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Polarization of Monochromatic Waves. Background of the Jones Matrix Methods. The Jones Calculus

1.1 Homogeneous Waves in Isotropic Media

1.1.1 Plane Waves

Light is an electromagnetic radiation with frequencies ν lying in the range from $\sim 4 \times 10^{14}$ to $\sim 8 \times 10^{14}$ Hz. An elementary model of light is a plane monochromatic wave. The electric field of a plane monochromatic wave can be represented, in complex form, as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}, \tag{1.1}$$

where $\omega = 2\pi\nu$ is the circular frequency and $\mathbf{k}$ is the wave vector of the wave, $\mathbf{r}$ is a position vector, and t is time. If the wave propagates in an isotropic nonabsorbing medium with refractive index n and is homogeneous (see Section 8.1.2), the vector $\mathbf{k}$ can be expressed as

$$\mathbf{k} = \frac{\omega}{c} n\mathbf{l}, \tag{1.2}$$

where $\mathbf{l}$ is the wave normal, a unit vector perpendicular to the wavefronts of the wave and indicating its propagation direction; c is the velocity of light in vacuum (free space). In this case, the wave is strictly transverse, satisfying the condition

$$\mathbf{l} \cdot \mathbf{E}_0 = 0. \tag{1.3}$$

The phase velocity of the wave is

$$c_n = \frac{\omega}{|\mathbf{k}|} = \frac{c}{n}. \tag{1.4}$$

Modeling and Optimization of LCD Optical Performance, First Edition.
Dmitry A. Yakovlev, Vladimir G. Chigrinov and Hoi-Sing Kwok.

Website Companion: www.wiley.com/go/yakovlev/modelinglcd

The true wavelength (λ_{true}) of the wave in the medium is defined as

$$\lambda_{\text{true}} \equiv c_n \tau,$$

where

$$\tau = \frac{1}{\nu} = \frac{2\pi}{\omega}$$

is the temporal period of the wave. Along with the true wavelength, one can associate with this wave the so-called *wavelength in free space*, defined as follows:

$$\lambda \equiv c\tau = \frac{c}{\nu} = \frac{2\pi c}{\omega}. \tag{1.5}$$

Throughout this book, speaking on monochromatic fields or monochromatic components of polychromatic fields, we will use the term "wavelength" only in the latter sense (often omitting "in free space"). Also, we will use the parameter

$$k_0 \equiv \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{1.6}$$

called the *wave number in free space*. In terms of λ and k_0, equation (1.1) can be rewritten as follows:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \mathrm{e}^{i(k_0 n\mathbf{lr} - \omega t)} = \mathbf{E}_0 \mathrm{e}^{i\left(\frac{2\pi}{\lambda} n\mathbf{lr} - \omega t\right)}. \tag{1.7}$$

The field (1.1) must satisfy the following wave equation [1]:

$$\nabla \times (\nabla \times \mathbf{E}) - k_0^2 \varepsilon \mathbf{E} = \widehat{\mathbf{0}}, \tag{1.8}$$

where ε is the electric permittivity of the medium, ∇ is the nabla operator, and $\widehat{\mathbf{0}}$ is the null vector. Throughout this book, we use the Gaussian system of units and consider only media that are nonmagnetic (i.e., having their magnetic permeability μ equal to 1) at optical frequencies. Substituting (1.1) into (1.8) gives the equation

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + k_0^2 \varepsilon \mathbf{E} = \widehat{\mathbf{0}}, \tag{1.9a}$$

which can be rewritten as

$$\mathbf{k} \cdot (\mathbf{k} \cdot \mathbf{E}) - \mathbf{k}^2 \mathbf{E} + k_0^2 \varepsilon \mathbf{E} = \widehat{\mathbf{0}}, \tag{1.9b}$$

where $\mathbf{k}^2 \equiv \mathbf{k} \cdot \mathbf{k}$. Scalarly multiplying any of these equations by $\mathbf{k}$, we see that these equations include the condition

$$\mathbf{k} \cdot \mathbf{E} = 0; \tag{1.10}$$

this condition may also be derived from the Maxwell equation $\nabla(\varepsilon\mathbf{E}) = 0$. We should note that condition (1.10) is valid for inhomogeneous waves of the form (1.1) as well (see Sections 8.1.2 and 9.2). In the case of a homogeneous wave, condition (1.10) is tantamount to (1.3). In view of (1.10), equation (1.9b) can be reduced to the following one:

$$\left(k_0^2\varepsilon - \mathbf{k}^2\right)\mathbf{E} = \widehat{\mathbf{0}}. \tag{1.11}$$

This equation requires that

$$\sqrt{\mathbf{k}^2} = k_0\sqrt{\varepsilon}. \tag{1.12}$$

In the case of a homogeneous wave, equation (1.12) leads to (1.2) with

$$n = \sqrt{\varepsilon}. \tag{1.13}$$

With complex n and ε, equations (1.1)–(1.3) and (1.13) can be used to describe homogeneous waves propagating in absorbing media (see Section 8.1.2).

1.1.2 Polarization. Jones Vectors

Polarization Parameters

Let us consider a plane wave satisfying (1.3). We introduce a rectangular right-handed Cartesian system (x, y, z) with the z-axis codirectional with the wave normal $\mathbf{l}$. Denote the unit vectors indicating the positive directions of the axes x, y, and z by $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$. Using this coordinate system, we can represent the electric field of the wave as follows:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(z,t) = \left(\boldsymbol{x}\tilde{E}_x(z) + \boldsymbol{y}\tilde{E}_y(z)\right)\mathrm{e}^{-i\omega t} \tag{1.14a}$$

or

$$\mathbf{E}(\mathbf{r},t) = \left(\boldsymbol{x}\left|\tilde{E}_x(z)\right|\mathrm{e}^{i\delta_x} + \boldsymbol{y}\left|\tilde{E}_y(z)\right|\mathrm{e}^{i\delta_y}\right)\mathrm{e}^{-i\omega t}, \tag{1.14b}$$

where $\tilde{E}_x$ and $\tilde{E}_y$ are the scalar complex amplitudes, and δ_x and δ_y are the phases of the x-component and the y-component of the field. The quantity

$$\chi = \frac{\tilde{E}_y}{\tilde{E}_x} = \frac{|\tilde{E}_y|}{|\tilde{E}_x|}\mathrm{e}^{i\delta}, \tag{1.15}$$

where $\delta = \delta_y - \delta_x$, fully describes the state of polarization (SOP) of the wave. For completely polarized waves, which we consider here, the SOP is essentially the shape, orientation, and sense of the trajectory that is described with time by the end of the true electric vector [Re($\mathbf{E}$)] associated with a given point in space ($\mathbf{r}$). It is well known that in general such a trajectory is an ellipse. With the help of Figure 1.1, we present basic parameters used for description of the SOP of completely polarized waves [1–3]:

1. The *azimuth* (*orientation angle*) γ_e of a polarization ellipse is defined as the angle between the positive direction of the x-axis and the major axis of the ellipse (Figure 1.1).
2. The *ellipticity* e_e is defined as

$$e_\mathrm{e} = \pm\frac{b}{a}, \tag{1.16}$$

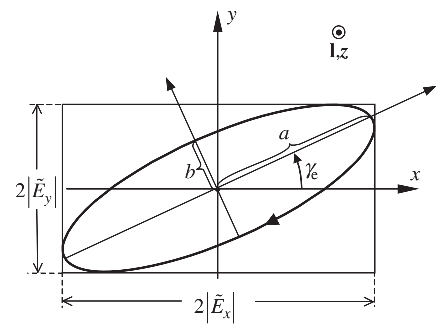

Figure 1.1 A polarization ellipse

where a and b are the lengths of the semimajor axis and semiminor axis of the ellipse, respectively. The ellipticity is taken positive if the polarization is right-handed and negative if the polarization is left-handed. The *handedness* of the polarization ellipse determines the sense in which the ellipse is described. In the literature, different conventions on the handedness of polarization are used. In this book, we use the convention adopted in the books [1, 2, 4]: the polarization is called *right-handed* if the polarization ellipse is described in the *clockwise* sense when looking against the direction of propagation of the light [this is the case in Figure 1.1 where the z-axis and the wave normal $\mathbf{l}$ are directed out of the page, toward the viewer] and *left-handed* otherwise. For a linearly polarized wave, $e_e = 0$. For right- and left-circularly polarized waves, e_e equals 1 and –1, respectively.

3. The *ellipticity angle* υ_e is defined by

$$e_e = \tan \upsilon_e. \tag{1.17}$$

The values of υ_e lie between $-\pi/4$ (left circular polarization) and $\pi/4$ (right circular polarization).

The azimuth γ_e and ellipticity angle υ_e are related to the complex polarization parameter χ as follows:

$$\cos 2\gamma_e = \frac{1-|\chi|^2}{\sqrt{\left(1-|\chi|^2\right)^2 + (2\,\mathrm{Re}\chi)^2}}, \quad \sin 2\gamma_e = \frac{2\,\mathrm{Re}\chi}{\sqrt{\left(1-|\chi|^2\right)^2 + (2\,\mathrm{Re}\chi)^2}}, \tag{1.18}$$

$$\sin 2\upsilon_e = -\frac{2\,\mathrm{Im}(\chi)}{1+|\chi|^2}. \tag{1.19}$$

Thus, given χ, the parameters γ_e, υ_e, and e_e can be calculated by formulas (1.18), (1.19), and (1.17). Note that for linearly polarized waves χ is purely real, while for circular polarizations it is purely imaginary ($\chi = -i$ for the right circular polarization and $\chi = i$ for the left circular polarization). We stress that relations (1.18) and (1.19) and all other relations for polarization parameters presented in this book correspond to the above choice of the convention on handedness and of the time factor in complex representation ($e^{-i\omega t}$).

The spatial evolution of the amplitudes $\tilde{E}_x$ and $\tilde{E}_y$ in (1.14) can be described by the following equations:

$$\tilde{E}_x(z) = \tilde{E}_x(z')e^{ik_0 n(z-z')}, \quad \tilde{E}_y(z) = \tilde{E}_y(z')e^{ik_0 n(z-z')}, \tag{1.20}$$

where z' is any given value of z. Even if the wave propagates in an absorbing medium (with complex n) and, consequently, is damped, its parameter χ is independent of z. This means that χ and the other polarization parameters listed above are spatially invariant and characterize the wave as a whole, that is, they are global characteristics of the wave.

Jones Vectors

The column

$$\tilde{\boldsymbol{J}}(z) = \begin{pmatrix} \tilde{E}_x(z) \\ \tilde{E}_y(z) \end{pmatrix} \tag{1.21}$$

represents a Jones vector of the wave (1.14). Different kinds of Jones vectors are used in practice. Some of them are considered in Section 5.4 and Chapter 8. Definition (1.21) corresponds to one of those kinds. The Jones vector defined by (1.21) is a local characteristic of the wave, being dependent on z. According to (1.20), its values for two arbitrary values of z, z' and z'' ($z'' > z'$), are related by

$$\tilde{\boldsymbol{J}}(z'') = \mathrm{e}^{ik_0 n(z''-z')}\tilde{\boldsymbol{J}}(z'). \tag{1.22}$$

This relation can be rewritten as

$$\tilde{\boldsymbol{J}}(z'') = \boldsymbol{t}_{is,n}(z', z'')\tilde{\boldsymbol{J}}(z'), \tag{1.23}$$

where

$$\boldsymbol{t}_{is,n}(z', z'') = \begin{pmatrix} e^{ik_0 n(z''-z')} & 0 \\ 0 & e^{ik_0 n(z''-z')} \end{pmatrix}. \tag{1.24}$$

The 2×2 matrix appearing here is a simple example of the Jones matrix.

If the medium where the wave propagates is nonabsorbing, the Jones vector $\tilde{\boldsymbol{J}}(z)$ can be represented as

$$\tilde{\boldsymbol{J}}(z) = a_\delta(z) a_\mathrm{I}\, \boldsymbol{J}, \tag{1.25}$$

where

$$\boldsymbol{J} = \begin{pmatrix} J_x \\ J_y \end{pmatrix} \tag{1.26}$$

is a spatially invariant Jones vector of the wave (see Section 5.4.3), a_δ is a scalar complex phase coefficient of unit magnitude ($a_\delta a_\delta^* = 1$), and a_I is a real coefficient that makes the following relation valid:

$$I = \boldsymbol{J}^\dagger\, \boldsymbol{J}, \tag{1.27}$$

where I represents a quantity (usually called *intensity*) that is regarded as a measure of irradiance (see Section 5.2) for waves in a particular problem or a method; the symbol † denotes the Hermitian conjugation operation (see Section 5.1.1). It is clear that, given $\boldsymbol{J}$, the complex polarization parameter χ of the wave can be calculated by the formula

$$\chi = \frac{J_y}{J_x}. \tag{1.28}$$

The use of such "global" and "fitted-to-intensity" [see (1.27)] Jones vectors for waves propagating in isotropic nonabsorbing media is a feature of the classical Jones calculus (JC) [5] (see Section 1.4). In JC, the quantity conventionally introduced to characterize irradiance is called *intensity*. Equation (1.27) is a standard expression for the intensity of a wave in terms of its Jones vector in this method. For many problems, the "global" Jones vector $\boldsymbol{J}$ of a wave contains all the information about the wave that is required for solving the problem, while the factors a_δ and a_I can be eliminated from the calculations. These factors are absent in standard algorithms based on JC. One should remember the differences between the vectors $\tilde{\boldsymbol{J}}$ and $\boldsymbol{J}$ when trying to use JC in combination with rigorous techniques derived from electromagnetic theory. Moreover, dealing with Jones vectors like $\tilde{\boldsymbol{J}}$, one should recognize that in many cases the use of the quantity

$$\tilde{I} = \tilde{\boldsymbol{J}}^\dagger \tilde{\boldsymbol{J}} = |\tilde{E}_x|^2 + |\tilde{E}_y|^2 \tag{1.29}$$

as a measure of irradiance is not justified. We will consider this issue in detail in Section 5.4. Here we restrict ourselves to the following example. Suppose that we use as *intensity* I FEFD irradiance (see Section 5.2), which is allowed by electromagnetic theory. In this case, the intensity I of the wave is expressed in terms of $\tilde{I}$ as follows:

$$I = \frac{cn}{8\pi}\tilde{I}. \tag{1.30}$$

As seen from (1.30), waves of equal $\tilde{I}$, propagating in media with different refractive indices, will have different "true" intensities I. Note that the coefficient a_I [see (1.25)] in this case is given by

$$a_\mathrm{I} = \sqrt{\frac{8\pi}{cn}}. \tag{1.31}$$

Polarization Jones Vector

Both the "global" and "fitted-to-intensity" Jones vector $\boldsymbol{J}$ and the local Jones vector $\tilde{\boldsymbol{J}}(z)$ can be represented as the product of a scalar factor and a unit vector

$$\boldsymbol{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix}, \tag{1.32}$$

unit in the sense that

$$\boldsymbol{j}^\dagger \boldsymbol{j} = 1. \tag{1.33}$$

The vector $\boldsymbol{j}$ carries information only on the polarization state of the wave ($\chi = j_y/j_x$) and may be called the *polarization Jones vector* (see Section 5.4.3). In solving practical problems, the polarization Jones vectors are often used to specify the polarization state of light incident on an optical system. Table 1.1 shows typical choices of the polarization vectors for different polarization states. The simplest choice of the vector $\boldsymbol{J}$ for incident light is

$$\boldsymbol{J} = \sqrt{I}\,\boldsymbol{j}. \tag{1.34}$$

A vector $\boldsymbol{J}'$ and the vector $\boldsymbol{J}'' = a\boldsymbol{J}'$, where a is a complex number of unit magnitude, can be regarded as equivalent apart from their phases. As a rule, when calculations for an optical system are performed in terms of "global" Jones vectors, the phases of these vectors are unimportant and can be assigned and transformed arbitrarily, owing to which there is a certain degree of freedom in choice of the vectors $\boldsymbol{j}$ and $\boldsymbol{J}$ for incident light and the Jones matrices describing the interaction of light with optical elements. In particular, this allows using reduced forms of Jones matrices for some kinds of elements (see, e.g., Sections 1.3.5 and 1.3.6), which simplifies the calculations.

Table 1.1 Variants of polarization Jones vectors for various polarization states

Polarization	Polarization Jones vector $\boldsymbol{j}$
Arbitrary elliptical	$\boldsymbol{j}_E(\gamma_e, \upsilon_e) \equiv \begin{pmatrix} \cos\gamma_e \cos\upsilon_e + i\sin\gamma_e \sin\upsilon_e \\ \sin\gamma_e \cos\upsilon_e - i\cos\gamma_e \sin\upsilon_e \end{pmatrix}$
Linear	$\boldsymbol{j}_P(\gamma_e) \equiv \begin{pmatrix} \cos\gamma_e \\ \sin\gamma_e \end{pmatrix}$
Right circular	$\boldsymbol{j}_R \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$
Left circular	$\boldsymbol{j}_L \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$

Stokes Parameters

In many cases, it is convenient to use Stoke vectors as state characteristics of light. Stokes vector is a 4 × 1 column composed of the so-called Stokes parameters, four real quantities characterizing the intensity and polarization state of light. In this subsection we present some useful expressions for Stokes parameters of monochromatic plane waves in terms of the polarization parameters considered above. Definitions for different kinds of Stokes vectors are given in Section 5.3. In particular, in Section 5.3 we define two types of Stokes vectors for plane waves. The Stokes vectors of these types for a wave are simply related. In view of this, we consider here Stokes vectors of only one of these types, namely, intensity-based Stokes vectors.

Using the x-axis as the polarization reference axis (see Section 5.3), after substitution of (1.14) into (5.80) it is easy to obtain the following expression for the intensity-based Stokes vector of the wave (1.14):

$$S_{(I)} \equiv \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{cn}{8\pi} \begin{pmatrix} |\tilde{E}_x|^2 + |\tilde{E}_y|^2 \\ |\tilde{E}_x|^2 - |\tilde{E}_y|^2 \\ 2\,\mathrm{Re}\left(\tilde{E}_x \tilde{E}_y^*\right) \\ 2\,\mathrm{Im}\left(\tilde{E}_x \tilde{E}_y^*\right) \end{pmatrix}. \tag{1.35}$$

Since $\tilde{E}_x \tilde{E}_y^* = |\tilde{E}_x||\tilde{E}_y| \mathrm{e}^{-i\delta}$, we may rewrite this expression as follows:

$$S_{(I)} = \frac{cn}{8\pi} \begin{pmatrix} |\tilde{E}_x|^2 + |\tilde{E}_y|^2 \\ |\tilde{E}_x|^2 - |\tilde{E}_y|^2 \\ 2|\tilde{E}_x||\tilde{E}_y| \cos\delta \\ -2|\tilde{E}_x||\tilde{E}_y| \sin\delta \end{pmatrix}. \tag{1.36}$$

Another useful expression for $S_{(I)}$ can be obtained by using the following representation of the vector $\tilde{\boldsymbol{J}}(z)$:

$$\tilde{\boldsymbol{J}}(z) \equiv \begin{pmatrix} \tilde{E}_x(z) \\ \tilde{E}_y(z) \end{pmatrix} = a(z)\sqrt{\tilde{I}}\,\boldsymbol{j}_E(\gamma_e, \upsilon_e), \tag{1.37}$$

where a is a complex phase factor of unit magnitude and $\boldsymbol{j}_E(\gamma_e, \upsilon_e)$ is the polarization Jones vector given in Table 1.1. Substitution from (1.37) into (1.35) gives

$$S_{(I)} \equiv \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{cn}{8\pi} \begin{pmatrix} \tilde{I} \\ \tilde{I}\cos 2\gamma_e \cos 2\upsilon_e \\ \tilde{I}\sin 2\gamma_e \cos 2\upsilon_e \\ \tilde{I}\sin 2\upsilon_e \end{pmatrix} = \begin{pmatrix} I \\ I\cos 2\gamma_e \cos 2\upsilon_e \\ I\sin 2\gamma_e \cos 2\upsilon_e \\ I\sin 2\upsilon_e \end{pmatrix}, \tag{1.38}$$

where I is the intensity defined as the FEFD irradiance of the wave. This expression is convenient when there is a need to construct the Stokes vector for given γ_e and υ_e or, vice versa, to find γ_e and υ_e from calculated or measured Stokes parameters. Note that in the case of a quasimonochromatic partially polarized wave, its Stokes vector can be represented as

$$S_{(I)} \equiv \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I \\ I_p\cos 2\gamma_e \cos 2\upsilon_e \\ I_p\sin 2\gamma_e \cos 2\upsilon_e \\ I_p\sin 2\upsilon_e \end{pmatrix}, \tag{1.39}$$

where I is the total intensity of the wave and I_p is the intensity of the completely polarized component of the wave. The intensity I_p is expressed in terms of the Stokes parameters as follows:

$$I_p = \sqrt{S_1^2 + S_2^2 + S_3^2}, \tag{1.40}$$

which allows one to easily find γ_e and υ_e from a given Stokes vector in this case as well.

If the Jones vector $\boldsymbol{J}$ is defined by (1.25) with a_I given by (1.31), the vector $S_{(I)}$ is expressed in terms of the $\boldsymbol{J}$ components as follows:

$$S_{(I)} \equiv \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} |J_x|^2 + |J_y|^2 \\ |J_x|^2 - |J_y|^2 \\ 2\,\mathrm{Re}\left(J_x J_y^*\right) \\ 2\,\mathrm{Im}\left(J_x J_y^*\right) \end{pmatrix} = \begin{pmatrix} |J_x|^2 + |J_y|^2 \\ |J_x|^2 - |J_y|^2 \\ 2|J_x||J_y|\cos\delta \\ -2|J_x||J_y|\sin\delta \end{pmatrix}. \tag{1.41}$$

Poincaré Sphere

Let us introduce the normalized Stokes parameters

$$s_1 = \frac{S_1}{S_0}, \quad s_2 = \frac{S_2}{S_0}, \quad s_3 = \frac{S_3}{S_0}. \tag{1.42}$$

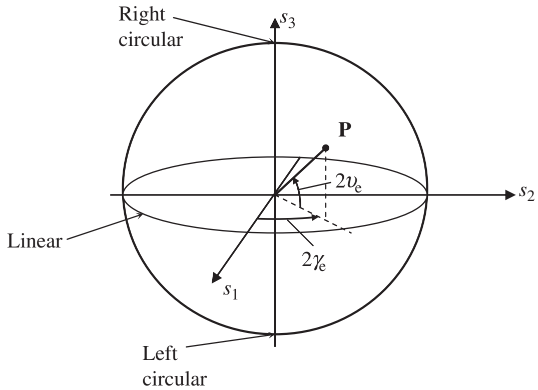

Figure 1.2 Representation of polarization states by points on the Poincaré sphere

According to (1.38), in the case of a completely polarized wave, these parameters can be expressed as follows:

$$s_1 = \cos 2\gamma_e \cos 2\upsilon_e, \quad s_2 = \sin 2\gamma_e \cos 2\upsilon_e, \quad s_3 = \sin 2\upsilon_e. \tag{1.43}$$

With γ_e and υ_e considered as free variables, equations (1.43) describe a unit sphere in a rectangular Cartesian coordinate system (s_1, s_2, s_3) (see Figure 1.2). This sphere is called the Poincaré sphere. The points of this sphere represent all possible SOPs of completely polarized light. The north and south poles on the Poincaré sphere represent the right and left circular polarizations, respectively. The equator represents linear polarization states and all the other points on the sphere represent elliptical polarization states. All left-handed polarization states are on the southern hemisphere, and the northern hemisphere corresponds to right-handed polarizations.

1.1.3 Coordinate Transformation Rules for Jones Vectors. Orthogonal Polarizations. Decomposition of a Wave into Two Orthogonally Polarized Waves

Coordinate Transformation Rules for Cartesian Jones Vectors

Let $\boldsymbol{x}'$ and $\boldsymbol{y}'$ be unit vectors directed along mutually orthogonal axes x' and y' perpendicular to the axis z. Using the reference frame (x', y', z) instead of (x, y, z), we can represent the wave (1.14) as

$$\mathbf{E}(\mathbf{r}, t) = \left(\boldsymbol{x}'\tilde{E}_{x'}(z) + \boldsymbol{y}'\tilde{E}_{y'}(z)\right) \mathrm{e}^{-i\omega t}. \tag{1.44}$$

According to (1.44) and (1.14a),

$$\boldsymbol{x}\tilde{E}_x + \boldsymbol{y}\tilde{E}_y = \boldsymbol{x}'\tilde{E}_{x'} + \boldsymbol{y}'\tilde{E}_{y'}. \tag{1.45}$$

Scalarly multiplying (1.45) by $\boldsymbol{x}'$ and $\boldsymbol{y}'$, we obtain the following equations:

$$\begin{aligned} \tilde{E}_{x'} &= (\boldsymbol{x}'\boldsymbol{x})\tilde{E}_x + (\boldsymbol{x}'\boldsymbol{y})\tilde{E}_y, \\ \tilde{E}_{y'} &= (\boldsymbol{y}'\boldsymbol{x})\tilde{E}_x + (\boldsymbol{y}'\boldsymbol{y})\tilde{E}_y. \end{aligned} \tag{1.46}$$

Introducing the column vector

$$\tilde{\boldsymbol{J}}' = \begin{pmatrix} \tilde{E}_{x'} \\ \tilde{E}_{y'} \end{pmatrix} \tag{1.47}$$

and the matrix

$$\boldsymbol{R}_{xy\to x'y'} = \begin{pmatrix} \boldsymbol{x}'\boldsymbol{x} & \boldsymbol{x}'\boldsymbol{y} \\ \boldsymbol{y}'\boldsymbol{x} & \boldsymbol{y}'\boldsymbol{y} \end{pmatrix}, \tag{1.48}$$

we may write (1.46) in matrix form

$$\begin{pmatrix} \tilde{E}_{x'} \\ \tilde{E}_{y'} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}'\boldsymbol{x} & \boldsymbol{x}'\boldsymbol{y} \\ \boldsymbol{y}'\boldsymbol{x} & \boldsymbol{y}'\boldsymbol{y} \end{pmatrix} \begin{pmatrix} \tilde{E}_{x} \\ \tilde{E}_{y} \end{pmatrix} \tag{1.49}$$

or

$$\tilde{\boldsymbol{J}}' = \boldsymbol{R}_{xy\to x'y'}\tilde{\boldsymbol{J}}. \tag{1.50}$$

Considering the space of Jones vectors as a space of states of a wave where each Jones vector represents a unique state, we may say that the columns $\tilde{\boldsymbol{J}}$ and $\tilde{\boldsymbol{J}}'$ represent the same Jones vector (as they describe the same state) referred to different bases. Relation (1.49) represents the law of transformation of the elements of this Jones vector under the change of basis $(x, y) \to (x', y')$. In view of this, it would be more correct to rewrite relation (1.50) as follows:

$$\tilde{\boldsymbol{J}}_{x'y'} = \boldsymbol{R}_{xy\to x'y'}\tilde{\boldsymbol{J}}_{xy} \tag{1.51}$$

with obvious notation.

If the system (x', y', z), like the system (x, y, z), is right-handed (as in Figure 1.3), the matrix $\boldsymbol{R}_{xy\to x'y'}$ can be expressed as

$$\boldsymbol{R}_{xy\to x'y'} = \widehat{R}_{\mathrm{C}}(\phi), \tag{1.52}$$

where ϕ is the angle between the axes x and x' (Figure 1.3), and $\widehat{R}_{\mathrm{C}}$ is the rotation matrix defined as

$$\widehat{R}_{\mathrm{C}}(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{1.53}$$

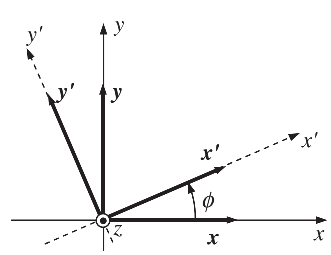

Figure 1.3 Reference frames (x, y, z) and (x', y', z)

for any α. Thus, in this case, the law of coordinate transformation can be expressed by the relation

$$\tilde{\boldsymbol{J}}_{x'y'} = \widehat{R}_{\mathrm{C}}(\phi)\tilde{\boldsymbol{J}}_{xy}. \tag{1.54}$$

For the inverse change $(x', y') \rightarrow (x, y)$,

$$\tilde{\boldsymbol{J}}_{xy} = \widehat{R}_{\mathrm{C}}(\phi)^{-1}\tilde{\boldsymbol{J}}_{x'y'} = \widehat{R}_{\mathrm{C}}(-\phi)\tilde{\boldsymbol{J}}_{x'y'}. \tag{1.55}$$

Expression (1.48) for the coordinate transformation matrix $\boldsymbol{R}_{xy\rightarrow x'y'}$ is valid irrespective of the handedness of the systems (x, y, z) and (x', y', z). For example, if the system (x, y, z) is, as before, right-handed, choosing the axes x' and y' so that $\boldsymbol{x}' = \boldsymbol{x}$ and $\boldsymbol{y}' = -\boldsymbol{y}$, we will obtain a left-handed system (x', y', z). In this case, equation (1.48) gives

$$\boldsymbol{R}_{xy\rightarrow x'y'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.56}$$

We should note that many formulas presented in this book, in particular in the previous section, are valid for right-handed coordinate systems only. In this book, we deal with left-handed systems very rarely, and it is always stated; if the handedness of a coordinate system is not specified, this system is assumed to be right-handed.

Orthogonal Polarizations

Two waves propagating in the same direction are said to be *orthogonally polarized* if their ellipses of polarization have the same shape but mutually orthogonal major axes and are traced in opposite senses (Figure 1.4). The right circular polarization is orthogonal with respect to the left circular polarization. For a wave with $\gamma_{\mathrm{e}} = \gamma'_{\mathrm{e}}$, $\upsilon_{\mathrm{e}} = \upsilon'_{\mathrm{e}}$, and $\chi = \chi'$, where γ'_{e}, υ'_{e}, and χ' are arbitrary, a wave with the corresponding orthogonal polarization will have $\gamma_{\mathrm{e}} = \gamma'_{\mathrm{e}} \pm \pi/2$, $\upsilon_{\mathrm{e}} = -\upsilon'_{\mathrm{e}}$, and $\chi = -1/\chi'^{*}$ [2]. By checking that

$$\boldsymbol{j}_E(\gamma'_{\mathrm{e}} \pm \pi/2, -\upsilon'_{\mathrm{e}})^{\dagger}\, \boldsymbol{j}_E(\gamma'_{\mathrm{e}}, \upsilon'_{\mathrm{e}}) = 0,$$

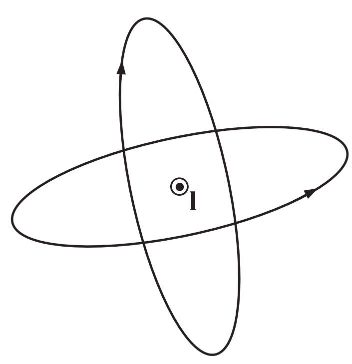

Figure 1.4 Polarization ellipses of mutually orthogonal polarizations

where $\boldsymbol{j}_E$ is the polarization vector defined in Table 1.1, it is easy to verify that the polarization Jones vectors of two orthogonally polarized waves, these vectors being denoted by $\boldsymbol{j}$ and $\boldsymbol{j}_{\text{ort}}$, are orthogonal in the sense that

$$\boldsymbol{j}_{\text{ort}}^{\dagger}\boldsymbol{j} = \boldsymbol{j}^{\dagger}\boldsymbol{j}_{\text{ort}} = 0. \tag{1.57}$$

It is clear that the Jones vectors of the other above-mentioned kinds ($\boldsymbol{J}$ and $\tilde{\boldsymbol{J}}$) for these waves will also be orthogonal in the same sense ($\tilde{\boldsymbol{J}}_{\text{ort}}^{\dagger}\tilde{\boldsymbol{J}} = 0$, $\boldsymbol{J}_{\text{ort}}^{\dagger}\boldsymbol{J} = 0$).

Decomposition of a Wave into Two Orthogonally Polarized Waves

The equation for the electric field of the wave (1.14) can be rewritten in the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_{(x)}(\mathbf{r}, t) + \mathbf{E}_{(y)}(\mathbf{r}, t), \tag{1.58}$$

where

$$\mathbf{E}_{(x)}(\mathbf{r}, t) = \boldsymbol{x}\,(\boldsymbol{x}\mathbf{E}(\mathbf{r}, t)) = \boldsymbol{x}\tilde{E}_x(z)\mathrm{e}^{-i\omega t},$$

$$\mathbf{E}_{(y)}(\mathbf{r}, t) = \boldsymbol{y}\,(\boldsymbol{y}\mathbf{E}(\mathbf{r}, t)) = \boldsymbol{y}\tilde{E}_y(z)\mathrm{e}^{-i\omega t}.$$

$\mathbf{E}_{(x)}(\mathbf{r}, t)$ and $\mathbf{E}_{(y)}(\mathbf{r}, t)$ represent linearly polarized plane waves, each satisfying the wave equation (1.8). These waves have mutually orthogonal polarizations: the field $\mathbf{E}_{(x)}(\mathbf{r}, t)$ vibrates along a line parallel to $\boldsymbol{x}$, while the field $\mathbf{E}_{(y)}(\mathbf{r}, t)$ oscillates along a line parallel to $\boldsymbol{y}$. Thus, we can regard the representation (1.58) as a decomposition of the wave $\mathbf{E}(\mathbf{r}, t)$ into two waves with given mutually orthogonal polarizations. A similar decomposition can be performed with the use of any other pair of orthogonal polarizations.

Let

$$\boldsymbol{j}_1 = \begin{pmatrix} j_{1x} \\ j_{1y} \end{pmatrix} \text{ and } \boldsymbol{j}_2 = \begin{pmatrix} j_{2x} \\ j_{2y} \end{pmatrix}$$

be a pair of mutually orthogonal polarization Jones vectors ($\boldsymbol{j}_1{}^{\dagger}\boldsymbol{j}_2 = 0$). Introduce the vectors

$$\hat{\mathbf{e}}_1 = j_{1x}\boldsymbol{x} + j_{1y}\boldsymbol{y},$$

$$\hat{\mathbf{e}}_2 = j_{2x}\boldsymbol{x} + j_{2y}\boldsymbol{y},$$

which are three-dimensional analogs of the vectors $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$. The vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are unit vectors in the sense that

$$\hat{\mathbf{e}}_j^*\hat{\mathbf{e}}_j = 1, \quad j = 1, 2, \tag{1.59}$$

and mutually orthogonal in the sense that

$$\hat{\mathbf{e}}_1^*\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_2^*\hat{\mathbf{e}}_1 = 0. \tag{1.60}$$

Using these vectors, we can represent the wave (1.14) as follows:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t), \tag{1.61}$$

where

$$\begin{aligned}\mathbf{E}_1(\mathbf{r},t) &= \hat{\mathbf{e}}_1\left(\hat{\mathbf{e}}_1^*\mathbf{E}(\mathbf{r},t)\right) = \hat{\mathbf{e}}_1\tilde{A}_1(z)\mathrm{e}^{-i\omega t},\\ \mathbf{E}_2(\mathbf{r},t) &= \hat{\mathbf{e}}_2\left(\hat{\mathbf{e}}_2^*\mathbf{E}(\mathbf{r},t)\right) = \hat{\mathbf{e}}_2\tilde{A}_2(z)\mathrm{e}^{-i\omega t},\\ \tilde{A}_j(z) &= \tilde{A}_j(z')\mathrm{e}^{ik_0 n(z-z')}, \quad j=1,2.\end{aligned} \tag{1.62}$$

$\mathbf{E}_1(\mathbf{r},t)$ and $\mathbf{E}_2(\mathbf{r},t)$ represent waves with polarizations $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$, respectively. The column

$$\tilde{\boldsymbol{J}}_{j_1 j_2} = \begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix} \tag{1.63}$$

is yet another representation of the Jones vector of the wave. From the relation

$$\boldsymbol{x}\tilde{E}_x + \boldsymbol{y}\tilde{E}_y = \hat{\mathbf{e}}_1\tilde{A}_1 + \hat{\mathbf{e}}_2\tilde{A}_2 = \left(j_{1x}\boldsymbol{x} + j_{1y}\boldsymbol{y}\right)\tilde{A}_1 + \left(j_{2x}\boldsymbol{x} + j_{2y}\boldsymbol{y}\right)\tilde{A}_2$$

it follows that

$$\begin{pmatrix}\tilde{E}_x\\ \tilde{E}_y\end{pmatrix} = \begin{pmatrix}\boldsymbol{x}\hat{\mathbf{e}}_1 & \boldsymbol{x}\hat{\mathbf{e}}_2\\ \boldsymbol{y}\hat{\mathbf{e}}_1 & \boldsymbol{y}\hat{\mathbf{e}}_2\end{pmatrix}\begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix} = \begin{pmatrix}j_{1x} & j_{2x}\\ j_{1y} & j_{2y}\end{pmatrix}\begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{j}_1 & \boldsymbol{j}_2\end{pmatrix}\begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix} = \boldsymbol{j}_1\tilde{A}_1 + \boldsymbol{j}_2\tilde{A}_2. \tag{1.64}$$

The column $\tilde{\boldsymbol{J}}_{j_1 j_2}$ can be expressed in terms of the column $\tilde{\boldsymbol{J}}_{xy}$ as follows:

$$\begin{pmatrix}\tilde{A}_1\\ \tilde{A}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{j}_1 & \boldsymbol{j}_2\end{pmatrix}^{-1}\begin{pmatrix}\tilde{E}_x\\ \tilde{E}_y\end{pmatrix} = \begin{pmatrix}\boldsymbol{j}_1^\dagger\\ \boldsymbol{j}_2^\dagger\end{pmatrix}\begin{pmatrix}\tilde{E}_x\\ \tilde{E}_y\end{pmatrix}. \tag{1.65}$$

It is clear that the Cartesian Jones vectors $\tilde{\boldsymbol{J}}_{xy}$ and $\tilde{\boldsymbol{J}}_{x'y'}$ can also be defined in the same way as the vector $\tilde{\boldsymbol{J}}_{j_1 j_2}$: the vector $\tilde{\boldsymbol{J}}_{xy}$ corresponds to the choice

$$\boldsymbol{j}_1 = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \boldsymbol{j}_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$$

($\hat{\mathbf{e}}_1 = \boldsymbol{x}$, $\hat{\mathbf{e}}_2 = \boldsymbol{y}$), and the vector $\tilde{\boldsymbol{J}}_{x'y'}$ to

$$\boldsymbol{j}_1 = \begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \quad \boldsymbol{j}_2 = \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}$$

($\hat{\mathbf{e}}_1 = \boldsymbol{x}'$, $\hat{\mathbf{e}}_2 = \boldsymbol{y}'$) in the coordinate system (x, y, z,).

The representation of wave fields in terms of basis wave modes (basis eigenwaves) is widely used in rigorous methods of polarization optics and optics of stratified media (see Chapter 8). State vectors introduced in the same manner as $\tilde{\boldsymbol{J}}_{j_1 j_2}$ [see (1.61)–(1.63)] are natural elements of these methods, where they are employed for description of homogeneous waves propagating in isotropic media as well as homogeneous waves propagating along the optic axis in uniaxial media. Choosing the basis polarization vectors in such a way that the Jones vector can be treated as a Cartesian Jones vector referred to a right-handed coordinate system makes it possible to use the formulas relating the components of Cartesian Jones vectors and the polarization ellipse parameters of Section 1.1.2 in such calculations.

General Coordinate Transformation Rules for Jones Vectors

The column $\tilde{\boldsymbol{J}}_{j_1\,j_2}$ [see (1.63)] is a particular representation of the Jones vector of the wave; to introduce this column we used the polarization basis $(\boldsymbol{j}_1, \boldsymbol{j}_2)$ [or, what is the same, $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2)$]. Let $(\boldsymbol{j}'_1, \boldsymbol{j}'_2)$ [$(\widehat{\mathbf{e}}'_1, \widehat{\mathbf{e}}'_2)$] be another polarization basis [with $\boldsymbol{j}'^{\dagger}_1 \boldsymbol{j}'_2 = 0$ $(\widehat{\mathbf{e}}'^{*}_1 \widehat{\mathbf{e}}'_2 = 0)$], and let the column $\tilde{\boldsymbol{J}}_{j'_1 j'_2}$ represent the same Jones vector in this new basis. One can show that

$$\tilde{\boldsymbol{J}}_{j'_1\,j'_2} = \begin{pmatrix} \boldsymbol{j}'^{\dagger}_1 \boldsymbol{j}_1 & \boldsymbol{j}'^{\dagger}_1 \boldsymbol{j}_2 \\ \boldsymbol{j}'^{\dagger}_2 \boldsymbol{j}_1 & \boldsymbol{j}'^{\dagger}_2 \boldsymbol{j}_2 \end{pmatrix} \tilde{\boldsymbol{J}}_{j_1\,j_2} \tag{1.66}$$

or, equivalently,

$$\tilde{\boldsymbol{J}}_{j'_1\,j'_2} = \begin{pmatrix} \widehat{\mathbf{e}}'^{*}_1 \widehat{\mathbf{e}}_1 & \widehat{\mathbf{e}}'^{*}_1 \widehat{\mathbf{e}}_2 \\ \widehat{\mathbf{e}}'^{*}_2 \widehat{\mathbf{e}}_1 & \widehat{\mathbf{e}}'^{*}_2 \widehat{\mathbf{e}}_2 \end{pmatrix} \tilde{\boldsymbol{J}}_{j_1\,j_2}. \tag{1.67}$$

Relation (1.66) can readily be derived by using (1.64) and (1.65).

1.2 Interface Optics for Isotropic Media

Many problems of LCD optics involve considering the optical effect of interfaces. In this book, we will deal with interfaces of different kinds—from interfaces between isotropic media to those between arbitrary anisotropic media. The simplest problem, the problem on reflection and transmission of a plane monochromatic wave incident on a plane interface between isotropic media, is considered in detail in many textbooks (e.g., [1, 4]). In Section 1.2.1, we present, without derivation, the basic laws and formulas relating to this problem. In Section 1.2.2, we use this problem to show some options of modern variants of the Jones matrix method.

1.2.1 Fresnel's Formulas. Snell's Law

Let a homogeneous plane monochromatic wave propagating in an isotropic homogeneous nonabsorbing medium with refractive index n_1 be obliquely incident at angle β_{inc} on a plane surface of another isotropic homogeneous nonabsorbing medium with refractive index n_2. First we consider the case when $n_1 < n_2$, which is illustrated by Figure 1.5. In this case, at any β_{inc}, the reflected and transmitted fields will be homogeneous plane waves. Considering amplitude relations between the incident, reflected, and transmitted waves, it is convenient to decompose each of these waves into two linearly polarized constituents: the wave with its electric field vector parallel to the plane of incidence, it is the so-called *p-polarized* component, and the wave with electric field vector perpendicular to the plane of incidence, it is the so-called *s-polarized* component (*the plane of incidence* is the plane containing the incident light wave vector and a normal to the interface). One can use the following variant of decomposition of the electric fields of the incident, reflected, and transmitted wave fields:

$$\begin{aligned} &\text{Incident wave:} \quad \mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \left[\mathbf{e}_{\text{p}}^{(\text{inc})} A_{\text{p}}^{(\text{inc})}(\mathbf{r}) + \mathbf{e}_{\text{s}}^{(\text{inc})} A_{\text{s}}^{(\text{inc})}(\mathbf{r})\right] \mathrm{e}^{-i\omega t}, \\ &\text{Reflected wave:} \quad \mathbf{E}_{\text{ref}}(\mathbf{r}, t) = \left[\mathbf{e}_{\text{p}}^{(\text{ref})} A_{\text{p}}^{(\text{ref})}(\mathbf{r}) + \mathbf{e}_{\text{s}}^{(\text{ref})} A_{\text{s}}^{(\text{ref})}(\mathbf{r})\right] \mathrm{e}^{-i\omega t}, \\ &\text{Transmitted wave:} \quad \mathbf{E}_{\text{tr}}(\mathbf{r}, t) = \left[\mathbf{e}_{\text{p}}^{(\text{tr})} A_{\text{p}}^{(\text{tr})}(\mathbf{r}) + \mathbf{e}_{\text{s}}^{(\text{tr})} A_{\text{s}}^{(\text{tr})}(\mathbf{r})\right] \mathrm{e}^{-i\omega t}, \end{aligned} \tag{1.68}$$

where $\mathbf{e}_{\text{p}}^{(\text{inc})}$, $\mathbf{e}_{\text{s}}^{(\text{inc})}$, $\mathbf{e}_{\text{p}}^{(\text{ref})}$, $\mathbf{e}_{\text{s}}^{(\text{ref})}$, $\mathbf{e}_{\text{p}}^{(\text{tr})}$, and $\mathbf{e}_{\text{s}}^{(\text{tr})}$ are unit real vectors which specify vibration directions of the electric fields of the p- and s-components of the waves and are oriented as indicated in Figure 1.5, and

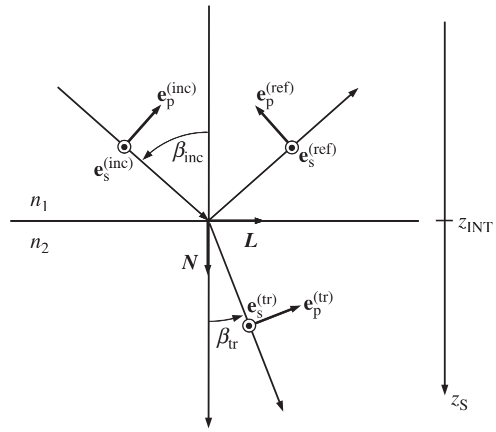

Figure 1.5 Transmission and reflection at a plane interface between isotropic media. Geometry of the problem

$A_{\mathrm{p}}^{(\mathrm{inc})}, A_{\mathrm{s}}^{(\mathrm{inc})}, A_{\mathrm{p}}^{(\mathrm{ref})}, A_{\mathrm{s}}^{(\mathrm{ref})}, A_{\mathrm{p}}^{(\mathrm{tr})}$, and $A_{\mathrm{s}}^{(\mathrm{tr})}$ are the scalar complex amplitudes of these components. The spatial evolution of the scalar amplitudes in the regions where the corresponding waves exist can be described by the equations

$$\begin{gathered} A_j^{(\mathrm{inc})}(\mathbf{r}) = A_j^{(\mathrm{inc})}(\mathbf{r}')\mathrm{e}^{ik_0\mathbf{m}_{\mathrm{inc}}(\mathbf{r}'-\mathbf{r})}, \quad A_j^{(\mathrm{ref})}(\mathbf{r}) = A_j^{(\mathrm{ref})}(\mathbf{r}'')\mathrm{e}^{ik_0\mathbf{m}_{\mathrm{ref}}(\mathbf{r}''-\mathbf{r})}, \\ A_j^{(\mathrm{tr})}(\mathbf{r}) = A_j^{(\mathrm{tr})}(\mathbf{r}''')\mathrm{e}^{ik_0\mathbf{m}_{\mathrm{tr}}(\mathbf{r}'''-\mathbf{r})}, \\ j = \mathrm{s}, \mathrm{p}, \end{gathered} \tag{1.69}$$

where $\mathbf{m}_{\mathrm{inc}}$, $\mathbf{m}_{\mathrm{ref}}$, and $\mathbf{m}_{\mathrm{tr}}$ are the refraction vectors (see Section 8.1.2) of the incident, reflected, and transmitted waves, respectively. The refraction vectors are related to the corresponding wave vectors by the equations

$$\mathbf{m}_{\mathrm{inc}} = k_0^{-1}\mathbf{k}_{\mathrm{inc}}, \quad \mathbf{m}_{\mathrm{ref}} = k_0^{-1}\mathbf{k}_{\mathrm{ref}}, \quad \mathbf{m}_{\mathrm{tr}} = k_0^{-1}\mathbf{k}_{\mathrm{tr}}. \tag{1.70}$$

Using the quantities

$$\zeta \equiv \mathbf{m}_{\mathrm{inc}}\boldsymbol{L} = n_1 \sin\beta_{\mathrm{inc}}, \quad \mathbf{b} = \boldsymbol{L}\zeta, \quad \sigma_{\mathrm{inc}} \equiv \mathbf{m}_{\mathrm{inc}}\boldsymbol{N} = n_1 \cos\beta_{\mathrm{inc}}, \tag{1.71}$$

where $\boldsymbol{N}$ and $\boldsymbol{L}$ are unit vectors oriented as shown in Figure 1.5 ($\boldsymbol{N}$ is normal to the interface surface; $\boldsymbol{L}$ is tangent to this surface), one may represent the vector $\mathbf{m}_{\mathrm{inc}}$ as follows:

$$\mathbf{m}_{\mathrm{inc}} = \boldsymbol{L}n_1 \sin\beta_{\mathrm{inc}} + \boldsymbol{N}n_1 \cos\beta_{\mathrm{inc}} = \boldsymbol{L}\zeta + \boldsymbol{N}\sigma_{\mathrm{inc}} = \mathbf{b} + \boldsymbol{N}\sigma_{\mathrm{inc}}. \tag{1.72}$$

According to (1.12),

$$\mathbf{m}_{\mathrm{inc}}\mathbf{m}_{\mathrm{inc}} = n_1^2, \quad \mathbf{m}_{\mathrm{ref}}\mathbf{m}_{\mathrm{ref}} = n_1^2, \quad \mathbf{m}_{\mathrm{tr}}\mathbf{m}_{\mathrm{tr}} = n_2^2. \tag{1.73}$$

It follows from the symmetry of the problem (see Section 8.1.3) that the vectors $\mathbf{m}_{\mathrm{ref}}$ and $\mathbf{m}_{\mathrm{tr}}$ are coplanar with the vectors $\mathbf{m}_{\mathrm{inc}}$ and $\boldsymbol{N}$ and have their tangential components equal to the tangential component ($\mathbf{b} = \boldsymbol{L}\zeta$) of the vector $\mathbf{m}_{\mathrm{inc}}$, that is, the vectors $\mathbf{m}_{\mathrm{ref}}$ and $\mathbf{m}_{\mathrm{tr}}$ can be represented as follows:

$$\mathbf{m}_{\mathrm{ref}} = \boldsymbol{L}\zeta + \boldsymbol{N}\sigma_{\mathrm{ref}}, \quad \mathbf{m}_{\mathrm{tr}} = \boldsymbol{L}\zeta + \boldsymbol{N}\sigma_{\mathrm{tr}}. \tag{1.74}$$

According to (1.73) and (1.74), $\sigma_{\mathrm{ref}} = -\sigma_{\mathrm{inc}}$ and

$$\sigma_{\mathrm{tr}} = \sqrt{n_2^2 - \zeta^2}. \tag{1.75}$$

If n_2 is real and $\zeta < n_2$, as in the case under consideration, the vector $\mathbf{m}_{\mathrm{tr}}$ can be represented as

$$\mathbf{m}_{\mathrm{tr}} = \boldsymbol{L}n_2 \sin\beta_{\mathrm{tr}} + \boldsymbol{N}n_2 \cos\beta_{\mathrm{tr}}. \tag{1.76}$$

Then from the condition of equality of the tangential components of $\mathbf{m}_{\mathrm{inc}}$ and $\mathbf{m}_{\mathrm{tr}}$ it follows that

$$n_2 \sin\beta_{\mathrm{tr}} = n_1 \sin\beta_{\mathrm{inc}}, \tag{1.77}$$

which is the well-known *Snell's law*.

Let the plane of the interface coincide with the plane $z_S = z_{\mathrm{INT}}$ in a rectangular Cartesian coordinate system (x_S, y_S, z_S) with the z_S-axis directed as shown in Figure 1.5. From the requirement of continuity of the tangential components of the electric and magnetic fields across the interface surface (see Section 8.1.1), one can find that amplitudes of the p-polarized components of the transmitted and reflected waves depend only on the amplitude of the p-polarized component of the incident wave and the same is true for the s-polarized components and that the ratios

$$\begin{aligned} t_{\mathrm{pp}} &\equiv \frac{A_{\mathrm{p}}^{(\mathrm{tr})}(x_S, y_S, z_{\mathrm{INT}}+0)}{A_{\mathrm{p}}^{(\mathrm{inc})}(x_S, y_S, z_{\mathrm{INT}}-0)}, \quad t_{\mathrm{ss}} \equiv \frac{A_{\mathrm{s}}^{(\mathrm{tr})}(x_S, y_S, z_{\mathrm{INT}}+0)}{A_{\mathrm{s}}^{(\mathrm{inc})}(x_S, y_S, z_{\mathrm{INT}}-0)}, \\ r_{\mathrm{pp}} &\equiv \frac{A_{\mathrm{p}}^{(\mathrm{ref})}(x_S, y_S, z_{\mathrm{INT}}-0)}{A_{\mathrm{p}}^{(\mathrm{inc})}(x_S, y_S, z_{\mathrm{INT}}-0)}, \quad r_{\mathrm{ss}} \equiv \frac{A_{\mathrm{s}}^{(\mathrm{ref})}(x_S, y_S, z_{\mathrm{INT}}-0)}{A_{\mathrm{s}}^{(\mathrm{inc})}(x_S, y_S, z_{\mathrm{INT}}-0)}, \end{aligned} \tag{1.78}$$

where $z_S = z_{\mathrm{INT}} - 0$ and $z_S = z_{\mathrm{INT}} + 0$ stand for the sides of the plane $z_S = z_{\mathrm{INT}}$ facing the half-spaces $z_S < z_{\mathrm{INT}}$ and $z_S > z_{\mathrm{INT}}$ respectively (or for corresponding planes infinitely close to the plane $z_S = z_{\mathrm{INT}}$), are independent of x_S and y_S and can be expressed as follows:

$$t_{\mathrm{pp}} = \frac{2n_1 n_2 \cos\beta_{\mathrm{inc}}}{n_1\sqrt{n_2^2 - n_1^2 \sin^2\beta_{\mathrm{inc}}} + n_2^2 \cos\beta_{\mathrm{inc}}}, \tag{1.79}$$

$$t_{\mathrm{ss}} = \frac{2n_1 \cos\beta_{\mathrm{inc}}}{n_1 \cos\beta_{\mathrm{inc}} + \sqrt{n_2^2 - n_1^2 \sin^2\beta_{\mathrm{inc}}}}, \tag{1.80}$$

$$r_{\mathrm{pp}} = -\frac{n_1\sqrt{n_2^2 - n_1^2 \sin^2\beta_{\mathrm{inc}}} - n_2^2 \cos\beta_{\mathrm{inc}}}{n_1\sqrt{n_2^2 - n_1^2 \sin^2\beta_{\mathrm{inc}}} + n_2^2 \cos\beta_{\mathrm{inc}}}, \tag{1.81}$$

$$r_{\mathrm{ss}} = \frac{n_1 \cos\beta_{\mathrm{inc}} - \sqrt{n_2^2 - n_1^2 \sin^2\beta_{\mathrm{inc}}}}{n_1 \cos\beta_{\mathrm{inc}} + \sqrt{n_2^2 - n_1^2 \sin^2\beta_{\mathrm{inc}}}}. \tag{1.82}$$

The quantities t_{pp}, t_{ss}, r_{pp}, and r_{ss} are called *the amplitude transmission* and *reflection coefficients*. Expressions (1.79)–(1.82) are *the Fresnel formulas* written in a special form.

In the case under consideration (nonabsorbing media, $n_1 < n_2$), the coefficients t_{pp}, t_{ss}, r_{pp}, and r_{ss} have real values at any β_{inc}. At $\beta_{inc} \neq 0$, the amount of the reflected light and that of the transmitted light depend on the polarization state of the incident light.

Transmissivity and Reflectivity of the Interface

Let $E^{(inc)}(z_{INT} - 0)$ be the irradiance produced by the incident wave on the plane $z_S = z_{INT} - 0$, $E^{(ref)}(z_{INT} - 0)$ the irradiance produced by the reflected wave on the same plane, and $E^{(tr)}(z_{INT} + 0)$ the irradiance produced by the transmitted wave on the plane $z_S = z_{INT} + 0$ (note that we deal here with another kind of irradiance than FEFD irradiance used in Section 1.1.2; see Sections 5.2, 5.4.2, and 8.5). The quantities

$$T_I \equiv \frac{E^{(tr)}(z_{INT} + 0)}{E^{(inc)}(z_{INT} - 0)} \text{ and } R_I \equiv \frac{E^{(ref)}(z_{INT} - 0)}{E^{(inc)}(z_{INT} - 0)} \quad (1.83)$$

are called respectively the *transmissivity* and *reflectivity* of the interface. In the case under consideration, the irradiances entering into (1.83) can be expressed as follows:

$$E^{(inc)}(z_{INT} - 0) = \frac{cn_1 \cos\beta_{inc}}{8\pi}\left(\left|A_p^{(inc)}(x_S, y_S, z_{INT} - 0)\right|^2 + \left|A_s^{(inc)}(x_S, y_S, z_{INT} - 0)\right|^2\right), \quad (1.84a)$$

$$E^{(ref)}(z_{INT} - 0) = \frac{cn_1 \cos\beta_{inc}}{8\pi}\left(\left|A_p^{(ref)}(x_S, y_S, z_{INT} - 0)\right|^2 + \left|A_s^{(ref)}(x_S, y_S, z_{INT} - 0)\right|^2\right), \quad (1.84b)$$

$$E^{(tr)}(z_{INT} + 0) = \frac{cn_2 \cos\beta_{tr}}{8\pi}\left(\left|A_p^{(tr)}(x_S, y_S, z_{INT} + 0)\right|^2 + \left|A_s^{(tr)}(x_S, y_S, z_{INT} + 0)\right|^2\right) \quad (1.84c)$$

at arbitrary x_S and y_S. Using the above formulas, it is easy to find that if the incident wave is p-polarized,

$$T_I = T_{pp} \equiv \frac{n_2 \cos\beta_{tr}}{n_1 \cos\beta_{inc}}|t_{pp}|^2, \quad (1.85a)$$

$$R_I = R_{pp} \equiv |r_{pp}|^2 \quad (1.85b)$$

and, if the incident wave is s-polarized,

$$T_I = T_{ss} \equiv \frac{n_2 \cos\beta_{tr}}{n_1 \cos\beta_{inc}}|t_{ss}|^2, \quad (1.86a)$$

$$R_I = R_{ss} \equiv |r_{ss}|^2. \quad (1.86b)$$

Here we have denoted the transmissivities and reflectivities of the interface for a p-polarized incident wave by T_{pp} and R_{pp} and those for an s-polarized incident wave by T_{ss} and R_{ss}. As an illustration, Figure 1.6 shows the dependences of these transmissivities and reflectivities on the angle of incidence β_{inc} at $n_1 = 1$ (vacuum or air) and $n_2 = 1.5$ (e.g., glass).

At any polarization of the incident wave and at any β_{inc},

$$T_I + R_I = 1. \quad (1.87)$$

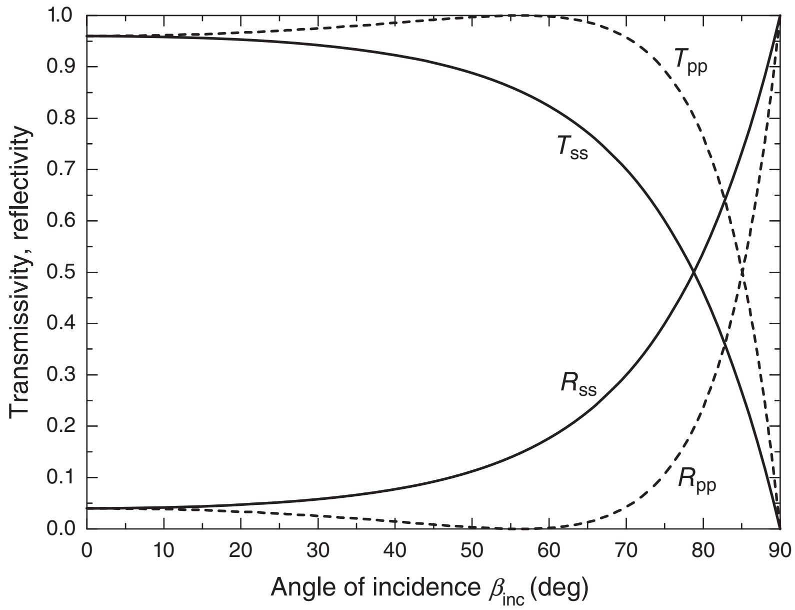

Figure 1.6 Transmissivities T_{pp} and T_{ss} and reflectivities R_{pp} and R_{ss} versus the angle of incidence β_{inc} at $n_1 = 1$ and $n_2 = 1.5$

The Brewster Angle

The angle

$$\beta_B = \arctan \frac{n_2}{n_1} \tag{1.88}$$

is called the *polarizing* or *Brewster angle*. As can be seen from (1.81), at $\beta_{inc} = \beta_B$ the coefficient r_{pp} is equal to zero, as is the reflectivity R_{pp} [see (1.85b)]. If $\beta_{inc} = \beta_B$, whatever the polarization of the incident wave, the reflected wave will be s-polarized. In the example illustrated by Figure 1.6 ($n_1 = 1$ and $n_2 = 1.5$), $\beta_B \approx 56.3°$.

The Case $n_1 > n_2$. Critical Angle

So far it has been assumed that $n_1 < n_2$. All the formulas presented above for the case $n_1 < n_2$ are also valid in the case $n_1 > n_2$ for $\beta_{inc} < \beta_c$, where

$$\beta_c = \arcsin \left(\frac{n_2}{n_1} \right) \tag{1.89}$$

is *the critical angle of total internal reflection*. At $\beta_{inc} > \beta_c$, in contrast to the case $\beta_{inc} < \beta_c$, the vector $\mathbf{m}_{tr}$ will be complex and have nonparallel real and imaginary parts [from (1.74) and (1.75) it is easy to see that $\mathrm{Re}(\mathbf{m}_{tr})$ and $\mathrm{Im}(\mathbf{m}_{tr})$ will be parallel to $\boldsymbol{L}$ and $\boldsymbol{N}$, respectively], that is, the transmitted wave will be *inhomogeneous* (see Section 8.1.2). In this case, decomposing the field $\mathbf{E}_{tr}$ [see (1.68)], we can use the same real vector $\mathbf{e}_{\mathrm{s}}^{(\mathrm{tr})}$ as in the above cases but cannot use a real vector $\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}$ since with a real $\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}$ $\mathbf{E}_{\mathrm{tr}}$ will not meet (1.10). To satisfy (1.10), one can take the following vector $\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}$:

$$\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})} = \frac{1}{\sqrt{\mathbf{m}_{\mathrm{tr}}\mathbf{m}_{\mathrm{tr}}^{*}}}\mathbf{e}_{\mathrm{s}}^{(\mathrm{tr})} \times \mathbf{m}_{\mathrm{tr}} \tag{1.90}$$

with $\mathbf{e}_{\mathrm{s}}^{(\mathrm{tr})}$ being chosen the same as in the previous cases (i.e., real, unit, and oriented as shown in Figure 1.5). The vector $\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}$ given by (1.90) is such that $\mathbf{m}_{\mathrm{tr}}\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})} = 0$, which is necessary for (1.10) to be satisfied, and unit in the sense that $\sqrt{\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})*}} = 1$. With the choice of $\mathbf{e}_{\mathrm{p}}^{(\mathrm{inc})}$, $\mathbf{e}_{\mathrm{s}}^{(\mathrm{inc})}$, $\mathbf{e}_{\mathrm{p}}^{(\mathrm{ref})}$, $\mathbf{e}_{\mathrm{s}}^{(\mathrm{ref})}$, and $\mathbf{e}_{\mathrm{s}}^{(\mathrm{tr})}$ as in Figure 1.5 and $\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}$ as in (1.90) [note that the vector $\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})}$ used above in the case of real $\mathbf{m}_{\mathrm{tr}}$ satisfies (1.90)], expressions (1.80)–(1.82) for the coefficients t_{ss}, r_{pp}, and r_{ss} remain valid in the case $\beta_{\mathrm{inc}} > \beta_{\mathrm{c}}$ (but these coefficients become complex), while the expression for t_{pp} takes a more general form, namely,

$$t_{\mathrm{pp}} = C_{n2}\frac{2n_1 n_2 \cos\beta_{\mathrm{inc}}}{n_1\sqrt{n_2^2 - n_1^2\sin^2\beta_{\mathrm{inc}}} + n_2^2\cos\beta_{\mathrm{inc}}}, \tag{1.91}$$

where

$$C_{n2} = \sqrt{C_{\beta n2}^{*}C_{\beta n2} + S_{\beta n2}^{*}S_{\beta n2}} \tag{1.92}$$

with

$$C_{\beta n2} = \frac{1}{n_2}\sqrt{n_2^2 - n_1^2\sin^2\beta_{\mathrm{inc}}}, \quad S_{\beta n2} = \left(\frac{n_1}{n_2}\right)\sin\beta_{\mathrm{inc}}.$$

As seen from these formulas, at $\beta_{\mathrm{inc}} < \beta_{\mathrm{c}}$, $C_{n2} = 1$ and expression (1.91) becomes identical to (1.79).

Total Internal Reflection (TIR)

In the case $\beta_{\mathrm{inc}} > \beta_{\mathrm{c}}$, it is convenient to rewrite expressions (1.81) and (1.82) as follows:

$$r_{\mathrm{pp}} = -\frac{in_1\sqrt{n_1^2\sin^2\beta_{\mathrm{inc}} - n_2^2} - n_2^2\cos\beta_{\mathrm{inc}}}{in_1\sqrt{n_1^2\sin^2\beta_{\mathrm{inc}} - n_2^2} + n_2^2\cos\beta_{\mathrm{inc}}}, \tag{1.93}$$

$$r_{\mathrm{ss}} = \frac{n_1\cos\beta_{\mathrm{inc}} - i\sqrt{n_1^2\sin^2\beta_{\mathrm{inc}} - n_2^2}}{n_1\cos\beta_{\mathrm{inc}} + i\sqrt{n_1^2\sin^2\beta_{\mathrm{inc}} - n_2^2}}. \tag{1.94}$$

It is easy to see from (1.93) and (1.94) that $|r_{\mathrm{pp}}|=|r_{\mathrm{ss}}|=1$. Since, as before, the incident and reflected waves are assumed to be homogeneous and the medium where they propagate to be nonabsorbing, expressions (1.84a) and (1.84b) and hence (1.85b) and (1.86b) remain applicable. According to (1.85b) and (1.86b), when $|r_{\mathrm{pp}}|=|r_{\mathrm{ss}}|=1$, $R_{\mathrm{pp}} = R_{\mathrm{ss}} = 1$, that is, *total reflection* takes place. Expression (1.84c) is not applicable when $\beta_{\mathrm{inc}} > \beta_{\mathrm{c}}$ because in this case the transmitted wave is inhomogeneous. One can show that at $\beta_{\mathrm{inc}} > \beta_{\mathrm{c}}$, $E^{(\mathrm{tr})} = 0$ and consequently $T_{\mathrm{pp}} = T_{\mathrm{ss}} = 0$ (although t_{pp} and t_{ss} are different from zero). Even at small deviations β_{inc} from β_{c} and n_2 from n_1, the transmitted wave, having an imaginary $\sigma_{\mathrm{tr}} = i\sqrt{n_1^2\sin^2\beta_{\mathrm{inc}} - n_2^2}$, has an appreciable amplitude only near the interface. Such waves are called *surface* or *evanescent waves*.

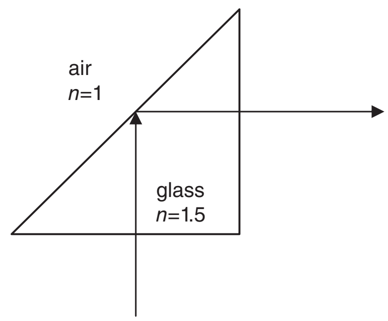

Figure 1.7 A prism reflector using the TIR phenomenon

At $\beta_{inc} > \beta_c$, r_{pp} and r_{ss}, being complex, are different in phase and the difference of the phases of r_{pp} and r_{ss} gradually changes with β_{inc}. This means that the phase shifts introduced into the p- and s-components of the reflected wave at reflection are different and that the difference of these phase shifts (and hence the shape of the polarization ellipse of the reflected light) can be controlled by choosing β_{inc}. The latter is used in polarization-transforming devices such as the Fresnel rhomb.

For a glass–air interface with $n_1 = 1.5$ and $n_2 = 1$, $\beta_c \approx 41.8°$. Therefore a right-angle glass prism can be used as a high-efficiency reflector as shown in Figure 1.7. Such a reflector may be almost lossless provided that the entrance and exit surfaces have antireflection coatings. The TIR phenomenon is used in many kinds of optical elements and devices. It is the principle of waveguides and optical fibers. In liquid crystal display applications, TIR is exploited in elements of backlight units, in projection systems, in beam steering, and so on. In Section 4.3, we will deal with an application of the TIR phenomenon in the intensity-modulating unit of an LCD.

Incidence of a Homogeneous Wave from a Nonabsorbing Medium on an Absorbing One

Formulas (1.80)–(1.82) and (1.91) can also be used for calculating the amplitude transmission and reflection coefficients in the case when the second medium is absorbing; in this case, n_2 is assumed to be complex. These formulas correspond to the choice of the vectors $\mathbf{e}_p^{(inc)}$, $\mathbf{e}_s^{(inc)}$, $\mathbf{e}_p^{(ref)}$, $\mathbf{e}_s^{(ref)}$, $\mathbf{e}_p^{(tr)}$, and $\mathbf{e}_s^{(tr)}$ in accordance with the same rules that were just used in the case of TIR. The transmitted wave in the absorbing medium will be inhomogeneous at any nonzero β_{inc} and has nonzero $\mathrm{Re}\,\sigma_{tr}$ and $\mathrm{Im}\,\sigma_{tr}$ at any β_{inc}.

1.2.2 Reflection and Transmission Jones Matrices for a Plane Interface between Isotropic Media

In all the above cases, the interaction of the incident light with the interface can be described by the relations

$$\tilde{\boldsymbol{J}}^{(tr)}(x_S, y_S, z_{INT} + 0) = \tilde{\boldsymbol{t}}_I \tilde{\boldsymbol{J}}^{(inc)}(x_S, y_S, z_{INT} - 0), \tag{1.95}$$

$$\tilde{\boldsymbol{J}}^{(ref)}(x_S, y_S, z_{INT} - 0) = \tilde{\boldsymbol{r}}_I \tilde{\boldsymbol{J}}^{(inc)}(x_S, y_S, z_{INT} - 0), \tag{1.96}$$

where

$$\tilde{\boldsymbol{J}}^{(inc)} = \begin{pmatrix} A_p^{(inc)} \\ A_s^{(inc)} \end{pmatrix}, \quad \tilde{\boldsymbol{J}}^{(tr)} = \begin{pmatrix} A_p^{(tr)} \\ A_s^{(tr)} \end{pmatrix}, \quad \tilde{\boldsymbol{J}}^{(ref)} = \begin{pmatrix} A_p^{(ref)} \\ A_s^{(ref)} \end{pmatrix} \tag{1.97}$$

are Jones vectors of the incident, transmitted, and reflected waves, and

$$\tilde{\boldsymbol{t}}_{\mathrm{I}} = \begin{pmatrix} t_{\mathrm{pp}} & 0 \\ 0 & t_{\mathrm{ss}} \end{pmatrix}, \quad \tilde{\boldsymbol{r}}_{\mathrm{I}} = \begin{pmatrix} r_{\mathrm{pp}} & 0 \\ 0 & r_{\mathrm{ss}} \end{pmatrix} \tag{1.98}$$

are the transmission and reflection Jones matrices of the interface corresponding to the representation (1.97) of the Jones vectors. The vectors $\tilde{\boldsymbol{J}}^{(\mathrm{inc})}$ and $\tilde{\boldsymbol{J}}^{(\mathrm{ref})}$ in all the considered cases as well as the vector $\tilde{\boldsymbol{J}}^{(\mathrm{tr})}$ when it characterizes a homogeneous wave are Jones vectors of the same kind as the vector $\tilde{\boldsymbol{J}}$ considered in Section 1.1.2. It is clear that the presented variant of transmission and reflection Jones matrices for the interface is not unique. Other kinds and representations of Jones matrices for interfaces may be more suitable in solving particular problems. For example, when considering transmission and reflection at an interface between nonabsorbing media in a situation where the waves in both media are homogeneous, it may be convenient to deal with the transmission and reflection matrices corresponding to the following Jones vectors:

$$\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})} = \sqrt{2n_1 \cos\beta_{\mathrm{inc}}}\,\tilde{\boldsymbol{J}}^{(\mathrm{inc})}, \quad \tilde{\boldsymbol{J}}_F^{(\mathrm{ref})} = \sqrt{2n_1 \cos\beta_{\mathrm{inc}}}\,\tilde{\boldsymbol{J}}^{(\mathrm{ref})}, \quad \tilde{\boldsymbol{J}}_F^{(\mathrm{tr})} = \sqrt{2n_2 \cos\beta_{\mathrm{tr}}}\,\tilde{\boldsymbol{J}}^{(\mathrm{tr})}. \tag{1.99}$$

We denote these Jones matrices by $\tilde{\boldsymbol{t}}_{\mathrm{I}(F)}$ and $\tilde{\boldsymbol{r}}_{\mathrm{I}(F)}$. From (1.95), (1.96) and the relations

$$\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}(x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{INT}}+0) = \tilde{\boldsymbol{t}}_{\mathrm{I}(F)}\,\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{INT}}-0), \tag{1.100}$$

$$\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})}(x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{INT}}-0) = \tilde{\boldsymbol{r}}_{\mathrm{I}(F)}\,\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{INT}}-0), \tag{1.101}$$

it follows that

$$\tilde{\boldsymbol{t}}_{\mathrm{I}(F)} = \frac{\sqrt{n_2 \cos\beta_{\mathrm{tr}}}}{\sqrt{n_1 \cos\beta_{\mathrm{inc}}}}\tilde{\boldsymbol{t}}_{\mathrm{I}}, \quad \tilde{\boldsymbol{r}}_{\mathrm{I}(F)} = \tilde{\boldsymbol{r}}_{\mathrm{I}}. \tag{1.102}$$

According to (1.84), (1.97), and (1.99), the irradiances $E^{(\mathrm{inc})}$, $E^{(\mathrm{ref})}$, and $E^{(\mathrm{tr})}$can be expressed as follows:

$$\begin{aligned}
E^{(\mathrm{inc})} &= \frac{cn_1 \cos\beta_{\mathrm{inc}}}{8\pi}\tilde{\boldsymbol{J}}^{(\mathrm{inc})\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{inc})} = \frac{c}{16\pi}\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}, \\
E^{(\mathrm{ref})} &= \frac{cn_1 \cos\beta_{\mathrm{inc}}}{8\pi}\tilde{\boldsymbol{J}}^{(\mathrm{ref})\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{ref})} = \frac{c}{16\pi}\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})}, \\
E^{(\mathrm{tr})} &= \frac{cn_2 \cos\beta_{\mathrm{tr}}}{8\pi}\tilde{\boldsymbol{J}}^{(\mathrm{tr})\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{tr})} = \frac{c}{16\pi}\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}.
\end{aligned} \tag{1.103}$$

Substitution of these expressions into (1.83) gives the following expressions for the transmissivity T_{I} and reflectivity R_{I} of the interface in terms of the Jones vectors:

$$T_{\mathrm{I}} = \left(\frac{n_2 \cos\beta_{\mathrm{tr}}}{n_1 \cos\beta_{\mathrm{inc}}}\right)\frac{\tilde{\boldsymbol{J}}^{(\mathrm{tr})}(\mathbf{r}_{\mathrm{INT}}^{+})^{\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{tr})}(\mathbf{r}_{\mathrm{INT}}^{+})}{\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})^{\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})} = \frac{\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}(\mathbf{r}_{\mathrm{INT}}^{+})^{\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}(\mathbf{r}_{\mathrm{INT}}^{+})}{\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})^{\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})}, \tag{1.104}$$

$$R_{\mathrm{I}} = \frac{\tilde{\boldsymbol{J}}^{(\mathrm{ref})}(\mathbf{r}_{\mathrm{INT}}^{-})^{\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{ref})}(\mathbf{r}_{\mathrm{INT}}^{-})}{\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})^{\dagger}\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})} = \frac{\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})}(\mathbf{r}_{\mathrm{INT}}^{-})^{\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})}(\mathbf{r}_{\mathrm{INT}}^{-})}{\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})^{\dagger}\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})}, \tag{1.105}$$

where $\mathbf{r}_{\mathrm{INT}}^{-} = (x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{INT}} - 0)$ and $\mathbf{r}_{\mathrm{INT}}^{+} = (x_{\mathrm{S}}, y_{\mathrm{S}}, z_{\mathrm{INT}} + 0)$. Defining the length $|\tilde{\boldsymbol{J}}|$ of a Jones vector $\tilde{\boldsymbol{J}}$ as

$$|\tilde{\boldsymbol{J}}| \equiv \sqrt{\tilde{\boldsymbol{J}}^{\dagger}\tilde{\boldsymbol{J}}}, \tag{1.106}$$

we can rewrite expressions (1.104) and (1.105) in the following form:

$$T_{\mathrm{I}} = \left(\frac{n_2 \cos \beta_{\mathrm{tr}}}{n_1 \cos \beta_{\mathrm{inc}}}\right) \frac{\left|\tilde{\boldsymbol{J}}^{(\mathrm{tr})}(\mathbf{r}_{\mathrm{INT}}^{+})\right|^2}{\left|\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})\right|^2} = \frac{\left|\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}(\mathbf{r}_{\mathrm{INT}}^{+})\right|^2}{\left|\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})\right|^2}, \tag{1.107}$$

$$R_{\mathrm{I}} = \frac{\left|\tilde{\boldsymbol{J}}^{(\mathrm{ref})}(\mathbf{r}_{\mathrm{INT}}^{-})\right|^2}{\left|\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})\right|^2} = \frac{\left|\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})}(\mathbf{r}_{\mathrm{INT}}^{-})\right|^2}{\left|\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})\right|^2}. \tag{1.108}$$

Denote a polarization Jones vector of the incident wave in the basis $(\mathbf{e}_{\mathrm{p}}^{(\mathrm{inc})}, \mathbf{e}_{\mathrm{s}}^{(\mathrm{inc})})$ by $\boldsymbol{j}^{(\mathrm{inc})}$. By definition, the vectors $\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})$ and $\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})$ are related to $\boldsymbol{j}^{(\mathrm{inc})}$ as follows:

$$\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-}) = a(\mathbf{r}_{\mathrm{INT}}^{-})\boldsymbol{j}^{(\mathrm{inc})}, \quad \tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-}) = a_F(\mathbf{r}_{\mathrm{INT}}^{-})\boldsymbol{j}^{(\mathrm{inc})}, \tag{1.109}$$

where $a(\mathbf{r}_{\mathrm{INT}}^{-})$ and $a_F(\mathbf{r}_{\mathrm{INT}}^{-})$ are scalar factors. Substitution from (1.109) into (1.95), (1.96), (1.100), and (1.101) gives expressions for the Jones vectors of the transmitted and reflected waves in terms of $\boldsymbol{j}^{(\mathrm{inc})}$. Substituting these expressions into (1.107) and (1.108) and using the fact that $|\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})|^2 = |a(\mathbf{r}_{\mathrm{INT}}^{-})|^2$ and $|\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(\mathbf{r}_{\mathrm{INT}}^{-})|^2 = |a_F(\mathbf{r}_{\mathrm{INT}}^{-})|^2$, we obtain the following expressions for the transmissivity and reflectivity: in terms of $\tilde{\boldsymbol{t}}_{\mathrm{I}}$ and $\tilde{\boldsymbol{r}}_{\mathrm{I}}$,

$$T_{\mathrm{I}} = \left(\frac{n_2 \cos \beta_{\mathrm{tr}}}{n_1 \cos \beta_{\mathrm{inc}}}\right) \left|\tilde{\boldsymbol{t}}_{\mathrm{I}}\boldsymbol{j}^{(\mathrm{inc})}\right|^2, \tag{1.110}$$

$$R_{\mathrm{I}} = \left|\tilde{\boldsymbol{r}}_{\mathrm{I}}\boldsymbol{j}^{(\mathrm{inc})}\right|^2 \tag{1.111}$$

and, in terms of $\tilde{\boldsymbol{t}}_{\mathrm{I}(F)}$ and $\tilde{\boldsymbol{r}}_{\mathrm{I}(F)}$,

$$T_{\mathrm{I}} = \left|\tilde{\boldsymbol{t}}_{\mathrm{I}(F)}\boldsymbol{j}^{(\mathrm{inc})}\right|^2, \tag{1.112}$$

$$R_{\mathrm{I}} = \left|\tilde{\boldsymbol{r}}_{\mathrm{I}(F)}\boldsymbol{j}^{(\mathrm{inc})}\right|^2. \tag{1.113}$$

Employing the Jones vectors and matrices labeled by the subscript $_F$, we include all the information required for finding T_{I}, apart from that contained in $\boldsymbol{j}^{(\mathrm{inc})}$, in the Jones matrix and can use the unified and algebraically simplest expressions for calculating the transmissivity and reflectivity from the corresponding Jones matrices. Note that we could introduce the vectors $\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}$, $\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}$, and $\tilde{\boldsymbol{J}}_F^{(\mathrm{ref})}$ as

$$\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})} = \begin{pmatrix} A_{\mathrm{p}}^{(\mathrm{inc})} \\ A_{\mathrm{s}}^{(\mathrm{inc})} \end{pmatrix}, \quad \tilde{\boldsymbol{J}}_F^{(\mathrm{tr})} = \begin{pmatrix} A_{\mathrm{p}}^{(\mathrm{tr})} \\ A_{\mathrm{s}}^{(\mathrm{tr})} \end{pmatrix}, \quad \tilde{\boldsymbol{J}}_F^{(\mathrm{ref})} = \begin{pmatrix} A_{\mathrm{p}}^{(\mathrm{ref})} \\ A_{\mathrm{s}}^{(\mathrm{ref})} \end{pmatrix} \tag{1.114}$$

[see (1.68)] by adopting the following normalization conditions for the basis vibration vectors:

$$\mathbf{e}_{\mathrm{p}}^{(\mathrm{inc})*}\mathbf{e}_{\mathrm{p}}^{(\mathrm{inc})} = \mathbf{e}_{\mathrm{s}}^{(\mathrm{inc})*}\mathbf{e}_{\mathrm{s}}^{(\mathrm{inc})} = \mathbf{e}_{\mathrm{p}}^{(\mathrm{ref})*}\mathbf{e}_{\mathrm{p}}^{(\mathrm{ref})} = \mathbf{e}_{\mathrm{s}}^{(\mathrm{ref})*}\mathbf{e}_{\mathrm{s}}^{(\mathrm{ref})} = \frac{1}{2n_1 \cos\beta_{\mathrm{inc}}}, \tag{1.115}$$

$$\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})*}\mathbf{e}_{\mathrm{p}}^{(\mathrm{tr})} = \mathbf{e}_{\mathrm{s}}^{(\mathrm{tr})*}\mathbf{e}_{\mathrm{s}}^{(\mathrm{tr})} = \frac{1}{2n_2 \cos\beta_{\mathrm{tr}}}. \tag{1.116}$$

Special normalizations of the basis vibration vectors, like this one, able to simplify a problem are considered in Chapters 8–12.

1.3 Wave Propagation in Anisotropic Media

Needless to say, the propagation of electromagnetic waves in optically anisotropic (birefringent) media and transmission characteristics of anisotropic layers are extremely important subjects to LCD optics. These subjects are considered in detail in Chapters 8 and 9, where we discuss rigorous methods of optics of stratified media applicable to both isotropic and anisotropic media. In the present section, we want to give an overview of basic features of light propagation in anisotropic media and shortly discuss transmission properties of anisotropic layers at normal incidence of light. The latter is directly concerned with the classical Jones matrix method (CJMM). In this section and almost everywhere in this book, we restrict our attention to anisotropic media that are nonmagnetic and nongyrotropic in the optical region.

1.3.1 Wave Equations

The basic difference of anisotropic media from isotropic ones from the standpoint of the Maxwell electromagnetic theory lies in relation between the electric field strength vector **E** and the electric displacement vector **D** (see Section 8.1.1). In the case of an arbitrary nongyrotropic medium, the vector **D** can be expressed in terms of the vector **E** as follows:

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}, \tag{1.117}$$

where $\boldsymbol{\varepsilon}$ is the permittivity tensor, $\boldsymbol{\varepsilon}$ being symmetric ($\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes the matrix transposition). If the medium is isotropic, the tensor $\boldsymbol{\varepsilon}$ can be represented as $\boldsymbol{\varepsilon} = \varepsilon\mathbf{U}$, where ε is a scalar (the permittivity coefficient) and **U** is the unit matrix. This, in particular, means that **D** is parallel to **E** and that the ratio $|\mathbf{D}|/|\mathbf{E}|$ is independent of the direction of **E**. In the case of an anisotropic medium, the representation $\boldsymbol{\varepsilon} = \varepsilon\mathbf{U}$ is not applicable, **D** and **E** may be unparallel, and the ratio $|\mathbf{D}|/|\mathbf{E}|$ depends on the **E** direction.

An analogue of equation (1.8) for the case of a homogeneous anisotropic medium is

$$\nabla \times (\nabla \times \mathbf{E}) - k_0^2 \boldsymbol{\varepsilon}\mathbf{E} = \widehat{\mathbf{0}}. \tag{1.118}$$

The wave vectors and vibration modes of the electric field of plane waves that can exist inside the anisotropic medium—such waves are called *natural waves*, *eigenwaves*, or *proper waves*—can be found from the equation

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + k_0^2 \boldsymbol{\varepsilon}\mathbf{E} = \widehat{\mathbf{0}} \tag{1.119}$$

which can be obtained by substituting (1.1) into (1.118). It is convenient to rewrite this equation in terms of the refraction vector $\mathbf{m} = \mathbf{k}/k_0$ and electric vibration vector $\mathbf{e}$ [$\mathbf{E}(\mathbf{r},t) = \mathbf{e}A(\mathbf{r},t)$, see (8.38) and definitions in Section 8.1.2]:

$$\mathbf{m} \times (\mathbf{m} \times \mathbf{e}) + \varepsilon\mathbf{e} = \hat{\mathbf{0}}. \qquad (1.120)$$

This equation can be written in the following form:

$$\mathbf{Q_E}\mathbf{e} = \hat{\mathbf{0}}. \qquad (1.121)$$

The matrix $\mathbf{Q_E}$ is expressed in terms of the elements of

$$\mathbf{m} \equiv \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \text{ and } \varepsilon \equiv \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}$$

as follows:

$$\mathbf{Q_E} = \mathbf{Q_m} + \varepsilon = \begin{pmatrix} \varepsilon_{11} - m_2^2 - m_3^2 & \varepsilon_{12} + m_1 m_2 & \varepsilon_{13} + m_1 m_3 \\ \varepsilon_{12} + m_1 m_2 & \varepsilon_{22} - m_1^2 - m_3^2 & \varepsilon_{23} + m_2 m_3 \\ \varepsilon_{13} + m_1 m_3 & \varepsilon_{23} + m_2 m_3 & \varepsilon_{33} - m_1^2 - m_2^2 \end{pmatrix}, \qquad (1.122)$$

where

$$\mathbf{Q_m} = \begin{pmatrix} -m_2^2 - m_3^2 & m_1 m_2 & m_1 m_3 \\ m_1 m_2 & -m_1^2 - m_3^2 & m_2 m_3 \\ m_1 m_3 & m_2 m_3 & -m_1^2 - m_2^2 \end{pmatrix}. \qquad (1.123)$$

In some cases, it is simpler to use the following form of equation (1.120):

$$\mathbf{Q_D}\mathbf{d} = \hat{\mathbf{0}}, \qquad (1.124)$$

where $\mathbf{d} = \varepsilon\mathbf{e}$ is the displacement vibration vector [$\mathbf{D}(\mathbf{r},t) = \mathbf{d}A(\mathbf{r},t)$, see (8.38)], and

$$\mathbf{Q_D} = \mathbf{Q_E}\varepsilon^{-1} = \mathbf{Q_m}\varepsilon^{-1} + \mathbf{U}. \qquad (1.125)$$

The vector $\mathbf{d}$ (as well as $\mathbf{D}$) of a plane wave is always orthogonal to its refraction vector $\mathbf{m}$ in the sense that

$$\mathbf{m} \cdot \mathbf{d} = 0, \qquad (1.126)$$

as it follows from the Maxwell equation $\nabla\mathbf{D} = 0$. According to (1.120),

$$\mathbf{d} = -\mathbf{m} \times (\mathbf{m} \times \mathbf{e}) = \mathbf{e}(\mathbf{m} \cdot \mathbf{m}) - \mathbf{m}(\mathbf{m} \cdot \mathbf{e}),$$

that is, the vector $\mathbf{d}$ is a linear combination of the vectors $\mathbf{e}$ and $\mathbf{m}$. If the wave is homogeneous and linearly polarized, this means simply that the vectors $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{m}$ are coplanar.

Equations (1.120) and (1.121) have a nontrivial solution only if

$$\det \mathbf{Q_E} = 0. \tag{1.127}$$

This condition can also be written as

$$\det \mathbf{Q_D} = 0. \tag{1.128}$$

From (1.127) or (1.128), the refraction vectors of natural waves are found.

In the next two sections we will consider some situations when the above equations are readily solved.

1.3.2 Waves in a Uniaxial Layer

In the case of a uniaxial medium with optic axis parallel to a unit vector **c**, the tensor ε can be represented as

$$\boldsymbol{\varepsilon} \equiv \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \varepsilon_\perp + \Delta\varepsilon c_1^2 & \Delta\varepsilon c_1 c_2 & \Delta\varepsilon c_1 c_3 \\ \Delta\varepsilon c_1 c_2 & \varepsilon_\perp + \Delta\varepsilon c_2^2 & \Delta\varepsilon c_2 c_3 \\ \Delta\varepsilon c_1 c_3 & \Delta\varepsilon c_2 c_3 & \varepsilon_\perp + \Delta\varepsilon c_3^2 \end{pmatrix}, \quad \Delta\varepsilon = \varepsilon_{||} - \varepsilon_\perp, \tag{1.129}$$

where $\varepsilon_\|$ and $\varepsilon_\perp$ are the principal permittivities of the medium ($\mathbf{D} = \varepsilon_\|\mathbf{E}$ if $\mathbf{E}\|\mathbf{c}$, and $\mathbf{D} = \varepsilon_\perp\mathbf{E}$ if $\mathbf{E}\perp\mathbf{c}$), and c_j (j = 1,2,3) are the elements of the vector $\mathbf{c} \equiv \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. The principal permittivities are related to the principal refractive indices of the medium, $n_\|$ and $n_\perp$, by

$$\varepsilon_{||} = n_{||}^2, \quad \varepsilon_\perp = n_\perp^2. \tag{1.130}$$

Ordinary and Extraordinary Waves

Natural waves in uniaxial media are divided into two classes: *ordinary waves* and *extraordinary waves*. The refraction vectors of the *ordinary* waves are independent of the optic axis orientation and satisfy the equation $\mathbf{m}\cdot\mathbf{m} = \varepsilon_\perp$. The refraction vectors of the *extraordinary* waves depend on the optic axis orientation and meet the equation $\mathbf{m}\cdot(\boldsymbol{\varepsilon}\mathbf{m}) = \varepsilon_\perp\varepsilon_\|$ (see Section 9.3). Let $\mathbf{m}_\mathrm{o}$ and $\mathbf{e}_\mathrm{o}$ be the refraction vector and an electric vibration vector of an *ordinary wave*, and let $\mathbf{m}_\mathrm{e}$ and $\mathbf{e}_\mathrm{e}$ be those of an extraordinary wave. In general, the vector $\mathbf{e}_\mathrm{o}$ satisfies the conditions $\mathbf{c}\cdot\mathbf{e}_\mathrm{o} = 0$ and $\mathbf{m}_\mathrm{o}\cdot\mathbf{e}_\mathrm{o} = 0$, while the vector $\mathbf{e}_\mathrm{e}$ can be represented as a linear combination of the vectors $\mathbf{m}_\mathrm{e}$ and **c**. If the medium is nonabsorbing and the waves are homogeneous (not evanescent), the vectors $\mathbf{m}_\mathrm{o}$ and $\mathbf{m}_\mathrm{e}$ are real (see Section 9.3). In this case, the electric field of the ordinary wave performs oscillations along a straight line perpendicular to **c** and $\mathbf{m}_\mathrm{o}$, while the electric field of the extraordinary wave vibrates along a straight line parallel to the plane spanned by the vectors $\mathbf{m}_\mathrm{e}$ and **c** (Figure 1.8). For homogeneous waves, the plane containing the wave normal and **c** is referred to as *the principal plane* [1].

If a natural wave is homogeneous, one can associate with it a refractive index (see Section 8.1.2). For a homogeneous wave, the vector **m** can be represented as $\mathbf{m} = n_\mathrm{w}\mathbf{l}$, where **l** is the wave normal and n_w is the refractive index for the wave. We will denote refractive indices for ordinary and extraordinary waves

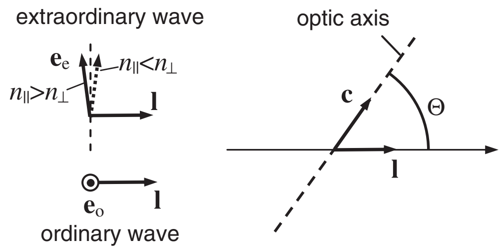

Figure 1.8 Homogeneous natural waves in a nonabsorbing uniaxial medium. **l** is the wave normal

by n_{o} and n_{e}, respectively. The refractive indices of a homogeneous ordinary wave and a homogeneous extraordinary wave can be expressed as

$$n_{\mathrm{o}} = \sqrt{\varepsilon_{\perp}} = n_{\perp}, \tag{1.131}$$

$$n_{\mathrm{e}} = \frac{\sqrt{\varepsilon_{\perp}\varepsilon_{||}}}{\sqrt{\varepsilon_{\perp} + \Delta\varepsilon \cos^2 \Theta}} = \frac{n_{||}n_{\perp}}{\sqrt{n_{||}^2 \cos^2 \Theta + n_{\perp}^2 \sin^2 \Theta}}, \tag{1.132}$$

where Θ is the angle between the wave normal of the extraordinary wave and the optic axis (Figure 1.8). At $\Theta = 90°$, $n_{\mathrm{e}} = n_{||}$. At $\Theta = 0$, $n_{\mathrm{e}} = n_{\perp}$, and the extraordinary wave turns into an ordinary one. Waves propagating along the optic axis ($\mathbf{m}||\mathbf{c}$) can have different polarizations (linear, elliptical, circular) as if the medium were isotropic.

Geometry of the Problem for a Layer

Let us consider a homogeneous uniaxial layer whose boundaries coincide with the planes $z_c = z_1$ and $z_c = z_2$ ($z_2 > z_1$) in a coordinate system (x_c, y_c, z_c) and whose optic axis is parallel to the x_c–z_c plane (Figure 1.9a). In this case, the vector **c** can be represented as

$$\mathbf{c} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}, \tag{1.133}$$

where θ is the angle between the x_c–y_c plane and the vector **c**, and, according to (1.129),

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{\perp} + \Delta\varepsilon\cos^2\theta & 0 & \Delta\varepsilon\cos\theta\sin\theta \\ 0 & \varepsilon_{\perp} & 0 \\ \Delta\varepsilon\cos\theta\sin\theta & 0 & \varepsilon_{\perp} + \Delta\varepsilon\sin^2\theta \end{pmatrix} \tag{1.134}$$

in the system (x_c, y_c, z_c). Let this layer be surrounded by a homogeneous nonabsorbing isotropic medium with refractive index n_1, and let a plane homogeneous wave with refraction vector $\mathbf{m}_{\mathrm{inc}}$ fall on this layer from the half-space $z_c < z_1$. As in Section 1.2.1, we represent the vector $\mathbf{m}_{\mathrm{inc}}$ as

$$\mathbf{m}_{\mathrm{inc}} = \boldsymbol{L}n_1 \sin\beta_{\mathrm{inc}} + \boldsymbol{N}n_1 \cos\beta_{\mathrm{inc}} = \boldsymbol{L}\zeta + \boldsymbol{N}\sigma_{\mathrm{inc}} \tag{1.135}$$

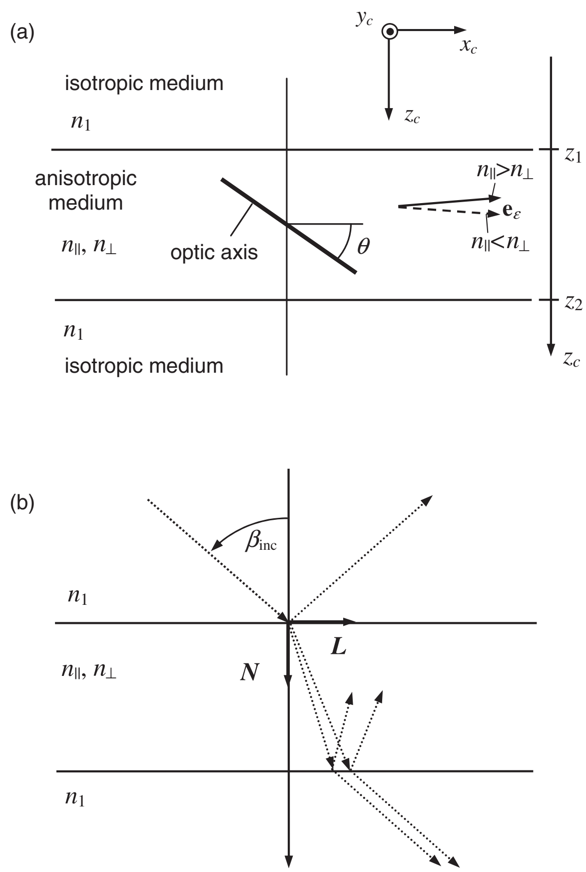

Figure 1.9 Geometry of the problem. The dotted arrows in sketch (b) show the directions of wave normals of the incident and induced waves

[see (1.72)], the unit vectors $\boldsymbol{L}$ and $\boldsymbol{N}$ being oriented as in Figure 1.5 (Figure 1.9b). The symmetry of the problem (see Section 8.1.3) implies that the refraction vector of any of natural waves produced by the incident wave inside or outside the layer will have the form $\mathbf{m} = \boldsymbol{L}\zeta + \boldsymbol{N}\sigma$, where $\zeta = n_1 \sin \beta_{inc}$. In particular, this means that all emergent waves in the half-space $z_c > z_2$, the components of the transmitted field, will have the same refraction vector, which allows considering any combination of these waves as a single plane wave. The same can be said about emergent waves propagating in the half-space $z_c < z_1$.

Normal Incidence

In the case of normal incidence ($\beta_{inc} = 0$), the refraction vectors of the waves propagating inside the layer, being represented in the system (x_c, y_c, z_c), will have the form

$$\mathbf{m} = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}. \tag{1.136}$$

With such **m**,

$$\mathbf{Q_m} = \begin{pmatrix} -\sigma^2 & 0 & 0 \\ 0 & -\sigma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.137}$$

and, according to (1.122) and (1.134),

$$\mathbf{Q_E} = \begin{pmatrix} \varepsilon_\perp + \Delta\varepsilon\cos^2\theta - \sigma^2 & 0 & \Delta\varepsilon\cos\theta\sin\theta \\ 0 & \varepsilon_\perp - \sigma^2 & 0 \\ \Delta\varepsilon\cos\theta\sin\theta & 0 & \varepsilon_\perp + \Delta\varepsilon\sin^2\theta \end{pmatrix}. \tag{1.138}$$

Equation (1.127) is a quartic equation in σ. It is easy to find that the roots of this equation with $\mathbf{Q_E}$ given by (1.138) are

$$\sigma_1 = \frac{\sqrt{\varepsilon_\perp\varepsilon_{||}}}{\sqrt{\varepsilon_\perp + \Delta\varepsilon\sin^2\theta}}, \quad \sigma_2 = \sqrt{\varepsilon_\perp}, \quad \sigma_3 = -\frac{\sqrt{\varepsilon_\perp\varepsilon_{||}}}{\sqrt{\varepsilon_\perp + \Delta\varepsilon\sin^2\theta}}, \quad \sigma_4 = -\sqrt{\varepsilon_\perp}. \tag{1.139}$$

The first two roots correspond to waves propagating in the $+z_c$-direction, and in particular to the waves transmitted through the frontal interface of the layer. The waves reflected from the rear interface will have $\sigma = \sigma_3$ and $\sigma = \sigma_4$. The roots σ_1 and σ_3 correspond to extraordinary waves, and σ_2 and σ_4 to ordinary waves. In the situation under consideration, be the uniaxial medium nonabsorbing or absorbing, the induced natural waves in the layer are homogeneous, which allows one to associate with each of them a refractive index. As seen from (1.139), (1.136), and (1.130), for both ordinary modes, as it must, $n_o = n_\perp$. For both extraordinary modes,

$$n_e = \frac{\sqrt{\varepsilon_\perp\varepsilon_{||}}}{\sqrt{\varepsilon_\perp + \Delta\varepsilon\sin^2\theta}} = \frac{n_{||}n_\perp}{\sqrt{n_{||}^2\sin^2\theta + n_\perp^2\cos^2\theta}}, \tag{1.140}$$

which conforms with (1.132)—in this example, the angle Θ can expressed as $\Theta = 90° - \theta$.

Substituting solutions (1.139) into (1.121), one can check that the electric vibration vectors of the ordinary waves must be chosen parallel to the y_c-axis, while those of the extraordinary waves must be perpendicular to the y_c-axis. It can also be seen that the electric vibration vector of any of the extraordinary waves can be represented as the product of the vector

$$\mathbf{e}_\varepsilon = \begin{pmatrix} \varepsilon_\perp + \Delta\varepsilon\sin^2\theta \\ 0 \\ -\Delta\varepsilon\cos\theta\sin\theta \end{pmatrix} \tag{1.141}$$

[in the system (x_c, y_c, z_c)] and a scalar. Note that the vector $\mathbf{e}_\varepsilon$ is parallel to the x_c-axis only when either $\cos\theta$ or $\sin\theta$ is equal to zero. In any other case, this vector is not perpendicular to the refraction vectors. If the medium is nonabsorbing, the vector $\mathbf{e}_\varepsilon$ is real and the electric fields of the extraordinary modes vibrate along a line parallel to $\mathbf{e}_\varepsilon$. The fact that the vibration direction of the electric field of such a wave is not perpendicular to its refraction vector, in particular, suggests that the direction of energy transfer by the wave—this direction is perpendicular to the electric field vector [see (8.16)]—is different from the direction of the refraction vector. For any homogeneous ordinary wave, the direction of energy transfer coincides with the direction of its refraction vector. A difference of the directions of energy transfer for an ordinary wave and an extraordinary wave having codirectional refraction vectors is a manifestation of the phenomenon of *double refraction* or *birefringence*. A difference in refractive indices for these waves is another manifestation of this phenomenon.

If the uniaxial medium is absorbing and the vector $\mathbf{e}_\varepsilon$ is not parallel to the x_c-axis, the vector $\mathbf{e}_\varepsilon$ is complex and $\mathrm{Re}\,\mathbf{e}_\varepsilon$ is in general not parallel to $\mathrm{Im}\mathbf{e}_\varepsilon$. It implies that the end of the true (real) electric field vector of the wave describes with time an ellipse in the plane parallel to the x_c–z_c plane. In contrast to the elliptically polarized waves considered in Section 1.1, for which the plane of the vibration ellipse is perpendicular to the refraction vector, in this case the vibration ellipse plane is parallel to the refraction vector. Really, we have dealt with waves having a similar polarization in some examples of Section 1.2.1. These are the "p-polarized" waves in the second medium in the cases where these waves are inhomogeneous (TIR mode, absorbing medium at oblique incidence). Such waves cannot be called linearly polarized. At the same time, the term "plane-polarized wave" as applied to them seems acceptable. The linearly polarized waves are also often called plane-polarized. Where convenient, we will also do so.

Thus, if the optic axis is not perpendicular to the layer boundaries, be the layer nonabsorbing or absorbing, all natural waves induced inside it by a normally incident plane wave are plane-polarized. The plane of polarization of the extraordinary waves is the x_c–z_c plane (the principal plane), and that of the ordinary waves is the y_c–z_c plane.

Oblique Incidence

When a plane wave falls obliquely from an isotropic medium on a plane interface with an anisotropic medium, it produces in general two transmitted waves with nonparallel wave normals in the anisotropic medium. This is one more manifestation of double refraction. As an illustration, returning to the uniaxial layer, we consider the simple situation when the plane of incidence is parallel to the x_c–z_c plane, that is, the optic axis is parallel to the plane of incidence. Let the vector $\boldsymbol{L}$ be codirectional with the positive x_c-axis (Figure 1.10). In this case, the refraction vectors of the natural waves induced in the layer, being represented in the system (x_c, y_c, z_c), have the form

$$\mathbf{m} = \begin{pmatrix} \zeta \\ 0 \\ \sigma \end{pmatrix}, \tag{1.142}$$

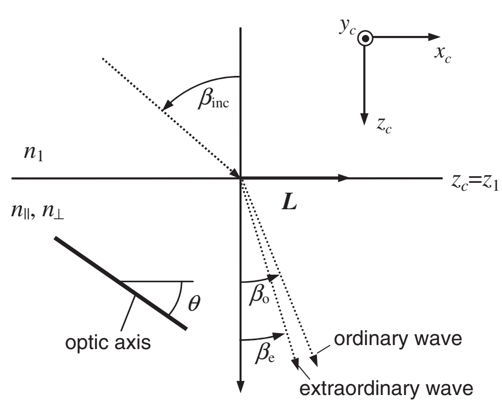

Figure 1.10 Double refraction at oblique incidence

where $\zeta = n_1 \sin\beta_{\rm inc}$. As seen from (1.123), (1.122), and (1.134), with such **m**,

$$\mathbf{Q_E} = \begin{pmatrix} \varepsilon_\perp + \Delta\varepsilon\cos^2\theta - \sigma^2 & 0 & \Delta\varepsilon\cos\theta\sin\theta + \zeta\sigma \\ 0 & \varepsilon_\perp - \zeta^2 - \sigma^2 & 0 \\ \Delta\varepsilon\cos\theta\sin\theta + \zeta\sigma & 0 & \varepsilon_\perp + \Delta\varepsilon\sin^2\theta - \zeta^2 \end{pmatrix}.$$

The solutions of (1.127) with this $\mathbf{Q_E}$ that correspond to waves propagating away from the frontal boundary of the layer are

$$\sigma_1 = \frac{-\zeta\Delta\varepsilon\cos\theta\sin\theta + \sqrt{(\zeta\Delta\varepsilon\cos\theta\sin\theta)^2 + \left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta\right)\left[\varepsilon_\perp\varepsilon_{||} - \zeta^2\left(\varepsilon_\perp + \Delta\varepsilon\cos^2\theta\right)\right]}}{\varepsilon_\perp + \Delta\varepsilon\sin^2\theta} \tag{1.143}$$

for extraordinary waves and

$$\sigma_2 = \sqrt{\varepsilon_\perp - \zeta^2} = \sqrt{n_\perp^2 - \zeta^2} \tag{1.144}$$

for ordinary waves. If the optic axis is parallel to the layer boundaries ($\theta = 0$), σ_1 can be expressed as follows:

$$\sigma_1 = \sqrt{\frac{\varepsilon_{||}\left(\varepsilon_\perp - \zeta^2\right)}{\varepsilon_\perp}} = \frac{n_{||}}{n_\perp}\sqrt{n_\perp^2 - \zeta^2}. \tag{1.145}$$

If the uniaxial medium is nonabsorbing and ζ is such that the radicands in the above expressions for σ_1 and σ_2 are positive (e.g., this is the case at any $\beta_{\rm inc}$ when $n_{||}$ and $n_\perp$ is greater than n_1), the corresponding waves are homogeneous. In this case, the angle of refraction for the transmitted extraordinary wave, $\beta_{\rm e}$, and that for the transmitted ordinary wave, $\beta_{\rm o}$ (see Figure 1.10), can be calculated by the formulas

$$\beta_{\rm e} = \arctan\frac{\zeta}{\sigma_1}, \quad \beta_{\rm o} = \arctan\frac{\zeta}{\sigma_2}. \tag{1.146}$$

As clearly seen from (1.144)–(1.146), the difference between $\beta_{\rm e}$ and $\beta_{\rm o}$ increases with increasing the ratio $\delta_n = |n_{||} - n_\perp|/n_\perp$. At δ_n values of the order of 0.1, which is typical of the liquid crystals used in LCDs, the difference between $\beta_{\rm e}$ and $\beta_{\rm o}$ may be appreciable. For example, taking $n_1 = 1$, $n_{||} = 1.7$, $n_\perp = 1.5$, $\theta = 0$, and $\beta_{\rm inc} = 60°$, we obtain $\beta_{\rm o} \approx 35.3°$ and $\beta_{\rm e} \approx 32°$.

1.3.3 A Simple Birefringent Layer and Its Principal Axes

A Biaxial Layer at Normal Incidence

As noted in the previous section, the natural waves induced in a uniaxial layer by a normally incident plane wave are in general plane-polarized, and each of these waves has its polarization plane coincident with one of two fixed, mutually perpendicular, planes. The same can be said about natural waves in a biaxial layer if the biaxial medium

(i) is nonabsorbing or
(ii) being absorbing has a plane of symmetry perpendicular to the layer boundaries.

To show this we refer to (1.124). If any of the two conditions is satisfied, there exists a coordinate system (x_c, y_c, z_c), with the z_c-axis perpendicular to the layer boundaries, such that the components $\bar{\varepsilon}_{12}$ and $\bar{\varepsilon}_{21}$ of the tensor $\boldsymbol{\varepsilon}^{-1} \equiv [\bar{\varepsilon}_{jk}]$ in this system are zero[1]. In this coordinate system, the matrix $\mathbf{Q_m}$ [see (1.123)] has the form (1.137), and, according to (1.125), the matrix $\mathbf{Q_D}$ can be written as follows:

$$\mathbf{Q_D} = \begin{pmatrix} 1 - \sigma^2\bar{\varepsilon}_{11} & 0 & -\sigma^2\bar{\varepsilon}_{13} \\ 0 & 1 - \sigma^2\bar{\varepsilon}_{22} & -\sigma^2\bar{\varepsilon}_{23} \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.147}$$

The roots of equation (1.128) in this case are

$$\sigma_1 = \frac{1}{\sqrt{\bar{\varepsilon}_{11}}}, \quad \sigma_2 = \frac{1}{\sqrt{\bar{\varepsilon}_{22}}}, \quad \sigma_3 = -\frac{1}{\sqrt{\bar{\varepsilon}_{11}}}, \quad \sigma_4 = -\frac{1}{\sqrt{\bar{\varepsilon}_{22}}}. \tag{1.148}$$

It is easily seen from (1.147) that at $\bar{\varepsilon}_{22} \neq \bar{\varepsilon}_{11}$, the displacement vibration vectors **d** for the waves with σ equal to σ_1 and σ_3 are parallel to the x_c-axis, and, consequently, the electric vibration vectors ($\mathbf{e} = \boldsymbol{\varepsilon}^{-1}\mathbf{d}$) of these waves lie in the x_c–z_c plane [recall that $\bar{\varepsilon}_{12} = \bar{\varepsilon}_{21} = 0$ in the system (x_c, y_c, z_c)], while the waves with σ equal to σ_2 and σ_4 have displacement vibration vectors parallel to the y_c-axis, and, consequently, their electric vibration vectors lie in the y_c–z_c plane, which is what we set out to prove.

A Simple Birefringent Layer and Its Principal Axes. Fast and Slow Axes

Thus, there is a broad class of homogeneous anisotropic layers such that any natural wave induced in the layer by a normally incident plane monochromatic wave is plane-polarized and has its polarization plane parallel to one of two fixed mutually perpendicular planes. Such layers will be called *simple birefringent layers*. The two fixed planes showing the possible orientations of the polarization planes of natural waves will be called the *basic planes of the layer*. Two mutually orthogonal axes each of which is parallel to the layer boundaries and one of the basic planes of the layer are called *the principal axes* of the layer. The principal axes of a layer should not be confused with the principal axes of the medium in the layer, although in many cases a principal axis of a layer is parallel to a principal axis of the medium. For example, the principal axis of a uniaxial medium is its optic axis. For the uniaxial layer considered in the previous section (see Figure 1.9a), one of the principal axes of the layer is parallel to the x_c-axis, and the other to the y_c-axis (as well as in the above example for a biaxial layer). If the optic axis is parallel to the layer boundaries, the former principal axis of the layer is parallel to its optic axis. A principal axis of a simple birefringent layer is called the *fast axis* or the *slow axis* according to whether the phase velocity of the natural waves with polarization plane parallel to this axis is greater or smaller than that of the natural waves whose polarization plane is perpendicular to this axis. For a layer of a nonabsorbing positive uniaxial medium ($n_\parallel > n_\perp$), the fast axis is perpendicular to the optic axis. For a layer of a nonabsorbing medium with negative birefringence ($n_\parallel < n_\perp$), the slow axis is perpendicular to the optic axis (in both cases, we assume that the optic axis is not perpendicular to the layer boundaries).

If the wave normally incident on a simple birefringent layer is linearly polarized along one of the principal axes of the layer, this wave induces in the layer only waves with polarization plane coincident with the polarization plane of the incident wave, and the wave transmitted by the layer has the same polarization state as the incident wave. The truth of this assertion can be proved by using the requirement of continuity of the tangential components of the electric and magnetic fields across interfaces (see Sections 8.1.1 and 12.2). This property of the simple birefringent layers is one of the cornerstones of the classical JC [5] where it is used in the mathematical description of the optical action of anisotropic

[1] In the presence of the plane of symmetry, the x_c-axis of such a system is parallel or perpendicular to this plane.

homogeneous layers (plates, films, etc.) functioning as linear retarders and linear polarizers in optical systems.

1.3.4 Transmission Jones Matrices of a Simple Birefringent Layer at Normal Incidence

Consider a simple birefringent layer sandwiched between nonabsorbing isotropic media. As in the above examples, we assume that the boundaries of the layer coincide with the planes $z_c = z_1$ and $z_c = z_2$ ($z_2 > z_1$) in a coordinate system (x_c, y_c, z_c) whose x_c-axis and y_c-axis are parallel to the principal axes of the layer. Let a plane monochromatic wave fall in the normal direction on the boundary $z_c = z_1$ of the layer from the medium of refractive index n_1. The refractive index of the medium beyond the layer will be denoted by n_2. Let reference frames (x, y, z) and (x', y', z) be introduced as in Section 1.1 (Figure 1.3) to represent the Jones vectors of the incident and transmitted waves. In the case under consideration, the z-axis is codirectional with the z_c-axis. Let the axes of the frame (x', y') be parallel to the principal axes of the layer (the x'-axis may be parallel to the x_c-axis or y_c-axis). We denote the Jones vectors—of the kind (1.21), referred to the system (x', y')—of the incident and transmitted waves by

$$\tilde{\boldsymbol{J}}'^{(\mathrm{inc})} \equiv \begin{pmatrix} \tilde{J}_{x'}^{(\mathrm{inc})} \\ \tilde{J}_{y'}^{(\mathrm{inc})} \end{pmatrix} \text{ and } \tilde{\boldsymbol{J}}'^{(\mathrm{tr})} \equiv \begin{pmatrix} \tilde{J}_{x'}^{(\mathrm{tr})} \\ \tilde{J}_{y'}^{(\mathrm{tr})} \end{pmatrix}, \tag{1.149}$$

respectively. Since the axes x' and y' are parallel to the principal axes, the components of the vector $\tilde{\boldsymbol{J}}'^{(\mathrm{tr})}$ are related to those of $\tilde{\boldsymbol{J}}'^{(\mathrm{inc})}$ by

$$\tilde{J}_{x'}^{(\mathrm{tr})}(z_2+0) = \tilde{t}_{\mathrm{L}x'}\tilde{J}_{x'}^{(\mathrm{inc})}(z_1-0), \quad \tilde{J}_{y'}^{(\mathrm{tr})}(z_2+0) = \tilde{t}_{\mathrm{L}y'}\tilde{J}_{y'}^{(\mathrm{inc})}(z_1-0), \tag{1.150}$$

where $\tilde{t}_{\mathrm{L}x'}$ and $\tilde{t}_{\mathrm{L}y'}$ are transmission coefficients depending on parameters of the layer and the wavelength λ. Here the Jones vectors are considered as functions of z_c. According to (1.150),

$$\tilde{\boldsymbol{J}}'^{(\mathrm{tr})}(z_2+0) = \tilde{\boldsymbol{t}}'_{\mathrm{L}}\tilde{\boldsymbol{J}}'^{(\mathrm{inc})}(z_1-0), \tag{1.151}$$

where

$$\tilde{\boldsymbol{t}}'_{\mathrm{L}} = \begin{pmatrix} \tilde{t}_{\mathrm{L}x'} & 0 \\ 0 & \tilde{t}_{\mathrm{L}y'} \end{pmatrix}. \tag{1.152}$$

The matrix $\tilde{\boldsymbol{t}}'_{\mathrm{L}}$ is the transmission Jones matrix of the layer, corresponding to the chosen kind and representation of the Jones vectors. Let us find the equivalent Jones matrix relating the input and output Jones vectors referred to the frame (x, y). Denote the Jones vectors of the incident and transmitted waves referred to the (x, y) frame by $\tilde{\boldsymbol{J}}^{(\mathrm{inc})}$ and $\tilde{\boldsymbol{J}}^{(\mathrm{tr})}$, respectively. According to (1.54),

$$\tilde{\boldsymbol{J}}'^{(\mathrm{inc})} = \widehat{R}_{\mathrm{C}}(\phi)\tilde{\boldsymbol{J}}^{(\mathrm{inc})}, \quad \tilde{\boldsymbol{J}}'^{(\mathrm{tr})} = \widehat{R}_{\mathrm{C}}(\phi)\tilde{\boldsymbol{J}}^{(\mathrm{tr})}, \tag{1.153}$$

where ϕ is the angle between the axes x and x' (see Figure 1.3). On substituting (1.153) into (1.151) and premultiplying the obtained equation by $\widehat{R}_{\mathrm{C}}(-\phi)$ [recall that $\widehat{R}_{\mathrm{C}}(-\phi) = \widehat{R}_{\mathrm{C}}(\phi)^{-1}$], we have

$$\tilde{\boldsymbol{J}}^{(\mathrm{tr})}(z_2+0) = \widehat{R}_{\mathrm{C}}(-\phi)\tilde{\boldsymbol{t}}'_{\mathrm{L}}\widehat{R}_{\mathrm{C}}(\phi)\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(z_1-0). \tag{1.154}$$

The transmission Jones matrix $\tilde{\boldsymbol{t}}_{\mathrm{L}}$ of the layer for the input and output Jones vectors represented in the system (x, y) is defined by the relation

$$\tilde{\boldsymbol{J}}^{(\mathrm{tr})}(z_2 + 0) = \tilde{\boldsymbol{t}}_{\mathrm{L}}\tilde{\boldsymbol{J}}^{(\mathrm{inc})}(z_1 - 0). \tag{1.155}$$

From (1.154) it is seen that the matrix $\tilde{\boldsymbol{t}}_{\mathrm{L}}$ can be expressed in terms of the matrix $\tilde{\boldsymbol{t}}'_{\mathrm{L}}$ as follows:

$$\tilde{\boldsymbol{t}}_{\mathrm{L}} = \widehat{R}_{\mathrm{C}}(-\phi)\tilde{\boldsymbol{t}}'_{\mathrm{L}}\widehat{R}_{\mathrm{C}}(\phi). \tag{1.156}$$

In principle, in modeling of a polarization system, the matrix $\tilde{\boldsymbol{t}}'_{\mathrm{L}}$ of an optical element can be defined in such a way as to take account of the whole variety of the optical effects involved in the process of light propagation through the layer, including multiple reflections from the boundaries of the layer. But usually, when employing the Jones matrix method, the multiple reflections are neglected and the transmitted light is considered as a result of the following sequence of operations: transmission of the frontal boundary of the layer → transmission of the bulk of the layer → transmission of the rear boundary of the layer. In this case, the amplitude transmission coefficients $\tilde{t}_{\mathrm{L}x'}$ and $\tilde{t}_{\mathrm{L}y'}$ [see (1.150)] of the layer can be expressed as follows:

$$\tilde{t}_{\mathrm{L}x'} = \tilde{t}_{\mathrm{LB}x'} \exp\left(ik_0 n_{\mathrm{w}x'} d\right), \quad \tilde{t}_{\mathrm{L}y'} = \tilde{t}_{\mathrm{LB}y'} \exp\left(ik_0 n_{\mathrm{w}y'} d\right), \tag{1.157}$$

where the factors $\tilde{t}_{\mathrm{LB}x'}$ and $\tilde{t}_{\mathrm{LB}y'}$ describe the transmission of the boundaries, $n_{\mathrm{w}x'}$ is the refractive index for the natural waves of the layer that have polarization planes parallel to the x'-axis, $n_{\mathrm{w}y'}$ is that for the natural waves whose polarization planes are parallel to the y'-axis, and $d = z_1 - z_2$ is the thickness of the layer. The z_c-dependences of the electric fields of the natural waves traveling inside the layer from the plane $z_c = z_1$ toward the plane $z_c = z_2$ are given by

$$\mathbf{E}_{(x')}(x_c, y_c, z_c, t) = \mathbf{E}_{(x')}(x_c, y_c, z_1 + 0, t) \exp\left[ik_0 n_{\mathrm{w}x'}\left(z_c - z_1\right)\right] \tag{1.158}$$

for a wave polarized in the plane parallel to the x'-axis and

$$\mathbf{E}_{(y')}(x_c, y_c, z_c, t) = \mathbf{E}_{(y')}(x_c, y_c, z_1 + 0, t) \exp\left[ik_0 n_{\mathrm{w}y'}\left(z_c - z_1\right)\right] \tag{1.159}$$

for a wave polarized in the plane parallel to the y'-axis, which explains the presence and the form of the exponential factors in (1.157). If in the above examples for the uniaxial and biaxial layers we direct the x'-axis along the x_c-axis, the refractive indices $n_{\mathrm{w}x'}$ and $n_{\mathrm{w}y'}$ can be expressed as

$$n_{\mathrm{w}x'} = n_{\mathrm{e}} = \frac{n_{||}n_{\perp}}{\sqrt{n_{||}^2 \sin^2\theta + n_{\perp}^2 \cos^2\theta}}, \quad n_{\mathrm{w}y'} = n_{\mathrm{o}} = n_{\perp} \tag{1.160}$$

for the uniaxial layer and

$$n_{\mathrm{w}x'} = \sigma_1, \quad n_{\mathrm{w}y'} = \sigma_2 \tag{1.161}$$

with σ_1 and σ_2 given by (1.148) for the biaxial layer.

The transmittance[2] (or the transmissivity) of the layer can be expressed as

$$T_{\mathrm{L}} = \frac{n_2}{n_1}\left|\tilde{\boldsymbol{t}}_{\mathrm{L}}\boldsymbol{j}^{(\mathrm{inc})}\right|^2 = \frac{n_2}{n_1}\left|\tilde{\boldsymbol{t}}'_{\mathrm{L}}\boldsymbol{j}'^{(\mathrm{inc})}\right|^2 \tag{1.162}$$

[cf. (1.110)], where $\boldsymbol{j}^{(\mathrm{inc})}$ is the polarization Jones vector of the incident wave referred to the frame (x, y) and $\boldsymbol{j}'^{(\mathrm{inc})}$ is the same vector but referred to the frame (x', y').

Let us consider analogous relations for other kinds of Jones vectors, namely, for the local "fitted-to-irradiance" (see Section 5.4.2) Jones vectors of the incident and transmitted waves defined by analogy with (1.99) as

$$\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})} = \sqrt{2n_1}\tilde{\boldsymbol{J}}^{(\mathrm{inc})}, \quad \tilde{\boldsymbol{J}}_F^{(\mathrm{tr})} = \sqrt{2n_2}\tilde{\boldsymbol{J}}^{(\mathrm{tr})} \tag{1.163}$$

and the "global" Jones vectors of these waves, $\boldsymbol{J}^{(\mathrm{inc})}$ and $\boldsymbol{J}^{(\mathrm{tr})}$, defined in the same way as the vector $\boldsymbol{J}$ in Section 1.1.2. Neglecting the multiple reflections, the matrix $\tilde{\boldsymbol{t}}_{\mathrm{L}(F)}$ such that

$$\tilde{\boldsymbol{J}}_F^{(\mathrm{tr})}(z_2+0) = \tilde{\boldsymbol{t}}_{\mathrm{L}(F)}\,\tilde{\boldsymbol{J}}_F^{(\mathrm{inc})}(z_1-0) \tag{1.164}$$

can be represented as follows:

$$\tilde{\boldsymbol{t}}_{\mathrm{L}(F)} = \widehat{R}_{\mathrm{C}}(-\phi)\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)}\widehat{R}_{\mathrm{C}}(\phi), \tag{1.165}$$

where

$$\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)} = \begin{pmatrix} \tilde{t}_{\mathrm{L}x'(F)} & 0 \\ 0 & \tilde{t}_{\mathrm{L}y'(F)} \end{pmatrix} \tag{1.166}$$

with

$$\tilde{t}_{\mathrm{L}x'(F)} = \tilde{t}_{\mathrm{LB}x'(F)}\exp\left(ik_0 n_{\mathrm{w}x'}d\right), \quad \tilde{t}_{\mathrm{L}y'(F)} = \tilde{t}_{\mathrm{LB}y'(F)}\exp\left(ik_0 n_{\mathrm{w}y'}d\right). \tag{1.167}$$

The transmittances of the layer for waves linearly polarized along its principal axes will be referred to as the *principal transmittances* of the layer. In the case under consideration, the principal transmittances can be expressed as $T_{\mathrm{L}x'} = \tilde{t}^*_{\mathrm{L}x'(F)}\tilde{t}_{\mathrm{L}x'(F)}$ and $T_{\mathrm{L}y'} = \tilde{t}^*_{\mathrm{L}y'(F)}\tilde{t}_{\mathrm{L}y'(F)}$. For any given polarization of the incident wave, the transmittance of the layer can be calculated by the formula

$$T_{\mathrm{L}} = \left|\tilde{\boldsymbol{t}}_{\mathrm{L}(F)}\boldsymbol{j}^{(\mathrm{inc})}\right|^2 = \left|\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)}\boldsymbol{j}'^{(\mathrm{inc})}\right|^2. \tag{1.168}$$

One of the principal transmittances is equal to the maximum value of T_{L} over all possible polarization states of the incident wave, and the other to the minimum one. The quantities $T_{\mathrm{LB}x'} \equiv \tilde{t}^*_{\mathrm{LB}x'(F)}\tilde{t}_{\mathrm{LB}x'(F)}$ and $T_{\mathrm{LB}y'} \equiv \tilde{t}^*_{\mathrm{LB}y'(F)}\tilde{t}_{\mathrm{LB}y'(F)}$ are equal to the products of the transmittances of the frontal and rear boundaries of the layer for the corresponding polarizations of the incident wave. If the layer is nonabsorbing, $\tilde{t}_{\mathrm{LB}x'(F)}$ and $\tilde{t}_{\mathrm{LB}y'(F)}$ are real. For absorbing layers, these coefficients are in general complex but most often have

[2] The term "transmittance" which is commonly used in CJMM corresponds to the treatment of the incident light as a beam of finite diameter (see Section 7.1). At the same time, CJMM uses a plane-wave approximation which involves the possibility to evaluate a transmittance as the corresponding transmissivity (see Section 7.1). The notion of transmittance is closer to practice than transmissivity and we will use it where convenient, even when this implies an approximation (as in this case).

very small imaginary parts and, to a good approximation, can be considered real (see examples in Section 12.2). Therefore, almost always, the matrix $\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)}$ can be represented as

$$\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)} = \begin{pmatrix} \sqrt{T_{\mathrm{LB}x'}} \exp\left(ik_0 n_{\mathrm{w}x'} d\right) & 0 \\ 0 & \sqrt{T_{\mathrm{LB}y'}} \exp\left(ik_0 n_{\mathrm{w}y'} d\right) \end{pmatrix}. \quad (1.169)$$

The representations (1.166) and (1.169) take into account polarization-dependent losses (*diattenuation*) at the interfaces. However, in most cases of practical interest the coefficients $\tilde{t}_{\mathrm{LB}x'(F)}$ and $\tilde{t}_{\mathrm{LB}y'(F)}$ are of the order of 1 and very close to each other (see Section 12.2), which allows one to neglect the diattenuation at the interfaces and to use the following approximation:

$$t_{\mathrm{LB}x'} = t_{\mathrm{LB}y'} = t_{\mathrm{LB}}, \quad (1.170)$$

where t_{LB} is the average over the actual values of $\sqrt{T_{\mathrm{LB}x'}}$ and $\sqrt{T_{\mathrm{LB}y'}}$. With this approximation, the matrix $\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)}$ can be written as

$$\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)} = t_{\mathrm{LB}} \begin{pmatrix} \exp\left(ik_0 n_{\mathrm{w}x'} d\right) & 0 \\ 0 & \exp\left(ik_0 n_{\mathrm{w}y'} d\right) \end{pmatrix}. \quad (1.171)$$

On omitting the factor t_{LB}, we arrive at the form of $\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)}$ usual for the classical JC, namely,

$$\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)} = \tilde{\boldsymbol{t}}'_{\mathrm{LU}}, \quad (1.172)$$

where

$$\tilde{\boldsymbol{t}}'_{\mathrm{LU}} \equiv \begin{pmatrix} \exp\left(ik_0 n_{\mathrm{w}x'} d\right) & 0 \\ 0 & \exp\left(ik_0 n_{\mathrm{w}y'} d\right) \end{pmatrix}. \quad (1.173)$$

Omission of the factor t_{LB} is often quite a reasonable step, but one should remember that, almost always, this step is far out of the rigorous theory. In the case of a nonabsorbing layer, one can avoid serious contradictions with the rigorous theory by using the matrix $\tilde{\boldsymbol{t}}'_{\mathrm{LU}}$ as the operator relating the *polarization* Jones vectors of the incident and transmitted waves:

$$\boldsymbol{j}'^{(\mathrm{tr})} = \tilde{\boldsymbol{t}}'_{\mathrm{LU}} \boldsymbol{j}'^{(\mathrm{inc})}, \quad (1.174)$$

where both vectors are referred to the frame (x', y'). In terms of the polarization Jones vectors of the incident and transmitted waves referred to the frame (x, y), respectively $\boldsymbol{j}^{(\mathrm{inc})}$ and $\boldsymbol{j}^{(\mathrm{tr})}$, the same relation can be written as

$$\boldsymbol{j}^{(\mathrm{tr})} = \tilde{\boldsymbol{t}}_{\mathrm{LU}} \boldsymbol{j}^{(\mathrm{inc})}, \quad (1.175)$$

where

$$\tilde{\boldsymbol{t}}_{\mathrm{LU}} = \widehat{R}_{\mathrm{C}}(-\phi) \tilde{\boldsymbol{t}}'_{\mathrm{LU}} \widehat{R}_{\mathrm{C}}(\phi). \quad (1.176)$$

Jones matrices relating polarization Jones vectors will be called *polarization Jones matrices*. Note that polarization Jones matrices are always unitary because polarization Jones vectors are unit in the sense (1.33) (see Section 5.1.3).

The Jones matrix $\boldsymbol{t}_{\mathrm{L}}$ intended for linking the "global" Jones vectors of the incident and transmitted waves,

$$\boldsymbol{J}^{(\mathrm{tr})} = \boldsymbol{t}_{\mathrm{L}}\, \boldsymbol{J}^{(\mathrm{inc})}, \tag{1.177}$$

can be taken equal to $\tilde{\boldsymbol{t}}_{\mathrm{L}(F)}$. Since the phase of "global" Jones vectors is unimportant, any other matrix $\boldsymbol{t}_{\mathrm{L}}$ representable as $\boldsymbol{t}_{\mathrm{L}} = a_{\mathrm{P}}\tilde{\boldsymbol{t}}_{\mathrm{L}(F)}$, where a_{P} is a complex number of unit magnitude ($|a_{\mathrm{P}}| = 1$), can also be used for this purpose. The same can be said about matrices relating polarization Jones vectors.

The above representations are used in constructing transmission Jones matrices of various polarization elements, in particular linear retarders and linear absorptive polarizers.

In closing, we note the following relations. In general, the coefficients $\tilde{t}_{\mathrm{LB}x'}$ and $\tilde{t}_{\mathrm{LB}y'}$ entering into (1.157) are related to the coefficients $\tilde{t}_{\mathrm{LB}x'(F)}$ and $\tilde{t}_{\mathrm{LB}y'(F)}$ as follows:

$$\tilde{t}_{\mathrm{LB}x'} = \sqrt{\frac{n_1}{n_2}}\tilde{t}_{\mathrm{LB}x'(F)}, \quad \tilde{t}_{\mathrm{LB}y'} = \sqrt{\frac{n_1}{n_2}}\tilde{t}_{\mathrm{LB}y'(F)}. \tag{1.178}$$

When the refractive indices n_1 and n_2 differ greatly from each other, the coefficients $\tilde{t}_{\mathrm{LB}x'}$ and $\tilde{t}_{\mathrm{LB}y'}$, even when the reflection losses are small and $\tilde{t}_{\mathrm{LB}x'(F)}$ and $\tilde{t}_{\mathrm{LB}y'(F)}$ are close to unity, may differ greatly from unity. For example, if $n_1 = 1$, $n_2 = 1.5$, and the principal refractive indices of the layer are real and about 1.5, the coefficients $\tilde{t}_{\mathrm{LB}x'(F)}$ and $\tilde{t}_{\mathrm{LB}y'(F)}$ will be about 0.98, while the coefficients $\tilde{t}_{\mathrm{LB}x'}$ and $\tilde{t}_{\mathrm{LB}y'}$ will be close to 0.8. If, with the same layer, $n_1 = 1.5$ and $n_2 = 1$, $\tilde{t}_{\mathrm{LB}x'}$ and $\tilde{t}_{\mathrm{LB}y'}$ will be about 1.2, while the coefficients $\tilde{t}_{\mathrm{LB}x'(F)}$ and $\tilde{t}_{\mathrm{LB}y'(F)}$ will be the same as in the previous case. The matrices $\tilde{\boldsymbol{t}}_{\mathrm{L}}$ and $\tilde{\boldsymbol{t}}_{\mathrm{L}(F)}$ are related by

$$\tilde{\boldsymbol{t}}_{\mathrm{L}} = \sqrt{\frac{n_1}{n_2}}\tilde{\boldsymbol{t}}_{\mathrm{L}(F)} \tag{1.179}$$

and are equal to each other at $n_1 = n_2$. Let (x'', y'') be a reference frame with the x''-axis parallel to the x'-axis and the y''-axis parallel to the y'-axis. Whatever the values of n_1 and n_2, the transmission Jones matrix of the layer for the local "fitted-to-irradiance" Jones vectors referred to the system (x'', y'') for the reverse propagation direction (that is, for the case when the incident wave normally falls on the layer from the half-space $z_c > z_2$) is equal to the matrix $\tilde{\boldsymbol{t}}'_{\mathrm{L}(F)}$. For the Jones matrices associated with the Jones vectors of the kind (1.21), such a relation will take place only at $n_1 = n_2$.

1.3.5 Linear Retarders

Linear retarders—retardation films and retardation plates—are common optical elements used to convert the polarization state of passing light. Retardation films are used in LCDs for color dispersion compensation and to improve the viewing angle characteristics. Detailed discussion of the standard applications of retarders in polarization optics and terminology connected with retarders can be found in the books [2, 6] and many others. Here we briefly discuss the action of linear retarders at normal incidence.

A simple linear retarder is a nonabsorbing birefringent layer. When light enters such a layer, in general it splits into two plane-polarized natural waves propagating through the layer with different phase velocities. These waves experience different phase retardation as they propagate through the layer and, upon exiting the layer, recombine into a new wave with a new polarization state. This is the operating principle of linear retarders. The most important characteristic of a retarder is the relative phase retardation

$$\Gamma = \frac{2\pi(n_{\mathrm{s}} - n_{\mathrm{f}})d}{\lambda}, \tag{1.180}$$

where n_s and n_f are the refractive indices for the natural waves with polarization planes parallel to the slow axis and to the fast axis, respectively; d is the thickness of the birefringent layer. If, at a given λ, $(n_s - n_f)d = \lambda/4$ and, consequently, $\Gamma = \pi/2$, the retarder is called *a quarter-wave plate (film)* for the given λ. The retarders with $(n_s - n_f)d = \lambda/2$ ($\Gamma = \pi$) are called *half-wave plates (films).*

Let the x'-axis of the frame (x', y') attached to the principal axes of a nonabsorbing simple birefringent layer be oriented along its slow axis. In this case, the polarization Jones matrix of the layer for the input and output Jones vectors referred to the frame (x', y') [see (1.174)] may be written as

$$\tilde{\boldsymbol{t}}'_{\text{LU}} = \begin{pmatrix} e^{i\frac{2\pi n_s d}{\lambda}} & 0 \\ 0 & e^{i\frac{2\pi n_f d}{\lambda}} \end{pmatrix} \tag{1.181}$$

or

$$\tilde{\boldsymbol{t}}'_{\text{LU}} = e^{i\frac{\pi(n_s+n_f)d}{\lambda}} \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}, \tag{1.182}$$

where

$$\delta \equiv \frac{\Gamma}{2} = \frac{\pi(n_s - n_f)d}{\lambda}. \tag{1.183}$$

Since the phase of a polarization Jones vector is inessential, we can omit the common exponential factor in expression (1.182) to deal with the mathematically simplest expression for $\tilde{\boldsymbol{t}}'_{\text{LU}}$:

$$\tilde{\boldsymbol{t}}'_{\text{LU}} = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}. \tag{1.184}$$

On substituting (1.184) into (1.176), we obtain

$$\tilde{\boldsymbol{t}}_{\text{LU}} = \widehat{R}_{\text{C}}(-\phi)\tilde{\boldsymbol{t}}'_{\text{LU}}\widehat{R}_{\text{C}}(\phi) = \begin{pmatrix} \cos\delta + i\sin\delta\cos 2\phi & i\sin\delta\sin 2\phi \\ i\sin\delta\sin 2\phi & \cos\delta - i\sin\delta\cos 2\phi \end{pmatrix}; \tag{1.185}$$

here ϕ can be treated as the angle between the x-axis of the frame (x, y), to which the input and output polarization Jones vectors are referred [see (1.175)], and the slow axis of the layer. This is a general expression for the polarization Jones matrix of the linear retarder in a reference frame arbitrarily oriented with respect to its principal axes.

Let us illustrate the ability of retarders to convert polarization by some examples, using the Jones matrix method.

Half-Wave Plate

In this case, $\delta = \pi/2$ and the matrix $\tilde{\boldsymbol{t}}'_{\text{LU}}$ can be written as

$$\tilde{\boldsymbol{t}}'_{\text{LU}} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.186}$$

Suppose that the frame (x, y) coincides with the frame (x', y'), so that $\tilde{\boldsymbol{t}}_{\text{LU}} = \tilde{\boldsymbol{t}}'_{\text{LU}}$. Assume that the incident wave has an arbitrary elliptical polarization. Taking the polarization vector $\boldsymbol{j}^{(\text{inc})}$ in the form

$$\boldsymbol{j}^{(\text{inc})} = \boldsymbol{j}_E(\gamma_{\text{inc}}, \upsilon_{\text{inc}}) \tag{1.187}$$

(see Table 1.1), where γ_{inc} and υ_{inc} are the values of the azimuth γ_{e} and ellipticity angle υ_{e} (see Section 1.1.2) of the incident wave, it is easy to find that

$$\boldsymbol{j}^{(\text{tr})} = \tilde{\boldsymbol{t}}_{\text{LU}}\boldsymbol{j}^{(\text{inc})} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\boldsymbol{j}_E(\gamma_{\text{inc}}, \upsilon_{\text{inc}}) = i\boldsymbol{j}_E(-\gamma_{\text{inc}}, -\upsilon_{\text{inc}}). \tag{1.188}$$

Since the vector $\boldsymbol{j}_E(-\gamma_{\text{inc}}, -\upsilon_{\text{inc}})$ represents just the same polarization state as the vector $\boldsymbol{j}^{(\text{tr})} = i\boldsymbol{j}_E(-\gamma_{\text{inc}}, -\upsilon_{\text{inc}})$, we may conclude that the transmitted wave will have an azimuth $\gamma_{\text{e}} = -\gamma_{\text{inc}}$ and an ellipticity angle $\upsilon_{\text{e}} = -\upsilon_{\text{inc}}$. If the incident wave is linearly polarized, the transmitted wave will also be linearly polarized, the polarization plane of the transmitted wave being the mirror image of that of the incident wave with respect to the x'–z plane. If the incident wave has the left circular polarization, the transmitted wave will have the right circular polarization and vice versa.

Quarter-Wave Plate

The main application of quarter-wave plates is in transforming linearly polarized light into circularly polarized one and vice versa. To illustrate these options, we again, for simplicity, assume that the frames (x, y) and (x', y') are coincident. For a quarter-wave plate, $\delta = \pi/4$ and the matrix $\tilde{\boldsymbol{t}}_{\text{LU}}$ can be represented as

$$\tilde{\boldsymbol{t}}_{\text{LU}} = \mathrm{e}^{i\frac{\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{1.189}$$

It is easy to verify that the polarization vectors from Table 1.1 satisfy the following relations:

$$\begin{gathered}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\boldsymbol{j}_P\left(\frac{\pi}{4}\right) = \boldsymbol{j}_R, \quad \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\boldsymbol{j}_P\left(-\frac{\pi}{4}\right) = \boldsymbol{j}_L, \\ \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\boldsymbol{j}_R = \boldsymbol{j}_P\left(-\frac{\pi}{4}\right), \quad \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\boldsymbol{j}_L = \boldsymbol{j}_P\left(\frac{\pi}{4}\right).\end{gathered} \tag{1.190}$$

As is seen from these relations, a quarter-wave plate can perform the following conversions:

$$P_{\pi/4} \to R, \quad P_{-\pi/4} \to L, \quad R \to P_{-\pi/4}, \quad L \to P_{\pi/4}, \tag{1.191}$$

where the symbols $P_{\pi/4}$, $P_{-\pi/4}$, R, and L denote respectively the linear polarization with $\gamma_{\text{e}} = \pi/4$, the linear polarization with $\gamma_{\text{e}} = -\pi/4$, the right circular polarization, and the left circular polarization.

1.3.6 Jones Matrices of Absorptive Polarizers. Ideal Polarizer

Absorptive polarizers are used in most kinds of liquid crystal displays. The main element of the usual absorptive polarizer is an absorbing anisotropic film exhibiting high diattenuation due to absorption anisotropy. In the spectral region where this film acts effectively as polarizer, one of the two principal transmittances of the film is close to zero, while the other is sufficiently high (ideally, equal to 1). The principal axis of the film corresponding to the higher principal transmittance is called *the transmission axis* of the polarizer [6].

A standard optical model of the polarizing film or the polarizer as a whole is a uniaxial layer whose optic axis is parallel to the layer boundaries (see Section 7.3). In the rigorous methods which are considered in Chapters 8–10, the specification of such a model includes the specification of the principal complex refractive indices of the layer. In calculations performed for the case of normal incidence using the classical Jones calculus, as a rule, simpler variants of specification of polarizers are used. Here we consider some of them.

Let the x'-axis of the reference frame (x', y') be parallel to the transmission axis of the layer being a model of the polarizer. We denote the principal transmittances of the layer by $t_{\|}$ and $t_{\perp}$, where $t_{\|}$ corresponds to the polarization along the transmission axis. Assuming that $\mathrm{Re}(n_{wx'}) = \mathrm{Re}(n_{wy'})$, in accordance with (1.171) we may write the matrix $\tilde{\boldsymbol{t}}'_{L(F)}$ of the layer as follows:

$$\tilde{\boldsymbol{t}}'_{L(F)} = \exp\left[ik_0 \,\mathrm{Re}\left(n_{wx'}\right) d\right] \begin{pmatrix} t_{LB} \exp\left[-k_0 \,\mathrm{Im}\left(n_{wx'}\right) d\right] & 0 \\ 0 & t_{LB} \exp\left[-k_0 \,\mathrm{Im}\left(n_{wy'}\right) d\right] \end{pmatrix}. \tag{1.192}$$

In this case, the principal transmittances of the layer can be expressed as

$$t_{||} = t_{LB}^2 \exp\left[-2k_0 \,\mathrm{Im}\left(n_{wx'}\right) d\right], \quad t_{\perp} = t_{LB}^2 \exp\left[-2k_0 \,\mathrm{Im}\left(n_{wy'}\right) d\right], \tag{1.193}$$

and consequently the matrix $\tilde{\boldsymbol{t}}'_{L(F)}$ can be represented as follows:

$$\tilde{\boldsymbol{t}}'_{L(F)} = \exp\left[ik_0 \,\mathrm{Re}\left(n_{wx'}\right) d\right] \begin{pmatrix} \sqrt{t_{||}} & 0 \\ 0 & \sqrt{t_{\perp}} \end{pmatrix}. \tag{1.194}$$

According to (1.194), the simplest variant of the Jones matrix of the polarizer for the “global” Jones vectors referred to the system (x', y') is

$$\boldsymbol{t}'_{L} = \begin{pmatrix} \sqrt{t_{||}} & 0 \\ 0 & \sqrt{t_{\perp}} \end{pmatrix} \tag{1.195}$$

[see the remark after (1.177)]. The corresponding Jones matrix for the “global” Jones vectors referred to the system (x, y) rotated with respect to the system (x', y') can be calculated by the formula

$$\boldsymbol{t}_{L} = \widehat{R}_C(-\phi)\boldsymbol{t}'_{L}\widehat{R}_C(\phi), \tag{1.196}$$

where ϕ is the angle between the x-axis and the x'-axis (the transmission axis of the polarizer). Thus, in this case, to specify the polarizer we need only the principal transmittances and orientation angle ϕ. It is sometimes convenient to represent the principal transmittances $t_{\|}$ and $t_{\perp}$ as follows:

$$t_{||} = C_p t_{||p}, \quad t_{\perp} = C_p t_{\perp p}, \tag{1.197}$$

where $t_{||p}$ and $t_{\perp p}$ are the principal bulk transmittances of the layer,

$$t_{||p} = \exp\left[-2k_0 \,\mathrm{Im}\left(n_{wx'}\right) d\right], \quad t_{\perp p} = \exp\left[-2k_0 \,\mathrm{Im}\left(n_{wy'}\right) d\right], \tag{1.198}$$

and $C_p = t_{LB}^2$ is a factor taking account of the reflection losses at the boundaries.

As a rule, the real parts of the refractive indices $n_{\mathrm{w}x'}$ and $n_{\mathrm{w}y'}$ of a real polarizing film are different. To take this circumstance into account one can use the following form of the matrix $\boldsymbol{t}'_{\mathrm{L}}$:

$$\boldsymbol{t}'_{\mathrm{L}} = \begin{pmatrix} \sqrt{t_{||}}\exp\left(i\delta_{\mathrm{w}}\right) & 0 \\ 0 & \sqrt{t_{\perp}}\exp\left(-i\delta_{\mathrm{w}}\right) \end{pmatrix}, \tag{1.199}$$

where $\delta_{\mathrm{w}} = \pi\left[\mathrm{Re}\left(n_{\mathrm{w}x'}\right) - \mathrm{Re}\left(n_{\mathrm{w}y'}\right)\right] d/\lambda$. Although the situation when $\mathrm{Re}n_{\mathrm{w}x'} \neq \mathrm{Re}n_{\mathrm{w}y'}$ is common, in solving typical problems for LCDs, as a rule, there is no need to use the representation (1.199) instead of (1.195) because the phase factors in (1.199) contribute nothing to the quantities to be estimated, such as the transmittance of the LCD panel, or their influence on the LCD characteristics is negligible.

The above matrices $\boldsymbol{t}'_{\mathrm{L}}$ at $t_{\perp} \neq 0$ describe partial polarizers. All real absorptive polarizers are partial ones. However, for many practical polarizers, $t_{\perp}$ is so small that it can be taken as zero in calculations. In such a case, the matrix $\boldsymbol{t}'_{\mathrm{L}}$ can be written as

$$\boldsymbol{t}'_{\mathrm{L}} = \begin{pmatrix} \sqrt{t_{||}} & 0 \\ 0 & 0 \end{pmatrix} = \sqrt{t_{||}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.200}$$

Often a still further idealized model of a linear polarizer is used. This model is *an ideal linear polarizer* whose matrix $\boldsymbol{t}'_{\mathrm{L}}$ is as follows:

$$\boldsymbol{t}'_{\mathrm{L}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.201}$$

The matrix $\boldsymbol{t}_{\mathrm{L}}$ (1.196) in this case can be written as

$$\boldsymbol{t}_{\mathrm{L}} = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}. \tag{1.202}$$

The concept of *an ideal polarizer* as an ideal device that transmits the light of a given polarization only, without losses, is applied to polarizers extracting an elliptical or a circular polarization as well [2].

With a given matrix $\boldsymbol{t}_{\mathrm{L}}$ of a polarizer, the transmittance of the polarizer for an incident wave with a given polarization Jones vectors $\boldsymbol{j}^{(\mathrm{inc})}$ can be calculated by the following general formula:

$$T_{\mathrm{L}} = \left|\boldsymbol{t}_{\mathrm{L}}\boldsymbol{j}^{(\mathrm{inc})}\right|^2. \tag{1.203}$$

In the case of *an ideal polarizer*, a simpler expression for the transmittance can be used:

$$T_{\mathrm{L}} = \left|\boldsymbol{j}^{\dagger}_{\mathrm{tp}}\boldsymbol{j}^{(\mathrm{inc})}\right|^2, \tag{1.204}$$

where $\boldsymbol{j}_{\mathrm{tp}}$ is the polarization Jones vector of waves that are transmitted by the polarizer. For example, the vector $\boldsymbol{j}_{\mathrm{tp}}$ for the ideal linear polarizer with matrix $\boldsymbol{t}_{\mathrm{L}}$ given by (1.202) can be expressed as $\boldsymbol{j}_{\mathrm{tp}} = \boldsymbol{j}_P(\phi)$ (see Table 1.1) in the system (x, y). Assuming that the light incident on this polarizer is linearly polarized and taking $\boldsymbol{j}^{(\mathrm{inc})} = \boldsymbol{j}_P(\gamma)$, we readily obtain from (1.204)

$$T_{\mathrm{L}} = \cos^2\gamma_{\phi}, \tag{1.205}$$

where $\gamma_\phi = \gamma - \phi$ is the angle between the transmission axis of the polarizer and the polarization direction of the incident light. Equation (1.205) expresses the familiar *law of Malus*. For a partial polarizer whose matrix $\boldsymbol{t}'_{\mathrm{L}}$ is expressed by (1.195) or (1.199) the dependence of the polarizer transmittance on γ_ϕ is as follows:

$$T_{\mathrm{L}} = t_{||} \cos^2 \gamma_\phi + t_\perp \sin^2 \gamma_\phi. \tag{1.206}$$

This expression can easily be derived by using the following representation of T_{L}:

$$T_{\mathrm{L}} = \left| \boldsymbol{t}'_{\mathrm{L}} \boldsymbol{j}_P(\gamma_\phi) \right|^2 = \boldsymbol{j}_P(\gamma_\phi)^\dagger \left(\boldsymbol{t}'^{\dagger}_{\mathrm{L}} \boldsymbol{t}'_{\mathrm{L}} \right) \boldsymbol{j}_P(\gamma_\phi). \tag{1.207}$$

1.4 Jones Calculus

The classical Jones matrix method (CJMM) includes two fundamental methods. The first method is a calculus for treatment of optical systems containing plane-parallel layers of anisotropic materials, homogeneous or with continuously varying parameters [5, 7, 8]. The second is a general method of description of the interaction of polarized light with nondepolarizing linear optical systems [9]: an action of the optical system is described by a 2 × 2 matrix ($\mathbf{t}$) relating the Jones vector of a wave incident on the system ($\mathbf{J}_{\mathrm{inc}}$) and the Jones vector of the wave emerging from the system that is considered as the result of this action with respect to the incident wave ($\mathbf{J}_{\mathrm{out}}$) as follows:

$$\mathbf{J}_{\mathrm{out}} = \mathbf{t}\mathbf{J}_{\mathrm{inc}}. \tag{1.208}$$

Jones matrices are adequate characteristics in any situation where waves incident on a system and emerging from it can be adequately represented by Jones vectors. For instance, in optics of stratified media, Jones matrices are commonly used to characterize transmission and reflection of such media, as transmission and reflection operators, including the case of oblique light incidence. If the incident and emergent waves are homogeneous and propagate in isotropic nonabsorbing media, they can be described by classical Cartesian Jones vectors. The description in terms of Jones vectors and Jones matrices is entirely consistent with electromagnetic theory. The rigorous methods discussed in Chapter 8 enable calculation of transmission and reflection Jones matrices of layered systems in strict accordance with this theory. The transmission and reflection Jones matrices for the interface between isotropic media in Section 1.2.2 are examples of exact Jones matrices.

In contrast to the matrix description [9], the Jones calculus (JC) is a semiempirical method and is limited to the case of normal incidence. This method was developed for calculating transmission characteristics of optical systems consisting of retarders and polarizers and other systems for which the transformation of the polarization state of the passing light by their elements is of paramount importance. In JC, the effect of an optical element of an optical system on a light beam is considered as a transformation of a plane wave incident on the element into a plane wave emerging from it and is characterized by a Jones matrix that relates Jones vectors of these waves. The action of an optical system consisting of two or more elements is considered as a chain of such transformations. JC is not strongly tied to electromagnetic theory and takes into account only basic functions of the elements and basic optical effects connected with performing these functions by the elements. In contrast to the electromagnetic methods where elements of an optical system are specified by their material parameters, in JC the elements are specified through description of their transfer characteristics which are specified using material parameters where it is convenient. When JC is used in solving optimization problems for finding optimal values of key parameters of polarization elements (such as orientation angles for polarizers, orientation angles and retardances for retarders, the twist angle and thickness for an LC layer), as a rule, the model system for direct analysis is composed of ideal elements such as an ideal polarizer, an ideal retarder, an ideal lossless LC cell, and so on.

JC has played and is playing a very important role in LCD optics. Many fundamental formulas in optics of liquid crystals and LCD optics were derived and many optimization problems were solved using this method. A lot of extremely useful and beautiful mathematics have appeared in polarization optics thanks to JC. That is why much attention in this book is given to this method and its applications in modeling and optimization of LCDs.

There are many points in JC that seem to be or really are inconsistent with the rigorous theory. On the other hand, based on the rigorous theory, one can prove that JC gives accurate results for many practical optical systems including LCDs. Starting from Maxwell's equations, using approximations that are fully justified in the context of electromagnetic theory, one may arrive at a technique which is mathematically (but not in every respect physically) equivalent to JC. This will be shown in Chapters 8, 11, and 12. The formal equivalence of JC and the more rigorous technique allows one to use the mathematical apparatus of JC, very rich and convenient, in the latter technique, or, what is practically the same, to use JC as it is but taking into account the amendments and refinements concerning the physical interpretation of some quantities and procedures involved in this method. Note that many helpful mathematical elements of JC are successfully used within the more rigorous method in considering both normal and oblique light incidence (see Chapter 11 and Section 12.4).

In this section, we consider some basic concepts of JC as well as some mathematical tricks useful when JC is applied to LCDs.

1.4.1 Basic Principles of the Jones Calculus

As has been said, in JC the action of an optical system is considered as a series of transformations to which the light is subjected as it passes through the system. Each of these elementary transformations is characterized by a Jones matrix. The Jones matrices are chosen in such a way that the output Jones vector for the Jones matrix describing the first or any intermediate transformation is the input Jones vector for the Jones matrix of the next transformation, which allows one to relate the Jones vector of the light incident on the system ($\mathbf{J}_{inc}$) and that of the light emerging from the system ($\mathbf{J}_{out}$) by the following chain of equations:

$$\mathbf{J}_1 = \mathbf{t}_1\mathbf{J}_{inc},\ \mathbf{J}_2 = \mathbf{t}_2\mathbf{J}_1, \dots,\ \mathbf{J}_{M-1} = \mathbf{t}_{M-1}\mathbf{J}_{M-2},\ \mathbf{J}_{out} = \mathbf{t}_M\mathbf{J}_{M-1}, \tag{1.209}$$

where M is the number of the elementary transformations and $\mathbf{t}_j$ is the Jones matrix of the jth transformation ($j = 1,2, \dots, M$). The substitutions of the expression for $\mathbf{J}_1$ in (1.209) (the first equation) into the second equation, of the obtained expression for $\mathbf{J}_2$ in terms of $\mathbf{J}_{inc}$ into the third equation, and so on lead to the following relation:

$$\mathbf{J}_{out} = \mathbf{t}_M\mathbf{t}_{M-1} \dots \mathbf{t}_2\mathbf{t}_1\mathbf{J}_{inc}. \tag{1.210}$$

Due to the associativity of the matrix product, this relation can be rewritten as

$$\mathbf{J}_{out} = (\mathbf{t}_M\mathbf{t}_{M-1} \dots \mathbf{t}_2\mathbf{t}_1)\mathbf{J}_{inc} \tag{1.211}$$

or

$$\mathbf{J}_{out} = \mathbf{t}_{sys}\mathbf{J}_{inc}, \tag{1.212}$$

where

$$\mathbf{t}_{sys} = \mathbf{t}_M\mathbf{t}_{M-1} \dots \mathbf{t}_2\mathbf{t}_1 \tag{1.213}$$

is the matrix that is regarded in JC as the Jones matrix of the system. Thus, the validity of (1.209) allows one to calculate the Jones matrix of the system by multiplying the Jones matrices of the elementary transformations in accordance with (1.213).

In Sections 1.3.4–1.3.6, we gave many expressions for Jones matrices of different optical elements, which can be used in such calculations. In all the cases considered in those sections, we assumed that the light incident on an element and the light emerging from the element propagate in isotropic media, so that we could legitimately use usual Cartesian Jones vectors to describe the waves regarded as the operand and the result of the transformation performed by the element. A peculiarity of the classical JC is that in any case the Jones matrix describing the transformation performed by an element is calculated as if the input and output media for this transformation (i.e., the medium from which the light falls on the element and the medium into which the transformed light passes leaving the element) were isotropic. Thus, for example, the transmission Jones matrix of a system consisting of two contiguous anisotropic layers is calculated as if there were an isotropic layer between the anisotropic layers but ignoring the effect of this intermediate isotropic layer on the passing light. It is clear that this approach is somewhat artificial. Some arguments for this approach from the standpoint of electromagnetic theory can be found in Chapter 12.

In principle, in considerations using the above algorithm, different kinds of Jones vectors (see Section 1.1.1) can be used. In the classical JC, the ordinary Jones vectors are assumed to be "fitted-to-intensity," the following relation between the Jones vector $\mathbf{J}$ and intensity I of a wave being adopted:

$$I = |\mathbf{J}|^2 \equiv \mathbf{J}^\dagger \mathbf{J}. \tag{1.214}$$

In the further consideration of JC and its applications, we will adhere to this convention and other prescriptions and principles of the classical variant of this method.

Standard Definition and Usual Representations of Transmittance in the Jones Calculus. Average Transmittance. "Unpolarized" Transmittance

The transmittance t of a device (a system or an element) is defined as

$$t \equiv I_{out}/I_{inc}, \tag{1.215}$$

where I_{inc} and I_{out} are the intensities of the light incident on the device and the light transmitted by the device, respectively. According to (1.214) and (1.215), the transmittance t can be expressed as

$$t = |\mathbf{J}_{out}|^2/|\mathbf{J}_{inc}|^2, \tag{1.216}$$

where $\mathbf{J}_{inc}$ and $\mathbf{J}_{out}$ are the Jones vectors of the incident light and transmitted light, respectively.

The substitution of the expression

$$\mathbf{J}_{out} = \mathbf{t}\mathbf{J}_{inc}, \tag{1.217}$$

where $\mathbf{t}$ is the Jones matrix of the device, into (1.216) gives the following expression for t:

$$t = |\mathbf{t}\mathbf{J}_{inc}|^2/|\mathbf{J}_{inc}|^2. \tag{1.218}$$

Yet another standard expression for the transmittance is

$$t = |\mathbf{t}\boldsymbol{j}_{inc}|^2, \tag{1.219}$$

where $\boldsymbol{j}_{\text{inc}}$ is the polarization Jones vector of the incident light ($|\boldsymbol{j}_{\text{inc}}| = 1$). We have dealt with expressions of this kind in the previous sections. The product $\mathbf{t}\,\boldsymbol{j}_{\text{inc}}$ is a normalized Jones vector whose squared norm is equal to the transmittance t.

Let $t_1 = |\mathbf{t}\,\boldsymbol{j}_1|^2$ and $t_2 = |\mathbf{t}\,\boldsymbol{j}_2|^2$ be the values of the transmittance of the device for two arbitrary mutually orthogonal polarizations of the incident light, described by polarization Jones vectors $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ ($\boldsymbol{j}_2{}^{\dagger}\boldsymbol{j}_1 = 0$). It is easy to verify that the magnitude of the *average transmittance* of the device defined as $t_{\text{avr}} = (t_1 + t_2)/2$ is independent of the choice of the pair of incident orthogonal polarizations and

$$t_{\text{avr}} = \frac{1}{2}\left(t_{11}^{*}t_{11} + t_{12}^{*}t_{12} + t_{21}^{*}t_{21} + t_{22}^{*}t_{22}\right) = \frac{1}{2}\,\|\mathbf{t}\|_{\text{E}}^{2}\,, \tag{1.220}$$

where t_{jk} are elements of the matrix $\mathbf{t}$ and $\|\mathbf{t}\|_{\text{E}}$ is the Euclidean norm of $\mathbf{t}$ (see Section 5.1.4). The *transmittance* of the device *for quasimonochromatic unpolarized incident light*, t_{unp}, according to prescriptions of JC, is calculated as t_{avr} in (1.220), that is, by the formula

$$t_{\text{unp}} = \frac{1}{2}\left(t_{11}^{*}t_{11} + t_{12}^{*}t_{12} + t_{21}^{*}t_{21} + t_{22}^{*}t_{22}\right). \tag{1.221}$$

The unpolarized quasimonochromatic incident wave can be represented as a superposition of two mutually incoherent quasimonochromatic orthogonally polarized waves of equal intensity, with polarization Jones vectors $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$. Denoting the transmittances of the device for these polarized constituents as t_1 and t_2, we may express t_{unp} as $t_{\text{unp}} = (t_1 + t_2)/2$. Then the assumption that the transmittances t_1 and t_2 can be calculated as $t_1 = |\mathbf{t}\,\boldsymbol{j}_1|^2$ and $t_2 = |\mathbf{t}\,\boldsymbol{j}_2|^2$, that is, just as in the case of monochromatic waves, leads us to (1.221).

Lossless Transformations and Transformations Without Diattenuation

Solving many problems is significantly simplified by using specific mathematical properties of Jones matrices describing certain kinds of transformations. Here we consider two important classes of transformations. One of them is the class of transformations for which the output light intensity is equal to the input light intensity whatever the SOP of the incident light. Such transformations are called *lossless*. Definition (1.214) of intensity determines that the Jones matrix describing such a transformation is a unitary matrix (see Section 5.1.3). Actually, let $\mathbf{t}$ be the Jones matrix of an operation, and let $\mathbf{J}_{\text{inc}}$ and $\mathbf{J}_{\text{out}} = \mathbf{t}\mathbf{J}_{\text{inc}}$ be the Jones vectors of the incident and output waves for this operation. Then, according to (1.214), the condition of equality of intensities of the incident and output waves can be written as

$$\mathbf{J}_{\text{out}}^{\dagger}\mathbf{J}_{\text{out}} = \mathbf{J}_{\text{inc}}^{\dagger}\mathbf{J}_{\text{inc}} \tag{1.222}$$

or

$$(\mathbf{t}\mathbf{J}_{\text{inc}})^{\dagger}\mathbf{t}\mathbf{J}_{\text{inc}} = \mathbf{J}_{\text{inc}}^{\dagger}\mathbf{J}_{\text{inc}}. \tag{1.223}$$

Using the identity $(\mathbf{t}\mathbf{J}_{\text{inc}})^{\dagger} = \mathbf{J}_{\text{inc}}{}^{\dagger}\mathbf{t}^{\dagger}$ [see (5.15)], we can rewrite (1.223) as follows:

$$\mathbf{J}_{\text{inc}}^{\dagger}(\mathbf{t}^{\dagger}\mathbf{t})\mathbf{J}_{\text{inc}} = \mathbf{J}_{\text{inc}}^{\dagger}\mathbf{J}_{\text{inc}}. \tag{1.224}$$

This relation holds at any $\mathbf{J}_{\text{inc}}$ only if

$$\mathbf{t}^{\dagger}\mathbf{t} = \mathbf{U}, \tag{1.225}$$

where $\mathbf{U}$ is the unit matrix. A square matrix $\mathbf{A}$ satisfying the condition $\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{U}$ is called *unitary*. A summary of properties of unitary matrices is given in Section 5.1.3. Devices that are assumed to perform lossless transformations are often called *lossless* or *unitary*.

Lossless transformations belong to the class of *transformations without diattenuation* (i.e., without polarization-dependent losses). A transformation can be called a transformation without diattenuation if the ratio of the output light intensity to the input light intensity is independent of the SOP of the incident light. This determining condition implies that at any $\mathbf{J}_{\text{inc}}$,

$$\mathbf{J}_{\text{inc}}^{\dagger}(\mathbf{t}^{\dagger}\mathbf{t})\mathbf{J}_{\text{inc}} = t_l \mathbf{J}_{\text{inc}}^{\dagger}\mathbf{J}_{\text{inc}}, \tag{1.226}$$

where $\mathbf{t}$ is the Jones matrix of the transformation, t_l is a real constant independent of $\mathbf{J}_{\text{inc}}$. In the presence of losses, $t_l < 1$. The transmittance t [see (1.216)] associated with this transformation in any case is equal to t_l. Relation (1.226) will hold at any $\mathbf{J}_{\text{inc}}$ only if

$$\mathbf{t}^{\dagger}\mathbf{t} = t_l \mathbf{U}. \tag{1.227}$$

Any matrix satisfying (1.227) can be represented as $\mathbf{t} = \varsigma\mathbf{t}_{\text{U}}$, where $\mathbf{t}_{\text{U}}$ is a unitary matrix and ς is a scalar factor such that $\varsigma\varsigma^* = t_l$. In this book, such matrices are referred to as STU matrices (see Section 5.1.3).

A chain of lossless transformations is a lossless transformation. The product of unitary matrices is always a unitary matrix. A chain of transformations without diattenuation is a transformation without diattenuation. The product of STU matrices is always an STU matrix.

An interesting feature of transformations without diattenuation is that under such transformations orthogonally polarized waves are converted into orthogonally polarized ones: if $\mathbf{t}$ is an STU matrix and $\mathbf{J}_{\text{inc1}}$ and $\mathbf{J}_{\text{inc2}}$ are arbitrary mutually orthogonal Jones vectors ($\mathbf{J}_{\text{inc1}}{}^{\dagger}\mathbf{J}_{\text{inc2}} = 0$), the vectors $\mathbf{J}_{\text{out1}} = \mathbf{t}\mathbf{J}_{\text{inc1}}$ and $\mathbf{J}_{\text{out2}} = \mathbf{t}\mathbf{J}_{\text{inc2}}$ will be also mutually orthogonal ($\mathbf{J}_{\text{out1}}{}^{\dagger}\mathbf{J}_{\text{out2}} = 0$) (see Section 5.1.3). This feature explains the following well-known property of transmissive devices (layers or layered systems) without diattenuation. If such a device is placed between linear polarizers (ideal or with zero transmittance for the unwanted polarization), the transmittance of the polarizer–device–polarizer system is invariant under rotations of the device about the axis of light propagation by 90°. Actually, due to the mentioned feature of transformations without diattenuation, such a rotation changes only the handedness of the polarization ellipse of the light emerging from the device. The transmittance of a linear polarizer is independent of the handedness of the polarization of light incident on it. Therefore, the intensity of the light transmitted by the second polarizer will remain unchanged after the rotation of the device.

Many practical optical elements and systems whose purpose is to convert the SOP of light with minimal losses (wave plates, polarization rotators, LC layers in most kinds of LCDs, compensation systems in LCDs, etc.) can be considered to a good approximation as devices that transmit light, at normal incidence, without diattenuation.

Idealized Systems in the Jones Calculus. Unitary Systems

As a rule, the object for JC is an idealized system whose transmittance multiplied by a certain attenuation factor is considered to be equal to the transmittance of a real (realistic) lossy system of interest. The attenuation factor may take account of absorption losses in isotropic layers of the lossy system, reflection losses, and some other kinds of losses. Almost always, the losses on the polarization-converting elements that are considered to perform transformations without diattenuation are taken into account in the attenuation factor, so that these elements are represented in the idealized system by lossless elements. An idealized system consisting of only lossless elements is clearly lossless. Such systems are called *unitary systems*. Representing the Jones matrix of a realistic lossy system with negligible diattenuation in the form $\varsigma\mathbf{t}_{\text{U}}$, where $\mathbf{t}_{\text{U}}$ is a unitary matrix and ς is a scalar, we can use the matrix $\mathbf{t}_{\text{U}}$ as operator relating polarization Jones vectors of the incident and emergent waves (we have used this in Section 1.3.5). If

$\mathbf{t}_U$ is the Jones matrix of a unitary system associated with the lossy system, we may regard this unitary system as a model system that transforms polarization in the same manner as the realistic lossy system. The concept of a unitary system is widely used in LCD optics (see Chapters 2, 3, 6, and 12).

The typical idealized optical system for JC is a sequence of elements each of which is able to convert the polarization state of light. The effect of spaces between the elements is usually disregarded, because, as a rule, there is no need to trace the changes in the absolute phase of the passing light.

1.4.2 Three Useful Theorems for Transmissive Systems

The usual model of an inhomogeneous LC layer is a pile of homogeneous birefringent layers (see Sections 2.1 and 11.1.1). The standard idealized model of a transmissive LCD to treat by means of JC is also a pile of homogeneous anisotropic layers. In this section, we present three theorems showing how the transmission Jones matrix of such a system changes under certain transformations of the system. Applied to inhomogeneous LC layers, these theorems are useful when there is a need to compare the optical properties of similar layers whose structures (LC director fields) are mapped into each other by a rotation, a reflection, or the inversion (see, e.g., [10]). For systems invariant under any of the transformations considered here, by using these theorems, it is easy to find restrictions imposed by this invariance on the Jones matrices of these systems. Knowledge of such restrictions simplifies solving some optimization problems for LCDs (see Chapter 6).

Consider a system S consisting of N simple birefringent layers (Figure 1.11) (say, a system of linear polarizers and linear retarders) whose boundaries are perpendicular to an axis z. The effect of spaces between the layers will be ignored here. Let the elements of the system (birefringent layers) be numbered as shown in Figure 1.11, and let a light wave $\vec{\mathbf{X}}_i$ propagating in the positive z direction be incident on the system (Figure 1.11a). We can calculate the Jones matrix of the system,

$$\vec{\boldsymbol{t}}_S \equiv \begin{pmatrix} \vec{t}_{S11} & \vec{t}_{S12} \\ \vec{t}_{S21} & \vec{t}_{S22} \end{pmatrix}, \tag{1.228}$$

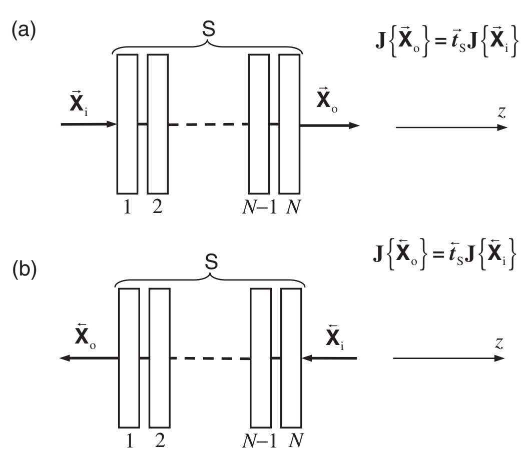

Figure 1.11 A transmissive system of birefringent layers. $\mathbf{J}\{\mathbf{X}\}$ stands for the Jones vector of a wave $\mathbf{X}$. Parts (a) and (b) show the two cases compared in Jones's reversibility theorem

as

$$\vec{\boldsymbol{t}}_{\mathrm{S}} = \vec{\boldsymbol{t}}_N \vec{\boldsymbol{t}}_{N-1} \ldots \vec{\boldsymbol{t}}_2 \vec{\boldsymbol{t}}_1, \tag{1.229}$$

where $\vec{\boldsymbol{t}}_j$ is a Jones matrix of layer number j, if the input reference frame[3] of the matrix $\vec{\boldsymbol{t}}_j$ (at j=2, … , N) is the same as the output reference frame of the matrix $\vec{\boldsymbol{t}}_{j-1}$. This condition will be satisfied if we use a fixed frame as the input and output one for all the matrices $\vec{\boldsymbol{t}}_j$. Take the frame (x_{I}, y_{I}) of a rectangular right-handed Cartesian system (x_{I}, y_{I}, z_{I}) with the z_{I}-axis codirectional with the z-axis as such a fixed frame. Let (x'_j, y'_j) be a frame whose axes are parallel to the principal axes of the jth layer, and let

$$\boldsymbol{t}'_j = \begin{pmatrix} t_{xj} & 0 \\ 0 & t_{yj} \end{pmatrix} \tag{1.230}$$

be the transmission Jones matrix of the jth layer for Jones vectors referred to the frame (x'_j, y'_j). Then the matrices $\vec{\boldsymbol{t}}_j$ can be represented as

$$\vec{\boldsymbol{t}}_j = \widehat{R}_{\mathrm{C}}(-\phi_j)\boldsymbol{t}'_j\widehat{R}_{\mathrm{C}}(\phi_j), \tag{1.231}$$

where ϕ_j is the angle between the axes x_{I} and x'_j.

Note that at any ϕ_j, the matrix $\vec{\boldsymbol{t}}_j$ is symmetric, that is,

$$\vec{\boldsymbol{t}}_j = \vec{\boldsymbol{t}}_j^{\mathrm{T}}. \tag{1.232}$$

Actually, according to (1.231),

$$\vec{\boldsymbol{t}}_j^{\mathrm{T}} = \left(\widehat{R}_{\mathrm{C}}(-\phi_j)\boldsymbol{t}'_j\widehat{R}_{\mathrm{C}}(\phi_j)\right)^{\mathrm{T}}. \tag{1.233}$$

Using matrix identity (5.14), the relation $\widehat{R}_{\mathrm{C}}(\phi)^{\mathrm{T}} = \widehat{R}_{\mathrm{C}}(-\phi)$, and the fact that $\boldsymbol{t}'^{\mathrm{T}}_j = \boldsymbol{t}'_j$, we can rewrite this expression as

$$\vec{\boldsymbol{t}}_j^{\mathrm{T}} = \widehat{R}_{\mathrm{C}}(\phi_j)^{\mathrm{T}}\boldsymbol{t}'^{\mathrm{T}}_j\widehat{R}_{\mathrm{C}}(-\phi_j)^{\mathrm{T}} = \widehat{R}_{\mathrm{C}}(-\phi_j)\boldsymbol{t}'_j\widehat{R}_{\mathrm{C}}(\phi_j). \tag{1.234}$$

Comparing (1.234) and (1.231), we see that $\vec{\boldsymbol{t}}_j^{\mathrm{T}} = \vec{\boldsymbol{t}}_j$.

Theorem 1.1 The Jones matrix $\vec{\boldsymbol{t}}_{\mathrm{S}'}$ of a system S′ that can be obtained from the system S by the permutation of the elements that provides the inverse order of the elements and, possibly, by rotating some elements by 180° about the z-axis is related to the Jones matrix of the system S as follows:

$$\vec{\boldsymbol{t}}_{\mathrm{S}'} = \vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}. \tag{1.235}$$

[3] Considering a Jones matrix, we will call the reference frames to which the input and output Jones vectors for this matrix are referred respectively the *input frame* and *output frame* of this Jones matrix. A frame that is used as both the input one and the output one for a Jones matrix will be called *the input and output frame*.

Proof. The rotation of any element about the z-axis by 180° does not change the Jones matrix of this element. Therefore, in any case, the matrix $\vec{t}_{S'}$ can be expressed in terms of the Jones matrices of the elements of the system S as

$$\vec{t}_{S'} = \vec{t}_1\vec{t}_2 \ldots \vec{t}_{N-1}\vec{t}_N. \tag{1.236}$$

Using the fact that all the matrices $\vec{t}_j$ are symmetric and identity (5.14), we can transform this expression as follows:

$$\vec{t}_{S'} = \vec{t}_1^{\,T}\vec{t}_2^{\,T} \ldots \vec{t}_{N-1}^{\,T}\vec{t}_N^{\,T} = \left(\vec{t}_N\vec{t}_{N-1} \ldots \vec{t}_2\vec{t}_1\right)^{T}. \tag{1.237}$$

As is seen from (1.237) and (1.229), the matrix $\vec{t}_{S'}$ is really equal to $\vec{t}_S^{\,T}$.

If the system S is such that $\vec{t}_N = \vec{t}_1, \vec{t}_{N-1} = \vec{t}_2$, and so on, the inversion of the order of its elements will give a system whose Jones matrix is equal to $\vec{t}_S$. It follows from Theorem 1.1 that the matrix $\vec{t}_S$ in this case satisfies the condition $\vec{t}_S = \vec{t}_S^{\,T}$, that is, it is symmetric.

Going to the next theorem, denote the values of the azimuthal angles ϕ_j and matrices $\vec{t}_j$ (j=1,2, … , N) for the system S by $\phi_j^{(S)}$ and $\vec{t}_j^{(S)}$ respectively. With this notation, the matrix $\vec{t}_S$ is expressed as

$$\vec{t}_S = \vec{t}_N^{(S)}\vec{t}_{N-1}^{(S)} \ldots \vec{t}_2^{(S)}\vec{t}_1^{(S)}, \tag{1.238}$$

where

$$\vec{t}_j^{(S)} = \hat{R}_C\left(-\phi_j^{(S)}\right)t'_j\hat{R}_C\left(\phi_j^{(S)}\right). \tag{1.239}$$

Theorem 1.2 Suppose that a system S′ consists of the same layers as the system S and their order is the same as in S, but the layers are rotated about the z-axis so that for the jth layer (j = 1,2, … , N) the angle ϕ_j is equal to $-\phi_j^{(S)}$ or $-\phi_j^{(S)} + 180°$. Then the Jones matrices of the systems S′ and S are related by

$$\vec{t}_{S'} = \mathbf{I}_1\vec{t}_S\mathbf{I}_1, \tag{1.240}$$

where

$$\mathbf{I}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.241}$$

Note that $\mathbf{I}_1\mathbf{I}_1 = \mathbf{U}$, where, as before, $\mathbf{U}$ is the unit matrix, that is, $\mathbf{I}_1^{-1} = \mathbf{I}_1$. According to (1.240),

$$\vec{t}_{S'} = \begin{pmatrix} \vec{t}_{S11} & -\vec{t}_{S12} \\ -\vec{t}_{S21} & \vec{t}_{S22} \end{pmatrix}.$$

Proof. The Jones matrix $\vec{t}_j^{(S')}$ of the jth layer of the system S′ for Jones vectors referred to the frame (x_I, y_I), whether ϕ_j for this layer be equal to $-\phi_j^{(S)}$ or $-\phi_j^{(S)} + 180°$, can be expressed as follows:

$$\vec{t}_j^{(S')} = \hat{R}_C\left(\phi_j^{(S)}\right)t'_j\hat{R}_C\left(-\phi_j^{(S)}\right). \tag{1.242}$$

It is easy to check that, at any ϕ, $\widehat{R}_{\mathrm{C}}(\phi) = \mathbf{I}_1\widehat{R}_{\mathrm{C}}(-\phi)\,\mathbf{I}_1$. Using this relation, we can rewrite (1.242) as

$$\vec{\boldsymbol{t}}_j^{(\mathrm{S}')} = \mathbf{I}_1\widehat{R}_{\mathrm{C}}\left(-\phi_j^{(\mathrm{S})}\right)\mathbf{I}_1\boldsymbol{t}'_j\mathbf{I}_1\widehat{R}_{\mathrm{C}}\left(\phi_j^{(\mathrm{S})}\right)\mathbf{I}_1. \tag{1.243}$$

Since $\boldsymbol{t}'_j$ is a diagonal matrix, $\mathbf{I}_1\boldsymbol{t}'_j\mathbf{I}_1 = \boldsymbol{t}'_j$. Consequently, from (1.243) we have

$$\vec{\boldsymbol{t}}_j^{(\mathrm{S}')} = \mathbf{I}_1\widehat{R}_{\mathrm{C}}\left(-\phi_j^{(\mathrm{S})}\right)\boldsymbol{t}'_j\widehat{R}_{\mathrm{C}}\left(\phi_j^{(\mathrm{S})}\right)\mathbf{I}_1. \tag{1.244}$$

From (1.244) and (1.239), we see that

$$\vec{\boldsymbol{t}}_j^{(\mathrm{S}')} = \mathbf{I}_1\vec{\boldsymbol{t}}_j^{(\mathrm{S})}\mathbf{I}_1. \tag{1.245}$$

In the case under consideration, the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}'}$ is expressed in terms of the matrices $\vec{\boldsymbol{t}}_j^{(\mathrm{S}')}$ as follows:

$$\vec{\boldsymbol{t}}_{\mathrm{S}'} = \vec{\boldsymbol{t}}_N^{(\mathrm{S}')}\vec{\boldsymbol{t}}_{N-1}^{(\mathrm{S}')}\ldots\vec{\boldsymbol{t}}_2^{(\mathrm{S}')}\vec{\boldsymbol{t}}_1^{(\mathrm{S}')}. \tag{1.246}$$

On substituting from (1.245) into (1.246), we obtain

$$\vec{\boldsymbol{t}}_{\mathrm{S}'} = \mathbf{I}_1\vec{\boldsymbol{t}}_N^{(\mathrm{S})}\mathbf{I}_1\mathbf{I}_1\vec{\boldsymbol{t}}_{N-1}^{(\mathrm{S})}\mathbf{I}_1\ldots\mathbf{I}_1\vec{\boldsymbol{t}}_2^{(\mathrm{S})}\mathbf{I}_1\mathbf{I}_1\vec{\boldsymbol{t}}_1^{(\mathrm{S})}\mathbf{I}_1 = \mathbf{I}_1\left(\vec{\boldsymbol{t}}_N^{(\mathrm{S})}\vec{\boldsymbol{t}}_{N-1}^{(\mathrm{S})}\ldots\vec{\boldsymbol{t}}_2^{(\mathrm{S})}\vec{\boldsymbol{t}}_1^{(\mathrm{S})}\right)\mathbf{I}_1, \tag{1.247}$$

where we have made use of the property $\mathbf{I}_1\mathbf{I}_1 = \mathbf{U}$.

Theorem 1.3 Suppose that a system S'' differs from a system S' that satisfies the conditions of the previous theorem only in that it has the inverse order of elements, and, consequently, the Jones matrix of the system S'', $\vec{\boldsymbol{t}}_{\mathrm{S}''}$, can be expressed in terms of the matrices $\vec{\boldsymbol{t}}_j^{(\mathrm{S}')}$ as follows:

$$\vec{\boldsymbol{t}}_{\mathrm{S}''} = \vec{\boldsymbol{t}}_1^{(\mathrm{S}')}\vec{\boldsymbol{t}}_2^{(\mathrm{S}')}\ldots\vec{\boldsymbol{t}}_{N-1}^{(\mathrm{S}')}\vec{\boldsymbol{t}}_N^{(\mathrm{S}')}. \tag{1.248}$$

Then the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}''}$ is related to the Jones matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}$ of the system S by

$$\vec{\boldsymbol{t}}_{\mathrm{S}''} = \mathbf{I}_1\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\mathbf{I}_1. \tag{1.249}$$

According to (1.249),

$$\vec{\boldsymbol{t}}_{\mathrm{S}''} = \begin{pmatrix} \vec{t}_{\mathrm{S}11} & -\vec{t}_{\mathrm{S}21} \\ -\vec{t}_{\mathrm{S}12} & \vec{t}_{\mathrm{S}22} \end{pmatrix}. \tag{1.250}$$

Proof. By Theorem 1.1, $\vec{\boldsymbol{t}}_{\mathrm{S}''} = \vec{\boldsymbol{t}}_{\mathrm{S}'}^{\mathrm{T}}$. According to Theorem 1.2, $\vec{\boldsymbol{t}}_{\mathrm{S}'} = \mathbf{I}_1\vec{\boldsymbol{t}}_{\mathrm{S}}\mathbf{I}_1$. Therefore,

$$\vec{\boldsymbol{t}}_{\mathrm{S}''} = \left(\mathbf{I}_1\vec{\boldsymbol{t}}_{\mathrm{S}}\mathbf{I}_1\right)^{\mathrm{T}} = \mathbf{I}_1^{\mathrm{T}}\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\mathbf{I}_1^{\mathrm{T}} = \mathbf{I}_1\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\mathbf{I}_1.$$

Note that a system S'' satisfying the conditions of Theorem 1.3 can be obtained by the rotation of the system S by 180° about an axis parallel to the x_{I}-axis. Thus, Theorem 1.3 makes clear how the Jones matrix of a system of birefringent layers is transformed under such a rotation. Starting from Theorem 1.3, by means of standard basis transformations, it is easy to find the rule of transformation of the Jones matrix of such a system under the 180° rotation of this system about a given axis perpendicular to the light propagation direction for the case of an arbitrary orientation of this axis with respect to the axes of the reference frame for the Jones matrix.

If the rotation of the system S by 180° about an axis parallel to the x_{I}-axis maps the system S into itself, that is, yields a system that is equivalent to S in its initial state, then, according to Theorem 1.3,

$$\vec{\boldsymbol{t}}_{\mathrm{S}} = \mathbf{I}_1 \vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}} \mathbf{I}_1, \tag{1.251}$$

which implies the following form of the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}$:

$$\vec{\boldsymbol{t}}_{\mathrm{S}} = \begin{pmatrix} \vec{t}_{\mathrm{S}11} & \vec{t}_{\mathrm{S}12} \\ -\vec{t}_{\mathrm{S}12} & \vec{t}_{\mathrm{S}22} \end{pmatrix}. \tag{1.252}$$

Applying this conclusion to the standard model of an inhomogeneous LC layer as a pile of homogeneous uniaxial layers with a varying, from layer to layer, orientation of the optic axis (see Section 11.1.1), one can readily show that the transmission Jones matrix of an LC layer that is invariant with respect to the 180° rotation about an axis parallel to the layer boundaries (this kind of symmetry is typical of LC layers of practical LCDs, see Figure 6.7 and Section 6.2.3) has the form (1.252) if the axis x_{I} of a reference frame (x_{I}, y_{I}) which is used as the input and output one for this Jones matrix is parallel to the symmetry axis (axis C_2 in Figure 6.7).

Certainly, the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}$ has the form (1.252) not only when the system S is symmetrical in the mentioned sense. For any variant of S for which $\boldsymbol{t}'_{N-j+1} = \boldsymbol{t}'_j$ and $\phi^{(\mathrm{S})}_{N-j+1}$ is equal to $-\phi^{(\mathrm{S})}_j$ or $-\phi^{(\mathrm{S})}_j \pm 180°$ ($j = 1,2,\dots,N$), the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}$ will be of the form (1.252).

For completeness, we must also mention here the following obvious relation. If a system S′ is composed of the same elements as the system S, arranged in the same order, but these elements are rotated about the z-axis so that for them the angles ϕ_j are equal to $\phi_j^{(\mathrm{S})} + \alpha_{\mathrm{R}}$ or $\phi_j^{(\mathrm{S})} + \alpha_{\mathrm{R}} + 180°$, where α_{R} is a fixed angle, the matrices $\vec{\boldsymbol{t}}_{\mathrm{S}'}$ and $\vec{\boldsymbol{t}}_{\mathrm{S}}$ are related by

$$\vec{\boldsymbol{t}}_{\mathrm{S}'} = \widehat{R}_{\mathrm{C}}(-\alpha_{\mathrm{R}}) \vec{\boldsymbol{t}}_{\mathrm{S}} \widehat{R}_{\mathrm{C}}(\alpha_{\mathrm{R}}). \tag{1.253}$$

1.4.3 Reciprocity Relations. Jones's Reversibility Theorem

In the previous section, we supposed that light is incident on the system S in the positive direction of the z-axis. Denote the transmission Jones matrix of this system for light incident on this system from the other side in the opposite direction (Figure 1.11b) by $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$. Using Theorem 1.3 of the previous section, we can easily determine the relation between the matrices $\vec{\boldsymbol{t}}_{\mathrm{S}}$ and $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$. Let a system S″ be identical to the system S rotated by 180° about an axis parallel to the x_{I}-axis and let a coordinate system (x_{R}, y_{R}, z_{R}) whose frame (x_{R}, y_{R}) is used as the input and output basis of the matrix $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$ be identical to the system (x_{I}, y_{I}, z_{I}) rotated by 180° about the x_{I}-axis (Figure 1.12a). With this choice of the frame (x_{R}, y_{R}) the matrix $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$ is obviously equal to the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}''}$ which is referred to the frame (x_{I}, y_{I}). Since the system S″ satisfies the conditions of Theorem 1.3, $\vec{\boldsymbol{t}}_{\mathrm{S}''} = \mathbf{I}_1 \vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}} \mathbf{I}_1$ and, consequently,

$$\overleftarrow{\boldsymbol{t}}_{\mathrm{S}} = \mathbf{I}_1 \vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}} \mathbf{I}_1.$$

Considering three choices of the basis (x_{R}, y_{R}, z_{R}) that are shown in Figure 1.12 and named C1, C2, and C3, the relationship between the matrices $\vec{\boldsymbol{t}}_{\mathrm{S}}$ and $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$ can be expressed as follows:

$$\overleftarrow{\boldsymbol{t}}_{\mathrm{S}} = \mathbf{U}_{\mathrm{r}} \vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}} \mathbf{U}_{\mathrm{r}}, \tag{1.254}$$

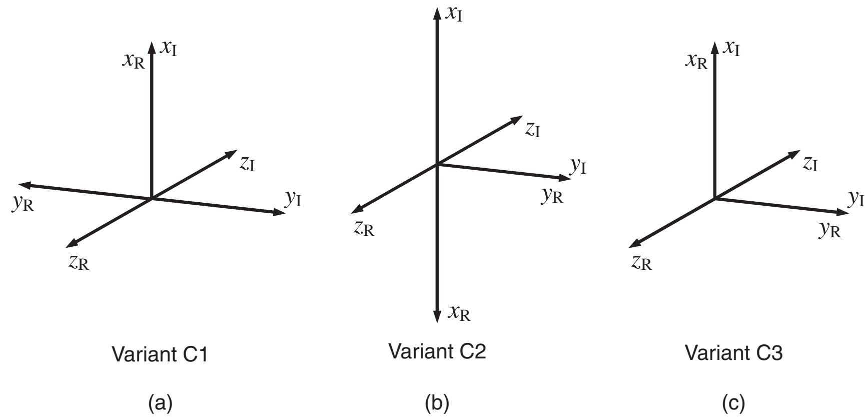

Figure 1.12 Three choices of reference frames for the Jones vectors of waves propagating in opposite directions. The axes z_I and z_R indicate the propagation directions of the waves. The system (x_I, y_I, z_I) is right-handed. The system (x_R, y_R, z_R) is right-handed in the cases C1 and C2 and left-handed in the case C3

where

$$\mathbf{U}_r = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{in the case C1} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{in the case C2} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{in the case C3.} \end{cases} \tag{1.255}$$

In the cases C1 and C2, the matrix $\overleftarrow{\boldsymbol{t}}_S$ is expressed in terms of the elements of $\vec{\boldsymbol{t}}_S$ as

$$\overleftarrow{\boldsymbol{t}}_S = \begin{pmatrix} \vec{t}_{S11} & -\vec{t}_{S21} \\ -\vec{t}_{S12} & \vec{t}_{S22} \end{pmatrix}. \tag{1.256}$$

In the case C3,

$$\overleftarrow{\boldsymbol{t}}_S = \begin{pmatrix} \vec{t}_{S11} & \vec{t}_{S21} \\ \vec{t}_{S12} & \vec{t}_{S22} \end{pmatrix}. \tag{1.257}$$

Equations that relate a transfer characteristic of an optical system to the characteristic of the same kind but for the reverse passage of light through the system, such as (1.254), are usually called *reciprocity relations*.

In the literature, reciprocity relations for Jones matrices of polarization devices are often written in the form $\overleftarrow{\boldsymbol{t}} = \vec{\boldsymbol{t}}^{\,T}$ (see, e.g., [11]) and correspond to the situation when the input reference frame for the matrix $\vec{\boldsymbol{t}}$ is the same (geometrically) as the output reference frame for the matrix $\overleftarrow{\boldsymbol{t}}$ and vice versa (as is the case in the above example for the variant C3). Devices for which such a reciprocity relation holds are sometimes called *reciprocal*. The usual polarization elements of LCDs—LC layer, film polarizers, and compensation films—are reciprocal devices. A layer of an isotropic medium with natural optical activity can also be considered as a reciprocal optical element. An example of a polarization-converting device that is not reciprocal is a Faraday rotator.

Jones's Reversibility Theorem

Certainly, relation (1.254) can be deduced by using the only requirement to the elements of the system S—each of them must be reciprocal. Actually, assuming that the elements of the system S are reciprocal, we can express the matrix $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$ as follows:

$$\begin{aligned}\overleftarrow{\boldsymbol{t}}_{\mathrm{S}} &= \left(\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_1^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}\right)\left(\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_2^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}\right)\ldots\left(\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_{N-1}^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}\right)\left(\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_N^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}\right)\\ &= \mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_1^{\mathrm{T}}\left(\mathbf{U}_{\mathrm{r}}\mathbf{U}_{\mathrm{r}}\right)\vec{\boldsymbol{t}}_2^{\mathrm{T}}\left(\mathbf{U}_{\mathrm{r}}\mathbf{U}_{\mathrm{r}}\right)\ldots\left(\mathbf{U}_{\mathrm{r}}\mathbf{U}_{\mathrm{r}}\right)\vec{\boldsymbol{t}}_{N-1}^{\mathrm{T}}\left(\mathbf{U}_{\mathrm{r}}\mathbf{U}_{\mathrm{r}}\right)\vec{\boldsymbol{t}}_N^{\mathrm{T}}\mathbf{U}_{\mathrm{r}},\end{aligned}$$

where the product $\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_j^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}$ represents the Jones matrix of the jth element for the reverse direction of light propagation. For all the three variants of $\mathbf{U}_{\mathrm{r}}$, $\mathbf{U}_{\mathrm{r}}\mathbf{U}_{\mathrm{r}} = \mathbf{U}$. Consequently,

$$\overleftarrow{\boldsymbol{t}}_{\mathrm{S}} = \mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_1^{\mathrm{T}}\vec{\boldsymbol{t}}_2^{\mathrm{T}}\ldots\vec{\boldsymbol{t}}_{N-1}^{\mathrm{T}}\vec{\boldsymbol{t}}_N^{\mathrm{T}}\mathbf{U}_{\mathrm{r}} = \mathbf{U}_{\mathrm{r}}\left(\vec{\boldsymbol{t}}_N\vec{\boldsymbol{t}}_{N-1}\ldots\vec{\boldsymbol{t}}_2\vec{\boldsymbol{t}}_1\right)^{\mathrm{T}}\mathbf{U}_{\mathrm{r}} = \mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\mathbf{U}_{\mathrm{r}},$$

which shows that the system S is reciprocal. The statement that a system composed of reciprocal elements is reciprocal expresses the essence of *Jones's reversibility theorem* [5, 11].

In Section 8.6.2, we consider analogous reciprocity relations of the rigorous electromagnetic theory of light propagation in stratified media.

The reciprocity relations for Jones matrices are used, for example, in calculations for reflective devices, and in particular RLCDs (see, e.g., [12]).

Application to Reflective Devices

Consider a reflective device consisting of a transmissive system S and a specular reflector (mirror) R which reflects the light transmitted by the system S back to S (Figure 1.13). Denote the transmission Jones matrices of the system S for the propagation directions toward the reflector and from it by $\vec{\boldsymbol{t}}_{\mathrm{S}}$ and $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$, respectively. The Jones matrix describing reflection from the mirror will be denoted by $\boldsymbol{r}_{\mathrm{R}}$. Let a frame $(x_{\mathrm{I}}, y_{\mathrm{I}})$, chosen as in the above consideration, be used as the input and output reference frame for the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}$ and the input reference frame for the matrix $\boldsymbol{r}_{\mathrm{R}}$, and let a reference frame $(x_{\mathrm{R}}, y_{\mathrm{R}})$ be used as the input and output one for the matrix $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$ and the output one for the matrix $\boldsymbol{r}_{\mathrm{R}}$. Then we can express

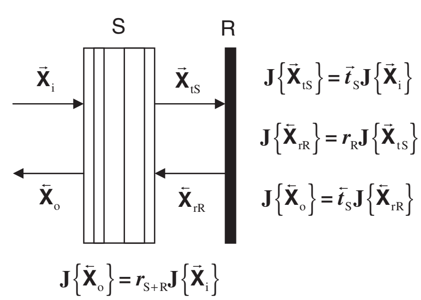

Figure 1.13 A reflective system. Geometry and notation

the Jones matrix $\boldsymbol{r}_{\mathrm{S+R}}$ that relates the Jones vector of the wave $\vec{\mathbf{X}}_{\mathrm{i}}$ incident on the reflective system and that of the wave $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$ emerging from this system (see Figure 1.13) as follows:

$$\boldsymbol{r}_{\mathrm{S+R}} = \overleftarrow{\boldsymbol{t}}_{\mathrm{S}}\boldsymbol{r}_{\mathrm{R}}\vec{\boldsymbol{t}}_{\mathrm{S}}. \tag{1.258}$$

For the three variants of the frame $(x_{\mathrm{R}}, y_{\mathrm{R}})$ shown in Figure 1.12, the Jones matrix of the reflector can be represented as

$$\boldsymbol{r}_{\mathrm{R}} = r_{\mathrm{R}}\mathbf{U}_{\mathrm{r}}, \tag{1.259}$$

where $r_{\mathrm{R}} = \sqrt{R_{\mathrm{R}}}$ with R_{R} being the reflectivity of the reflector, and $\mathbf{U}_{\mathrm{r}}$, as before, is the matrix defined by (1.255). In the case of an ideal lossless reflector, one can take

$$\boldsymbol{r}_{\mathrm{R}} = \mathbf{U}_{\mathrm{r}}. \tag{1.260}$$

Using the reciprocity relation $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}} = \mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}$ and (1.259), we can modify expression (1.258) as follows:

$$\boldsymbol{r}_{\mathrm{S+R}} = \overleftarrow{\boldsymbol{t}}_{\mathrm{S}}\boldsymbol{r}_{\mathrm{R}}\vec{\boldsymbol{t}}_{\mathrm{S}} = \left(\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\mathbf{U}_{\mathrm{r}}\right)\left(r_{\mathrm{R}}\mathbf{U}_{\mathrm{r}}\right)\vec{\boldsymbol{t}}_{\mathrm{S}} = r_{\mathrm{R}}\mathbf{U}_{\mathrm{r}}\left(\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\vec{\boldsymbol{t}}_{\mathrm{S}}\right). \tag{1.261}$$

Thus, one can compute the matrix $\boldsymbol{r}_{\mathrm{S+R}}$ without dealing with the matrix $\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}$. Note that the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\vec{\boldsymbol{t}}_{\mathrm{S}}$ is symmetric, as is the matrix $\boldsymbol{r}_{\mathrm{S+R}}$ in the case C3. In the cases C1 and C2, the off-diagonal elements of $\boldsymbol{r}_{\mathrm{S+R}}$ are equal but opposite in sign.

The following theorem is also useful in considering RLCDs.

1.4.4 Theorem of Polarization Reversibility for Systems Without Diattenuation

Let $\mathbf{X}_{\mathrm{d}}$ and $\mathbf{X}_{\mathrm{r}}$ be plane monochromatic waves of the same frequency propagating in an isotropic medium in opposite directions. We will say that the polarization of the wave $\mathbf{X}_{\mathrm{r}}$ is *reverse* with respect to the polarization of the wave $\mathbf{X}_{\mathrm{d}}$, or that the waves $\mathbf{X}_{\mathrm{d}}$ and $\mathbf{X}_{\mathrm{r}}$ are *reversely polarized*, if the shape and orientation of the polarization ellipses of these waves are identical, but these ellipses are described in opposite senses (Figure 1.14). Note that the handedness of the polarization ellipses of waves with mutually reverse polarizations is the same (recall that oppositely propagating waves are compared here). For example, the waves $\mathbf{X}_{\mathrm{d}}$ and $\mathbf{X}_{\mathrm{r}}$ can be called reversely polarized if they both have the right circular polarization or left circular polarization. If the waves $\mathbf{X}_{\mathrm{d}}$ and $\mathbf{X}_{\mathrm{r}}$ are linearly polarized and have the same polarization plane, they can also be called reversely polarized. If waves $\mathbf{X}_{\mathrm{d}}$ and $\mathbf{X}_{\mathrm{r}}$ have mutually

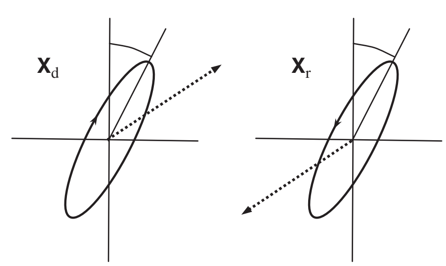

Figure 1.14 Reversely polarized waves. The dotted arrows show the propagation directions of the waves

reverse polarizations and reference frames to which the Jones vectors of these waves, $\mathbf{J}\{\mathbf{X}_{\mathrm{d}}\}$ and $\mathbf{J}\{\mathbf{X}_{\mathrm{r}}\}$, are referred, are chosen as in Figure 1.12, the relationship between these vectors can be expressed as follows:

$$\mathbf{J}\{\mathbf{X}_{\mathrm{d}}\} = k\mathbf{U}_{\mathrm{r}}\mathbf{J}\{\mathbf{X}_{\mathrm{r}}\}^*, \tag{1.262}$$

where k is a scalar factor depending on the intensities and phases of the waves; the matrix $\mathbf{U}_{\mathrm{r}}$ is defined in (1.255).

If a reciprocal system is free of polarization-dependent losses, for this system a theorem, which we will call *the theorem of polarization reversibility*, is valid [11, 12]. With the notation of Figure 1.11 for the incident ($\vec{\mathbf{X}}_{\mathrm{i}}$, $\overleftarrow{\mathbf{X}}_{\mathrm{i}}$) and transmitted ($\vec{\mathbf{X}}_{\mathrm{o}}$, $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$) waves, this theorem can be formulated as follows: whatever the polarization of $\vec{\mathbf{X}}_{\mathrm{i}}$, if the polarization of $\overleftarrow{\mathbf{X}}_{\mathrm{i}}$ is reverse with respect to the polarization of $\vec{\mathbf{X}}_{\mathrm{o}}$, the polarization of $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$ will be reverse with respect to that of $\vec{\mathbf{X}}_{\mathrm{i}}$. This theorem can be proved in the following way.

Suppose that the polarization of the wave $\overleftarrow{\mathbf{X}}_{\mathrm{i}}$ is reverse to that of the wave $\vec{\mathbf{X}}_{\mathrm{o}}$. By making use of (1.262), we can express the Jones vector of $\overleftarrow{\mathbf{X}}_{\mathrm{i}}$ as follows:

$$\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{i}}\right\} = k\mathbf{U}_{\mathrm{r}}\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{o}}\right\}^*. \tag{1.263}$$

By definition,

$$\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{o}}\right\} = \vec{\boldsymbol{t}}_{\mathrm{S}}\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{i}}\right\}, \tag{1.264}$$

$$\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{o}}\right\} = \overleftarrow{\boldsymbol{t}}_{\mathrm{S}}\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{i}}\right\}. \tag{1.265}$$

On substituting (1.263) into (1.265), we have

$$\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{o}}\right\} = k\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}\mathbf{U}_{\mathrm{r}}\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{o}}\right\}^*. \tag{1.266}$$

According to (1.264) and identity (5.13),

$$\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{o}}\right\}^* = \vec{\boldsymbol{t}}_{\mathrm{S}}^*\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{i}}\right\}^*.$$

Substitution of this expression into (1.266) leads to the following relation:

$$\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{o}}\right\} = k\overleftarrow{\boldsymbol{t}}_{\mathrm{S}}\mathbf{U}_{\mathrm{r}}\vec{\boldsymbol{t}}_{\mathrm{S}}^*\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{i}}\right\}^*. \tag{1.267}$$

Using (1.254), we can rewrite this relation as follows:

$$\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{o}}\right\} = k\mathbf{U}_{\mathrm{r}}\left(\vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\vec{\boldsymbol{t}}_{\mathrm{S}}^*\right)\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{i}}\right\}^*. \tag{1.268}$$

Since the system under consideration is free of polarization-dependent losses, the matrix $\vec{\boldsymbol{t}}_{\mathrm{S}}$ satisfies the relations

$$\vec{\boldsymbol{t}}_{\mathrm{S}}^{\dagger}\vec{\boldsymbol{t}}_{\mathrm{S}} = (\vec{\boldsymbol{t}}_{\mathrm{S}}^{\dagger}\vec{\boldsymbol{t}}_{\mathrm{S}})^* = \vec{\boldsymbol{t}}_{\mathrm{S}}^{\mathrm{T}}\vec{\boldsymbol{t}}_{\mathrm{S}}^* = t_{\mathrm{S}}\mathbf{U}, \tag{1.269}$$

where t_{S} is the transmittance of the system. From (1.268) and (1.269), we obtain

$$\mathbf{J}\left\{\overleftarrow{\mathbf{X}}_{\mathrm{o}}\right\} = k'\mathbf{U}_{\mathrm{r}}\mathbf{J}\left\{\vec{\mathbf{X}}_{\mathrm{i}}\right\}^{*}, \tag{1.270}$$

where $k' = kt_{\mathrm{S}}$. Relation (1.270) [cf. (1.262)] shows that with the chosen polarization of $\vec{\mathbf{X}}_{\mathrm{i}}$, the polarization of $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$ is really reverse to that of $\vec{\mathbf{X}}_{\mathrm{i}}$.

This theorem explains the following properties of reflective systems without diattenuation, which are very important in considering single-polarizer reflective LCDs and transflective LCDs.

Two Important Properties of Reflective Systems Without Polarization-Dependent Losses

To present these properties, we return to the problem illustrated by Figure 1.13 and assume that the system S is free of diattenuation.

Property 1 Suppose that the wave $\vec{\mathbf{X}}_{\mathrm{i}}$ incident on the reflective system is linearly polarized and the transmitted wave $\vec{\mathbf{X}}_{\mathrm{tS}}$ is also linearly polarized. Then the reflected wave $\overleftarrow{\mathbf{X}}_{\mathrm{rR}}$ is linearly polarized and has the same polarization plane as $\vec{\mathbf{X}}_{\mathrm{tS}}$, that is, the waves $\overleftarrow{\mathbf{X}}_{\mathrm{rR}}$ and $\vec{\mathbf{X}}_{\mathrm{tS}}$ are reversely polarized. According to the theorem of polarization reversibility, the wave $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$ in this case is linearly polarized and has the same polarization plane as the incident wave $\vec{\mathbf{X}}_{\mathrm{i}}$.

Property 2 Let the wave $\vec{\mathbf{X}}_{\mathrm{i}}$ incident on the system be linearly polarized and let the transmitted wave $\vec{\mathbf{X}}_{\mathrm{tS}}$ be circularly polarized. In this case, the output wave $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$ will be linearly polarized and have a polarization plane perpendicular to that of $\vec{\mathbf{X}}_{\mathrm{i}}$. To elucidate this situation, we assume, for definiteness, that the wave $\vec{\mathbf{X}}_{\mathrm{tS}}$ has the right circular polarization. In this case, the reflected wave $\overleftarrow{\mathbf{X}}_{\mathrm{rR}}$ will have the left circular polarization. It follows from the theorem of polarization reversibility that if the wave $\overleftarrow{\mathbf{X}}_{\mathrm{rR}}$ had the right circular polarization, the output wave $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$ would have the linear polarization and the same polarization plane as $\vec{\mathbf{X}}_{\mathrm{i}}$. However, $\overleftarrow{\mathbf{X}}_{\mathrm{rR}}$ has polarization orthogonal to the right circular one, and the wave $\overleftarrow{\mathbf{X}}_{\mathrm{o}}$, being linearly polarized, will have its polarization plane orthogonal to the polarization plane of $\vec{\mathbf{X}}_{\mathrm{i}}$, which is clear in view of the fact that transformations without diattenuation convert orthogonally polarized waves into orthogonally polarized ones (see item *Lossless transformations and transformations without diattenuation* in Section 1.4.1).

1.4.5 Particular Variants of Application of the Jones Calculus. Cartesian Jones Vectors for Wave Fields in Anisotropic Media

Reduced Transmittance of a System

When dealing with optical devices in which the input and output elements are polarizers (e.g., double-polarizer LCDs, single-polarizer reflective LCDs, reflective LCDs with polarizing beam splitters), the following approach is often used.

A scheme of light passage through an idealized system used in considering such a device can be written as follows: input polarizer → polarization-converting system → output polarizer. As for LCDs, typical elements of polarization-converting systems (PCSs), along with LC layer, are compensation films (retarders) and reflector in the case of reflective LCDs. On the assumption that the polarizers are ideal, the transmittance T defined as

$$T \equiv I_{\mathrm{out}}/I_{\mathrm{incPCS}}, \tag{1.271}$$

where I_{incPCS} is the intensity of the light incident on the PCS and I_{out} is the intensity of the light emerging from the output polarizer, is considered as a key characteristic of the system. This kind of transmittance will be referred to as *reduced transmittance*. The reduced transmittance can be expressed in terms of the Jones matrix of the PCS, $\boldsymbol{t}_{\text{PCS}}$, as

$$T = \left| \boldsymbol{j}_{\text{tp2}}^{\dagger} \boldsymbol{t}_{\text{PCS}} \boldsymbol{j}_{\text{tp1}} \right|^2, \tag{1.272}$$

where $\boldsymbol{j}_{\text{tp1}}$ and $\boldsymbol{j}_{\text{tp2}}$ are the polarization Jones vectors of waves that are transmitted by the input polarizer and the output polarizer, respectively [cf. (1.204)]. Sometimes, equation (1.272) is directly used for computation of T. When using this expression, it should be remembered that the vectors $\boldsymbol{j}_{\text{tp1}}$ and $\boldsymbol{j}_{\text{tp2}}$ must be referred, respectively, to the input and output reference frames of the matrix $\boldsymbol{t}_{\text{PCS}}$. In Chapter 6, we give convenient explicit expressions for the reduced transmittance in terms of orientation angles of the polarizers for different kinds of LCDs and present optimization methods using these expressions.

Unimodular Representation of Unitary Jones Matrices

In Chapters 2 and 3 and some other places of this book, PCSs of LCDs are considered as systems of lossless optical elements, that is, as unitary systems. The absence of losses allows one to calculate the Jones matrices of PCSs dealing with only unitary Jones matrices. Such calculations as well as further analysis and calculations are simplified when all elements of the PCS are represented by unimodular unitary (UU) Jones matrices, because such matrices are simple in form and their product is a matrix of a simple form (see Section 5.1.3). Any optical element that can be represented by a unitary Jones matrix can be represented in such calculations by a UU Jones matrix that describes the same transformation of polarization. In the most compact and convenient variants of representation of Jones matrices for lossless elements, these matrices are unimodular (see, e.g., expressions (1.184) and (1.185) for wave plates). The product of UU matrices is a UU matrix. Therefore, the Jones matrix of a unitary system that is calculated as the product of UU Jones matrices is also a UU matrix. By definition, the determinant of any unimodular matrix is equal to 1 or –1. All UU 2×2 matrices of determinant 1 have the form (5.31). This is the case, for example, for rotation matrices $\widehat{R}_{\text{C}}$ [see (1.53)] and Jones matrices for wave plates given by (1.184) and (1.185). The product of such UU matrices is always a matrix of determinant 1, that is, a matrix of the form (5.31). All UU 2×2 matrices of determinant -1 have the form (5.33). This is the case, for example, for the reflection Jones matrix of a lossless reflector given by (1.260) in the cases C1 and C2. In calculations involving such Jones matrices, the sign of the determinant of the resultant matrix of the system and, consequently, the form of this matrix can be predicted by using property (5.17) of determinants. In the context of the optical equivalence theorem that is presented in Section 3.1, it is important that any unitary system can be represented by a UU Jones matrix with determinant 1 (the multiplication of a UU 2×2 matrix by the imaginary unit gives a UU matrix with the opposite sign of the determinant) and that only three real parameters are in general required to fully specify such a matrix [(see (5.32)].

Cartesian Jones Vector for a Wave Field Propagating in an Anisotropic Medium

So far we dealt only with Cartesian Jones vectors that describe waves propagating in isotropic media, for example, in an isotropic medium surrounding an optical system or in isotropic spaces between optical elements. In the classical JC, Cartesian Jones vectors are used to characterize wave fields propagating inside anisotropic regions as well. In particular, this variant of description underlies the differential JC [8] (see Sections 2.1 and 11.1.1) which is used for treatment of inhomogeneous layers whose local optical parameters are continuous functions of spatial coordinates, such as inhomogeneous LC layers. The use of Cartesian Jones vectors for describing wave fields propagating in anisotropic media raises some questions. To explain, we return to the example illustrated by Figure 1.9a.

Suppose that the light falls on the uniaxial layer in the normal direction and is polarized so that both ordinary and extraordinary waves are induced. In accordance with the classical JC, the state of the wave field consisting of the forward propagating ordinary and extraordinary waves at an arbitrary point inside the layer can be described by a "fitted-to-intensity" Cartesian Jones vector $\boldsymbol{J} = (J_{x''}\ J_{y''})^{\mathrm{T}}$ referred to an arbitrary rectangular coordinate system (x'', y'', z'') with the z''-axis directed along the wave normal of the incident wave. A Cartesian Jones vector of any kind is a column composed of two Cartesian components of a vector collinear to the complex electric field strength vector of the wave field to be characterized. In our case, the wave field to be characterized is a superposition of two waves and its electric field strength vector, we denote it by $\mathbf{E}_{\mathrm{e+o}}$, is equal to $\mathbf{E}_{\mathrm{e}} + \mathbf{E}_{\mathrm{o}}$, where $\mathbf{E}_{\mathrm{e}}$ and $\mathbf{E}_{\mathrm{o}}$ are the electric fields strength vectors of the extraordinary wave and ordinary wave, respectively. By definition, we have

$$J_{x''} = b(\boldsymbol{x}''\mathbf{E}_{\mathrm{e+o}}), \quad J_{y''} = b(\mathbf{y}''\mathbf{E}_{\mathrm{e+o}}), \tag{1.273}$$

where $\boldsymbol{x}''$ and $\boldsymbol{y}''$ are unit vectors along the axes x'' and y'', and b is a complex coefficient. If the optic axis of the layer is parallel to its boundaries and, consequently, the vector $\mathbf{E}_{\mathrm{e+o}}$ is perpendicular to the z''-axis, the Jones vector $\boldsymbol{J}$ characterizes the wave field to the same extent as the Jones vector, of the same kind, characterizing a wave propagating in an isotropic medium. However, there is a serious difference. The contributions of the extraordinary and ordinary components into the intensity, with any reasonable choice of the physical quantity considered as intensity (see Sections 5.2 and 5.4), depend on their phase velocities which are different. Therefore the ratio of $|\boldsymbol{J}|^2$ to the intensity is dependent on $\boldsymbol{J}$. This means that the vector $\boldsymbol{J}$ cannot be "fitted-to-intensity" in principle. This vector can be considered to be approximately "fitted-to-intensity" only when the principal refractive indices of the anisotropic medium are very close to each other or, more precisely, when $|n_{\|} - n_{\perp}| \ll n_{\|}, n_{\perp}$. Thus, defining a Cartesian Jones vector as in (1.273) and postulating that this vector is "fitted-to-intensity," we thereby restrict the consideration to the case of a weakly anisotropic medium. The assumption that the medium is weakly anisotropic also allows us to disregard the fact that at $\theta \neq 0, 90°$ the field $\mathbf{E}_{\mathrm{e}}$ has a nonzero z''-component [see (1.141)], since at $|n_{\|} - n_{\perp}| \ll n_{\|}, n_{\perp}$ this component is very small compared with the transverse constituent of $\mathbf{E}_{\mathrm{e}}$. Note that liquid crystals in most display applications cannot be considered as a weakly anisotropic medium.

It is possible to remove the mentioned restriction by using another, somewhat artificial, definition of Cartesian Jones vector for anisotropic media. To illustrate this, we proceed with the above example.

To define the Cartesian vector $\boldsymbol{J}(\xi)$ at points of a plane $z_c = \xi$ inside the uniaxial layer, we may imagine that we replaced the rest of the layer beyond this plane by an isotropic medium and let the light pass the boundary $z_c = \xi$ without losses. Then we may take as $\boldsymbol{J}(\xi)$ the Jones vector of the emergent wave just beyond the plane $z_c = \xi$. It is clear that this kind of definition of Jones vectors is applicable in considering inhomogeneous layers as well. We should note that this definition, where the Jones vector characterizes the wave field inside the anisotropic medium indirectly, is to the greatest extent consistent with the standard apparatus of JC developed for considering continuously inhomogeneous media, which is used in LCD optics for calculating Jones matrices for inhomogeneous LC layers (see Sections 2.1 and 11.1.1).

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