl equations in (1.21) yields

$$\int_{\Gamma} \mathbf{H} \cdot \mathbf{u}_t d\Gamma = \int_{S} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{u}_n dS, \int_{\Gamma} \mathbf{E} \cdot \mathbf{u}_t d\Gamma = -\int_{S} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{u}_n dS \tag{1.31}$$

If S is a closed surface, applying Gauss's theorem to the two divergence equations in (1.21) gives

$$\int_{S} \mathbf{D} \cdot \mathbf{u}_n dS = \int_{V} \rho dV, \int_{S} \mathbf{B} \cdot \mathbf{u}_n dS = 0. \tag{1.32}$$

□

Remark 1.9: The boundary conditions on the surface between two different media can be easily obtained from (1.31) and (1.32), and they are

$$\begin{aligned} \mathbf{u}_n \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_s, \\ \mathbf{u}_n \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0, \\ \mathbf{u}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \rho_s, \\ \mathbf{u}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0, \end{aligned} \tag{1.33}$$

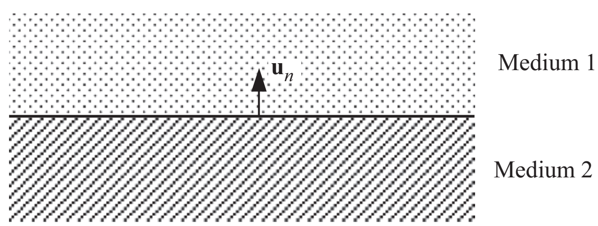

Figure 1.4 Interface between two different media

where $\mathbf{u}_n$ is the unit normal of the boundary directed from medium 2 to medium 1 as shown in Figure 1.4; $\mathbf{J}_s$ and ρ_s are the surface current density and surface charge density respectively. These boundary conditions can also be obtained from the differential form of Maxwell equations in the sense of generalized functions (see Chapter 2). □

1.2.2 Constitutive Relations

Maxwell equations are a set of seven equations involving 16 unknowns (that is five vector functions $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}, \mathbf{J}$ and one scalar function ρ and the last equation of (1.21) is not independent). To determine the fields, nine more equations are needed, and they are given

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by the **generalized constitutive relations**:

$$\mathbf{D} = f_1(\mathbf{E}, \mathbf{H}), \mathbf{B} = f_2(\mathbf{E}, \mathbf{H})$$

together with the **generalized Ohm's law**:

$$\mathbf{J} = f_3(\mathbf{E}, \mathbf{H})$$

if the medium is conducting. The constitutive relations establish the connections between field quantities and reflect the properties of the medium, and they are totally independent of the Maxwell equations. In most cases, the constitutive relations can be expressed as

$$\begin{aligned} D_i(\mathbf{r}, t) &= \sum_{j=x,y,z} [a_i^j(\mathbf{r}) E_j(\mathbf{r}, t) + b_i^j(\mathbf{r}) H_j(\mathbf{r}, t)] \\ &\quad \sum_{j=x,y,z} [(G_i^j * E_j)(\mathbf{r}, t) + (K_i^j * H_j)(\mathbf{r}, t)], \\ B_i(\mathbf{r}, t) &= \sum_{j=x,y,z} [c_i^j(\mathbf{r}) E_j(\mathbf{r}, t) + d_i^j(\mathbf{r}) H_j(\mathbf{r}, t)] \\ &= \sum_{j=x,y,z} [(L_i^j * E_j)(\mathbf{r}, t) + (F_i^j * H_j)(\mathbf{r}, t)], \end{aligned}$$

where $i = x, y, z$; $*$ denotes the convolution with respect to time; $a_i^j, b_i^j, c_i^j, d_i^j$ are independent of time; and $G_i^j, K_i^j, L_i^j, F_i^j$ are functions of $(\mathbf{r}, t)$. The medium defined by the above equations is called **bianisotropic**. An **anisotropic medium** is defined by

$$\begin{aligned} D_i(\mathbf{r}, t) &= \sum_{j=x,y,z} [a_i^j(\mathbf{r}) E_j(\mathbf{r}, t) + (G_i^j * E_j)(\mathbf{r}, t)], \\ B_i(\mathbf{r}, t) &= \sum_{j=x,y,z} [d_i^j(\mathbf{r}) H_j(\mathbf{r}, t) + (F_i^j * H_j)(\mathbf{r}, t)]. \end{aligned}$$

A **biisotropic medium** is defined by

$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) &= a(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + b(\mathbf{r}) \mathbf{H}(\mathbf{r}, t) \\ &\quad + (G * \mathbf{E})(\mathbf{r}, t) + (K * \mathbf{H})(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) &= c(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + d(\mathbf{r}) \mathbf{H}(\mathbf{r}, t) \\ &\quad + (L * \mathbf{E})(\mathbf{r}, t) + (F * \mathbf{H})(\mathbf{r}, t) \end{aligned}$$

where a, b, c, d are independent of time and G, K, L, F are functions of $(\mathbf{r}, t)$. An **isotropic medium** is defined by

$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) &= a(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + (G * \mathbf{E})(\mathbf{r}, t), \\ \mathbf{B}(\mathbf{r}, t) &= d(\mathbf{r}) \mathbf{H}(\mathbf{r}, t) + (F * \mathbf{H})(\mathbf{r}, t). \end{aligned}$$

For monochromatic fields, the constitutive relations for a bianisotropic medium are usually expressed by

$$\mathbf{D} = \overset{\leftrightarrow}{\varepsilon} \cdot \mathbf{E} + \overset{\leftrightarrow}{\xi} \cdot \mathbf{H}, \mathbf{B} = \overset{\leftrightarrow}{\varsigma} \cdot \mathbf{E} + \overset{\leftrightarrow}{\mu} \cdot \mathbf{H}.$$

For an anisotropic medium, both $\overset{\leftrightarrow}{\xi}$ and $\overset{\leftrightarrow}{\varsigma}$ vanish.

Remark 1.10: The effects of the current $\mathbf{J} = \mathbf{J}_{imp} + \mathbf{J}_{ind}$ can be included in the constitutive relations by introducing a new vector $\mathbf{D}''$ such that

$$\mathbf{D}''(\mathbf{r}, t) = \int_{-\infty}^{t} \mathbf{J}(\mathbf{r}, t')dt' + \mathbf{D}(\mathbf{r}, t).$$

Thus (1.21) can be written as

$$\begin{aligned}
&\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}''(\mathbf{r}, t)}{\partial t}, \\
&\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \\
&\nabla \cdot \mathbf{D}''(\mathbf{r}, t) = 0, \\
&\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0.
\end{aligned}$$

So the current source has been absorbed in the displacement current $\partial \mathbf{D}''(\mathbf{r}, t)/\partial t$, and the Maxwell equations are defined in a lossless and source-free region. □

The constitutive relations are often written as

$$\begin{aligned}
\mathbf{D}(\mathbf{r}, t) &= \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) + \cdots, \\
\mathbf{B}(\mathbf{r}, t) &= \mu_0 [\mathbf{H}(\mathbf{r}, t) + \mathbf{M}(\mathbf{r}, t) + \cdots],
\end{aligned} \tag{1.34}$$

where **M** is the magnetization vector and **P** is the polarization vector. Equations (1.34) may contain higher order terms, which have been omitted since in most cases only the magnetization and polarization vectors are significant. The vectors **M** and **P** reflect the effects of the Lorentz force on elemental particles in the medium and therefore they depend on both **E** and **B** in general. Since the elemental particles in the medium have finite masses and are mutually interacting, **M** and **P** are also functions of time derivatives of **E** and **B** as well as their magnitudes. The same applies for the current density $\mathbf{J}_{ind}$.

A detailed study of magnetization and polarization process belongs to the subject of quantum mechanics. However, a macroscopic description of electromagnetic properties of the medium is simple as compared to the microscopic description. When the field quantities are replaced by their respective volume averages, the effects of the complicated array of atoms and electrons constituting the medium may be represented by a few parameters. The macroscopic description is satisfactory only when the large-scale effects of the presence of the medium are considered, and the details of the physical phenomena occurring on an atomic scale can be ignored. Since the averaging process is linear, any linear relation between the microscopic fields remains valid for the macroscopic fields.

In most cases, **M** is only dependent on the magnetic field **B** and its time derivatives while **P** and **J** depend only on the electric field **E** and its time derivatives. If these dependences are linear, the medium is said to be **linear**. These linear dependences are usually expressed as

$$\begin{aligned} \mathbf{D} &= \tilde{\varepsilon}\mathbf{E} + \tilde{\varepsilon}_1 \frac{\partial \mathbf{E}}{\partial t} + \tilde{\varepsilon}_2 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \cdots, \\ \mathbf{B} &= \tilde{\mu}\mathbf{H} + \tilde{\mu}_1 \frac{\partial \mathbf{H}}{\partial t} + \tilde{\mu}_2 \frac{\partial^2 \mathbf{H}}{\partial t^2} + \cdots, \\ \mathbf{J}_{ind} &= \tilde{\sigma}\mathbf{E} + \tilde{\sigma}_1 \frac{\partial \mathbf{E}}{\partial t} + \tilde{\sigma}_2 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \cdots, \end{aligned} \tag{1.35}$$

where all the scalar coefficients are constants. For the monochromatic fields, the first two expressions of (1.35) reduce to

$$\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}$$

where

$$\begin{aligned} \varepsilon &= \varepsilon' - j\varepsilon'', & \mu &= \mu' - j\mu'', \\ \varepsilon' &= \tilde{\varepsilon} - \omega^2 \tilde{\varepsilon}_2 + \cdots, & \mu' &= \tilde{\mu} - \omega^2 \tilde{\mu}_2 + \cdots, \\ \varepsilon'' &= -\omega \tilde{\varepsilon}_1 + \omega^3 \tilde{\varepsilon}_3 - \cdots, & \mu'' &= -\omega \tilde{\mu}_1 + \omega^3 \tilde{\mu}_3 - \cdots. \end{aligned} \tag{1.36}$$

The parameters ε' and ε'' are real and are called **capacitivity** and **dielectric loss factor** respectively. The parameters μ' and μ'' are real and are called **inductivity** and **magnetic loss factor** respectively.

Remark 1.11: According to the transformation of electromagnetic fields under the Lorentz transform (see Chapter 9), the constitutive relations depend on the reference systems. □

1.2.3 Wave Equations

The electromagnetic wave equations are second-order partial differential equations that describe the propagation of electromagnetic waves through a medium. If the medium is homogeneous and isotropic and non-dispersive, we have $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$, where μ and ε are constants. On elimination of **E** or **H** in the generalized Maxwell equations, we obtain

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} &= -\nabla \times \mathbf{J}_m - \mu \frac{\partial \mathbf{J}}{\partial t}, \\ \nabla \times \nabla \times \mathbf{H} + \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} &= \nabla \times \mathbf{J} - \varepsilon \frac{\partial \mathbf{J}_m}{\partial t}. \end{aligned} \tag{1.37}$$

These are known as the **wave equations**. Making use of $\nabla \cdot \mathbf{E} = -\rho/\varepsilon$ and $\nabla \cdot \mathbf{H} = -\rho_m/\mu$, the equations become

$$\begin{aligned}\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\mathbf{E} &= \nabla \times \mathbf{J}_m + \mu\frac{\partial \mathbf{J}}{\partial t} + \nabla\left(\frac{\rho}{\varepsilon}\right),\\ \left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\mathbf{H} &= -\nabla \times \mathbf{J} + \varepsilon\frac{\partial \mathbf{J}_m}{\partial t} + \nabla\left(\frac{\rho_m}{\mu}\right).\end{aligned} \tag{1.38}$$

In a source-free region, Equations (1.38) reduce to homogeneous equations, which have non-trivial solutions. The existence of the non-trivial solutions in a source-free region indicates the possibility of a self-sustaining electromagnetic field outside the source region. For the time-harmonic fields, Equations (1.37) and (1.38) respectively reduce to

$$\begin{aligned}\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} &= -\nabla \times \mathbf{J}_m - j\omega\mu\mathbf{J},\\ \nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} &= \nabla \times \mathbf{J} - j\omega\varepsilon\mathbf{J}_m,\end{aligned} \tag{1.39}$$

and

$$\begin{aligned}(\nabla^2 + k^2)\mathbf{E} &= \nabla \times \mathbf{J}_m + j\omega\mu\mathbf{J} - \frac{\nabla(\nabla \cdot \mathbf{J})}{j\omega\varepsilon},\\ (\nabla^2 + k^2)\mathbf{H} &= -\nabla \times \mathbf{J} + j\omega\varepsilon\mathbf{J}_m - \frac{\nabla(\nabla \cdot \mathbf{J}_m)}{j\omega\mu},\end{aligned} \tag{1.40}$$

where $k = \omega\sqrt{\mu\varepsilon}$. It can be seen that the source terms on the right-hand side of (1.37) and (1.40) are very complicated. To simplify the analysis, the electromagnetic potential functions may be introduced (see Section 2.6.1). The wave equations may be used to solve the following three different field problems:

1. Electromagnetic fields in source-free region: wave propagations in space and waveguides, wave oscillation in cavity resonators, etc.
2. Electromagnetic fields generated by known source distributions: antenna radiations, excitations in waveguides and cavity resonators, etc.
3. Interaction of field and sources: wave propagation in plasma, coupling between electron beams and propagation mechanism, etc.

In a source-free region, Equations (1.39) and (1.40) become

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0, \nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} = 0, \tag{1.41}$$

and

$$(\nabla^2 + k^2)\mathbf{E} = 0, (\nabla^2 + k^2)\mathbf{H} = 0, \tag{1.42}$$

respectively. It should be noted that Equation (1.41) is not equivalent to Equation (1.42). The former implies

$$\nabla \cdot \mathbf{E} = 0, \nabla \cdot \mathbf{H} = 0 \tag{1.43}$$

but the latter does not. Therefore the solutions of (1.41) satisfy Maxwell equations while those of (1.42) may not. For example, $\mathbf{E} = \mathbf{u}_z e^{-jkz}$ is a solution of (1.42) but it does not satisfy $\nabla \cdot \mathbf{E} = 0$. So it is not a solution of Maxwell equations. For this reason, it is imperative that one must incorporate (1.42) with (1.43). This can be accomplished by solving one of the equations in (1.42) to get one field quantity, say **E**, and then using Maxwell equations to get the other field quantity **H**. Such an approach guarantees that the fields satisfy (1.43).

If the medium is inhomogeneous and anisotropic so that $\mathbf{D} = \overset{\leftrightarrow}{\varepsilon} \cdot \mathbf{E}$ and $\mathbf{B} = \overset{\leftrightarrow}{\mu} \cdot \mathbf{H}$, the wave equations for the time-harmonic fields are

$$\begin{aligned} &\nabla \times \overset{\leftrightarrow}{\mu}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \overset{\leftrightarrow}{\varepsilon} \cdot \mathbf{E}(\mathbf{r}) = -j\omega \mathbf{J}(\mathbf{r}) - \nabla \times \overset{\leftrightarrow}{\mu}^{-1} \cdot \mathbf{J}_m, \\ &\nabla \times \overset{\leftrightarrow}{\varepsilon}^{-1} \cdot \nabla \times \mathbf{H}(\mathbf{r}) - \omega^2 \overset{\leftrightarrow}{\mu} \cdot \mathbf{H}(\mathbf{r}) = -j\omega \mathbf{J}_m(\mathbf{r}) + \nabla \times \overset{\leftrightarrow}{\varepsilon}^{-1} \cdot \mathbf{J}. \end{aligned} \tag{1.44}$$

1.2.4 Dispersion

If the speed of the wave propagation and the wave attenuation in a medium depend on the frequency, the medium is said to be dispersive. Dispersion arises from the fact that the polarization and magnetization and the current density cannot follow the rapid changes of the electromagnetic fields, which implies that the electromagnetic energy can be absorbed by the medium. Thus, dissipation or absorption always occurs whenever the medium shows the dispersive effects. In reality, all media show some dispersive effects. The medium can be divided into normal dispersive and anomalous dispersive. A **normal dispersive medium** refers to the situation where the refractive index increases as the frequency increases. Most naturally occurring transparent media exhibit normal dispersion in the visible range of electromagnetic spectrum. In an **anomalous dispersive medium**, the refractive index decreases as frequency increases. The dispersive effects are usually recognized by the existence of elementary solutions (plane wave solution) of Maxwell equations in a source-free region

$$A(\mathbf{k}) e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})} \tag{1.45}$$

where $A(\mathbf{k})$ is the amplitude, **k** is the wave vector and ω is the frequency. When the elementary solutions are introduced into Maxwell equations, it will be found that **k** and ω must be related by an equation

$$f(\omega, \mathbf{k}) = 0. \tag{1.46}$$

This is called the **dispersion equation**. The plane wave $e^{j\omega t - j\mathbf{k} \cdot \mathbf{r}}$ has four-dimensional space-time orthorgonality properties, and is a solution of Maxwell equations in a source-free region when it satisfies the dispersion relation. It can be assumed that the frequency can be expressed in terms of the wave vector by solving the above dispersion equation

$$\omega = W(\mathbf{k}). \tag{1.47}$$

In general, a number of such solutions exist, which give different functions $W(\mathbf{k})$. Each solution is called a mode. To ensure that the solution $e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}}$ is a plane wave, some restrictions must be put on the solution of dispersion equation, which are (Whitham, 1974)

$$\det\left|\frac{\partial^2 W}{\partial k_i \partial k_j}\right| \neq 0,\ W(\mathbf{k}) \text{ is real.} \tag{1.48}$$

These conditions have excluded all non-dispersive waves. A medium is called dispersive if there are solutions of (1.45) and (1.47) that satisfy (1.48). This definition applies to uniform medium. For a non-uniform medium, the definition of dispersive waves can be generalized to allow more general separable solutions of Maxwell equations, such as $A(\mathbf{k}, \mathbf{r})e^{j\omega t}$, where $A(\mathbf{k}, \mathbf{r})$ is an oscillatory function (for example, a Bessel function). It is hard to give a general definition of dispersion of waves. Roughly speaking, the dispersive effects may be expected whenever oscillations in space are coupled with oscillations in time.

If (1.45) is an elementary solution for a linear equation, then formally

$$\varphi(\mathbf{r}, t) = \int_{-\infty}^{\infty} A(\mathbf{k})e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}d\mathbf{k} \tag{1.49}$$

is also a solution of the linear equation. The arbitrary function $A(\mathbf{k})$ may be chosen to satisfy the initial or boundary condition. If there are n modes with n different choices of $W(\mathbf{k})$, there will be n terms like (1.49) with n arbitrary functions $A(\mathbf{k})$. For a single linear differential equation with constant coefficients, there is a one-to-one correspondence between the equation and the dispersion relation. We only need to consider the following correspondences:

$$\frac{\partial}{\partial t} \leftrightarrow j\omega,\ \nabla \leftrightarrow -j\mathbf{k},$$

which yield a polynomial dispersion relation. More complicated dispersion relation may be obtained for other different type of differential equations.

Example 1.3: To find the dispersion relation of the medium, the plane wave solutions may be assumed for Maxwell equations as follows

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}[\mathbf{E}(\mathbf{r}, \omega)e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}}], \text{ etc.} \tag{1.50}$$

Similar expressions hold for other quantities. In the following, the wave vector $\mathbf{k}$ is allowed to be a complex vector and there is no impressed source inside the medium. Introducing (1.50) into (1.26) and using the calculation $\nabla e^{-j\mathbf{k}\cdot\mathbf{r}} = -j\mathbf{k}e^{-j\mathbf{k}\cdot\mathbf{r}}$, we obtain

$$-j\mathbf{k} \times \mathbf{H}(\mathbf{r}, \omega) + \nabla \times \mathbf{H}(\mathbf{r}, \omega) = j\omega\mathbf{D}(\mathbf{r}, \omega) + \mathbf{J}_{con}(\mathbf{r}, \omega),$$
$$-j\mathbf{k} \times \mathbf{E}(\mathbf{r}, \omega) + \nabla \times \mathbf{E}(\mathbf{r}, \omega) = -j\omega\mathbf{B}(\mathbf{r}, \omega).$$

In most situations, the complex amplitudes of the fields are slowly varying functions of space coordinates. The above equations may reduce to

$$\begin{aligned} \mathbf{k} \times \mathbf{H}(\mathbf{r}, \omega) &= -\omega \mathbf{D}(\mathbf{r}, \omega) + j \mathbf{J}_{con}(\mathbf{r}, \omega), \\ \mathbf{k} \times \mathbf{E}(\mathbf{r}, \omega) &= \omega \mathbf{B}(\mathbf{r}, \omega). \end{aligned} \tag{1.51}$$

If the medium is isotropic, dispersive and lossy, we may write

$$\mathbf{J}_{con} = \sigma \mathbf{E}, \mathbf{D} = (\varepsilon' - j\varepsilon'')\mathbf{E}, \mathbf{B} = (\mu' - j\mu'')\mathbf{H}.$$

Substituting these equations into (1.51) yields

$$\mathbf{k} \cdot \mathbf{k} = \omega^2 (\mu' - j\mu'')[\varepsilon' - j(\varepsilon'' + \sigma/\omega)].$$

Assuming $\mathbf{k} = \mathbf{u}_k(\beta - j\alpha)$, then

$$\beta - j\alpha = \omega \sqrt{(\mu' - j\mu'')[\varepsilon' - j(\varepsilon'' + \sigma/\omega)]}$$

from which we may find that

$$\beta = \frac{\omega}{\sqrt{2}} \sqrt{(A^2 + B^2)^{1/2} + A}, \alpha = \frac{\omega}{\sqrt{2}} \sqrt{(A^2 + B^2)^{1/2} - A}$$

where $A = \mu'\varepsilon' - \mu''(\varepsilon'' + \sigma/\omega)$, $\mathrm{B} = \mu''\varepsilon' + \mu'(\varepsilon'' + \sigma/\omega)$. □

1.3 Theorems for Electromagnetic Fields

A number of theorems can be derived from Maxwell equations, and they usually bring deep physical insight into the problems. When applied properly, these theorems can simplify the problems dramatically.

1.3.1 Superposition Theorem

Superposition theorem applies to all linear systems. Suppose that the impressed current source $\mathbf{J}_{imp}$ can be expressed as a linear combination of independent impressed current sources $\mathbf{J}_{imp}^k$ $(k = 1, 2, \cdots, n)$

$$\mathbf{J}_{imp} = \sum_{k=1}^{n} a_k \mathbf{J}_{imp}^k,$$

where a_k $(k = 1, 2, \cdots, n)$ are arbitrary constants. If $\mathbf{E}^k$ and $\mathbf{H}^k$ are fields produced by the source $\mathbf{J}_{imp}^k$, the **superposition theorem** for electromagnetic fields asserts that the fields

$\mathbf{E} = \sum_{k=1}^{n} a_k \mathbf{E}^k$ and $\mathbf{H} = \sum_{k=1}^{n} a_k \mathbf{H}^k$ are a solution of Maxwell equations produced by the source $\mathbf{J}_{imp}$.

1.3.2 Compensation Theorem

The compensation theorem in network theory is well known, which says that any component in the network can be substituted by an ideal current generator with the same current intensity as in the element. Similarly the **compensation theorem** for electromagnetic fields states that the influences of the medium on the electromagnetic fields can be substituted by the equivalent impressed sources. Let **E**, **H**, **M**, **P** and $\mathbf{J}_{ind}$ be the field quantities induced by the impressed current $\mathbf{J}_{imp}$, which satisfy the Maxwell equations (1.21) and the constitutive relations (1.34) with $\mathbf{J} = \mathbf{J}_{ind} + \mathbf{J}_{imp}$. Suppose that the medium is arbitrary and we can write

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2, \mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2, \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2, \tag{1.52}$$

Then Equation (1.34) becomes

$$\begin{aligned} \mathbf{D} &= (\varepsilon_0 \mathbf{E} + \mathbf{P}_1) + \mathbf{P}_2 = \mathbf{D}_1 + \mathbf{P}_2, \\ \mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{M}_1) + \mu_0 \mathbf{M}_2 = \mathbf{B}_1 + \mu_0 \mathbf{M}_2, \end{aligned}$$

with $\mathbf{B}_1 = \mu_0(\mathbf{H} + \mathbf{M}_1)$ and $\mathbf{D}_1 = (\varepsilon_0 \mathbf{E} + \mathbf{P}_1)$. Accordingly, the Maxwell equations (1.21) can be written as

$$\begin{aligned} \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}_1}{\partial t} + \mathbf{J}_1 + (\mathbf{J}_{imp} + \mathbf{J}'_{imp}), \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}_1(\mathbf{r}, t)}{\partial t} - \mathbf{J}'_{m,imp}, \\ \nabla \cdot \mathbf{D}_1 &= \rho + \rho', \rho' = -\nabla \cdot \mathbf{P}_2, \\ \nabla \cdot \mathbf{B}_1 &= \rho_m, \rho_m = -\mu_0 \nabla \cdot \mathbf{M}_2, \end{aligned} \tag{1.53}$$

where the new impressed electric current $\mathbf{J}'_{imp} = \mathbf{J}_2 + \partial \mathbf{P}_2/\partial t$ and magnetic current $\mathbf{J}'_{m,imp} = \mu_0 \partial \mathbf{M}_2/\partial t$ have been introduced to represent the influences of the medium partly or completely, depending on how the division is made in (1.52). Equations (1.53) are the mathematical formulation of compensation theorem. Note that both impressed electric and magnetic current source are needed to replace the medium, and the magnetic current density and the magnetic charge density satisfy the continuity equation $\nabla \cdot \mathbf{J}'_{m,imp} = -\partial \rho_m/\partial t$.

1.3.3 Conservation of Electromagnetic Energy

The law of **conservation of electromagnetic energy** is known as the **Poynting theorem**, named after the English physicist John Henry Poynting (1852–1914). It can be found from

(1.21) that

$$-\mathbf{J}_{imp}\cdot\mathbf{E}-\mathbf{J}_{ind}\cdot\mathbf{E}=\nabla\cdot\mathbf{S}+\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}. \tag{1.54}$$

In a region V bounded by S, the integral form of (1.54) is

$$-\int_V \mathbf{J}_{imp}\cdot\mathbf{E}dV=\int_V \mathbf{J}_{ind}\cdot\mathbf{E}dV+\int_S \mathbf{S}\cdot\mathbf{u}_n dS+\int_V\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)dV, \tag{1.55}$$

where $\mathbf{u}_n$ is the unit outward normal of S, and $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ is the **Poynting vector** representing the electromagnetic power-flow density measured in watts per square meter (W/m^2). It will be assumed that this explanation holds for all media. Thus, the left-hand side of Equation (1.55) stands for the power supplied by the impressed current source. The first term on the right-hand side is the work done per second by the electric field to maintain the current in the conducting part of the system. The second term denotes the electromagnetic power flowing out of S. The last term can be interpreted as the work done per second by the impressed source to establish the fields. The energy density w required to establish the electromagnetic fields may be defined as follows

$$dw=\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)dt. \tag{1.56}$$

Assuming all the sources and fields are zero at $t=-\infty$, we have

$$w=w_e+w_m, \tag{1.57}$$

where w_e and w_m are the **electric field energy density** and **magnetic field energy density** respectively

$$w_e=\frac{1}{2}\mathbf{E}\cdot\mathbf{D}+\int_{-\infty}^{t}\frac{1}{2}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}-\mathbf{D}\cdot\frac{\partial\mathbf{E}}{\partial t}\right)dt,$$

$$w_m=\frac{1}{2}\mathbf{H}\cdot\mathbf{B}+\int_{-\infty}^{t}\frac{1}{2}\left(\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-\mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial t}\right)dt.$$

Equation (1.55) can be written as

$$-\int_V \mathbf{J}_{imp}\cdot\mathbf{E}dV=\int_V \mathbf{J}_{ind}\cdot\mathbf{E}dV+\int_S \mathbf{S}\cdot\mathbf{u}_n dS+\frac{\partial}{\partial t}\int_V (w_e+w_m)dV. \tag{1.58}$$

In general, the energy density w does not represent the stored energy density in the fields: the energy temporarily located in the fields and completely recoverable when the fields are reduced to zero. The energy density w given by (1.57) can be considered as the stored energy density

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only if the medium is lossless (that is, $\nabla \cdot \mathbf{S} = 0$). If medium is isotropic and time-invariant, we have

$$w_e = \frac{1}{2}\mathbf{E} \cdot \mathbf{D},\, w_m = \frac{1}{2}\mathbf{H} \cdot \mathbf{B}.$$

If the fields are time-harmonic, the Poynting theorem takes the following form

$$\begin{aligned} -\frac{1}{2}\int_V \mathbf{E} \cdot \bar{\mathbf{J}}_{imp} dV = \frac{1}{2}\int_V \mathbf{E} \cdot \bar{\mathbf{J}}_{ind} dV + \int_S \frac{1}{2}(\mathbf{E} \times \bar{\mathbf{H}}) \cdot \mathbf{u}_n dS \\ + j2\omega \int_V \left(\frac{1}{4}\mathbf{B} \cdot \bar{\mathbf{H}} - \frac{1}{4}\mathbf{E} \cdot \bar{\mathbf{D}}\right) dV, \end{aligned} \tag{1.59}$$

where the bar denotes complex conjugate. The time averages of the Poynting vector, energy densities over one period of the sinusoidal wave $e^{j\omega t}$, denoted T, are

$$\begin{aligned} \bar{\bar{\mathbf{S}}} &= \frac{1}{T}\int_0^T \mathbf{E} \times \bar{\mathbf{H}} dt = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \bar{\mathbf{H}}), \\ \frac{1}{T}\int_0^T \frac{1}{2}\mathbf{E} \cdot \mathbf{D} dt &= \frac{1}{4}\mathrm{Re}(\mathbf{E} \cdot \bar{\mathbf{D}}), \\ \frac{1}{T}\int_0^T \frac{1}{2}\mathbf{H} \cdot \mathbf{B} dt &= \frac{1}{4}\mathrm{Re}(\mathbf{H} \cdot \bar{\mathbf{B}}), \end{aligned}$$

where the double line indicates the time average.

1.3.4 Conservation of Electromagnetic Momentum

The force acting on a charged particle by electromagnetic fields is given by the Lorentz force equation

$$\mathbf{F}(\mathbf{r}, t) = q[\mathbf{E}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)],$$

where $\mathbf{v}$ is the velocity of the particle. Let m be the mass of the particle and $\mathbf{G}_p = m\mathbf{v}$ its momentum. By Newton's law, we have

$$\frac{d\mathbf{G}_p(\mathbf{r}, t)}{dt} = q[\mathbf{E}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)]. \tag{1.60}$$

Let $W_p = m\mathbf{v} \cdot \mathbf{v}/2$ denote the kinetic energy of the particle. It follows from (1.60) that

$$\frac{dW_p(\mathbf{r}, t)}{dt} = q\mathbf{v}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t). \tag{1.61}$$

For a continuous charge distribution ρ, Equations (1.60) and (1.61) should be changed to

$$\frac{d\mathbf{g}_p}{dt} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \mathbf{f}, \tag{1.62}$$

$$\frac{dw_p}{dt} = \mathbf{J} \cdot \mathbf{E}, \tag{1.63}$$

where $\mathbf{J} = \rho\mathbf{v}$; $\mathbf{g}_p = \rho_m\mathbf{v}$ and $w_p = \rho_m\mathbf{v} \cdot \mathbf{v}/2$ are the density of momentum and density of kinetic energy of the charge distribution respectively, and ρ_m is the mass density. Equations (1.62) and (1.63) indicate that the charged system gains energy and momentum from the electromagnetic fields if $dw_p/dt > 0$ and $d\mathbf{g}_p/dt > 0$ or releases energy and momentum to the electromagnetic fields if $dw_p/dt < 0$ and $d\mathbf{g}_p/dt < 0$. From the conservation laws of energy and momentum, it may be concluded that electromagnetic fields have energy and momentum. From the Maxwell equations and Lorentz force equation in free space, we obtain

$$\begin{aligned}\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} &= \mathbf{E}\nabla \cdot \mathbf{D} + \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}\right) \times \mathbf{B} \\ &= -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\mathbf{E} \times \mathbf{H}\right) + \nabla \cdot \left[\varepsilon_0\mathbf{E}\mathbf{E} + \mu_0\mathbf{H}\mathbf{H} - \frac{1}{2}\left(\varepsilon_0\mathbf{E} \cdot \mathbf{E} + \mu_0\mathbf{H} \cdot \mathbf{H}\right)\overleftrightarrow{\mathbf{I}}\right],\end{aligned}$$

where $c = 1/\sqrt{\mu_0\varepsilon_0}$, **EE** and **HH** are dyads. By means of (1.62), the above equation can be written as

$$\nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{\partial}{\partial t}(\mathbf{g} + \mathbf{g}_p) = 0. \tag{1.64}$$

where $\overleftrightarrow{\mathbf{T}} = \varepsilon_0\mathbf{E}\mathbf{E} + \mu_0\mathbf{H}\mathbf{H} - \frac{1}{2}(\varepsilon_0\mathbf{E} \cdot \mathbf{E} + \mu_0\mathbf{H} \cdot \mathbf{H})\overleftrightarrow{\mathbf{I}}$ is referred to as the **Maxwell stress tensor** and $\mathbf{g} = \mathbf{E} \times \mathbf{H}/c^2$ is known as the **electromagnetic momentum density**. The integral form of (1.64) over a region V bounded by S is

$$\frac{\partial}{\partial t}\int_V (\mathbf{g} + \mathbf{g}_p)dV = \int_S \mathbf{u}_n \cdot \overleftrightarrow{\mathbf{T}}dS, \tag{1.65}$$

Equation (1.65) indicates that the increase of total momentum (the electromagnetic momentum plus the momentum of the charged system) inside V per unit time is equal to the force acting on the fields inside V through the boundary S by the fields outside S. For this reason, $\mathbf{u}_n \cdot \overleftrightarrow{\mathbf{T}}$ may be interpreted as the force per unit area acting on the surface. We can also interpret $-\mathbf{u}_n \cdot \overleftrightarrow{\mathbf{T}}$ as the momentum flow density into S and call $-\overleftrightarrow{\mathbf{T}}$ the **electromagnetic momentum flow density tensor** or the **electromagnetic energy-momentum tensor**.

1.3.5 Conservation of Electromagnetic Angular Momentum

It follows from (1.64) that

$$\nabla \cdot (\mathbf{r} \times \overleftrightarrow{\mathbf{T}}) + \frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{g} + \mathbf{r} \times \mathbf{g}_p) = 0.$$

The integral form of the above equation over a region V bounded by S is

$$\frac{\partial}{\partial t} \int_V (\mathbf{r} \times \mathbf{g} + \mathbf{r} \times \mathbf{g}_p) dV = -\int_S \mathbf{u}_n \cdot (\mathbf{r} \times \overleftrightarrow{\mathbf{T}}) dS.$$

Here $\mathbf{r} \times \mathbf{g}$ may be interpreted as the **electromagnetic angular momentum density** and $\mathbf{r} \times \overleftrightarrow{\mathbf{T}}$ as the **electromagnetic angular momentum flow density tensor**.

Remark 1.12: The quantities of a dynamic system that do not change with time play an important role in theoretical physics. These conserved quantities can be the energy, momentum, and angular momentum. Noether's theorem, named after the German mathematician Amalie Emmy Noether (1882–1935), states that the conservation laws are the consequences of continuous symmetry transformations under which the action integral of the system is left invariant. For example, time translation symmetry gives conservation of energy; space translation symmetry gives conservation of momentum; rotation symmetry gives conservation of angular momentum. □

1.3.6 Uniqueness Theorems

It is important to know the conditions under which the solution of Maxwell equations is unique. Let us consider a multiple-connected region V bounded by $S = \sum_{i=0}^{N} S_i$, as shown in Figure 1.5. Assume that the medium inside V is linear, isotropic and time-invariant, and it may contain some impressed source $\mathbf{J}_{imp}$. So we have $\mathbf{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}$, and $\mathbf{J}_{ind} = \sigma\mathbf{E}$. Let $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ be two solutions of Maxwell equations. Then the difference fields $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$ and $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2$ are a solution of the Maxwell equations free of impressed source. The requirements that the difference fields must be identically zero are the conditions

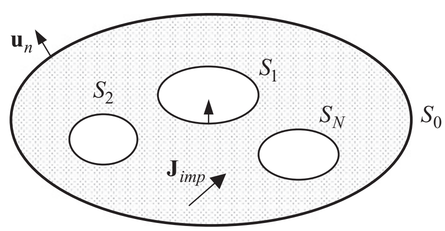

Figure 1.5 Multiple-connected region

for uniqueness that we seek. According to the Poynting theorem in the time domain, we write

$$\int_V \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dV + \int_V \sigma \left| \mathbf{E} \right|^2 dV = -\int_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{u}_n dS, \tag{1.66}$$

where $\mathbf{u}_n$ is the unit outward normal of S. If $\mathbf{E}_1 = \mathbf{E}_2$ or $\mathbf{H}_1 = \mathbf{H}_2$ holds on the boundary S for $t > 0$, the above equation reduces to

$$\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \right) dV = -\int_V \sigma \left| \mathbf{E} \right|^2 dV.$$

Suppose that the source is turned on at $t = 0$. Taking the integration with respect to time over $[0, t]$ yields

$$\int_V \left(\frac{1}{2} \varepsilon \left| \mathbf{E}(\mathbf{r}, t) \right|^2 + \frac{1}{2} \mu \left| \mathbf{H}(\mathbf{r}, t) \right|^2 \right) dV - \int_V \left(\frac{1}{2} \varepsilon \left| \mathbf{E}(\mathbf{r}, 0) \right|^2 + \frac{1}{2} \mu \left| \mathbf{H}(\mathbf{r}, 0) \right|^2 \right) dV$$
$$= -\int_0^t dt \int_V \sigma \left| \mathbf{E} \right|^2 dV.$$

If $\mathbf{E}_1(\mathbf{r}, 0) = \mathbf{E}_2(\mathbf{r}, 0)$ and $\mathbf{H}_1(\mathbf{r}, 0) = \mathbf{H}_2(\mathbf{r}, 0)$ hold in V, the second term on the left-hand side vanishes. Since the right-hand side is a negative number while the left-hand side is a positive number, this is possible only when $\mathbf{E}_1(\mathbf{r}, t) = \mathbf{E}_2(\mathbf{r}, t)$ and $\mathbf{H}_1(\mathbf{r}, t) = \mathbf{H}_2(\mathbf{r}, t)$ for all $t > 0$.

If the region extends to infinity ($S_0 \to \infty$), we can assume that $\mathbf{E}_1 = \mathbf{E}_2$ or $\mathbf{H}_1 = \mathbf{H}_2$ on the boundary $\sum_{i=1}^{N} S_i$ for $t > 0$, and $\mathbf{E}_1(\mathbf{r}, 0) = \mathbf{E}_2(\mathbf{r}, 0)$ and $\mathbf{H}_1(\mathbf{r}, 0) = \mathbf{H}_2(\mathbf{r}, 0)$ in V. It follows from (1.66) that

$$\int_V \left(\frac{1}{2} \varepsilon \left| \mathbf{E}(\mathbf{r}, t) \right|^2 + \frac{1}{2} \mu \left| \mathbf{H}(\mathbf{r}, t) \right|^2 \right) dV$$
$$= -\int_0^t dt \int_V \sigma \left| \mathbf{E} \right|^2 dV - \int_0^t dt \int_{S_0} \frac{1}{\eta_0} \left| \mathbf{E} \right|^2 dS. \tag{1.67}$$

Here $\eta_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the wave impedance in free space. Equation (1.67) implies $\mathbf{E}_1(\mathbf{r}, t) = \mathbf{E}_2(\mathbf{r}, t)$ and $\mathbf{H}_1(\mathbf{r}, t) = \mathbf{H}_2(\mathbf{r}, t)$ for all $t > 0$. Note that the preceding discussions are valid even if σ is zero. Thus the following uniqueness theorem for electromagnetic fields in time domain has been proved.

Theorem 1.1 ***Uniqueness theorem for time-domain fields:*** *Suppose that the electromagnetic sources are turned on at $t = 0$. The electromagnetic fields in a region are uniquely determined by the sources within the region, the initial electric field and the initial magnetic field at $t = 0$ inside the region, together with the tangential electric field (or the tangential magnetic field) on the boundary for $t > 0$, or together with the tangential electric field on part of the boundary and the tangential magnetic field on the rest of the boundary for $t > 0$.* □

We now derive the uniqueness theorem in the frequency domain. Let $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ be two solutions of the time-harmonic Maxwell equations. For the difference fields $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$ and $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2$, we may use the Poynting theorem in the frequency domain to write

$$\int_S \frac{1}{2}(\mathbf{E} \times \bar{\mathbf{H}}) \cdot \mathbf{u}_n dS + j2\omega \int_V \left(\frac{1}{4}\mathbf{B} \cdot \bar{\mathbf{H}} - \frac{1}{4}\mathbf{E} \cdot \bar{\mathbf{D}}\right) dV = -\frac{1}{2}\int_V \sigma \left|\mathbf{E}\right|^2 dV. \tag{1.68}$$

If $\mathbf{E}_1 = \mathbf{E}_2$ or $\mathbf{H}_1 = \mathbf{H}_2$ holds on S, the first term on the left-hand side vanishes and we have

$$j\omega \int_V \frac{1}{2}\mu \left|\mathbf{H}\right|^2 dV - j\omega \int_V \frac{1}{2}\bar{\varepsilon} \left|\mathbf{E}\right|^2 dV + \frac{1}{2}\int_V \sigma \left|\mathbf{E}\right|^2 dV = 0.$$

This implies

$$\begin{aligned}
\omega \int_V \frac{1}{2}\mathrm{Re}\,\varepsilon \left|\mathbf{E}\right|^2 dV - \omega \int_V \frac{1}{2}\mathrm{Re}\,\mu \left|\mathbf{H}\right|^2 dV &= 0,\\
\omega \int_V \frac{1}{2}\mathrm{Im}\,\varepsilon \left|\mathbf{E}\right|^2 dV + \omega \int_V \frac{1}{2}\mathrm{Im}\,\mu \left|\mathbf{H}\right|^2 dV &= \frac{1}{2}\int_V \sigma \left|\mathbf{E}\right|^2 dV.
\end{aligned}$$

For a dissipative medium, we have $\mathrm{Im}\,\varepsilon < 0$ and $\mathrm{Im}\,\mu < 0$. It is easy to see that if one of the following two conditions is met

$$\mathrm{Im}\,\varepsilon < 0, \mathrm{Im}\,\mu < 0, \tag{1.69}$$

$$\sigma > 0, \tag{1.70}$$

then the difference fields $\mathbf{E}$ and $\mathbf{H}$ vanish in V, which implies that the fields in V can be uniquely determined. Therefore, a loss must be assumed for time-harmonic fields in order to obtain the uniqueness.

In an unbounded region where $S_0 \to \infty$, we may assume that $\mu \to \mu_0$, $\varepsilon \to \varepsilon_0$ on S_0. Thus (1.68) may be written as

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{S_i} \frac{1}{2}(\mathbf{E} \times \bar{\mathbf{H}}) \cdot \mathbf{u}_n dS + j\omega \int_V \frac{1}{2}\mu \left|\mathbf{H}\right|^2 dV - j\omega \int_V \frac{1}{2}\bar{\varepsilon} \left|\mathbf{E}\right|^2 dV\\
&\quad = -\int_{S_0} \frac{1}{2\eta_0} \left|\mathbf{E}\right|^2 dS - \frac{1}{2}\int_V \sigma \left|\mathbf{E}\right|^2 dV.
\end{aligned} \tag{1.71}$$

If $\mathbf{E}_1 = \mathbf{E}_2$ or $\mathbf{H}_1 = \mathbf{H}_2$ holds on $\sum_{i=1}^{N} S_i$, the first term on the left-hand side of (1.71) vanishes and (1.71) reduces to

$$j\omega \int_V \frac{1}{2}\mu \left|\mathbf{H}\right|^2 dV - j\omega \int_V \frac{1}{2}\bar{\varepsilon} \left|\mathbf{E}\right|^2 dV = -\int_{S_0} \frac{1}{2\eta_0} \left|\mathbf{E}\right|^2 dS - \frac{1}{2}\int_V \sigma \left|\mathbf{E}\right|^2 dV. \quad (1.72)$$

This leads to

$$\begin{aligned} &\omega \int_V \frac{1}{2}\mathrm{Re}\,\varepsilon \left|\mathbf{E}\right|^2 dV - \omega \int_V \frac{1}{2}\mathrm{Re}\,\mu \left|\mathbf{H}\right|^2 dV = 0, \\ &\omega \int_V \frac{1}{2}\mathrm{Im}\,\varepsilon \left|\mathbf{E}\right|^2 dV + \omega \int_V \frac{1}{2}\mathrm{Im}\,\mu \left|\mathbf{H}\right|^2 dV = \int_{S_0} \frac{1}{2\eta_0} \left|\mathbf{E}\right|^2 dS + \frac{1}{2}\int_V \sigma \left|\mathbf{E}\right|^2 dV. \end{aligned} \quad (1.73)$$

The difference fields vanish in the infinite region if either condition (1.69) or (1.70) is satisfied. We can further show that the difference fields vanish in the infinite region where radiation exists, even if the medium is lossless. Assuming that the medium is lossless, the second equation of (1.73) implies

$$\int_{S_0} \frac{1}{2\eta_0} \left|\mathbf{E}\right|^2 dS = 0,\ S_0 \to \infty.$$

It follows that

$$\left|\mathbf{E}\right|^2 = 0,\ S_0 \to \infty. \quad (1.74)$$

This relation implies $\mathbf{E} = \mathbf{H} = 0$ in the region V, which can be proved as follows. Consider a sufficiently large sphere that contains all the impressed sources and inhomogeneities. The fields on the sphere may be expanded in terms of the spherical vector wavefunctions as follows (see Section 4.3)

$$\begin{aligned} \mathbf{E} &= -\sum_{n,m,l} \left(\alpha_{nml}^{(2)} \mathbf{M}_{nml}^{(2)} + \beta_{nml}^{(2)} \mathbf{N}_{nml}^{(2)}\right), \\ \mathbf{H} &= \frac{1}{j\eta_0} \sum_{n,m,l} \left(\alpha_{nml}^{(2)} \mathbf{N}_{nml}^{(2)} + \beta_{nml}^{(2)} \mathbf{M}_{nml}^{(2)}\right). \end{aligned}$$

A simple calculation gives

$$\left|\mathbf{E}\right|^2 = \frac{1}{k_0^2} \sum_{n,m,l} N_{nm}^2 \left(\left|\alpha_{nml}^{(2)}\right|^2 + \left|\beta_{nml}^{(2)}\right|^2\right), \quad (1.75)$$

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where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ and N_{nm} is a constant. Combining (1.74) and (1.75), we obtain $\alpha_{nml}^{(2)} = \beta_{nml}^{(2)} = 0$. As a result, the fields outside a sufficiently large sphere are identically zero. By the analyticity of the electromagnetic fields, one must have $\mathbf{E} = \mathbf{H} = 0$ in the region V. Consequently the uniqueness theorem for time-harmonic field may be stated as follows.

Theorem 1.2 ***Uniqueness theorem for time-harmonic fields:*** *For a region that contains the dissipation loss or radiation loss, the electromagnetic fields are uniquely determined by the sources within the region, together with the tangential electric field (or the tangential magnetic field) on the boundary, or together with the tangential electric field on part of the boundary and the tangential magnetic field on the rest of the boundary.* □

The uniqueness for time-harmonic fields is guaranteed if the system has radiation loss, regardless whether the medium is lossy or not. This property has been widely validated by the study of antenna radiation problems, in which the surrounding medium is often assumed to be lossless.

Remark 1.13: The uniqueness for time-harmonic fields fails for a system that contains no dissipation loss and radiation loss. The uniqueness in a lossless medium is usually obtained by considering the fields in a lossless medium to be the limit of the corresponding fields in a lossy medium as the loss goes to zero, which is based on an assumption that the limit of a unique solution is also unique. However, this limiting process may lead to physically unacceptable solutions (see Section 3.3.2 and Section 8.2.1). Note that there is no need to introduce loss for a unique solution in the time-domain analysis. □

Example 1.4 (Image principle): To solve the boundary value problem with a perfect electric conductor, one can use the image principle that is based on the uniqueness theorem. The perfect electric conductor may be removed by introducing an ‘image’ of the original field source. The image is constructed in such a way so that the tangential component of the total electric field produced by the original source and its image vanishes on the perfect electric conductor. For example, an electric current element parallel to an infinitely large conducting plane has an image that is positioned symmetrically relative to the conducting plane and has a reverse orientation. The images for electric and magnetic current elements placed near the conducting plane are shown in Figure 1.6. □

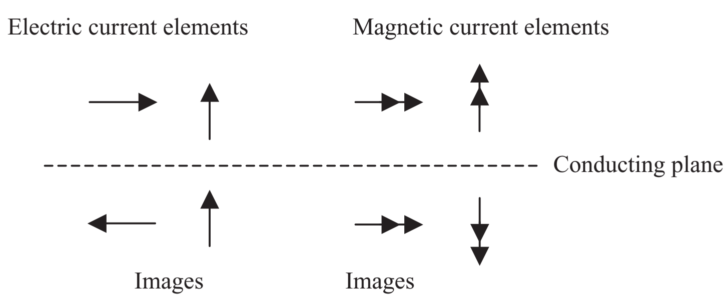

Figure 1.6 Image principle

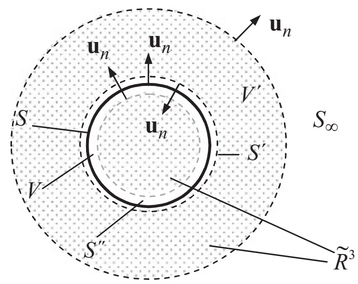

Figure 1.7 Equivalence theorem

1.3.7 Equivalence Theorems

It is known that there is no answer to the question of whether field or source is primary. The equivalence principles indicate that the distinction between the field and source is kind of blurred. Let V be an arbitrary region bounded by S; let S' be a closed surface pressed tightly over S from outside; let S'' be a closed surface pressed tightly to S from inside; let V' be the domain outside S'. A large closed surface S_∞ encloses S' as shown in Figure 1.7. Two sources that produce the same fields inside a region are said to be equivalent within that region. Similarly, two electromagnetic fields $\{\mathbf{E}_1, \mathbf{D}_1, \mathbf{H}_1, \mathbf{B}_1\}$ and $\{\mathbf{E}_2, \mathbf{D}_2, \mathbf{H}_2, \mathbf{B}_2\}$ are said to be equivalent inside a region if they both satisfy the Maxwell equations and are equal in that region.

The main application of the equivalence theorem is to find equivalent sources to replace the influences of substance (the medium is homogenized), so that the formulae for retarding potentials can be used. The equivalent sources may be located inside S (equivalent volume sources) or on S (equivalent surface sources). The most general form of the equivalent principles is as follows.

General equivalence principle: Let us consider two electromagnetic field problems in two different media:

$$\text{Problem 1}: \begin{cases} \nabla \times \mathbf{H}_1(\mathbf{r}, t) = \partial \mathbf{D}_1(\mathbf{r}, t)/\partial t + \mathbf{J}_1(\mathbf{r}, t), \\ \nabla \times \mathbf{E}_1(\mathbf{r}, t) = -\partial \mathbf{B}_1(\mathbf{r}, t)/\partial t - \mathbf{J}_{m1}(\mathbf{r}, t), \\ \nabla \cdot \mathbf{D}_1(\mathbf{r}, t) = \rho_1(\mathbf{r}, t), \nabla \cdot \mathbf{B}_1(\mathbf{r}, t) = \rho_{m1}(\mathbf{r}, t), \\ \mathbf{D}_1(\mathbf{r}, t) = \varepsilon_1(\mathbf{r})\mathbf{E}_1(\mathbf{r}, t), \mathbf{B}_1(\mathbf{r}, t) = \mu_1(\mathbf{r})\mathbf{H}_1(\mathbf{r}, t) \end{cases}$$

$$\text{Problem 2}: \begin{cases} \nabla \times \mathbf{H}_2(\mathbf{r}, t) = \partial \mathbf{D}_2(\mathbf{r}, t)/\partial t + \mathbf{J}_2(\mathbf{r}, t), \\ \nabla \times \mathbf{E}_2(\mathbf{r}, t) = -\partial \mathbf{B}_2(\mathbf{r}, t)/\partial t - \mathbf{J}_{m2}(\mathbf{r}, t), \\ \nabla \cdot \mathbf{D}_2(\mathbf{r}, t) = \rho_2(\mathbf{r}, t), \nabla \cdot \mathbf{B}_2(\mathbf{r}, t) = \rho_{m2}(\mathbf{r}, t), \\ \mathbf{D}_2(\mathbf{r}, t) = \varepsilon_2(\mathbf{r})\mathbf{E}_2(\mathbf{r}, t), \mathbf{B}_2(\mathbf{r}, t) = \mu_2(\mathbf{r})\mathbf{H}_2(\mathbf{r}, t). \end{cases}$$

If a new set of electromagnetic fields $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ satisfying

$$\begin{cases} \nabla \times \mathbf{H}(\mathbf{r}, t) = \partial \mathbf{D}(\mathbf{r}, t)/\partial t + \mathbf{J}(\mathbf{r}, t), \\ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial \mathbf{B}(\mathbf{r}, t)/\partial t - \mathbf{J}_m(\mathbf{r}, t), \\ \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t), \nabla \cdot \mathbf{B}(\mathbf{r}, t) = \rho_m(\mathbf{r}, t), \\ \mathbf{D}(\mathbf{r}, t) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t) = \mu(\mathbf{r})\mathbf{H}(\mathbf{r}, t), \end{cases} \tag{1.76}$$

is constructed in such a way that the sources of the fields $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ and the parameters of the medium satisfy

$$\begin{cases} \mathbf{J} = \mathbf{J}_1, \mathbf{J}_m = \mathbf{J}_{m1} \\ \rho = \rho_1, \rho_m = \rho_{m1}, \mathbf{r} \in V \\ \mu = \mu_1, \varepsilon = \varepsilon_2 \end{cases}; \begin{cases} \mathbf{J} = \mathbf{J}_2, \mathbf{J}_m = \mathbf{J}_{m2} \\ \rho = \rho_2, \rho_m = \rho_{m2}, \mathbf{r} \in R^3 - V \\ \mu = \mu_2, \varepsilon = \varepsilon_2 \end{cases}$$

and

$$\begin{cases} \mathbf{J} = \mathbf{u}_n \times (\mathbf{H}_{2+} - \mathbf{H}_{1-}) \\ \mathbf{J}_m = -\mathbf{u}_n \times (\mathbf{E}_{2+} - \mathbf{E}_{1-}) \\ \rho = \mathbf{u}_n \cdot (\mathbf{D}_{2+} - \mathbf{D}_{1-}) \\ \rho_m = \mathbf{u}_n \cdot (\mathbf{B}_{2+} - \mathbf{B}_{1-}) \end{cases}, \mathbf{r} \in S$$

where $\mathbf{u}_n$ is the unit outward normal to S, and the subscripts $+$ and - signify the values obtained as S is approached from outside S and inside S respectively, then we have

$$\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\} = \{\mathbf{E}_1, \mathbf{D}_1, \mathbf{H}_1, \mathbf{B}_1\}, \mathbf{r} \in V$$
$$\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\} = \{\mathbf{E}_2, \mathbf{D}_2, \mathbf{H}_2, \mathbf{B}_2\}, \mathbf{r} \in R^3 - V$$

To prove this theorem, we only need to show that the difference fields

$$\begin{cases} \delta\mathbf{E} = \mathbf{E} - \mathbf{E}_1 \\ \delta\mathbf{H} = \mathbf{H} - \mathbf{H}_1 \\ \delta\mathbf{D} = \mathbf{D} - \mathbf{D}_1 \\ \delta\mathbf{B} = \mathbf{B} - \mathbf{B}_1 \end{cases}, \mathbf{r} \in V; \begin{cases} \delta\mathbf{E} = \mathbf{E} - \mathbf{E}_2 \\ \delta\mathbf{H} = \mathbf{H} - \mathbf{H}_2 \\ \delta\mathbf{D} = \mathbf{D} - \mathbf{D}_2 \\ \delta\mathbf{B} = \mathbf{B} - \mathbf{B}_2 \end{cases}, \mathbf{r} \in R^3 - V$$

in the shadowed region bounded by $S' + S'' + S_\infty$, denoted by $\tilde{R}^3$, are identically zero. The difference fields satisfy

$$\begin{aligned} &\nabla \times \delta\mathbf{H}(\mathbf{r}, t) = \partial \delta\mathbf{D}(\mathbf{r}, t)/\partial t, \\ &\nabla \times \delta\mathbf{E}(\mathbf{r}, t) = -\partial \delta\mathbf{B}(\mathbf{r}, t)/\partial t, \\ &\nabla \cdot \delta\mathbf{D}(\mathbf{r}, t) = 0, \nabla \cdot \delta\mathbf{B}(\mathbf{r}, t) = 0, \\ &\delta\mathbf{D}(\mathbf{r}, t) = \varepsilon_\delta(\mathbf{r})\mathbf{E}(\mathbf{r}, t), \\ &\delta\mathbf{B}(\mathbf{r}, t) = \mu_\delta(\mathbf{r})\delta\mathbf{H}(\mathbf{r}, t), \end{aligned} \tag{1.77}$$

where $\varepsilon_\delta = \varepsilon_1$, $\mu_\delta = \mu_1$ for $\mathbf{r} \in V$ and $\varepsilon_\delta = \varepsilon_2$, $\mu_\delta = \mu_2$ for $\mathbf{r} \in R^3 - V$. From

$$\mathbf{u}_n \times (\mathbf{H}_{2+} - \mathbf{H}_{1-}) = \mathbf{u}_n \times (\mathbf{H}_+ - \mathbf{H}_-),$$
$$\mathbf{u}_n \times (\mathbf{E}_{2+} - \mathbf{E}_{1-}) = \mathbf{u}_n \times (\mathbf{E}_+ - \mathbf{E}_-),$$

we can find $\mathbf{u}_n \times \delta\mathbf{E}_+ = \mathbf{u}_n \times \delta\mathbf{E}_-$ and $\mathbf{u}_n \times \delta\mathbf{H}_+ = \mathbf{u}_n \times \delta\mathbf{H}_-$, which imply that the tangential components of $\delta\mathbf{E}$ and $\delta\mathbf{H}$ are continuous on S. It follows from (1.77) that

$$-\nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}) = \frac{1}{2}\frac{\partial}{\partial t}(\varepsilon_\delta \left|\delta\mathbf{E}\right|^2 + \mu_\delta \left|\delta\mathbf{H}\right|^2).$$

Taking the integration over the shadowed region $\tilde{R}^3$ yields

$$-\int\limits_{S'+S''+S_\infty} (\delta\mathbf{E} \times \delta\mathbf{H}) \cdot \mathbf{u}_n dS = \frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\tilde{R}^3} (\varepsilon_\delta \left|\delta\mathbf{E}\right|^2 + \mu_\delta \left|\delta\mathbf{H}\right|^2) dV.$$

If all the fields are produced after a finite moment $t_0 > -\infty$, one may take the integration with respect to time from $-\infty$ to t

$$\begin{aligned} &-\int\limits_{-\infty}^{t} dt \int\limits_{S'+S''+S_\infty} [\delta\mathbf{E}(\mathbf{r}, t) \times \delta\mathbf{H}(\mathbf{r}, t)] \cdot \mathbf{u}_n dS \\ &\quad = \frac{1}{2}\int\limits_{\tilde{R}^3} (\varepsilon_\delta \left|\delta\mathbf{E}(\mathbf{r}, t)\right|^2 + \mu_\delta \left|\delta\mathbf{H}(\mathbf{r}, t)\right|^2) dV. \end{aligned} \tag{1.78}$$

When S' and S'' approach S, the values of $\delta\mathbf{E}(\mathbf{r}, t) \times \delta\mathbf{H}(\mathbf{r}, t)$ on S' and S'' tend to be the same since $\delta\mathbf{E}(\mathbf{r}, t) \times \delta\mathbf{H}(\mathbf{r}, t)$ is continuous on S. Thus

$$\int\limits_{S'+S''} [\delta\mathbf{E}(\mathbf{r}, t) \times \delta\mathbf{H}(\mathbf{r}, t)] \cdot \mathbf{u}_n dS = 0.$$

The electromagnetic wave travels at finite speed. It is thus possible to choose S_∞ to be large enough so that

$$\int\limits_{S_\infty} [\delta\mathbf{E}(\mathbf{r}, t) \times \delta\mathbf{H}(\mathbf{r}, t)] \cdot \mathbf{u}_n dS = 0.$$

Consequently, Equation (1.78) reduces to

$$\int\limits_{\tilde{R}^3} (\varepsilon_\delta \left|\delta\mathbf{E}(\mathbf{r}, t)\right|^2 + \mu_\delta \left|\delta\mathbf{H}(\mathbf{r}, t)\right|^2) dV = 0,$$

which implies $\delta\mathbf{E}(\mathbf{r}, t) = 0$ and $\delta\mathbf{H}(\mathbf{r}, t) = 0$. The proof is completed.

By the equivalence principle, the magnetic current $\mathbf{J}_m$ and magnetic charge ρ_m, introduced in the generalized Maxwell equations, are justified in the sense of equivalence. The difference between the compensation theorem and equivalence theorem is that the compensation implies replacement of induced sources or part of them by the imaginary impressed sources at the same locations. Equivalence implies replacement of any sources (impressed and/or induced) by another set of impressed sources, usually distributed in a different location.

If $\mathbf{E}_1 = \mathbf{D}_1 = \mathbf{H}_1 = \mathbf{B}_1 = \mathbf{J}_1 = \mathbf{J}_{m1} = 0$ in the general equivalence theorem, we can choose $\mu = \mu_2$, $\varepsilon = \varepsilon_2$ in (1.76) inside S. If all the sources for Problem 2 are contained inside S, the following sources

$$\begin{cases} \mathbf{J}_s = \mathbf{u}_n \times \mathbf{H}_{2+}, \mathbf{J}_{ms} = -\mathbf{u}_n \times \mathbf{E}_{2+} \\ \rho_s = \mathbf{u}_n \cdot \mathbf{D}_{2+}, \rho_{ms} = \mathbf{u}_n \cdot \mathbf{B}_{2+} \end{cases}, \ \mathbf{r} \in S$$

produce the electromagnetic fields $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ in (1.76). In other words, the above sources generate the fields $\{\mathbf{E}_2, \mathbf{D}_2, \mathbf{H}_2, \mathbf{B}_2\}$ in $R^3 - V$ and a zero field in V. Thus we have:

Theorem 1.3 ***Schelkunoff–Love equivalence****: (named after the American mathematician Sergei Alexander Schelkunoff, 1897–1992; and the English mathematician Augustus Edward Hough Love, 1863–1940):* Let $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ be the electromagnetic fields with source confined in S. The following surface sources

$$\begin{cases} \mathbf{J}_s = \mathbf{u}_n \times \mathbf{H}, \mathbf{J}_{ms} = -\mathbf{u}_n \times \mathbf{E} \\ \rho_s = \mathbf{u}_n \cdot \mathbf{D}, \rho_{ms} = \mathbf{u}_n \cdot \mathbf{B} \end{cases}, \ \mathbf{r} \in S \tag{1.79}$$

produce the same fields $\{\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}\}$ outside S and a zero field inside S. □

It must be mentioned that the electromagnetic fields generated by a single electric source or a single magnetic source will never be zero within a finite region if the medium is homogeneous. The fields can be made to vanish inside a region only if both electric source and magnetic source exist so that the fields generated by both sources cancel each other in the region. In other words, only the solution of the generalized Maxwell equations can be zero within a finite region of homogeneous space. However, the solution of Maxwell equations can be zero within a finite region if the medium is inhomogeneous. Since the sources in (1.79) produce a zero field inside S, the interior of S may be filled with a perfect electric conductor. By use of the Lorentz reciprocity theorem (see Example 1.6), it can be shown that the surface electric current pressed tightly on the perfect conductor does not produce fields. As a result, only the surface magnetic current is needed in (1.79). Similarly, the interior of S may be filled with a perfect magnetic conductor, and in this case the surface magnetic current does not produce fields and only the surface electric current is needed in (1.79). In both cases, one cannot directly apply the vector potential formula even if the medium outside S is homogeneous.

Example 1.5 (An aperture problem): A general aperture coupling problem between two regions a and b is shown in Figure 1.8 (a). The impressed electric current $\mathbf{J}_{imp}$ and magnetic current $\mathbf{J}_{m,imp}$ are assumed to be located in region a only and there is no source in region b. The conductors in region b are assumed to be extended to infinity. By equivalent principle, the original problem can be separated into two equivalent problems as shown in Figure 1.8 (b). In region a, the fields are produced by the impressed sources $\mathbf{J}_{imp}$ and $\mathbf{J}_{m,imp}$, and the equivalent magnetic current $\mathbf{J}_{ms} = -\mathbf{u}_n \times \mathbf{E}$ over the aperture region S_a, with the aperture covered by an

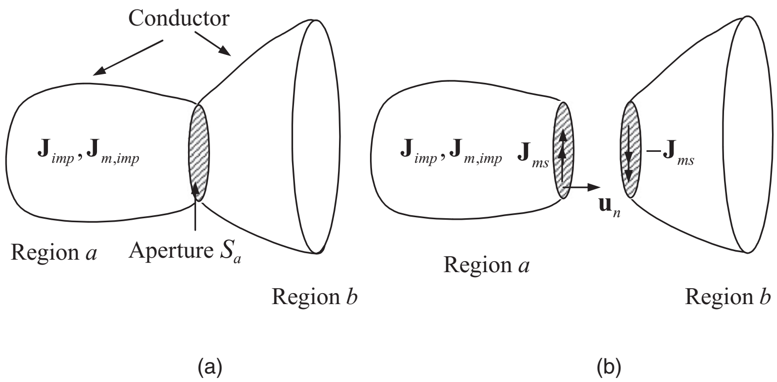

Figure 1.8 An aperture problem

electric conductor. In region b, the field is produced by the equivalent magnetic current $-\mathbf{J}_{ms}$ (the minus sign ensures that the tangential electrical field is continuous across the aperture). The tangential magnetic field in region a over the aperture, denoted $\mathbf{H}_t^a$, can be decomposed into two parts (Harrington and Mautz, 1976)

$$\mathbf{H}_t^a = \mathbf{H}_t^i + \mathbf{H}_t^m(\mathbf{J}_{ms}),$$

where $\mathbf{H}_t^i$ is due to the impressed source and $\mathbf{H}_t^m(\mathbf{J}_{ms})$ due to the equivalent source $\mathbf{J}_{ms}$, both being calculated with the aperture covered by an electric conductor. If $\mathbf{H}_t^b(-\mathbf{J}_{ms})$ denotes the tangential magnetic field in region b over the aperture, then the condition that the tangential magnetic field must be continuous across the aperture yields

$$\mathbf{H}_t^b(-\mathbf{J}_{ms}) = \mathbf{H}_t^i + \mathbf{H}_t^m(\mathbf{J}_{ms}).$$

This can be used to determine the magnetic current $\mathbf{J}_{ms}$. □

1.3.8 Reciprocity

A linear system is said to be reciprocal if the response of the system with a particular load and a source is the same as the response when the source and the load are interchanged. The earliest study of reciprocity can be traced back to the work done by the English physicist Lord Rayleigh (1842–1919) in 1894 and the work by Lorentz in 1895. The reciprocity theorems are the most important analytical tools in the simplification and solution of various practical problems (Rumsey, 1961; Monteath, 1973; Richmond, 1961).

A linear system is characterized in an abstract way by a known source f, a response u and a system operator $\hat{L}$

$$\hat{L}(u) = f. \tag{1.80}$$

The system operator $\hat{L}$ is not unique for a given linear system and it depends on how the source and the response are defined. In what follows, it is assumed that $\hat{L}$ is a linear partial differential operator, and both f and u are defined in a region V with boundary S. For arbitrary functions u_1 and u_2, the following identity can be easily derived using integration by parts (Courant and Hilbert, 1953)

$$\int_V u_2 \hat{L}(u_1) dV = \int_V u_1 \hat{L}^*(u_2) dV + T(u_1, u_2; S) \tag{1.81}$$

for a time-independent system or

$$\int_{T_1}^{T_2} dt \int_V u_2 \hat{L}(u_1) dV = \int_{T_1}^{T_2} dt \int_V u_1 \hat{L}^*(u_2) dV + T(u_1, u_2; S, T_1, T_2) \tag{1.82}$$

for a time-dependent system. In (1.81) and (1.82), $\hat{L}^*$ is known as **formal adjoint** of $\hat{L}$; $T(u_1, u_2; S)$ and $T(u_1, u_2; S, T_1, T_2)$ are bilinear forms (boundary terms); and $[T_1, T_2]$ is an arbitrary time interval.

Equations (1.81) and (1.82) may be interpreted as **Huygens' principle**, named after the Dutch physicist Christiaan Huygens (1629–1695). For a time-independent system, Huygens' principle states that, given a source inside a hypothetical surface S, there is a certain source spreading over S, which gives the same field outside S as the original source inside S. For a time-dependent system, it states that the position of a wavefront and the magnitude of the wave at each point of the wavefront may be determined by the wavefront at any earlier time. Huygens' principle can be traced back to 1690 when Huygens published his classical work *Treatise on Light* (Huygens, 1690). Huygens was not able to formulate his principle precisely at that time. A number of famous scientists have worked in this area and elaborated this principle since then. It should be mentioned that different authors use the term 'Huygens' principle' with different meanings. The best-known representation of Huygens' principle is to express the field at some observation point in terms of a surface integral over a closed surface separating the observation point from the source. Such an expression can easily be obtained from (1.81) or (1.82), which in general gives a relationship between some volume integrals defined in the region V and some surface integrals defined on the boundary S. The idea behind Huygens' principle could apply not only to electromagnetics but also to any branch of physics, such as gravitation, elasticity, acoustics and many more (Rumsey, 1959).

One can consider three situations: (1) $\hat{L}^* = \hat{L}$ ($\hat{L}$ is formally self adjoint); (2) $\hat{L}^* \neq \hat{L}$; and (3) $\hat{L}^* = -\hat{L}$ ($\hat{L}$ is skew adjoint). For the first two situations, one can choose u_2 as a solution of the adjoint system $\hat{L}^*(u_2) = f_2$, where f_2 is a known source function. If the boundary term in (1.81) or (1.82) can be made to vanish, we have

$$\int_V u_2 f_1 dV = \int_V u_1 f_2 dV, \int_{T_1}^{T_2} dt \int_V u_2 f_1 dV = \int_{T_1}^{T_2} dt \int_V u_1 f_2 dV, \tag{1.83}$$

$$T(u_1, u_2; S) = 0, T(u_1, u_2; S, T_1, T_2) = 0. \tag{1.84}$$

Equations (1.83) are called reciprocity theorems of **Rayleigh-Carson form**, and Equations (1.84) are called reciprocity theorems of **Lorentz form**. These two forms are equivalent. For

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the situation $\hat{L}^* = -\hat{L}$, we may choose u_2 as a solution of the original system $\hat{L}(u_2) = f_2$. If the boundary term in (1.81) or (1.82) can be made to vanish, then

$$\int_V u_2 f_1 dV = -\int_V u_1 f_2 dV, \int_{T_1}^{T_2} dt \int_V u_2 f_1 dV = -\int_{T_1}^{T_2} dt \int_V u_1 f_2 dV. \tag{1.85}$$

The above relations may be called skew-reciprocity theorems of Rayleigh-Carson form.

The quantity $\int_V u_2 f_1 dV$ is called the **reaction** of field u_2 on source f_1 (Rumsey, 1954). Equations (1.83) (or (1.85)) simply state that the reaction of field u_2 on source f_1 is equal to the reaction (or the negative reaction) of field u_1 on source f_2. Apparently this kind of relations exists in various fields of physics and engineering. The concept of reaction is very useful and it can be used to answer some difficult questions with simplicity. If f_1 is a testing source of unit strength, the reaction $\int_V u_2 f_1 dV$ gives the numerical value of response u_2 at the point of the testing source. Thus, a method can be established to solve various boundary value problems based on the reaction, which is basically a theory of measurement rather than a field theory (Rumsey, 1963).

A number of reciprocity theorems for electromagnetic fields in both time domain and frequency domain can be derived by choosing different forms of the operator $\hat{L}$. But most of them are useless. Only one of the reciprocity theorems in frequency domain will be discussed here, which states that all possible time-harmonic fields of the same frequency are to some extent interrelated. Suppose that the sources $\mathbf{J}_1(\mathbf{r})$ and $\mathbf{J}_{m1}(\mathbf{r})$ give rise to the fields $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{H}_1(\mathbf{r})$. Then the Maxwell equations in an isotropic medium can be rewritten in the operator form $\hat{L}(u_1) = f_1$ with

$$\hat{L} = \begin{bmatrix} -j\omega\varepsilon\cdot & \nabla\times \\ \nabla\times & j\omega\mu\cdot \end{bmatrix}, u_1 = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{bmatrix}, f_1 = \begin{bmatrix} \mathbf{J}_1 \\ -\mathbf{J}_{m1} \end{bmatrix}.$$

For an arbitrary $u_2 = [\mathbf{E}_2, \mathbf{H}_2]^T$ (the superscript T stands for the transpose operation), the formal adjoint of $\hat{L}$ and the boundary term may be found through integration by parts as

$$\hat{L}^* = \hat{L},$$
$$T(u_1, u_2; S) = \int_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{u}_n dS,$$

where $\mathbf{u}_n$ is the outward unit normal to S. If $u_2 = [\mathbf{E}_2, \mathbf{H}_2]^T$ is a solution of the transposed system $\hat{L}^*(u_2) = f_2$ with $f_2 = [\mathbf{J}_2, -\mathbf{J}_{m2}]^T$, Equation (1.81) becomes

$$\int_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{J}_{m1}) dV = \int_V (\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{J}_{m2}) dV + \int_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{u}_n dS. \tag{1.86}$$

If both sources are outside S, the surface integral in (1.86) is zero. If both sources are inside S, it can be shown that the surface integral is also zero by using the radiation condition. Therefore we obtain the Lorentz form of reciprocity

$$\int_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{u}_n dS = 0,$$

and the Rayleigh-Carson form of reciprocity

$$\int_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{J}_{m1}) dV = \int_V (-\mathbf{H}_1 \cdot \mathbf{J}_{m2} + \mathbf{E}_1 \cdot \mathbf{J}_2) dV \tag{1.87}$$

If the surface S only contains the sources $\mathbf{J}_1(\mathbf{r})$ and $\mathbf{J}_{m1}(\mathbf{r})$, Equation (1.86) becomes

$$\int_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{J}_{m1}) dV = \int_S (\mathbf{E}_2 \cdot \mathbf{u}_n \times \mathbf{H}_1 - \mathbf{H}_2 \cdot \mathbf{E}_1 \times \mathbf{u}_n) dS.$$

This is the familiar form of Huygens' principle. The electromagnetic reciprocity theorem can also be generalized to an anisotropic medium (Kong, 1990; Tai, 1961; Harrington, 1958).

Example 1.6: An interesting application of the reciprocity theorem is to prove that a surface electric (or magnetic) current pressed tightly on a perfect electric (or magnetic conductor) does not radiate. Let $\mathbf{J}_{s1}$ be a surface electric current pressed tightly on a perfect electric conductor, which generates electromagnetic fields $\mathbf{E}_1$ and $\mathbf{H}_1$. Now remove the surface electric current $\mathbf{J}_{s1}$ and place an arbitrary current source $\mathbf{J}_2$ in space that produces electromagnetic fields $\mathbf{E}_2$ and $\mathbf{H}_2$. According to (1.87), we have $\int_V \mathbf{E}_2 \cdot \mathbf{J}_{s1} dV = \int_V \mathbf{E}_1 \cdot \mathbf{J}_2 dV$, where V denotes the region outside the conductor. Since $\mathbf{E}_2$ only has a normal component on the surface of the conductor while $\mathbf{J}_{s1}$ is a tangential vector, the left side of the above equation must be zero. Thus, we have $\int_V \mathbf{E}_1 \cdot \mathbf{J}_2 dV = 0$. For $\mathbf{J}_2$ is arbitrary, we obtain $\mathbf{E}_1 = 0$. □

1.4 Wavepackets

A time-domain field can be expressed as the superposition of individual plane waves of the form $e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}}$. Each plane wave travels with a **phase velocity** defined by $\mathbf{v}_p = \mathbf{u}_k \omega / |\mathbf{k}|$, where $\mathbf{u}_k = \mathbf{k}/|\mathbf{k}|$ and $\mathbf{k}$ is the wave vector. The phase velocity is the velocity at which the points of constant phase move in the medium. It is well known that, to transmit energy or a signal, the waves must come in a range of frequencies to form a wavepacket. The **wavepacket** was first introduced by Schrödinger and is used to represent a small group of plane waves. There are two different ways of building wavepackets. A waveform is called a **spatial wavepacket** (or **paraxial approximation**) if it is monochromatic and is confined to a narrow region of space along the path of propagation. A spatial wavepacket is basically a beam of wave. A waveform is called a **temporal wavepacket** (or **narrow-band approximation**) if it propagates in only one direction and its frequency spectrum is confined to a narrow band around a central frequency.

The propagation of wavepackets in an absorbing medium was studied for the first time by the German physicist, Arnold Johannes Wilhelm Sommerfeld (1868–1951), and the French physicist, Léon Nicolas Brillouin (1889–1969) (Brillouin, 1960). The group velocity, signal velocity and energy velocity are important quantities for characterizing the propagation of wavepackets and there are certain relationships among them. The speed of each frequency component is the phase velocity while the speed of the envelope of the wavepacket is called **group velocity**. The velocity at which the main part of the wavepacket propagates is called **signal velocity**. When a signal propagates in a dispersive medium, it does not retain its original form. At certain depth of the medium, very weak signal components appear at first and are called **forerunners** or **fronts** whose speed is always equal to the light speed in vacuum. The **energy velocity** of the wavepacket is defined as the ratio of the Poynting vector to the energy density.

1.4.1 Spatial Wavepacket and Temporal Wavepacket

By definition, a spatial wavepacket can be represented by

$$\begin{aligned}\mathbf{F}(\mathbf{r},t) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\tilde{\mathbf{F}}(\mathbf{k})e^{j\omega(\mathbf{k})t-j\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}\\ &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{F}(\boldsymbol{\xi},0)e^{j\omega(\mathbf{k})t-j\mathbf{k}\cdot(\mathbf{r}-\boldsymbol{\xi})}d\boldsymbol{\xi}d\mathbf{k},\end{aligned} \tag{1.88}$$

where $\tilde{\mathbf{F}}(\mathbf{k})$ is given by $\tilde{\mathbf{F}}(\mathbf{k}) = \int_{-\infty}^{\infty}\mathbf{F}(\mathbf{r},0)e^{j\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}$ and ω is the angular frequency of the wavepacket. Since $\mathbf{F}(\mathbf{r},0)$ is narrow-band in $\mathbf{k}$-space, a rapid phase variation $e^{-j\mathbf{k}_c\cdot\mathbf{r}}$ may be factored out so that one may write $\mathbf{F}(\mathbf{r},0) = \mathbf{A}(\mathbf{r},0)e^{-j\mathbf{k}_c\cdot\mathbf{r}}$, where $\mathbf{k}_c$ is the central wave vector and $\mathbf{A}(\mathbf{r},0)$ is the complex envelope that describes the slowly varying transverse beam profile or the spatial modulation as the wave propagates. If the dispersion of the medium is not strong, we may use the first-order approximation for the dispersion relation

$$\omega(\mathbf{k}) \approx \omega_c + \delta\mathbf{k}\cdot\nabla\omega(\mathbf{k}_c),$$

where $\omega_c = \omega(\mathbf{k}_c)$ and $\delta\mathbf{k} = \mathbf{k} - \mathbf{k}_c$. As a result, Equation (1.88) can be approximated by

$$\begin{aligned}\mathbf{F}(\mathbf{r},t) &= \int_{-\infty}^{\infty}\mathbf{F}(\boldsymbol{\xi},0)e^{j[\omega_c t-\mathbf{k}_c\cdot(\mathbf{r}-\boldsymbol{\xi})]}\delta\left[\nabla\omega(\mathbf{k}_c)t-(\mathbf{r}-\boldsymbol{\xi})\right]d\boldsymbol{\xi}\\ &= \mathbf{F}(\mathbf{r}-\mathbf{v}_g t,0)e^{j(\omega_c t-\mathbf{k}_c\cdot\mathbf{v}_g t)},\end{aligned}$$

where $\mathbf{v}_g = \nabla\omega(\mathbf{k}_c)$ is defined as the group velocity. Making use of $\mathbf{F}(\mathbf{r},0) = \mathbf{A}(\mathbf{r},0)e^{-j\mathbf{k}_c\cdot\mathbf{r}}$ we have

$$\mathbf{F}(\mathbf{r},t) = \mathbf{A}(\mathbf{r}-\mathbf{v}_g t,0)e^{j(\omega_c t-\mathbf{k}_c\cdot\mathbf{r})}. \tag{1.89}$$

Hence the group velocity represents the speed of the envelope of the wavepacket. In a medium where the dispersion is not strong, the shape of the envelope of the wavepacket does not change very much as it propagates. When the wavepacket propagates in a highly dispersive medium, the shape of the envelope of the wavepacket will not remain the same. The phase of the wavepacket will change as the propagation distance and time increase. As a result, the concept of group velocity is no longer valid in a highly dispersive medium.

By definition, an arbitrary temporal wavepacket may be expressed as

$$\begin{aligned}\mathbf{F}(\mathbf{r}, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathbf{F}}(\omega) e^{j[\omega t - \mathbf{k}(\omega)\cdot\mathbf{r}]} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{F}(0, t') e^{j[\omega(t-t') - \mathbf{k}(\omega)\cdot\mathbf{r}]} dt' d\omega,\end{aligned} \tag{1.90}$$

where $\tilde{\mathbf{F}}(\omega) = \int_{-\infty}^{\infty} \mathbf{F}(0, t) e^{-j\omega t} dt$. The narrow-band approximation assumes that the frequency spectrum of the time variation is confined to a narrow band around a carrier. Therefore $\mathbf{F}(0, t)$ is a bandpass signal and can be written as $\mathbf{F}(0, t) = \mathbf{A}(0, t)e^{j\omega_c t}$, where ω_c is the carrier frequency and $\mathbf{A}(0, t)$ is a slowly varying function of time. If the dispersion of the medium is not very strong we may make the first order approximation

$$\mathbf{k}(\omega) \approx \mathbf{k}_c + \delta\omega \frac{d\mathbf{k}(\omega_c)}{d\omega},$$

where $\delta\omega = \omega - \omega_c$ and $\mathbf{k}_c$ is the wave vector at the carrier frequency ω_c. Hence (1.90) can be expressed as

$$\begin{aligned}\mathbf{F}(\mathbf{r}, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{F}(0, t') e^{j\omega_c(t-t') - j\mathbf{k}_c\cdot\mathbf{r} + j\delta\omega\left[(t-t') - \frac{d\mathbf{k}(\omega_c)}{d\omega}\cdot\mathbf{r}\right]} dt' d\delta\omega \\ &= \mathbf{F}\left(0, t - \frac{d\mathbf{k}(\omega_c)}{d\omega}\cdot\mathbf{r}\right) e^{j\omega_c \frac{d\mathbf{k}(\omega_c)}{d\omega}\cdot\mathbf{r} - j\mathbf{k}_c\cdot\mathbf{r}} \\ &= \mathbf{A}\left(0, t - \frac{d\mathbf{k}(\omega_c)}{d\omega}\cdot\mathbf{r}\right) e^{j\omega_c t - j\mathbf{k}_c\cdot\mathbf{r}}.\end{aligned} \tag{1.91}$$

If the wavepacket propagates mainly in $\mathbf{k}_c$ direction, that is, $\mathbf{k} \approx \mathbf{k}_c$, we have $d\mathbf{k}(\omega_c)/d\omega = \mathbf{u}_{k_c}/v_g$, where $\mathbf{u}_{k_c}$ is the unit vector along the direction of $\mathbf{k}_c$, and $v_g = d\omega(\omega_c)/dk$ is the group velocity. Equation (1.91) indicates that the propagation velocity of the envelope of a temporal wavepacket is equal to group velocity. Again, the shape of the envelope as well as the phase of the wavepacket will change as it propagates in a strongly dispersive medium, and the concept of group velocity becomes invalid.

An analogy exists between the spatial diffraction of beams and temporal dispersion of pulses. It can be shown that the equation that describes how a wavepacket spreads in time due to dispersion is equivalent to the equation for the transverse spreading due to diffraction. The temporal imaging technique is based on this space–time analogy (Kolner, 1994).

1.4.2 Signal Velocity and Group Velocity

According to Sommerfeld and Brillouin, the signal velocity represents the velocity of the main part of the signal. Thus we may define signal velocity as $\mathbf{v}_s = d\mathbf{r}_p/dt$, where $\mathbf{r}_p$ is the position of the main part of the wavepacket $\mathbf{F}$, defined by (Vichnevetsky, 1988)

$$\mathbf{r}_p(t) = \frac{\int_{-\infty}^{\infty} \mathbf{r} \left|\mathbf{F}(\mathbf{r}, t)\right|^2 d\mathbf{r}}{\int_{-\infty}^{\infty} \left|\mathbf{F}(\mathbf{r}, t)\right|^2 d\mathbf{r}}. \tag{1.92}$$

It should be understood that the concept of signal velocity of a wavepacket is useful only when the dispersion of the medium is not very strong (so that the first-order approximation for the dispersion relation is valid). Substituting (1.89) into (1.92) and using the transformation $\mathbf{r} = \mathbf{u} + \mathbf{v}_g t$, we obtain

$$\mathbf{r}_p(t) = \frac{\int_{-\infty}^{\infty} \mathbf{r} \left|\mathbf{A}(\mathbf{r} - \mathbf{v}_g t, 0)\right|^2 d\mathbf{r}}{\int_{-\infty}^{\infty} \left|\mathbf{A}(\mathbf{r} - \mathbf{v}_g t, 0)\right|^2 d\mathbf{r}} = \frac{\int_{-\infty}^{\infty} (\mathbf{u} + \mathbf{v}_g t) \left|\mathbf{A}(\mathbf{u}, 0)\right|^2 d\mathbf{u}}{\int_{-\infty}^{\infty} \left|\mathbf{A}(\mathbf{u}, 0)\right|^2 d\mathbf{u}}.$$

By taking the time derivative of the above equation, we obtain $\mathbf{v}_s = \mathbf{v}_g$, and the signal velocity of a spatial wavepacket is equal to the group velocity. Similarly substituting (1.91) into (1.92) and making use of the transformation $\mathbf{r} = \mathbf{u} + v_g t \mathbf{u}_{k_c}$ yields

$$\mathbf{r}_p(t) = \frac{\int_{-\infty}^{\infty} \mathbf{r} \left|\mathbf{A}(0, t - \mathbf{u}_{k_c} \cdot \mathbf{r}/v_g)\right|^2 d\mathbf{r}}{\int_{-\infty}^{\infty} \left|\mathbf{A}(0, t - \mathbf{u}_{k_c} \cdot \mathbf{r}/v_g)\right|^2 d\mathbf{r}} = \frac{\int_{-\infty}^{\infty} (\mathbf{u} + v_g t \mathbf{u}_{k_c}) \left|\mathbf{A}(0, -\mathbf{u}_{k_c} \cdot \mathbf{u}/v_g)\right|^2 d\mathbf{u}}{\int_{-\infty}^{\infty} \left|\mathbf{A}(0, -\mathbf{u}_{k_c} \cdot \mathbf{u}/v_g)\right|^2 d\mathbf{u}}.$$

Hence $\mathbf{v}_s = v_g \mathbf{u}_{k_c} = \mathbf{v}_g$, and the signal velocity of a temporal wavepacket is equal to the group velocity.

1.4.3 Energy Density for Wavepackets

An expression for the electromagnetic energy density that does not involve any medium properties (such as isotropic, anisotropic) is useful. Such an expression exists for a monochromatic wave (Tonning, 1960), and can be generalized to the wavepackets, which are more realistic in applications. In order to find the general expression of the energy density for a wavepacket, we have to assume that the dispersion of the medium is not very strong so that the first-order approximation of the dispersion is valid. From (1.89) and (1.91), the fields for both spatial and temporal wavepackets can be expressed as

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}[\mathbf{E}_{en}(\mathbf{r}, t; \omega_c) e^{j\omega_c t}], \text{ etc.} \tag{1.93}$$

where $\mathbf{E}_{en}$ etc. are the envelopes and they are slowly varying functions of time compared to $e^{j\omega_c t}$, and ω_c is the angular frequency of the monochromatic paraxial wave or the carrier wave frequency for the narrow-band signal. We cannot apply (1.57) to a wavepacket directly because the fields might not be zero at $t = -\infty$. To make use of (1.57), the standard way is to introduce a damping mechanism for the fields first so that all the fields are zero at $t = -\infty$, and then let the damping tend to zero after the calculation is finished. To this end, we can introduce a complex frequency $\tilde{\omega} = -j\alpha + \omega_c$ to replace the real frequency ω_c, where α is a small positive number (the damping factor). Equation (1.93) may be rewritten as

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathrm{Re}[\tilde{\mathbf{E}}_{en}(\mathbf{r}, t; \tilde{\omega})e^{j\tilde{\omega}t}], \text{ etc.} \tag{1.94}$$

which approach zero when $t \to -\infty$ and approach the corresponding real fields as $\alpha \to 0$. Assuming that the fields are analytic functions of frequency, the following first-order expansion can be made

$$\tilde{\mathbf{E}}_{en} \approx \mathbf{E}_{en} - j\alpha \frac{\partial \mathbf{E}_{en}}{\partial \omega_c}, \text{ etc.}$$

for α is assumed to be small. A simple calculation shows that

$$\begin{aligned}
&\tilde{\mathbf{E}} \cdot \frac{\partial \tilde{\mathbf{D}}}{\partial t} - \tilde{\mathbf{D}} \cdot \frac{\partial \tilde{\mathbf{E}}}{\partial t} + \tilde{\mathbf{H}} \cdot \frac{\partial \tilde{\mathbf{B}}}{\partial t} - \tilde{\mathbf{B}} \cdot \frac{\partial \tilde{\mathbf{H}}}{\partial t} \\
&\quad = -\omega_c e^{2\alpha t} \mathrm{Im}(\bar{\mathbf{E}}_{en} \cdot \mathbf{D}_{en} + \bar{\mathbf{H}}_{en} \cdot \mathbf{B}_{en}) \\
&\qquad + \alpha\omega_c e^{2\alpha t} \mathrm{Re}\left(\bar{\mathbf{E}}_{en} \cdot \frac{\partial \mathbf{D}_{en}}{\partial \omega_c} - \mathbf{D}_{en} \cdot \frac{\partial \bar{\mathbf{E}}_{en}}{\partial \omega_c} + \bar{\mathbf{H}}_{en} \cdot \frac{\partial \mathbf{B}_{en}}{\partial \omega_c} - \mathbf{B}_{en} \cdot \frac{\partial \bar{\mathbf{H}}_{en}}{\partial \omega_c}\right).
\end{aligned}$$

From the lossless condition $\overline{\overline{\nabla \cdot (\mathbf{E} \times \mathbf{H})}} = 0$ and Maxwell equations, we obtain

$$\mathrm{Im}(\mathbf{E}_{en} \cdot \bar{\mathbf{D}}_{en} - \bar{\mathbf{H}}_{en} \cdot \mathbf{B}_{en}) = -\mathrm{Im}(\bar{\mathbf{E}}_{en} \cdot \mathbf{D}_{en} + \bar{\mathbf{H}}_{en} \cdot \mathbf{B}_{en}) = 0. \tag{1.95}$$

Thus the quantity $\bar{\mathbf{E}}_{en} \cdot \mathbf{D}_{en} + \bar{\mathbf{H}}_{en} \cdot \mathbf{B}_{en}$ is real. Consequently the integral of (1.57) can be expressed as

$$\begin{aligned}
&\int_{-\infty}^{t} \frac{1}{2}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t}\right) dt \\
&\quad = \lim_{\alpha \to 0} \frac{1}{2} \int_{-\infty}^{t} \tilde{\mathbf{E}} \cdot \frac{\partial \tilde{\mathbf{D}}}{\partial t} - \tilde{\mathbf{D}} \cdot \frac{\partial \tilde{\mathbf{E}}}{\partial t} + \tilde{\mathbf{H}} \cdot \frac{\partial \tilde{\mathbf{B}}}{\partial t} - \tilde{\mathbf{B}} \cdot \frac{\partial \tilde{\mathbf{H}}}{\partial t} dt \\
&\quad = \frac{\omega_c}{4} \mathrm{Re}\left(\bar{\mathbf{E}}_{en} \cdot \frac{\partial \mathbf{D}_{en}}{\partial \omega_c} - \bar{\mathbf{D}}_{en} \cdot \frac{\partial \mathbf{E}_{en}}{\partial \omega_c} + \bar{\mathbf{H}}_{en} \cdot \frac{\partial \mathbf{B}_{en}}{\partial \omega_c} - \bar{\mathbf{B}}_{en} \cdot \frac{\partial \mathbf{H}_{en}}{\partial \omega_c}\right).
\end{aligned} \tag{1.96}$$

The quantity in the bracket is real. Actually taking the derivative of (1.95) with respect to the frequency gives

$$\mathrm{Im}\left(\bar{\mathbf{E}}_{en}\cdot\frac{\partial \mathbf{D}_{en}}{\partial\omega_c}+\mathbf{D}_{en}\cdot\frac{\partial \bar{\mathbf{E}}_{en}}{\partial\omega_c}+\bar{\mathbf{H}}_{en}\cdot\frac{\partial \mathbf{B}_{en}}{\partial\omega_c}+\mathbf{B}_{en}\cdot\frac{\partial \bar{\mathbf{H}}_{en}}{\partial\omega_c}\right)=0.$$

This relation still holds if we take the complex conjugate of the second and the fourth term and change their sign simultaneously. Since the envelopes can be considered as constants over one period of the carrier wave $e^{j\omega_c t}$, the time average of (1.96) over one period of the carrier wave is

$$\begin{aligned}&\overline{\overline{\frac{1}{2}\int_{-\infty}^{t}\left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}-\mathbf{B}\cdot\frac{\partial \mathbf{H}}{\partial t}+\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}-\mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial t}\right)dt}}\\&\quad=\frac{\omega_c}{4}\left(\bar{\mathbf{E}}_{en}\cdot\frac{\partial \mathbf{D}_{en}}{\partial\omega_c}-\bar{\mathbf{D}}_{en}\cdot\frac{\partial \mathbf{E}_{en}}{\partial\omega_c}+\bar{\mathbf{H}}_{en}\cdot\frac{\partial \mathbf{B}_{en}}{\partial\omega_c}-\bar{\mathbf{B}}_{en}\cdot\frac{\partial \mathbf{H}_{en}}{\partial\omega_c}\right).\end{aligned}\tag{1.97}$$

The above expression can be interpreted as the energy density related to dispersion and will be denoted $\bar{w}_d$. Similarly the time average of the rest of (1.57) over one period of the carrier wave is

$$\begin{aligned}\frac{1}{2}(\overline{\overline{\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}}})&=\frac{1}{4}\mathrm{Re}(\mathbf{E}_{en}\cdot\bar{\mathbf{D}}_{en}+\mathbf{H}_{en}\cdot\bar{\mathbf{B}}_{en})\\&=\frac{1}{4}(\mathbf{E}_{en}\cdot\bar{\mathbf{D}}_{en}+\mathbf{H}_{en}\cdot\bar{\mathbf{B}}_{en}).\end{aligned}\tag{1.98}$$

It follows from (1.57), (1.97) and (1.98) that the time average of the energy density over one period of the carrier wave $e^{j\omega_c t}$ can be expressed as

$$\begin{aligned}\overline{\overline{w}}&=\frac{1}{4}(\bar{\mathbf{E}}_{en}\cdot\mathbf{D}_{en}+\bar{\mathbf{H}}_{en}\cdot\mathbf{B})\\&\quad+\frac{\omega_c}{4}\left(\bar{\mathbf{E}}_{en}\cdot\frac{\partial \mathbf{D}_{en}}{\partial\omega_c}-\bar{\mathbf{D}}_{en}\cdot\frac{\partial \mathbf{E}_{en}}{\partial\omega_c}+\bar{\mathbf{H}}_{en}\cdot\frac{\partial \mathbf{B}_{en}}{\partial\omega_c}-\bar{\mathbf{B}}_{en}\cdot\frac{\partial \mathbf{H}_{en}}{\partial\omega_c}\right)\\&=\frac{1}{4}\left[\bar{\mathbf{E}}_{en}\cdot\frac{\partial(\omega_c\mathbf{D}_{en})}{\partial\omega_c}-\omega_c\bar{\mathbf{D}}_{en}\cdot\frac{\partial \mathbf{E}_{en}}{\partial\omega_c}+\bar{\mathbf{H}}_{en}\cdot\frac{\partial(\omega_c\mathbf{B}_{en})}{\partial\omega_c}-\omega_c\bar{\mathbf{B}}_{en}\cdot\frac{\partial \mathbf{H}_{en}}{\partial\omega_c}\right].\end{aligned}\tag{1.99}$$

Taking the time average of (1.55) over one period of the carrier wave $e^{j\omega_c t}$, the Poynting theorem in a lossless medium without impressed sources becomes

$$\int_S \overline{\overline{\mathbf{S}}}\cdot\mathbf{u}_n dS+\int_V\frac{\partial\overline{\overline{w}}}{\partial t}dV=0,$$

where $\overline{\overline{\mathbf{S}}}=\mathrm{Re}(\mathbf{E}_{en}\times\bar{\mathbf{H}}_{en})/2$ is the time average of the Poynting vector over one period of the wave $e^{j\omega_c t}$ and the calculation $\overline{\overline{\partial w/\partial t}}=\partial\overline{\overline{w}}/\partial t$ has been used. Note that $\partial\overline{\overline{w}}/\partial t=0$ for a monochromatic wave. As a special case, let us consider an isotropic medium defined by $\mathbf{D}_{en} = \varepsilon \mathbf{E}_{en}$, $\mathbf{B}_{en} = \mu \mathbf{H}_{en}$. In this case, Equation (1.99) reduces to the well-known expression

$$\overline{\overline{w}} = \frac{1}{4}\left[\frac{\partial(\omega_c \varepsilon)}{\partial \omega_c}|\mathbf{E}_{en}|^2 + \frac{\partial(\omega_c \mu)}{\partial \omega_c}|\mathbf{H}_{en}|^2\right].$$

1.4.4 Energy Velocity and Group Velocity

The energy velocity is defined as the ratio of the Poynting vector to the energy density, that is, $\mathbf{v}_e = \overline{\overline{\mathbf{S}}}/\overline{\overline{w}}$. If the dispersion of the medium is not very strong, a spatial or temporal wavepacket in its first-order approximation can be expressed as

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}[\mathbf{E}_0(\mathbf{r}, t; \omega_c) e^{j\omega_c t - j\mathbf{k}_c \cdot \mathbf{r}}], \text{ etc.} \tag{1.100}$$

where the fast phase variation $e^{-j\mathbf{k}_c \cdot \mathbf{r}}$ of the fields has been factored out. The new envelopes $\mathbf{E}_0$, etc. are slowly varying functions of both spatial coordinates and time. Introducing (1.100) into Maxwell equations in a source-free and lossless region and using the calculation $\nabla e^{-j\mathbf{k}_c \cdot \mathbf{r}} = -j\mathbf{k}_c e^{-j\mathbf{k}_c \cdot \mathbf{r}}$, we obtain

$$\begin{aligned} \mathbf{k}_c \times \mathbf{H}_0 + j\nabla \times \mathbf{H}_0 &= -\omega_c \mathbf{D}_0, \\ \mathbf{k}_c \times \mathbf{E}_0 + j\nabla \times \mathbf{E}_0 &= \omega_c \mathbf{B}_0. \end{aligned}$$

Since $\mathbf{E}_0$ and $\mathbf{H}_0$ are slowly varying function of space, we can let $\nabla \times \mathbf{E}_0 \approx 0$ and $\nabla \times \mathbf{H}_0 \approx 0$. Thus the above equation may be rewritten as

$$\begin{aligned} \mathbf{k}_c \times \mathbf{H}_{en} + \omega_c \mathbf{D}_{en} &\approx 0, \\ \mathbf{k}_c \times \mathbf{E}_{en} - \omega_c \mathbf{B}_{en} &\approx 0. \end{aligned} \tag{1.101}$$

By letting $\mathbf{k}_c = k_{cx}\mathbf{u}_x + k_{cy}\mathbf{u}_y + k_{cz}\mathbf{u}_z$ and taking the derivative of (1.101) with respect to k_{cx}, we obtain

$$\begin{aligned} \mathbf{u}_x \times \mathbf{H}_{en} + \left[\mathbf{k}_c \times \frac{\partial \mathbf{H}_{en}}{\partial \omega_c} + \frac{\partial(\omega_c \mathbf{D}_{en})}{\partial \omega_c}\right]\frac{\partial \omega_c}{\partial k_{cx}} &\approx 0, \\ \mathbf{u}_x \times \mathbf{E}_{en} + \left[\mathbf{k}_c \times \frac{\partial \mathbf{E}_{en}}{\partial \omega_c} - \frac{\partial(\omega_c \mathbf{B}_{en})}{\partial \omega_c}\right]\frac{\partial \omega_c}{\partial k_{cx}} &\approx 0. \end{aligned}$$

Multiplying the first equation by $-\bar{\mathbf{E}}_{en}$ and second by $\bar{\mathbf{H}}_{en}$ and adding the resultant equations and using (1.99), we get

$$\begin{aligned} \mathbf{u}_x \cdot \bar{\bar{\mathbf{S}}} &= \frac{\partial \omega_c}{\partial k_{cx}} \cdot \frac{1}{4}\left[\bar{\mathbf{E}}_{en} \cdot \frac{\partial(\omega \mathbf{D}_{en})}{\partial \omega_c} - \omega_c \bar{\mathbf{D}}_{en} \cdot \frac{\partial \mathbf{E}_{en}}{\partial \omega_c} + \bar{\mathbf{H}}_{en} \cdot \frac{\partial(\omega_c \mathbf{B}_{en})}{\partial \omega_c} - \omega_c \bar{\mathbf{B}}_{en} \cdot \frac{\partial \mathbf{H}_{en}}{\partial \omega_c}\right] \\ &= \frac{\partial \omega_c}{\partial k_{cx}} \bar{w}. \end{aligned}$$

Similarly we have $\mathbf{u}_y \cdot \bar{\bar{\mathbf{S}}} = \bar{\bar{w}} \partial\omega_c \big/ \partial k_{cy}$ and $\mathbf{u}_z \cdot \bar{\bar{\mathbf{S}}} = \bar{\bar{w}} \partial\omega_c \big/ \partial k_{cz}$. Therefore

$$\mathbf{v}_g = \frac{\bar{\bar{\mathbf{S}}}}{\bar{\bar{w}}} = \nabla\omega_c(\mathbf{k}_c) = \mathbf{v}_e.$$

This indicates that the group velocity is always equal to the energy velocity for a spatial wavepacket.

Taking the derivative of (1.101) with respect to the frequency, we obtain

$$\begin{aligned}
\frac{d\mathbf{k}_c}{d\omega_c} \times \mathbf{H}_{en} + \mathbf{k}_c \times \frac{\partial \mathbf{H}_{en}}{\partial\omega_c} + \frac{\partial(\omega_c \mathbf{D}_{en})}{\partial\omega_c} \approx 0, \\
\frac{d\mathbf{k}_c}{d\omega_c} \times \mathbf{E}_{en} + \mathbf{k}_c \times \frac{\partial \mathbf{E}_{en}}{\partial\omega_c} - \frac{\partial(\omega_c \mathbf{B}_{en})}{\partial\omega_c} \approx 0.
\end{aligned}$$

Multiplying the first equation by $-\bar{\mathbf{E}}_{en}$ and second by $\bar{\mathbf{H}}_{en}$ and adding the resultant equations and using (1.99) yields

$$\frac{d\mathbf{k}_c}{d\omega_c} \cdot \bar{\bar{\mathbf{S}}} = \frac{1}{4}\left[\bar{\mathbf{E}}_{en} \cdot \frac{\partial(\omega \mathbf{D}_{en})}{\partial\omega_c} - \omega_c \bar{\mathbf{D}}_{en} \cdot \frac{\partial \mathbf{E}_{en}}{\partial\omega_c} + \bar{\mathbf{H}}_{en} \cdot \frac{\partial(\omega_c \mathbf{B}_{en})}{\partial\omega_c} - \omega_c \bar{\mathbf{B}}_{en} \cdot \frac{\partial \mathbf{H}_{en}}{\partial\omega_c}\right] = \bar{\bar{w}}.$$

It follows that

$$\mathbf{v}_e \cdot \frac{d\mathbf{k}_c}{d\omega_c} \approx 1 \text{ or } \mathbf{v}_e \cdot \mathbf{u}_{k_c} \approx v_g.$$

The above equation shows that the projection of the energy velocity in the direction of wave propagation is always equal to the group velocity for a temporal wavepacket.

Remark 1.14: In deriving the electromagnetic energy density for a wavepacket in a general lossless medium, a damping mechanism (that is, the small parameter α) has been introduced. This process appears to be a bit contrived. Nonetheless it is required by the uniqueness theorem for solutions of Maxwell equations. In a steady state, the information about the initial condition of the field has been lost and many possible solutions may exist. Introducing the loss is equivalent to introducing causality. □

Remark 1.15: One of the essential assumptions in special relativity is that the light speed is the greatest speed at which energy, information and signals can be transmitted. This is also the requirement of causality. Sommerfeld and Brillouin were the first to note that group velocity could be faster than light in the regions of anomalous dispersion. Some experiments in recent years have shown that the group velocity can exceed the light speed c or even become negative (for example, Wong, 2000). In all these experiments, the wavepackets experience a very strong dispersion when they travel in the medium, and the concept of group velocity that relies on the first-order approximation of dispersion relation is actually invalid. □

Giving an exact definition for the propagation velocity of wavepackets in a highly dispersive medium is essentially difficult. Several definitions have been proposed for various specific situations (Fushchych, 1998; Diener, 1998).

1.4.5 Narrow-band Stationary Stochastic Vector Field

As a linear modulation technique, an easy way to translate the spectrum of low-pass or baseband signal to a higher frequency is to multiply or heterodyne the baseband signal with a carrier wave. A narrowband bandpass stochastic vector field **F** (**modulated signal**) in the time domain can be expressed as

$$\mathbf{F}(\mathbf{r},t)=\begin{cases}\mathbf{a}(\mathbf{r},t)\cos[\omega_c t+\varphi(\mathbf{r},t)],\\ \mathbf{x}(\mathbf{r},t)\cos\omega_c t-\mathbf{y}(\mathbf{r},t)\sin\omega_c t,\\ \operatorname{Re}\mathbf{F}_{en}(\mathbf{r},t)e^{j\omega_c t},\end{cases}$$

where $\omega_c = 2\pi f_c$, $\mathbf{a}(\mathbf{r},t)$ and $\varphi(\mathbf{r},t)$ are the carrier frequency, envelope and phase of the modulated signal respectively, and

$$\begin{aligned}\mathbf{F}_{en}(\mathbf{r},t)&=\mathbf{x}(\mathbf{r},t)+j\mathbf{y}(\mathbf{r},t),\\ \mathbf{x}(\mathbf{r},t)&=\mathbf{a}(\mathbf{r},t)\cos\varphi(\mathbf{r},t),\\ \mathbf{y}(\mathbf{r},t)&=\mathbf{a}(\mathbf{r},t)\sin\varphi(\mathbf{r},t).\end{aligned}$$

Here $\mathbf{F}_{en}(\mathbf{r},t)$, $\mathbf{x}(\mathbf{r},t)$ and $\mathbf{y}(\mathbf{r},t)$ are the complex envelope, in-phase component, and quadrature component of the modulated signal respectively. The complex envelope $\mathbf{F}_{en}(\mathbf{r},t)$ is a slowly varying function of time compared to $e^{j\omega_c t}$. It is easy to show that the complex envelopes of the electromagnetic fields satisfy the time-harmonic Maxwell equations

$$\begin{aligned}\nabla\times\mathbf{H}_{en}(\mathbf{r},t)&=j\omega_c\varepsilon\mathbf{E}_{en}(\mathbf{r},t)+\mathbf{J}_{en}(\mathbf{r},t),\\ \nabla\times\mathbf{E}_{en}(\mathbf{r},t)&=-j\omega_c\mu\mathbf{H}_{en}(\mathbf{r},t).\end{aligned}\tag{1.102}$$

Therefore the theoretical results about the time-harmonic fields can be applied to the complex envelopes of the fields. Let $\langle\mathbf{F}\rangle$ denote the ensemble average of **F**. For a stationary and ergodic vector field **F**, the ensemble average equals the time average, that is, $\langle\mathbf{F}\rangle=\overline{\overline{\mathbf{F}}}=\lim\limits_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathbf{F}(t)dt$. For a stationary and ergodic electromagnetic field, we may take the ensemble average of (1.102) to get

$$\begin{aligned}\nabla\times\overline{\overline{\mathbf{H}_{en}(\mathbf{r})}}&=j\omega_c\varepsilon\overline{\overline{\mathbf{E}_{en}(\mathbf{r})}}+\overline{\overline{\mathbf{J}_{en}(\mathbf{r})}},\\ \nabla\times\overline{\overline{\mathbf{E}_{en}(\mathbf{r})}}&=-j\omega_c\mu\overline{\overline{\mathbf{H}_{en}(\mathbf{r})}}.\end{aligned}$$

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Hence the theoretical results about the time-harmonic fields can also be applied to the ensemble averages of the complex envelopes of the fields.

> All the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers.
>
> —James Maxwell