1

Principal Laws and Methods in Electrical Machine Design

1.1 Electromagnetic Principles

A comprehensive command of electromagnetic phenomena relies fundamentally on Maxwell's equations. The description of electromagnetic phenomena is relatively easy when compared with various other fields of physical sciences and technology, since all the field equations can be written as a single group of equations. The basic quantities involved in the phenomena are the following five vector quantities and one scalar quantity:

Electric field strength	$\boldsymbol{E}$	[V/m]
Magnetic field strength	$\boldsymbol{H}$	[A/m]
Electric flux density	$\boldsymbol{D}$	[C/m^2]
Magnetic flux density	$\boldsymbol{B}$	[V s/m^2], [T]
Current density	$\boldsymbol{J}$	[A/m^2]
Electric charge density, dQ/dV	ρ	[C/m^3]

The presence of an electric and magnetic field can be analysed from the force exerted by the field on a charged object or a current-carrying conductor. This force can be calculated by the Lorentz force (Figure 1.1), a force experienced by an infinitesimal charge dQ moving at a speed $\boldsymbol{v}$. The force is given by the vector equation

$$\mathrm{d}\boldsymbol{F} = \mathrm{d}Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \mathrm{d}Q\boldsymbol{E} + \frac{\mathrm{d}Q}{\mathrm{d}t}\mathrm{d}\boldsymbol{l} \times \boldsymbol{B} = \mathrm{d}Q\boldsymbol{E} + i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}. \tag{1.1}$$

In principle, this vector equation is the basic equation in the computation of the torque for various electrical machines. The latter part of the expression in particular, formulated with a current-carrying element of a conductor of the length d$\boldsymbol{l}$, is fundamental in the torque production of electrical machines.

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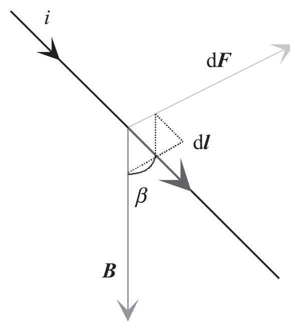

Figure 1.1 Lorentz force d$\boldsymbol{F}$ acting on a differential length d$\boldsymbol{l}$ of a conductor carrying an electric current i in the magnetic field $\boldsymbol{B}$. The angle β is measured between the conductor and the flux density vector $\boldsymbol{B}$. The vector product $i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}$ may now be written in the form $i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{B} = i\mathrm{d}lB\sin\beta$

Example 1.1: Calculate the force exerted on a conductor 0.1 m long carrying a current of 10 A at an angle of 80° with respect to a field density of 1 T.

Solution: Using (1.1) we get directly for the magnitude of the force

$$F = |i\boldsymbol{l} \times \boldsymbol{B}| = 10\,\mathrm{A} \cdot 0.1\,\mathrm{m} \cdot \sin 80^\circ \cdot 1\,\mathrm{Vs/m^2} = 0.98\,\mathrm{V\,A\,s/m} = 0.98\,\mathrm{N}.$$

In electrical engineering theory, the other laws, which were initially discovered empirically and then later introduced in writing, can be derived from the following fundamental laws presented in complete form by Maxwell. To be independent of the shape or position of the area under observation, these laws are presented as differential equations.

A current flowing from an observation point reduces the charge of the point. This law of conservation of charge can be given as a divergence equation

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}, \tag{1.2}$$

which is known as the continuity equation of the electric current.

Maxwell's actual equations are written in differential form as

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.3}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{1.4}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{1.5}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{1.6}$$

The curl relation (1.3) of an electric field is Faraday's induction law that describes how a changing magnetic flux creates an electric field around it. The curl relation (1.4) for magnetic field strength describes the situation where a changing electric flux and current produce magnetic field strength around them. This is Ampère's law. Ampère's law also yields a law for conservation of charge (1.2) by a divergence Equation (1.4), since the divergence of the curl is identically zero. In some textbooks, the curl operation may also be expressed as $\nabla \times \boldsymbol{E} = \mathrm{curl}\,\boldsymbol{E} = \mathrm{rot}\,\boldsymbol{E}$.

An electric flux always flows from a positive charge and passes to a negative charge. This can be expressed mathematically by the divergence Equation (1.5) of an electric flux. This law is also known as Gauss's law for electric fields. Magnetic flux, however, is always a circulating flux with no starting or end point. This characteristic is described by the divergence Equation (1.6) of the magnetic flux density. This is Gauss's law for magnetic fields. The divergence operation in some textbooks may also be expressed as $\nabla \cdot \boldsymbol{D} = \mathrm{div}\,\boldsymbol{D}$.

Maxwell's equations often prove useful in their integral form: Faraday's induction law

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{1.7}$$

states that the change of a magnetic flux Φ penetrating an open surface S is equal to a negative line integral of the electric field strength along the line l around the surface. Mathematically, an element of the surface S is expressed by a differential operator $\mathrm{d}\boldsymbol{S}$ perpendicular to the surface. The contour line l of the surface is expressed by a differential vector $\mathrm{d}\boldsymbol{l}$ parallel to the line.

Faraday's law together with Ampère's law are extremely important in electrical machine design. At its simplest, the equation can be employed to determine the voltages induced in the windings of an electrical machine. The equation is also necessary for instance in the determination of losses caused by eddy currents in a magnetic circuit, and when determining the skin effect in copper. Figure 1.2 illustrates Faraday's law. There is a flux Φ penetrating through a surface S, which is surrounded by the line l.

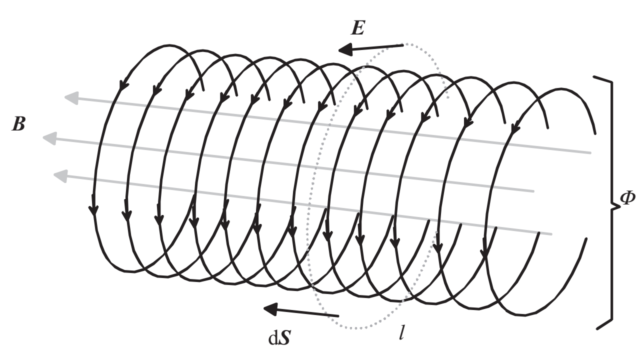

Figure 1.2 Illustration of Faraday's induction law. A typical surface S, defined by a closed line l, is penetrated by a magnetic flux Φ with a density $\boldsymbol{B}$. A change in flux density creates an electric current strength $\boldsymbol{E}$. The circles illustrate the behaviour of $\boldsymbol{E}$. $\mathrm{d}\boldsymbol{S}$ is a vector perpendicular to the surface S

The arrows in the circles point the direction of the electric field strength $\boldsymbol{E}$ in the case where the flux density $\boldsymbol{B}$ inside the observed area is increasing. If we place a short-circuited metal wire around the flux, we will obtain an integrated voltage $\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$ in the wire, and consequently also an electric current. This current creates its own flux that will oppose the flux penetrating through the coil.

If there are several turns N of winding (cf. Figure 1.2), the flux does not link all these turns ideally, but with a ratio of less than unity. Hence we may denote the effective turns of winding by $k_\mathrm{w}N$, ($k_\mathrm{w} < 1$). Equation (1.7) yields a formulation with an electromotive force e of a multi-turn winding. In electrical machines, the factor k_w is known as the winding factor (see Chapter 2). This formulation is essential to electrical machines and is written as

$$e = -k_\mathrm{w}N\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = -k_\mathrm{w}N\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}\Psi}{\mathrm{d}t}. \tag{1.8}$$

Here, we introduce the flux linkage $\Psi = k_\mathrm{w}N\Phi = LI$, one of the core concepts of electrical engineering. It may be noted that the inductance L describes the ability of a coil to produce flux linkage Ψ. Later, when calculating the inductance, the effective turns, the permeance Λ or the reluctance R_m of the magnetic circuit are needed ($L = (k_\mathrm{w}N)^2\Lambda = (k_\mathrm{w}N)^2/R_\mathrm{m}$).

Example 1.2: There are 100 turns in a coil having a cross-sectional area of 0.0001 m^2. There is an alternating peak flux density of 1 T linking the turns of the coil with a winding factor of $k_\mathrm{w} = 0.9$. Calculate the electromotive force induced in the coil when the flux density variation has a frequency of 100 Hz.

Solution: Using Equation (1.8) we get

$$\begin{aligned} e &= -\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -k_\mathrm{w}N\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -k_\mathrm{w}N\frac{\mathrm{d}}{\mathrm{d}t}\hat{B}S\sin\omega t \\ &= -0.9\cdot 100\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(1\frac{\mathrm{V\,s}}{\mathrm{m}^2}\cdot 0.0001\,\mathrm{m}^2\sin\frac{100}{\mathrm{s}}\cdot 2\pi t\right) \\ e &= -90\cdot 2\pi\,\mathrm{V}\cos\frac{200}{\mathrm{s}}\pi t = -565\,\mathrm{V}\cos\frac{200}{\mathrm{s}}\pi t. \end{aligned}$$

Hence, the peak value of the voltage is 565 V and the effective value of the voltage induced in the coil is $565\,\mathrm{V}/\sqrt{2} = 400\,\mathrm{V}$.

Ampère's law involves a displacement current that can be given as the time derivative of the electric flux ψ. Ampère's law

$$\oint_l \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{J}\cdot\mathrm{d}\boldsymbol{S} + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}\cdot\mathrm{d}\boldsymbol{S} = i\,(t) + \frac{\mathrm{d}\psi_\mathrm{e}}{\mathrm{d}t} \tag{1.9}$$

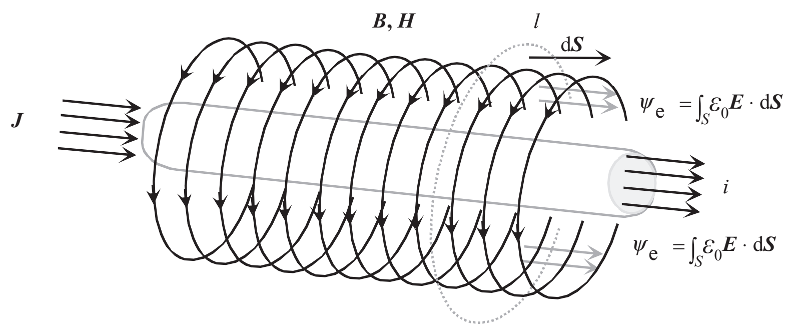

Figure 1.3 Application of Ampère's law in the surroundings of a current-carrying conductor. The line l defines a surface S, the vector d$\boldsymbol{S}$ being perpendicular to it

indicates that a current $i(t)$ penetrating a surface S and including the change of electric flux has to be equal to the line integral of the magnetic flux $\boldsymbol{H}$ along the line l around the surface S. Figure 1.3 depicts an application of Ampère's law.

The term

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}\cdot\mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}\psi_\mathrm{e}}{\mathrm{d}t}$$

in (1.9) is known as Maxwell's displacement current, which ultimately links the electromagnetic phenomena together. The displacement current is Maxwell's historical contribution to the theory of electromagnetism. The invention of displacement current helped him to explain the propagation of electromagnetic waves in a vacuum in the absence of charged particles or currents. Equation (1.9) is quite often presented in its static or quasi-static form, which yields

$$\oint_l \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{J}\cdot\mathrm{d}\boldsymbol{S} = \sum i\,(t) = \Theta\,(t)\,. \tag{1.10}$$

The term 'quasi-static' indicates that the frequency f of the phenomenon in question is low enough to neglect Maxwell's displacement current. The phenomena occurring in electrical machines meet the quasi-static requirement well, since, in practice, considerable displacement currents appear only at radio frequencies or at low frequencies in capacitors that are deliberately produced to take advantage of the displacement currents.

The quasi-static form of Ampère's law is a very important equation in electrical machine design. It is employed in determining the magnetic voltages of an electrical machine and the required current linkage. The instantaneous value of the current sum $\sum i\,(t)$ in Equation (1.10), that is the instantaneous value of current linkage Θ, can, if desired, be assumed to involve also the apparent current linkage of a permanent magnet $\Theta_\mathrm{PM} = H_\mathrm{c}' h_\mathrm{PM}$. Thus, the apparent current linkage of a permanent magnet depends on the calculated coercive force H_c' of the material and on the thickness h_PM of the magnetic material.

The corresponding differential form of Ampère's law (1.10) in a quasi-static state ($\mathrm{d}D/\mathrm{d}t$ neglected) is written as

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}. \tag{1.11}$$

The continuity Equation (1.2) for current density in a quasi-static state is written as

$$\nabla \cdot \boldsymbol{J} = 0. \tag{1.12}$$

Gauss's law for electric fields in integral form

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho_V \mathrm{d}V \tag{1.13}$$

indicates that a charge inside a closed surface S that surrounds a volume V creates an electric flux density $\boldsymbol{D}$ through the surface. Here $\int_V \rho_V \mathrm{d}V = q(t)$ is the instantaneous net charge inside the closed surface S. Thus, we can see that in electric fields, there are both sources and drains. When considering the insulation of electrical machines, Equation (1.13) is required. However, in electrical machines, it is not uncommon that charge densities in a medium prove to be zero. In that case, Gauss's law for electric fields is rewritten as

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = 0 \quad \text{or} \quad \nabla \cdot \boldsymbol{D} = 0 \Rightarrow \nabla \cdot \boldsymbol{E} = 0. \tag{1.14}$$

In uncharged areas, there are no sources or drains in the electric field either.

Gauss's law for magnetic fields in integral form

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0 \tag{1.15}$$

states correspondingly that the sum of a magnetic flux penetrating a closed surface S is zero; in other words, the flux entering an object must also leave the object. This is an alternative way of expressing that there is no source for a magnetic flux. In electrical machines, this means for instance that the main flux encircles the magnetic circuit of the machine without a starting or end point. Similarly, all other flux loops in the machine are closed. Figure 1.4 illustrates the surfaces S employed in integral forms of Maxwell's equations, and Figure 1.5, respectively, presents an application of Gauss's law for a closed surface S.

The permittivity, permeability and conductivity ε, μ and σ of the medium determine the dependence of the electric and magnetic flux densities and current density on the field strength. In certain cases, ε, μ and σ can be treated as simple constants; then the corresponding pair of quantities ($\boldsymbol{D}$ and $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$, or $\boldsymbol{J}$ and $\boldsymbol{E}$) are parallel. Media of this kind are called isotropic, which means that ε, μ and σ have the same values in different directions. Otherwise, the media have different values of the quantities ε, μ and σ in different directions, and may therefore be treated as tensors; these media are defined as anisotropic. In practice, the

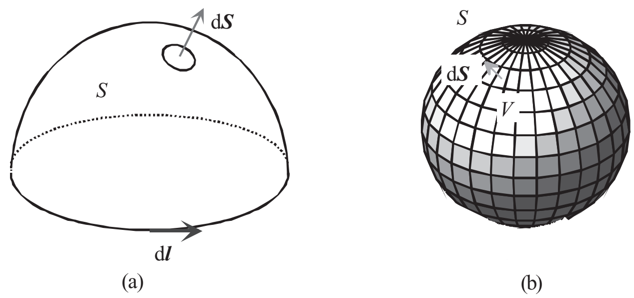

Figure 1.4 Surfaces for the integral forms of the equations for electric and magnetic fields. (a) An open surface S and its contour l, (b) a closed surface S, enclosing a volume V. d$\boldsymbol{S}$ is a differential surface vector that is everywhere normal to the surface

permeability in ferromagnetic materials is always a highly nonlinear function of the field strength H: $\mu = f(H)$.

The general formulations for the equations of a medium can in principle be written as

$$\boldsymbol{D} = f(\boldsymbol{E}), \tag{1.16}$$

$$\boldsymbol{B} = f(\boldsymbol{H}), \tag{1.17}$$

$$\boldsymbol{J} = f(\boldsymbol{E}). \tag{1.18}$$

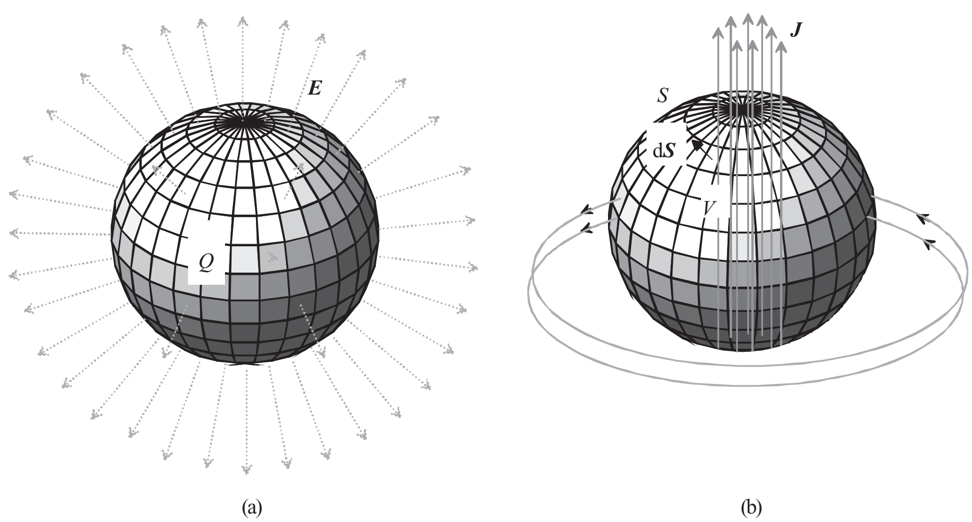

Figure 1.5 Illustration of Gauss's law for (a) an electric field and (b) a magnetic field. The charge Q inside a closed object acts as a source and creates an electric flux with the field strength $\boldsymbol{E}$. Correspondingly, a magnetic flux created by the current density $\boldsymbol{J}$ outside a closed surface S passes through the closed surface (penetrates into the sphere and then comes out). The magnetic field is thereby sourceless ($\text{div}\,\boldsymbol{B} = 0$)

The specific forms for the equations have to be determined empirically for each medium in question. By applying permittivity ε [F/m], permeability μ [V s/A m] and conductivity σ [S/m], we can describe materials by the following equations:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{1.19}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{1.20}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}. \tag{1.21}$$

The quantities describing the medium are not always simple constants. For instance, the permeability of ferromagnetic materials is strongly nonlinear. In anisotropic materials, the direction of flux density deviates from the field strength, and thus ε and μ can be tensors. In a vacuum the values are

$$\varepsilon_0 = 8.854 \cdot 10^{-12}\ \mathrm{F/m},\ \ \mathrm{A\,s/V\,m}\ \text{and}$$
$$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{H/m},\ \mathrm{V\,s/A\,m}.$$

Example 1.3: Calculate the electric field density D over an insulation layer 0.3 mm thick when the potential of the winding is 400 V and the magnetic circuit of the system is at earth potential. The relative permittivity of the insulation material is $\varepsilon_\mathrm{r} = 3$.

Solution: The electric field strength across the insulation is $E = 400\ \mathrm{V}/0.3\ \mathrm{mm} = 133\ \mathrm{kV/m}$. According to Equation (1.19), the electric field density is

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_\mathrm{r}\varepsilon_0 \boldsymbol{E} = 3 \cdot 8.854 \cdot 10^{-12}\ \mathrm{A\,s/V\,m} \cdot 133\ \mathrm{kV/m} = 3.54\ \mathrm{\mu A\,s/m^2}.$$

Example 1.4: Calculate the displacement current over the slot insulation of the previous example at 50 Hz when the insulation surface is 0.01 m^2.

Solution: The electric field over the insulation is $\psi_\mathrm{e} = DS = 0.0354\ \mathrm{\mu A\,s}$.

The time-dependent electric field over the slot insulation is

$$\psi_\mathrm{e}(t) = \hat{\psi}_\mathrm{e} \sin \omega t = 0.0354\ \mathrm{\mu A\,s}\ \sin 314t.$$

Differentiating with respect to time gives

$$\frac{\mathrm{d}\psi_\mathrm{e}(t)}{\mathrm{d}t} = \omega \hat{\psi}_\mathrm{e} \cos \omega t = 11\ \mathrm{\mu A} \cos 314t.$$

The effective current over the insulation is hence $11/\sqrt{2} = 7.86\ \mathrm{\mu A}$.

Here we see that the displacement current is insignificant from the viewpoint of the machine's basic functionality. However, when a motor is supplied by a frequency converter and the transistors create high frequencies, significant displacement currents may run across the insulation and bearing current problems, for instance, may occur.

1.2 Numerical Solution

The basic design of an electrical machine, that is the dimensioning of the magnetic and electric circuits, is usually carried out by applying analytical equations. However, accurate performance of the machine is usually evaluated using different numerical methods. With these numerical methods, the effect of a single parameter on the dynamical performance of the machine can be effectively studied. Furthermore, some tests, which are not even feasible in laboratory circumstances, can be virtually performed. The most widely used numerical method is the finite element method (FEM), which can be used in the analysis of two- or three-dimensional electromagnetic field problems. The solution can be obtained for static, time-harmonic or transient problems. In the latter two cases, the electric circuit describing the power supply of the machine is coupled with the actual field solution. When applying FEM in the electromagnetic analysis of an electrical machine, special attention has to be paid to the relevance of the electromagnetic material data of the structural parts of the machine as well as to the construction of the finite element mesh.

Because most of the magnetic energy is stored in the air gap of the machine and important torque calculation formulations are related to the air-gap field solution, the mesh has to be sufficiently dense in this area. The rule of thumb is that the air-gap mesh should be divided into three layers to achieve accurate results. In the transient analysis, that is in time-stepping solutions, the selection of the size of the time step is also important in order to include the effect of high-order time harmonics in the solution. A general method is to divide one time cycle into 400 steps, but the division could be even denser than this, in particular with high-speed machines.

There are five common methods to calculate the torque from the FEM field solution. The solutions are (1) the Maxwell stress tensor method, (2) Arkkio's method, (3) the method of magnetic coenergy differentiation, (4) Coulomb's virtual work and (5) the magnetizing current method. The mathematical torque formulations related to these methods will shortly be discussed in Sections 1.4 and 1.5.

The magnetic fields of electrical machines can often be treated as a two-dimensional case, and therefore it is quite simple to employ the magnetic vector potential in the numerical solution of the field. In many cases, however, the fields of the machine are clearly three dimensional, and therefore a two-dimensional solution is always an approximation. In the following, first, the full three-dimensional vector equations are applied.

The magnetic vector potential $\boldsymbol{A}$ is given by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}; \tag{1.22}$$

Coulomb's condition, required to define unambiguously the vector potential, is written as

$$\nabla \cdot \boldsymbol{A} = 0. \tag{1.23}$$

The substitution of the definition for the magnetic vector potential in the induction law (1.3) yields

$$\nabla \times \boldsymbol{E} = -\nabla \times \frac{\partial}{\partial t}\boldsymbol{A}. \tag{1.24}$$

Electric field strength can be expressed by the vector potential $\boldsymbol{A}$ and the scalar electric potential ϕ as

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \tag{1.25}$$

where ϕ is the reduced electric scalar potential. Because $\nabla \times \nabla\phi \equiv 0$, adding a scalar potential causes no problems with the induction law. The equation shows that the electric field strength vector consists of two parts, namely a rotational part induced by the time dependence of the magnetic field, and a nonrotational part created by electric charges and the polarization of dielectric materials.

Current density depends on the electric field strength

$$\boldsymbol{J} = \sigma\boldsymbol{E} = -\sigma\frac{\partial \boldsymbol{A}}{\partial t} - \sigma\nabla\phi. \tag{1.26}$$

Ampère's law and the definition for vector potential yield

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}\right) = \boldsymbol{J}. \tag{1.27}$$

Substituting (1.26) into (1.27) gives

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}\right) + \sigma\frac{\partial \boldsymbol{A}}{\partial t} + \sigma\nabla\phi = 0. \tag{1.28}$$

The latter is valid in areas where eddy currents may be induced, whereas the former is valid in areas with source currents $\boldsymbol{J} = \boldsymbol{J}_\mathrm{s}$, such as winding currents, and areas without any current densities $\boldsymbol{J} = 0$.

In electrical machines, a two-dimensional solution is often the obvious one; in these cases, the numerical solution can be based on a single component of the vector potential $\boldsymbol{A}$. The field solution ($\boldsymbol{B}$, $\boldsymbol{H}$) is found in an xy plane, whereas $\boldsymbol{J}$, $\boldsymbol{A}$ and $\boldsymbol{E}$ involve only the z-component. The gradient $\nabla\phi$ only has a z-component, since $\boldsymbol{J}$ and $\boldsymbol{A}$ are parallel to z, and (1.26) is valid. The reduced scalar potential is thus independent of x- and y-components. ϕ could be a linear function of the z-coordinate, since a two-dimensional field solution is independent of z. The assumption of two-dimensionality is not valid if there are potential differences caused by electric charges or by the polarization of insulators. For two-dimensional cases with eddy currents, the reduced scalar potential has to be set as $\phi = 0$.

In a two-dimensional case, the previous equation is rewritten as

$$-\nabla \cdot \left(\frac{1}{\mu}\nabla A_z\right) + \sigma\frac{\partial A_z}{\partial t} = 0. \tag{1.29}$$

Outside eddy current areas, the following is valid:

$$-\nabla \cdot \left(\frac{1}{\mu}\nabla A_z\right) = J_z. \tag{1.30}$$

The definition of vector potential yields the following components for flux density:

$$B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x}. \tag{1.31}$$

Hence, the vector potential remains constant in the direction of the flux density vector. Consequently, the iso-potential curves of the vector potential are flux lines. In the two-dimensional case, the following formulation can be obtained from the partial differential equation of the vector potential:

$$-\boldsymbol{k}\left[\frac{\partial}{\partial x}\left(\nu\frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial A_z}{\partial y}\right)\right] = \boldsymbol{k}J. \tag{1.32}$$

Here ν is the reluctivity of the material. This again is similar to the equation for a static electric field

$$\nabla \cdot (\nu\nabla \boldsymbol{A}) = -\boldsymbol{J}. \tag{1.33}$$

Further, there are two types of boundary conditions. Dirichlet's boundary condition indicates that a known potential, here the known vector potential

$$\boldsymbol{A} = \text{constant}, \tag{1.34}$$

can be achieved for a vector potential for instance on the outer surface of an electrical machine. The field is parallel to the contour of the surface. Similar to the outer surface of an electrical machine, also the centre line of the machine's pole can form a symmetry plane. Neumann's homogeneous boundary condition determined with the vector potential

$$\nu\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{n}} = 0 \tag{1.35}$$

can be achieved when the field meets a contour perpendicularly. Here $\boldsymbol{n}$ is the normal unit vector of a plane. A contour of this kind is for instance part of a field confined to infinite permeability iron or the centre line of the pole clearance.

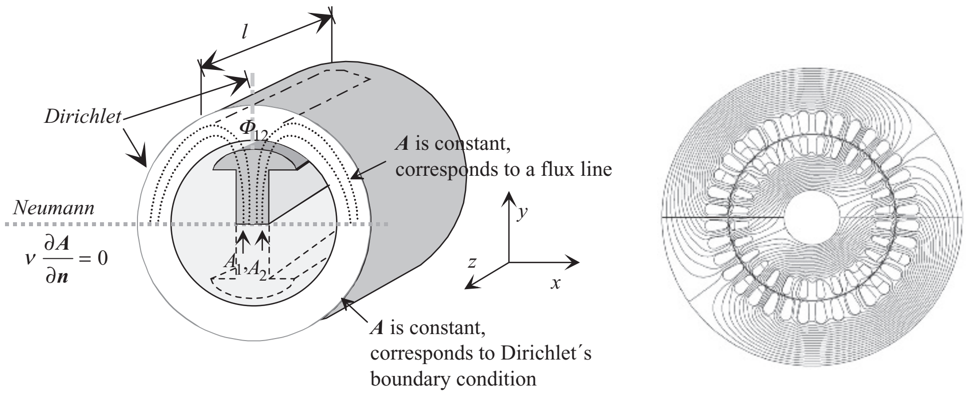

Figure 1.6 Left, a two-dimensional field and its boundary conditions for a salient-pole synchronous machine are illustrated. Here, the constant value of the vector potential $\boldsymbol{A}$ (e.g. the machine's outer contour) is taken as Dirichlet's boundary condition, and the zero value of the derivative of the vector potential with respect to normal is taken as Neumann's boundary condition. In the case of magnetic scalar potential, the boundary conditions with respect to potential would take opposite positions. Because of symmetry, the zero value of the normal derivative of the vector potential corresponds to the constant magnetic potential V_{m}, which in this case would be a known potential and thus Dirichlet's boundary condition. Right, a vector-potential-based field solution of a two-pole asynchronous machine assuming a two-dimensional field is presented

The magnetic flux penetrating a surface is easy to calculate with the vector potential. Stoke's theorem yields for the flux

$$\Phi = \int_{\mathrm{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \int_{\mathrm{S}} (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \oint_{\mathrm{l}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.36}$$

This is an integral around the contour l of the surface S. These phenomena are illustrated with Figure 1.6. In the two-dimensional case of the illustration, the end faces' share of the integral is zero, and the vector potential along the axis is constant. Consequently, for a machine of length l we obtain a flux

$$\Phi_{12} = l\,(A_1 - A_2)\,. \tag{1.37}$$

This means that the flux Φ_{12} is the flux between vector equipotential lines A_1 and A_2.

1.3 The Most Common Principles Applied to Analytic Calculation

The design of an electrical machine involves the quantitative determination of the magnetic flux of the machine. Usually, phenomena in a single pole are analysed. In the design of a magnetic circuit, the precise dimensions for individual parts are determined, the required current linkage for the magnetic circuit and also the required magnetizing current are calculated, and the magnitude of losses occurring in the magnetic circuit are estimated.

If the machine is excited with permanent magnets, the permanent magnet materials have to be selected and the main dimensions of the parts manufactured from these materials have to be determined. Generally, when calculating the magnetizing current for a rotating machine, the machine is assumed to run at no load: that is, there is a constant current flowing in the magnetizing winding. The effects of load currents are analysed later.

The design of a magnetic circuit of an electrical machine is based on Ampère's law (1.4) and (1.8). The line integral calculated around the magnetic circuit of an electrical machine, that is the sum of magnetic potential differences $\sum U_{\mathrm{m},i}$, is equal to the surface integral of the current densities over the surface S of the magnetic circuit. (The surface S here indicates the surface penetrated by the main flux.) In practice, in electrical machines, the current usually flows in the windings, the surface integral of the current density corresponding to the sum of these currents (flowing in the windings), that is the current linkage Θ. Now Ampère's law can be rewritten as

$$U_{\mathrm{m,tot}} = \sum U_{\mathrm{m},i} = \oint_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \oint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = \Theta = \sum i. \tag{1.38}$$

The sum of magnetic potential differences U_{m} around the complete magnetic circuit is equal to the sum of the magnetizing currents in the circuit, that is the current linkage Θ. In simple applications, the current sum may be given as $\sum i = k_{\mathrm{w}} N i$, where $k_{\mathrm{w}} N$ is the effective number of turns and i the current flowing in them. In addition to the windings, this current linkage may also involve the effect of the permanent magnets. In practice, when calculating the magnetic voltage, the machine is divided into its components, and the magnetic voltage U_{m} between points a and b is determined as

$$U_{\mathrm{m},ab} = \int_a^b \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.39}$$

In electrical machines, the field strength is often in the direction of the component in question, and thus Equation (1.39) can simply be rewritten as

$$U_{\mathrm{m},ab} = \int_a^b H \mathrm{d}l. \tag{1.40}$$

Further, if the field strength is constant in the area under observation, we get

$$U_{\mathrm{m},ab} = Hl. \tag{1.41}$$

In the determination of the required current linkage Θ of a machine's magnetizing winding, the simplest possible integration path is selected in the calculation of the magnetic voltages. This means selecting a path that encloses the magnetizing winding. This path is defined as the main integration path and it is also called the main flux path of the machine (see Chapter 3). In salient-pole machines, the main integration path crosses the air gap in the middle of the pole shoes.

Example 1.5: Consider a C-core inductor with a 1 mm air gap. In the air gap, the flux density is 1 T. The ferromagnetic circuit length is 0.2 m and the relative permeability of the core material at 1 T is $\mu_r = 3500$. Calculate the field strengths in the air gap and the core, and also the magnetizing current needed. How many turns N of wire carrying a 10 A direct current are needed to magnetize the choke to 1 T? Fringing in the air gap is neglected and the winding factor is assumed to be $k_w = 1$.

Solution: According to (1.20), the magnetic field strength in the air gap is

$$H_\delta = B_\delta / \mu_0 = 1\,\mathrm{V\,s/m^2} / \left(4\pi \cdot 10^{-7}\,\mathrm{V\,s/A\,m}\right) = 795\,\mathrm{kA/m}.$$

The corresponding magnetic field strength in the core is

$$H_{\mathrm{Fe}} = B_{\mathrm{Fe}} \big/ (\mu_r \mu_0) = 1\,\mathrm{V\,s/m^2} \big/ \left(3500 \cdot 4\pi \cdot 10^{-7}\,\mathrm{V\,s/A\,m}\right) = 227\,\mathrm{A/m}.$$

The magnetic voltage in the air gap (neglecting fringing) is

$$U_{\mathrm{m},\delta} = H_\delta \delta = 795\,\mathrm{kA/m} \cdot 0.001\,\mathrm{m} = 795\,\mathrm{A}.$$

The magnetic voltage in the core is

$$U_{\mathrm{m,Fe}} = H_{\mathrm{Fe}} l_{\mathrm{Fe}} = 227\,\mathrm{A/m} \cdot 0.2\,\mathrm{m} = 45\,\mathrm{A}.$$

The magnetomotive force (mmf) of the magnetic circuit is

$$\oint_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = U_{\mathrm{m,tot}} = \sum U_{\mathrm{m},i} = U_{\mathrm{m},\delta} + U_{\mathrm{m,Fe}} = 795\,\mathrm{A} + 45\,\mathrm{A} = 840\,\mathrm{A}.$$

The current linkage Θ of the choke has to be of equal magnitude with the mmf $U_{\mathrm{m,tot}}$,

$$\Theta = \sum i = k_w N i = U_{\mathrm{m,tot}}.$$

We get

$$N = \frac{U_{\mathrm{m,tot}}}{k_w i} = \frac{840\,\mathrm{A}}{1 \cdot 10\,\mathrm{A}} = 84\ \text{turns}.$$

In machine design, not only does the main flux have to be analysed, but also all the leakage fluxes of the machine have to be taken into account.

In the determination of the no-load curve of an electrical machine, the magnetic voltages of the magnetic circuit have to be calculated with several different flux densities. In practice, for the exact definition of the magnetizing curve, a computation program that solves the different magnetizing states of the machine is required.

According to their magnetic circuits, electrical machines can be divided into two main categories: in salient-pole machines, the field windings are concentrated pole windings, whereas in nonsalient-pole machines, the magnetizing windings are spatially distributed in the machine. The main integration path of a salient-pole machine consists for instance of the following components: a rotor yoke (yr), pole body (p2), pole shoe (p1), air gap (δ), teeth (d) and armature yoke (ya). For this kind of salient-pole machine or DC machine, the total magnetic voltage of the main integration path therefore consists of the following components

$$U_{\mathrm{m,tot}} = U_{\mathrm{m,yr}} + 2U_{\mathrm{m,p2}} + 2U_{\mathrm{m,p1}} + 2U_{\mathrm{m},\delta} + 2U_{\mathrm{m,d}} + U_{\mathrm{m,ya}}. \tag{1.42}$$

In a nonsalient-pole synchronous machine and induction motor, the magnetizing winding is contained in slots. Therefore both stator (s) and rotor (r) have teeth areas (d)

$$U_{\mathrm{m,tot}} = U_{\mathrm{m,yr}} + 2U_{\mathrm{m,dr}} + 2U_{\mathrm{m},\delta} + 2U_{\mathrm{m,ds}} + U_{\mathrm{m,ys}}. \tag{1.43}$$

With Equations (1.42) and (1.43), we must bear in mind that the main flux has to flow twice across the teeth area (or pole arc and pole shoe) and air gap.

In a switched reluctance (SR) machine, where both the stator and rotor have salient poles (double saliency), the following equation is valid:

$$U_{\mathrm{m,tot}} = U_{\mathrm{m,yr}} + 2U_{\mathrm{m,rp2}} + 2U_{\mathrm{m,rp1}}(\alpha) + 2U_{\mathrm{m},\delta}(\alpha) + 2U_{\mathrm{m,sp1}}(\alpha) + 2U_{\mathrm{m,sp2}} + U_{\mathrm{m,ys}}. \tag{1.44}$$

This equation proves difficult to employ, because the shape of the air gap in an SR machine varies constantly when the machine rotates. Therefore the magnetic voltage of both the rotor and stator pole shoes depends on the position of the rotor α.

The magnetic potential differences of the most common rotating electrical machines can be presented by equations similar to Equations (1.42)–(1.44).

In electrical machines constructed of ferromagnetic materials, only the air gap can be considered magnetically linear. All ferromagnetic materials are both nonlinear and often anisotropic. In particular, the permeability of oriented electrical steel sheets varies in different directions, being highest in the rolling direction and lowest in the perpendicular direction. This leads to a situation where the permeability of the material is, strictly speaking, a tensor.

The flux is a surface integral of the flux density. Commonly, in electrical machine design, the flux density is assumed to be perpendicular to the surface to be analysed. Since the area of a perpendicular surface $\boldsymbol{S}$ is S, we can rewrite the equation simply as

$$\Phi = \int B\mathrm{d}S. \tag{1.45}$$

Further, if the flux density B is also constant, we obtain

$$\Phi = BS. \tag{1.46}$$

Using the equations above, it is possible to construct a magnetizing curve for each part of the machine

$$\Phi_{ab} = f\left(U_{\mathrm{m},ab}\right),\ B = f\left(U_{\mathrm{m},ab}\right). \tag{1.47}$$

In the air gap, the permeability is constant $\mu = \mu_0$. Thus, we can employ magnetic conductivity, that is permeance Λ, which leads us to

$$\Phi_{ab} = \Lambda_{ab} U_{\mathrm{m},ab}. \tag{1.48}$$

If the air gap field is homogeneous, we get

$$\Phi_{ab} = \Lambda_{ab} U_{\mathrm{m},ab} = \frac{\mu_0 S}{\delta} U_{\mathrm{m},ab}. \tag{1.49}$$

Equations (1.38) and (1.42)–(1.44) yield the magnetizing curve for a machine

$$\Phi_\delta = f(\Theta), \quad B_\delta = f(\Theta), \tag{1.50}$$

where the term Φ_δ is the air-gap flux. The absolute value for flux density B_δ is the maximum flux density in the air gap in the middle of the pole shoe, when slotting is neglected. The magnetizing curve of the machine is determined in the order Φ_δ, $B_\delta \to B \to H \to U_\mathrm{m} \to \Theta$ by always selecting a different value for the air-gap flux Φ_δ, or for its density, and by calculating the magnetic voltages in the machine and the required current linkage Θ. With the current linkage, it is possible to determine the current *I* flowing in the windings. Correspondingly, with the air-gap flux and the winding, we can determine the electromotive force (emf) *E* induced in the windings. Now we can finally plot the actual no-load curve of the machine (Figure 1.7)

$$E = f(I). \tag{1.51}$$

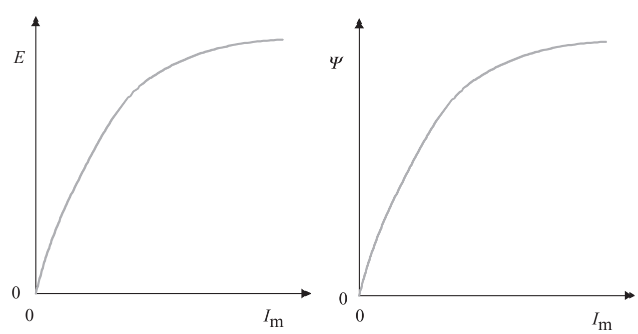

Figure 1.7 Typical no-load curve for an electrical machine expressed by the electromotive force E or the flux linkage Ψ as a function of the magnetizing current I_m. The E curve as a function of I_m has been measured when the machine is running at no load at a constant speed. In principle, the curve resembles a *BH* curve of the ferromagnetic material used in the machine. The slope of the no-load curve depends on the *BH* curve of the material, the (geometrical) dimensions and particularly on the length of the air gap

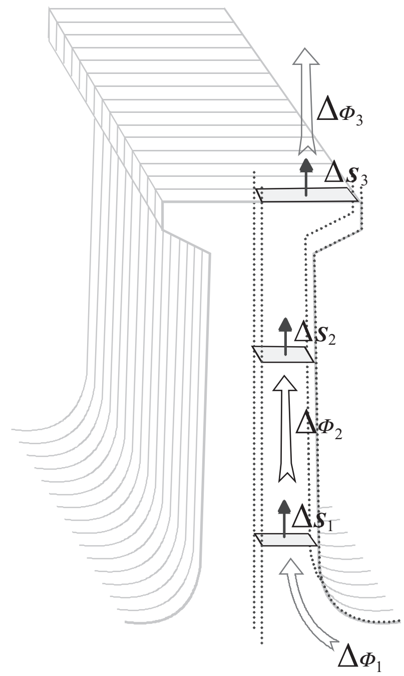

Figure 1.8 Laminated tooth and a coarse flux tube running in a lamination. The cross-sections of the tube are presented with surface vectors $\Delta\boldsymbol{S}_\mathrm{i}$. There is a flux $\Delta\Phi$ flowing in the tube. The flux tubes follow the flux lines in the magnetic circuit of the electrical machine. Most of the tubes constitute the main magnetic circuit, but a part of the flux tubes forms leakage flux paths. If a two-dimensional field solution is assumed, two-dimensional flux diagrams as shown in Figure 1.6 may replace the flux tube approach

1.3.1 Flux Line Diagrams

Let us consider areas with an absence of currents. A spatial magnetic flux can be assumed to flow in a flux tube. A flux tube can be analysed as a tube of a quadratic cross-section $\Delta\boldsymbol{S}$. The flux does not flow through the walls of the tube, and hence $\boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = 0$ is valid for the walls. As depicted in Figure 1.8, we can see that the corners of the flux tube form the flux lines.

When calculating a surface integral along a closed surface surrounding the surface of a flux tube, Gauss's law (1.15) yields

$$\oint \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = 0. \tag{1.52}$$

Since there is no flux through the side walls of the tube in Figure 1.8, Equation (1.52) can be rewritten as

$$\oint \boldsymbol{B}_1\cdot \mathrm{d}\Delta\boldsymbol{S}_1 = \oint \boldsymbol{B}_2\cdot \mathrm{d}\Delta\boldsymbol{S}_2 = \oint \boldsymbol{B}_3\cdot \mathrm{d}\Delta\boldsymbol{S}_3, \tag{1.53}$$

indicating that the flux of the flux tube is constant

$$\Delta\Phi_1 = \Delta\Phi_2 = \Delta\Phi_3 = \Delta\Phi. \tag{1.54}$$

A magnetic equipotential surface is a surface with a certain magnetic scalar potential V_{m}. When travelling along any route between two points a and b on this surface, we must get

$$\int_a^b \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = U_{\mathrm{m},ab} = V_{\mathrm{m}a} - V_{\mathrm{m}b} = 0. \tag{1.55}$$

When observing a differential route, this is valid only when $\boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$. For isotropic materials, the same result can be expressed as $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = 0$. In other words, the equipotential surfaces are perpendicular to the lines of flux.

If we select an adequately small area ΔS of the surface S, we are able to calculate the flux

$$\Delta\Phi = B\Delta S. \tag{1.56}$$

The magnetic potential difference between two equipotential surfaces that are close enough to each other (H is constant along the integration path l) is written as

$$\Delta U_{\mathrm{m}} = Hl. \tag{1.57}$$

The above equations give the permeance Λ of the cross-section of the flux tube

$$\Lambda = \frac{\Delta\Phi}{\Delta U_{\mathrm{m}}} = \frac{B \cdot \mathrm{d}S}{Hl} = \mu\frac{\mathrm{d}S}{l}. \tag{1.58}$$

The flux line diagram (Figure 1.9) comprises selected flux and potential lines. The selected flux lines confine flux tubes, which all have an equal flux $\Delta\Phi$. The magnetic voltage between the chosen potential lines is always the same, ΔU_{m}. Thus, the magnetic conductivity of each section of the flux tube is always the same, and the ratio of the distance of flux lines x to the distance of potential lines y is always the same. If we set

$$\frac{x}{y} = 1, \tag{1.59}$$

the field diagram forms, according to Figure 1.9, a grid of quadratic elements.

In a homogeneous field, the field strength H is constant at every point of the field. According to Equations (1.57) and (1.59), the distance of all potential and flux lines is thus always the same. In that case, the flux diagram comprises squares of equal size.

When constructing a two-dimensional orthogonal field diagram, for instance for the air gap of an electrical machine, certain boundary conditions have to be known to be able to draw the diagram. These boundary conditions are often solved based on symmetry, or also because the potential of a certain potential surface of the flux tube in Figure 1.8 is already known. For instance, if the stator and rotor length of the machine is l, the area of the flux tube can,

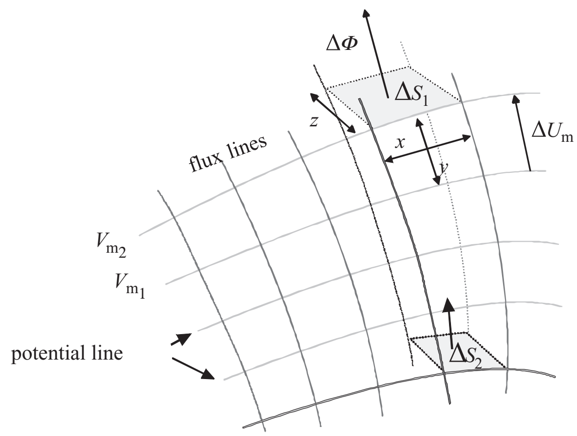

Figure 1.9 Flux lines and potential lines in a three-dimensional area with a flux flowing across an area where the length dimension z is constant. In principle, the diagram is thus two dimensional. Such a diagram is called an orthogonal field diagram

without significant error, be written as $\mathrm{d}S = l\,\mathrm{d}x$. The interface of the iron and air is now analysed according to Figure 1.10a. We get

$$\mathrm{d}\Phi_y = B_{y\delta} l \mathrm{d}x - B_{y\mathrm{Fe}} l \mathrm{d}x = 0 \Rightarrow B_{y\delta} = B_{y\mathrm{Fe}}. \tag{1.60}$$

Here, $B_{y\delta}$ and $B_{y\mathrm{Fe}}$ are the flux densities of air in the y-direction and of iron in the y-direction.

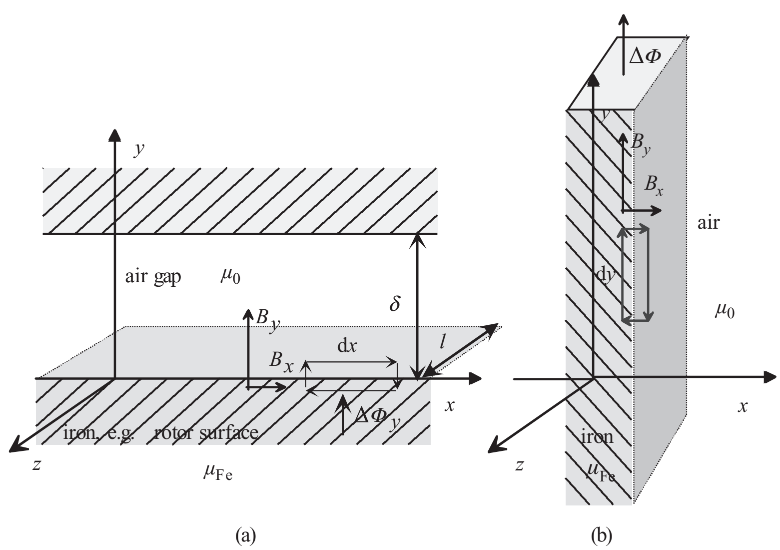

Figure 1.10 (a) Interface of air δ and iron Fe. The x-axis is tangential to the rotor surface. (b) Flux travelling on iron surface

In Figure 1.10a, the field strength has to be continuous in the x-direction on the iron–air interface. If we consider the interface in the x-direction and, based on Ampère's law, assume a section dx of the surface has no current, we get

$$H_{x\delta}\mathrm{d}x - H_{x\mathrm{Fe}}\mathrm{d}x = 0, \tag{1.61}$$

and thus

$$H_{x\delta} = H_{x\mathrm{Fe}} = \frac{B_{x\mathrm{Fe}}}{\mu_{\mathrm{Fe}}}. \tag{1.62}$$

By assuming that the permeability of iron is infinite, $\mu_{\mathrm{Fe}} \to \infty$, we get $H_{x\mathrm{Fe}} = H_{x\delta} = 0$ and thereby also $B_{x\delta} = \mu_0 H_{x\delta} = 0$.

Hence, if we set $\mu_{\mathrm{Fe}} \to \infty$, the flux lines leave the ferromagnetic material perpendicularly into the air. Simultaneously, the interface of iron and air forms an equipotential surface. If the iron is not saturated, its permeability is very high, and the flux lines can be assumed to leave the iron almost perpendicularly in currentless areas. In saturating areas, the interface of the iron and air cannot strictly be considered an equipotential surface. The magnetic flux and the electric flux refract on the interface.

In Figure 1.10b, the flux flows in the iron in the direction of the interface. If the iron is not saturated ($\mu_{\mathrm{Fe}} \to \infty$) we can set $B_x \approx 0$. Now, there is no flux passing from the iron into air. When the iron is about to become saturated ($\mu_{\mathrm{Fe}} \to 1$), a significant magnetic voltage occurs in the iron. Now, the air adjacent to the iron becomes an appealing route for the flux, and part of the flux passes into the air. This is the case for instance when the teeth of electrical machines saturate: a part of the flux flows across the slots of the machine, even though the permeability of the materials in the slot is in practice equal to the permeability of a vacuum.

The lines of symmetry in flux diagrams are either potential or field lines. When drawing a flux diagram, we have to know if the lines of symmetry are flux or potential lines. Figure 1.11 is an example of an orthogonal field diagram, in which the line of symmetry forms a potential line; this could depict for instance the air gap between the contour of an magnetizing pole of a DC machine and the rotor.

The solution of an orthogonal field diagram by drawing is best started at those parts of the geometry where the field is as homogeneous as possible. For instance, in the case of Figure 1.11, we start from the point where the air gap is at its narrowest. Now, the surface of the magnetizing pole and the rotor surface that is assumed to be smooth form the potential lines together with the surface between the poles. First, the potential lines are plotted at equal distances and, next, the flux lines are drawn perpendicularly to potential lines so that the area under observation forms a grid of quadratic elements. The length of the machine being l, each flux tube created this way carries a flux of $\Delta\Phi$.

With the field diagram, it is possible to solve various magnetic parameters of the area under observation. If n_Φ is the number (not necessarily an integer) of contiguous flux tubes carrying a flux $\Delta\Phi$, and ΔU_{m} is the magnetic voltage between the sections of a flux tube (n_U sections in sequence), the permeance of the entire air gap Λ_δ assuming that $\Delta b = \Delta\delta$ can be

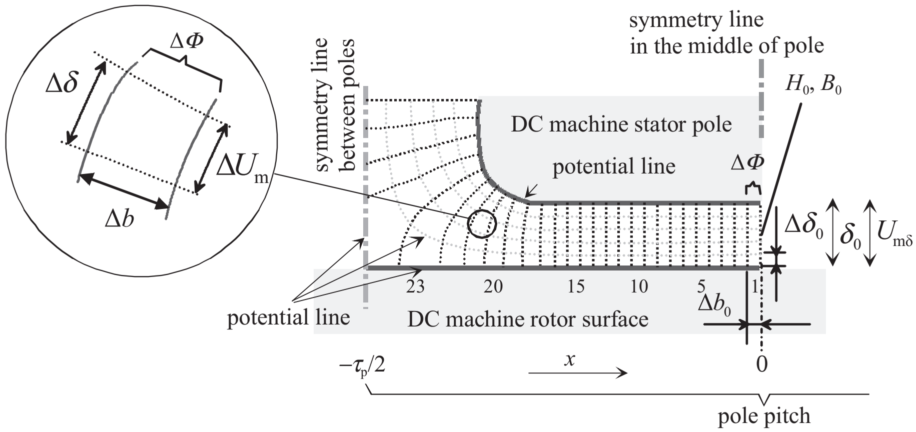

Figure 1.11 Drawing an orthogonal field diagram in an air gap of a DC machine in the edge zone of a pole shoe. Here, a differential equation for the magnetic scalar potential is solved by drawing. Dirichlet's boundary conditions for magnetic scalar potentials created on the surfaces of the pole shoe and the rotor and on the symmetry plane between the pole shoes. The centre line of the pole shoe is set at the origin of the coordinate system. At the origin, the element is dimensioned as $\Delta\delta_0$, Δb_0. The $\Delta\delta$ and Δb in different parts of the diagram have different sizes, but the $\Delta\Phi$ remains the same in all flux tubes. The pole pitch is τ_p. There are about 23.5 flux tubes from the pole surface to the rotor surface in the figure

written as

$$\Lambda_\delta = \frac{\Phi}{U_{m\delta}} = \frac{n_\Phi \Delta\Phi}{n_U \Delta U_m} = \frac{n_\Phi \Delta b l \mu_0 \dfrac{\Delta U_m}{\Delta\delta}}{n_U \Delta U_m} = \frac{n_\Phi}{n_U}\mu_0 l. \tag{1.63}$$

The magnetic field strength in the enlarged element of Figure 1.11 is

$$H = \frac{\Delta U_m}{\Delta\delta}, \tag{1.64}$$

and correspondingly the magnetic flux density

$$B = \mu_0 \frac{\Delta U_m}{\Delta\delta} = \frac{\Delta\Phi}{\Delta b l}. \tag{1.65}$$

With Equation (1.56), it is also possible to determine point by point the distribution of flux density on a potential line; in other words, on the surface of the armature or the magnetizing pole. With the notation in Figure 1.11, we get

$$\Delta\Phi_0 = B_0 \Delta b_0 l = \Delta\Phi(x) = B(x)\Delta b(x) l. \tag{1.66}$$

In the middle of the pole, where the air-gap flux is homogeneous, the flux density is

$$B_0 = \mu_0 H_0 = \mu_0 \frac{\Delta U_m}{\Delta\delta_0} = \mu_0 \frac{U_{m,\delta}}{\delta_0}. \tag{1.67}$$

Thus, the magnitude of flux density as a function of the x-coordinate is

$$B(x) = \frac{\Delta b_0}{\Delta b(x)} B_0 = \frac{\Delta b_0}{\Delta b(x)} \mu_0 \frac{U_{\mathrm{m},\delta}}{\delta_0}. \tag{1.68}$$

Example 1.6: What is the permeance of the main flux in Figure 1.11 when the air gap $\delta = 0.01$ m and the stator stack length is $l = 0.1$ m? How much flux is created with $\Theta_{\mathrm{f}} = 1000\,\mathrm{A}$?

Solution: In the centre of the pole, the orthogonal flux diagram is uniform and we see that $\Delta\delta_0$ and Δb_0 have the same size; $\Delta\delta_0 = \Delta b_0 = 2\,\mathrm{mm}$. The permeance of the flux tube in the centre of the pole is

$$\Lambda_0 = \mu_0 \frac{\Delta b_0 l}{\Delta\delta_0} = \mu_0 \frac{0.02\,\mathrm{m} \cdot 0.1\,\mathrm{m}}{0.02\,\mathrm{m}} = 4\pi \cdot 10^{-8} \frac{\mathrm{V\,s}}{\mathrm{A}}.$$

As we can see in Figure 1.11, about 23.5 flux tubes travel from half of the stator pole to the rotor surface. Each of these flux tubes transmits the same amount of flux, and hence the permeance of the whole pole seen by the main flux is

$$\Lambda = 2 \cdot 23.5 \cdot \Lambda_0 = 47 \cdot \Lambda_0 = 47 \cdot 4\pi \cdot 10^{-8} \frac{\mathrm{V\,s}}{\mathrm{A}} = 5.9 \frac{\mu\mathrm{V\,s}}{\mathrm{A}}.$$

If we have $\Theta_{\mathrm{f}} = 1000\,\mathrm{A}$ current linkage magnetizing the air gap, we get the flux

$$\Phi = \Lambda\Theta_{\mathrm{f}} = 5.9 \frac{\mu\mathrm{V\,s}}{\mathrm{A}} \cdot 1000\,\mathrm{A} = 5.9\,\mathrm{mV\,s}.$$

1.3.2 Flux Diagrams for Current-Carrying Areas

Let us first consider a situation in which an equivalent linear current density A [A/m] covers the area under observation. In principle, the linear current density corresponds to the surface current $\boldsymbol{J}_{\mathrm{s}} = \boldsymbol{n} \times \boldsymbol{H}_0$ induced in a conducting medium by an alternating field strength $\boldsymbol{H}_0$ outside the surface. The surface normal unit vector is denoted by $\boldsymbol{n}$.

In an electrical machine with windings, the 'artificial surface current', that is the local value of the linear current density A, may for instance be calculated as a current sum flowing in a slot divided by the slot pitch. Equivalent linear current density can be employed in approximation, because the currents flowing in the windings of electrical machines are usually situated close to the air gap, and the current linkages created by the currents excite mainly the air gaps. Thus, we can set $\mu = \mu_0$ in the observed area of equivalent linear current density. The utilization of equivalent linear current density simplifies the manual calculation of the machine by idealizing the potential surfaces, and does not have a crucial impact on the field diagram in the areas outside the area of linear current density. Figure 1.12 illustrates an equivalent linear current density.

The value for equivalent linear current density A is expressed per unit length in the direction of observation. The linear current density A corresponds to the tangential magnetic field

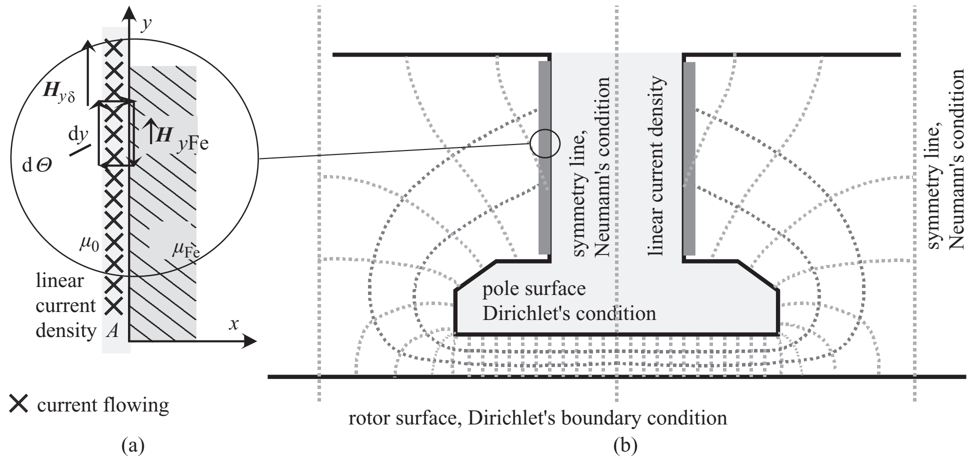

Figure 1.12 (a) General representation of linear current density A [A/m] and (b) its application to the field diagram of a magnetizing pole of a DC machine. It is important to note that in the area of the pole body, the potential lines now pass from air to iron. Dirichlet's boundary conditions indicate here a known equiscalar potential surface

strength $H_{y\delta}$. Assuming the permeability of iron to be infinite, Ampère's law yields for the element dy of Figure 1.12a

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \mathrm{d}\Theta = H_{y\mathrm{air}}\mathrm{d}y - H_{y\mathrm{Fe}}\mathrm{d}y = A\mathrm{d}y. \tag{1.69}$$

Further, this gives us

$$H_{y\mathrm{air}} = A \quad \text{and} \quad B_{y\mathrm{air}} = \mu_0 A. \tag{1.70}$$

Equation (1.70) indicates that in the case of Figure 1.12 we have a tangential flux density on the pole body surface. The tangential flux density makes the flux lines travel inclined to the pole body surface and not perpendicular to it as in currentless areas.

If we assume that the phenomenon is observed on the stator inner surface or on the rotor outer surface, the x-components may be regarded as tangential components and the y-components as normal components. In the air gap δ, there is a tangential field strength $\boldsymbol{H}_{x\delta}$ along the x-component, and a corresponding component of flux density $\boldsymbol{B}_{x\delta}$ created by the linear current density A. This is essential when considering the force density, the tangential stress $\sigma_{F\mathrm{tan}}$, that generates torque (Maxwell stresses will be discussed later). On iron surfaces with linear current density, the flux lines no longer pass perpendicularly from the iron to the air gap, as Figure 1.12 depicting the field diagram of a DC machine's magnetizing pole also illustrates. The influence of a magnetizing winding on the pole body is illustrated with the linear current density. Since the magnetizing winding is evenly distributed over the length of the pole body (the linear current density being constant), it can be seen that the potential changes linearly in the area of linear current density in the direction of the height of the pole.

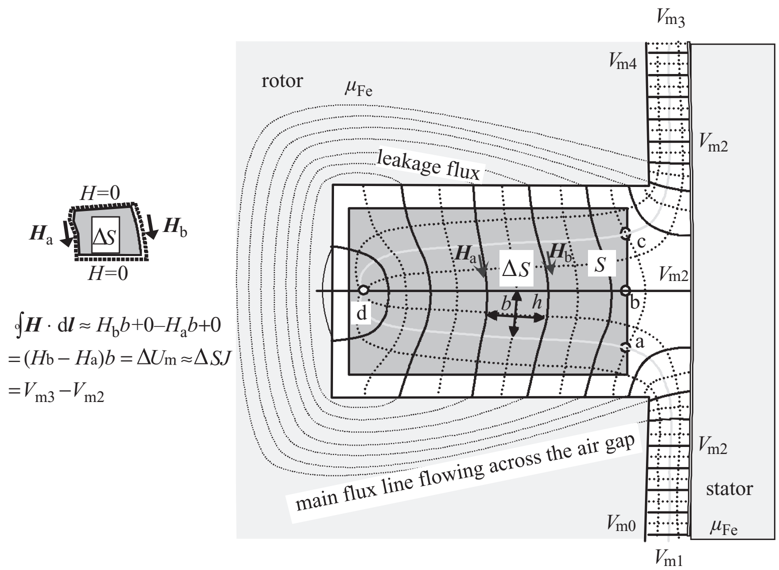

Figure 1.13 Current-carrying conductor in a slot and its field diagram. The illustration on the left demonstrates the closed line integral around the surface ΔS; also some flux lines in the iron are plotted. Note that the flux lines travelling across the slot depict leakage flux

As evidence of this we can see that in Figure 1.12 the potential lines starting in the air gap enter the area of linear current density at even distances.

In areas with current densities $\boldsymbol{J}$, the potential lines become gradient lines. This can be seen in Figure 1.13 at points a, b and c. We could assume that the figure illustrates for instance a nonsalient-pole synchronous machine field winding bar carrying a DC density. The magnetic potential difference between V_{m4} and V_{m0} equals the slot current.

The gradient lines meet the slot leakage flux lines orthogonally, which means that $\int \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$ along a gradient line. In the figure, we calculate a closed line integral around the area ΔS of the surface S

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = V_{m3} - V_{m2} = \int_{\Delta S} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}, \tag{1.71}$$

where we can see that when the current density $\boldsymbol{J}$ and the difference of magnetic potentials ΔU_m are constant, the area ΔS of the surface S also has to be constant. In other words, the selected gradient lines confine areas of equal size from the surface S with a constant current density. The gradient lines meet at a single point d, which is called an indifference point. If the current-carrying area is confined by an area with infinite permeability, the border line is a potential line and the indifference point is located on this border line. If the permeability of iron is not infinite, then d is located in the current-carrying area, as in Figure 1.13. If inside a current-carrying area the line integral is defined for instance around the area ΔS, we can see that the closer to the point d we get, the smaller become the distances between the gradient lines. In order to maintain the same current sum in the observed areas, the heights of the areas ΔS have to be changed.

Outside the current-carrying area, the following holds:

$$\Delta\Phi = \Lambda\Delta U_{\mathrm{m}} = \mu_0 l \frac{h}{b}\int_{\mathrm{S}} \boldsymbol{J}\cdot \mathrm{d}\boldsymbol{S}. \tag{1.72}$$

Inside the area under observation, when a closed line integral according to Equation (1.71) is written only for the area ΔS ($<S$), the flux of a flux tube in a current-carrying area becomes

$$\Delta\Phi' = \mu_0 l \left(\frac{h}{b}\right)' \int_{\Delta\mathrm{S}} \boldsymbol{J}\cdot \mathrm{d}\boldsymbol{S} \tag{1.73}$$

and thus, in that case, if $(h/b)' = (h/b)$, then in fact $\Delta\Phi' < \Delta\Phi$. If the current density $\boldsymbol{J}$ in the current-carrying area is constant, $\Delta\Phi' = \Delta\Phi\Delta S/S$ is valid. When crossing the boundary between a current-carrying slot and currentless iron, the flux of the flux tube cannot change. Therefore, the dimensions of the line grid have to be altered. When $\boldsymbol{J}$ is constant and $\Delta\Phi' = \Delta\Phi$, Equations (1.72) and (1.73) yield for the dimensions in the current-carrying area

$$\left(\frac{h}{b}\right)' = \frac{Sh}{\Delta Sb}. \tag{1.74}$$

This means that near the indifference point d the ratio (h/b) increases. An orthogonal field diagram can be drawn for a current-carrying area by correcting the equivalent linear current density by iterating the created diagram. For a current-carrying area, gradient lines are extended from potential lines up to the indifference point. Now, bearing in mind that the gradient lines have to divide the current-carrying area into sections of equal size, next the orthogonal flux lines are plotted by simultaneously paying attention to changing dimensions. The diagram is altered iteratively until Equation (1.74) is valid to the required accuracy.

1.4 Application of the Principle of Virtual Work in the Determination of Force and Torque

When investigating electrical equipment, the magnetic circuit of which changes form during operation, the easiest method is to apply the principle of virtual work in the estimation of force and torque. Examples of this kind of equipment are double-salient-pole reluctant machines, various relays and so on.

Faraday's induction law presents the voltage induced in the winding, which creates a current that tends to resist the changes in flux. The voltage equation for the winding is written as

$$u = Ri + \frac{\mathrm{d}\Psi}{\mathrm{d}t} = Ri + \frac{\mathrm{d}}{\mathrm{d}t}Li, \tag{1.75}$$

where u is the voltage connected to the coil terminals, R is the resistance of the winding and Ψ is the coil flux linkage, and L the self-inductance of the coil consisting of its magnetizing inductance and leakage inductance: $L = \Psi/i = N\Phi/i = N^2\Lambda = N^2/R_{\mathrm{m}}$ (see also Section 1.6).

If the number of turns in the winding is N and the flux is Φ, Equation (1.75) can be rewritten as

$$u = Ri + N\frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{1.76}$$

The required power in the winding is written correspondingly as

$$ui = Ri^2 + Ni\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{1.77}$$

and the energy

$$\mathrm{d}W = P\,\mathrm{d}t = Ri^2\,\mathrm{d}t + Ni\,\mathrm{d}\Phi. \tag{1.78}$$

The latter energy component $Ni\,\mathrm{d}\Phi$ is reversible, whereas $Ri^2\,\mathrm{d}t$ turns into heat. Energy cannot be created or destroyed, but may only be converted to different forms. In isolated systems, the limits of the energy balance can be defined unambiguously, which simplifies the energy analysis. The net energy input is equal to the energy stored in the system. This result, the first law of thermodynamics, is applied to electromechanical systems, where electrical energy is stored mainly in magnetic fields. In these systems, the energy transfer can be represented by the equation

$$\mathrm{d}W_{\mathrm{el}} = \mathrm{d}W_{\mathrm{mec}} + \mathrm{d}W_{\Phi} + \mathrm{d}W_{\mathrm{R}} \tag{1.79}$$

where

$\mathrm{d}W_{\mathrm{el}}$ is the differential electrical energy input,
$\mathrm{d}W_{\mathrm{mec}}$ is the differential mechanical energy output,
$\mathrm{d}W_{\Phi}$ is the differential change of magnetic stored energy,
$\mathrm{d}W_{\mathrm{R}}$ is the differential energy loss.

Here the energy input from the electric supply is written equal to the mechanical energy together with the stored magnetic field energy and heat loss. Electrical and mechanical energy have positive values in motoring action and negative values in generator action. In a magnetic system without losses, the change of electrical energy input is equal to the sum of the change of work done by the system and the change of stored magnetic field energy

$$\mathrm{d}W_{\mathrm{el}} = \mathrm{d}W_{\mathrm{mec}} + \mathrm{d}W_{\Phi}, \tag{1.80}$$

$$\mathrm{d}W_{\mathrm{el}} = ei\,\mathrm{d}t. \tag{1.81}$$

In the above, e is the instantaneous value of the induced voltage, created by changes in the energy in the magnetic circuit. Because of this electromotive force, the external electric circuit converts power into mechanical power by utilizing the magnetic field. This law of energy conversion combines a reaction and a counter-reaction in an electrical and mechanical

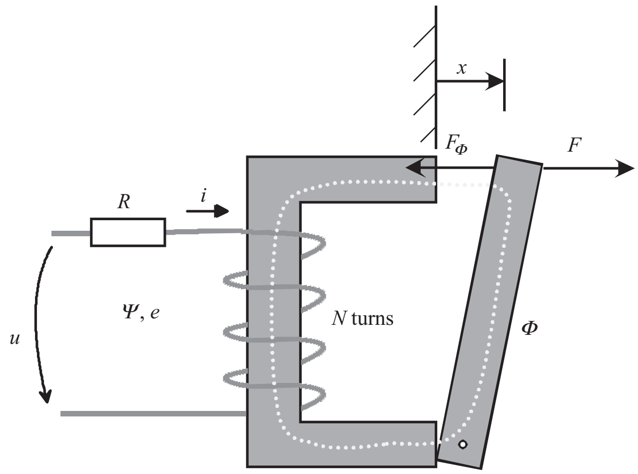

Figure 1.14 Electromagnetic relay connected to an external voltage source u. The mass of a moving yoke is neglected and the resistance of the winding is assumed to be concentrated in the external resistor R. There are N turns in the winding and a flux Φ flowing in the magnetic circuit; a flux linkage $\Psi \approx N\Phi$ is produced in the winding. The negative time derivative of the flux linkage is an emf, e. The force F pulls the yoke open. The force is produced by a mechanical source. A magnetic force F_Φ tries to close the air gap

system. The combination of Equations (1.80) and (1.81) yields

$$\mathrm{d}W_{\mathrm{el}} = ei\,\mathrm{d}t = \frac{\mathrm{d}\Psi}{\mathrm{d}t} i\,\mathrm{d}t = i\,\mathrm{d}\Psi = \mathrm{d}W_{\mathrm{mec}} + \mathrm{d}W_{\Phi}. \tag{1.82}$$

Equation (1.82) lays a foundation for the energy conversion principle. Next, its utilization in the analysis of electromagnetic energy converters is discussed.

As is known, a magnetic circuit (Figure 1.14) can be described by an inductance L determined from the number of turns of the winding, the geometry of the magnetic circuit and the permeability of the magnetic material. In electromagnetic energy converters, there are air gaps that separate the moving magnetic circuit parts from each other. In most cases – because of the high permeability of iron parts – the reluctance R_{m} of the magnetic circuit consists mainly of the reluctances of the air gaps. Thus, most of the energy is stored in the air gap. The wider the air gap, the more energy can be stored. For instance, in induction motors this can be seen from the fact that the wider is the gap, the higher is the magnetizing current needed.

According to Faraday's induction law, Equation (1.82) yields

$$\mathrm{d}W_{\mathrm{el}} = i\,\mathrm{d}\Psi. \tag{1.83}$$

The computation is simplified by neglecting for instance the magnetic nonlinearity and iron losses. The inductance of the device now depends only on the geometry and, in our example, on the distance x creating an air gap in the magnetic circuit. The flux linkage is thus a product of the varying inductance and the current

$$\Psi = L(x)i. \tag{1.84}$$

A magnetic force F_Φ is determined as

$$\mathrm{d}W_{\mathrm{mec}} = F_\Phi \,\mathrm{d}x. \tag{1.85}$$

From Equations (1.83) and (1.85), we may rewrite Equation (1.80) as

$$\mathrm{d}W_\Phi = i \,\mathrm{d}\Psi - F_\Phi \,\mathrm{d}x. \tag{1.86}$$

Since it is assumed that there are no losses in the magnetic energy storage, $\mathrm{d}W_\Phi$ is determined from the values of Ψ and x. $\mathrm{d}W_\Phi$ is independent of the integration path A or B, and the energy equation can be written as

$$W_\Phi\,(\Psi_0, x_0) = \int\limits_{\text{path A}} \mathrm{d}W_\Phi + \int\limits_{\text{path B}} \mathrm{d}W_\Phi. \tag{1.87}$$

With no displacement allowed ($\mathrm{d}x = 0$), Equations (1.86) now (1.87) yield

$$W_\Phi\,(\Psi, x_0) = \int\limits_0^\Psi i\,(\Psi, x_0)\mathrm{d}\Psi. \tag{1.88}$$

In a linear system, Ψ is proportional to current i, as in Equations (1.84) and (1.88). We therefore obtain

$$W_\Phi\,(\Psi, x_0) = \int\limits_0^\Psi i\,(\Psi, x_0)\mathrm{d}\Psi = \int\limits_0^\Psi \frac{\Psi}{L\,(x_0)}\mathrm{d}\Psi = \frac{1}{2}\frac{\Psi^2}{L\,(x_0)} = \frac{1}{2}L\,(x_0)\,i^2. \tag{1.89}$$

The magnetic field energy can also be represented by the energy density $w_\Phi = W_\Phi/V = BH/2$ [J/m^3] in a magnetic field integrated over the volume V of the magnetic field. This gives

$$W_\Phi = \int\limits_V \frac{1}{2}(H \cdot B)\,\mathrm{d}V. \tag{1.90}$$

Assuming the permeability of the magnetic medium constant and substituting $B = \mu H$ gives

$$W_\Phi = \int\limits_V \frac{1}{2}\frac{B^2}{\mu}\mathrm{d}V. \tag{1.91}$$

This yields the relation between the stored energy in a magnetic circuit and the electrical and mechanical energy in a system with a lossless magnetic energy storage. The equation for differential magnetic energy is expressed in partial derivatives

$$\mathrm{d}W_\Phi(\Psi, x) = \frac{\partial W_\Phi}{\partial \Psi}\mathrm{d}\Psi + \frac{\partial W_\Phi}{\partial x}\mathrm{d}x. \tag{1.92}$$

Since Ψ and x are independent variables, Equations (1.86) and (1.92) have to be equal at all values of $\mathrm{d}\Psi$ and $\mathrm{d}x$, which yields

$$i = \frac{\partial W_\Phi(\Psi, x)}{\partial \Psi}, \tag{1.93}$$

where the partial derivative is calculated by keeping x constant. The force created by the electromagnet at a certain flux linkage level Ψ can be calculated from the magnetic energy

$$F_\Phi = -\frac{\partial W_\Phi(\Psi, x)}{\partial x}. \tag{1.94a}$$

The minus sign is due to the coordinate system in Figure 1.14. The corresponding equation is valid for torque as a function of angular displacement θ while keeping flux linkage Ψ constant

$$T_\Phi = -\frac{\partial W_\Phi(\Psi, \theta)}{\partial \theta}. \tag{1.94b}$$

Alternatively, we may employ coenergy (see Figure 1.15a), which gives us the force directly as a function of current. The coenergy W'_Φ is determined as a function of i and x as

$$W'_\Phi(i, x) = i\Psi - W_\Phi(\Psi, x). \tag{1.95}$$

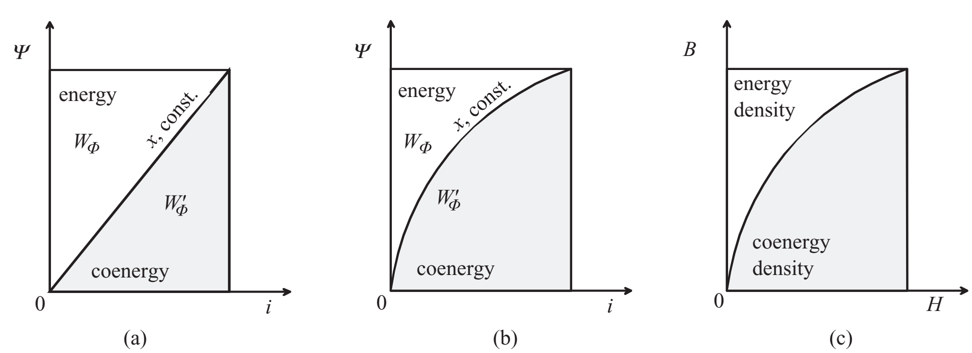

Figure 1.15 Determination of energy and coenergy with current and flux linkage (a) in a linear case (L is constant), (b) and (c) in a nonlinear case (L saturates as a function of current). If the figure is used to illustrate the behaviour of the relay in Figure 1.14, the distance x remains constant

In the conversion, it is possible to apply the differential of $i\Psi$

$$\mathrm{d}(i\Psi) = i\,\mathrm{d}\Psi + \Psi\,\mathrm{d}i. \tag{1.96}$$

Equation (1.95) now yields

$$\mathrm{d}W'_\Phi\,(i, x) = \mathrm{d}\,(i\Psi) - \mathrm{d}W_\Phi\,(\Psi, x)\,. \tag{1.97}$$

By substituting Equations (1.86) and (1.96) into Equation (1.97) we obtain

$$\mathrm{d}W'_\Phi\,(i, x) = \Psi\,\mathrm{d}i + F_\Phi\,\mathrm{d}x. \tag{1.98}$$

The coenergy W'_Φ is a function of two independent variables, i and x. This can be represented by partial derivatives

$$\mathrm{d}W'_\Phi\,(i, x) = \frac{\partial W'_\Phi}{\partial i}\mathrm{d}i + \frac{\partial W'_\Phi}{\partial x}\mathrm{d}x. \tag{1.99}$$

Equations (1.98) and (1.99) have to be equal at all values of di and dx. This gives us

$$\Psi = \frac{\partial W'_\Phi\,(i, x)}{\partial i}, \tag{1.100}$$

$$F_\Phi = \frac{\partial W'_\Phi(i, x)}{\partial x}. \tag{1.101a}$$

Correspondingly, when the current i is kept constant, the torque is

$$T_\Phi = \frac{\partial W'_\Phi(i, \theta)}{\partial \theta}. \tag{1.101b}$$

Equation (1.101) gives a mechanical force or a torque directly from the current i and displacement x, or from the angular displacement θ. The coenergy can be calculated with i and x

$$W'_\Phi\,(i_0, x_0) = \int_0^i \Psi\,(i, x_0)\mathrm{d}i. \tag{1.102}$$

In a linear system, Ψ and i are proportional, and the flux linkage can be represented by the inductance depending on the distance, as in Equation (1.84). The coenergy is

$$W'_\Phi\,(i, x) = \int_0^i L\,(x)i\,\mathrm{d}i = \frac{1}{2}L\,(x)\,i^2. \tag{1.103}$$

Using Equation (1.91), the magnetic energy can be expressed also in the form

$$W_\Phi = \int_V \frac{1}{2}\mu H^2 \mathrm{d}V. \tag{1.104}$$

In linear systems, the energy and coenergy are numerically equal, for instance $0.5Li^2 = 0.5\Psi^2/L$ or $(\mu/2)H^2 = (1/2\mu)B^2$. In nonlinear systems, Ψ and i or B and H are not proportional. In a graphical representation, the energy and coenergy behave in a nonlinear way according to Figure 1.15.

The area between the curve and flux linkage axis can be obtained from the integral $i\,\mathrm{d}\Psi$, and it represents the energy stored in the magnetic circuit W_Φ. The area between the curve and the current axis can be obtained from the integral $\Psi\,\mathrm{d}i$, and it represents the coenergy W'_Φ. The sum of these energies is, according to the definition,

$$W_\Phi + W'_\Phi = i\Psi. \tag{1.105}$$

In the device in Figure 1.14, with certain values of x and i (or Ψ), the field strength has to be independent of the method of calculation; that is, whether it is calculated from energy or coenergy – graphical presentation illustrates the case. The moving yoke is assumed to be in a position x so that the device is operating at the point a, Figure 1.16a. The partial derivative in Equation (1.92) can be interpreted as $\Delta W_\Phi/\Delta x$, the flux linkage Ψ being constant and $\Delta x \to 0$. If we allow a change Δx from position a to position b (the air gap becomes smaller), the stored energy change $-\Delta W_\Phi$ will be as shown in Figure 1.16a by the shaded area, and the energy thus becomes smaller in this case. Thus, the force F_Φ is the shaded area divided by Δx when $\Delta x \to 0$. Since the energy change is negative, the force will also act in the negative x-axis direction. Conversely, the partial derivative can be interpreted as $\Delta W'_\Phi/\Delta x$, i being constant and $\Delta x \to 0$.

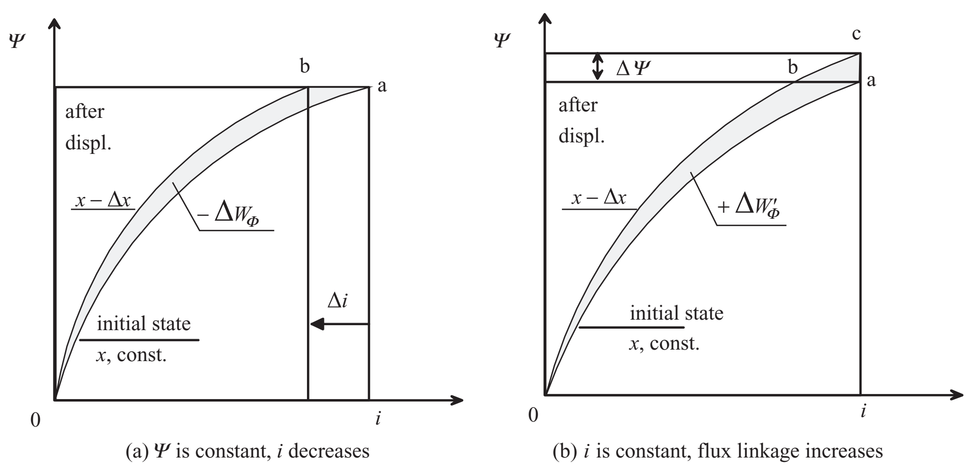

Figure 1.16 Influence of the change Δx on energy and coenergy: (a) the change of energy, when Ψ is constant; (b) the change of coenergy, when i is constant

The shaded areas in Figures 1.16a and b differ from each other by the amount of the small triangle abc, the two sides of which are Δi and $\Delta \Psi$. When calculating the limit, Δx is allowed to approach zero, and thereby the areas of the shaded sections also approach each other.

Equations (1.94) and (1.101) give the mechanical force or torque of electric origin as partial derivatives of the energy and coenergy functions $W_\Phi(\Psi, x)$ and $W'_\Phi(x, i)$.

Physically, the force depends on the magnetic field strength $\boldsymbol{H}$ in the air gap; this will be studied in the next section. According to the study above, the effects of the field can be represented by the flux linkage Ψ and the current i. The force or the torque caused by the magnetic field strength tends to act in all cases in the direction where the stored magnetic energy decreases with a constant flux, or the coenergy increases with a constant current. Furthermore, the magnetic force tends to increase the inductance and drive the moving parts so that the reluctance of the magnetic circuit finds its minimum value.

Using finite elements, torque can be calculated by differentiating the magnetic coenergy W' with respect to movement, and by maintaining the current constant:

$$T = l\frac{\mathrm{d}W'}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_V \int_0^H (\boldsymbol{B}\cdot \mathrm{d}\boldsymbol{H})\,\mathrm{d}V. \tag{1.106}$$

In numerical modelling, this differential is approximated by the difference between two successive calculations:

$$T = \frac{l\left(W'(\alpha+\Delta\alpha) - W'(\alpha)\right)}{\Delta\alpha}. \tag{1.107}$$

Here, l is the machine length and $\Delta\alpha$ represents the displacement between successive field solutions. The adverse effect of this solution is that it needs two successive calculations.

Coulomb's virtual work method in FEM is also based on the principle of virtual work. It gives the following expression for the torque:

$$T = \int_\Omega l\left[\left(-\boldsymbol{B}^{\mathrm{t}}\boldsymbol{J}^{-1}\right)\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}\varphi}H + \int_0^H B\,\mathrm{d}H\,|\boldsymbol{J}|^{-1}\frac{\mathrm{d}\,|\boldsymbol{J}|}{\mathrm{d}\varphi}\right]\mathrm{d}\Omega, \tag{1.108}$$

where the integration is carried out over the finite elements situated between fixed and moving parts, having undergone a virtual deformation. In Equation (1.108), l is the length, $\boldsymbol{J}$ denotes the Jacobian matrix, $\mathrm{d}\boldsymbol{J}/\mathrm{d}\varphi$ is its differential representing element deformation during the displacement $\mathrm{d}\varphi$, $|\boldsymbol{J}|$ is the determinant of $\boldsymbol{J}$ and $\mathrm{d}\,|\boldsymbol{J}|\,/\mathrm{d}\varphi$ is the differential of the determinant, representing the variation of the element volume during displacement $\mathrm{d}\varphi$. Coulomb's virtual work method is regarded as one of the most reliable methods for calculating the torque and it is favoured by many important commercial suppliers of FEM programs. Its benefit compared with the previous virtual work method is that only one solution is needed to calculate the torque.

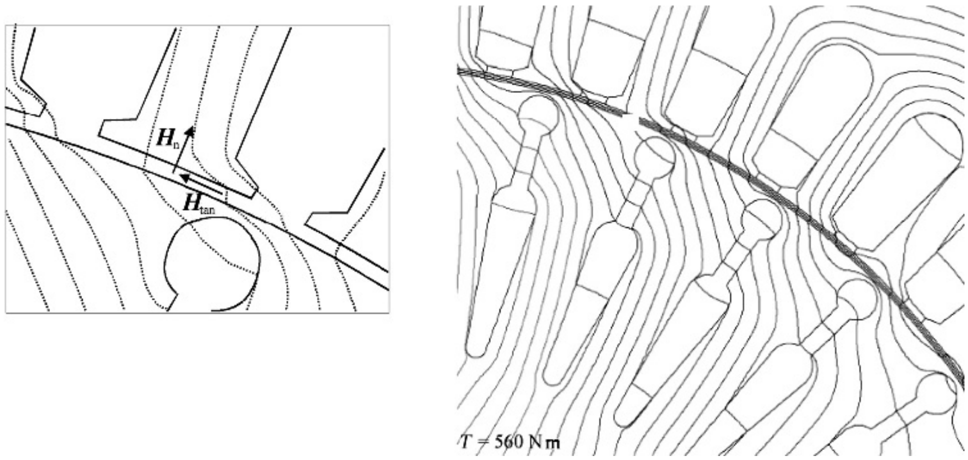

Figure 1.17 Flux solution of a loaded 30 kW, four-pole, 50 Hz induction motor, the machine rotating counterclockwise as a motor. The figure depicts a heavy overload. The tangential field strength in this case is very large and produces a high torque. The enlarged figure shows the tangential and normal components of the field strength in principle. Reproduced by permission of Janne Nerg

1.5 Maxwell's Stress Tensor; Radial and Tangential Stress

Maxwell's stress tensor is probably the most generic idea of producing magnetic stresses, forces and torque. We discussed previously that the linear current density A on a metal surface creates tangential field strength components on the metal surfaces. Such tangential field strength components are essential in both tangential stress generation and torque generation in rotating-field electrical machines.

In numerical methods, Maxwell's stress tensor is often employed in the calculation of forces and torque. The idea is based on Faraday's statement according to which stress occurs in the flux lines. Figures 1.17 depict the flux solution for an air gap of an asynchronous machine, when the machine is operating under a heavy load. Such a heavy load condition is selected in order to illustrate clearly the tangential routes of the flux lines. When we compare Figure 1.17 with Figure 1.6, we can see the remarkable difference in the behaviour of the flux lines in the vicinity of the air gap.

In the figures, the flux lines cross the air gap somewhat tangentially so that if we imagine the flux lines to be flexible, they cause a notable torque rotating the rotor counterclockwise. According to Maxwell's stress theory, the magnetic field strength between objects in a vacuum creates a stress σ_F on the object surfaces, given by

$$\sigma_F = \frac{1}{2}\mu_0 H^2. \tag{1.109}$$

The stress occurs in the direction of lines of force and creates an equal pressure perpendicularly to the lines. When the stress term is divided into its normal and tangential components with respect to the object in question, we obtain

$$\sigma_{F\mathrm{n}} = \frac{1}{2}\mu_0 \left(H_\mathrm{n}^2 - H_\mathrm{tan}^2\right), \tag{1.110}$$

$$\sigma_{F\mathrm{tan}} = \mu_0 H_\mathrm{n} H_\mathrm{tan}. \tag{1.111}$$

Considering torque production, the tangential component $\sigma_{F\mathrm{tan}}$ is of the greatest interest. The total torque exerted on the rotor can be obtained by integrating the stress tensor for instance over a cylinder Γ that confines the rotor. The cylinder is dimensioned exactly to enclose the rotor. The torque is obtained by multiplying the result by the radius of the rotor. Note that no steel may be left inside the surface to be integrated. The torque can be calculated by the following relationship:

$$T = \frac{l}{\mu_0}\int_\Gamma \boldsymbol{r} \times \left((\boldsymbol{B}\cdot\boldsymbol{n})\boldsymbol{B}\mathrm{d}S - \frac{B^2\boldsymbol{n}}{2}\right)\mathrm{d}\Gamma, \tag{1.112}$$

where l is the length, $\boldsymbol{B}$ is the flux density vector, $\boldsymbol{n}$ the normal unit vector in the elements and $\boldsymbol{r}$ the lever arm, in other words the vector which connects the rotor origin to the midpoint of the segment $\mathrm{d}\Gamma$. The former term contains the tangential force contributing to the torque. Since $\boldsymbol{n}$ and $\boldsymbol{r}$ are parallel the latter term does not contribute to the torque but represents the normal stress.

Maxwell's stress tensor illustrates well the fundamental principle of torque generation. Unfortunately, because of numerical inaccuracies, for instance in the FEM, the obtained torque must be employed with caution. Therefore, in the FEM analysis, the torque is often solved by other methods, for instance Arkkio's method, which is a variant of Maxwell's stress tensor and is based on integrating the torque over the whole volume of the air gap constituted by the layers of radii r_s and r_r. The method has been presented with the following expression for the torque:

$$T = \frac{l}{\mu_0\left(r_\mathrm{s} - r_\mathrm{r}\right)}\int_S rB_\mathrm{n}B_\mathrm{tan}\mathrm{d}S, \tag{1.113}$$

in which l is the length, B_n and B_tan denote the radial and tangential flux densities in the elements of surface S and formed between radii r_r and r_s. $\mathrm{d}S$ is the surface of one element.

The magnetizing current method is yet another variant of Maxwell's stress method used in FEM solvers. This method is based on the calculation of the magnetizing current and the flux density over the element edges that constitute the boundary between the iron or permanent magnet and the air. Here the torque can be determined by the following expression:

$$T = \frac{l}{\mu_0}\int_{\Gamma_\mathrm{c}} \left\{\boldsymbol{r} \times \left[\left(B_\mathrm{tan,Fe}^2 - B_\mathrm{tan,air}^2\right)\boldsymbol{n} - \left(B_\mathrm{tan,Fe}B_\mathrm{tan,air} - B_\mathrm{n,air}^2\right)\boldsymbol{t}\right]\right\}\mathrm{d}\Gamma_\mathrm{c}, \tag{1.114}$$

where l is the machine length, Γ_c denotes all the interfaces between the iron or permanent magnet and the air, and $\mathrm{d}\Gamma_\mathrm{c}$ is the length of the element edge located at the boundary. The vector $\boldsymbol{r}$ is the lever arm, in other words the vector connecting the rotor origin to the midpoint of $\mathrm{d}\Gamma_\mathrm{c}$. B_tan and B_n denote the tangential and normal flux densities with respect to $\mathrm{d}\Gamma_\mathrm{c}$. The subscript Fe refers to the iron or permanent magnet. The normal unit vector is $\boldsymbol{n}$ and tangential unit vector $\boldsymbol{t}$.

Equation (1.70) states that a linear current density A creates a tangential field strength in an electrical machine: $H_{\mathrm{tan},\delta} = A$ and $B_{\mathrm{tan},\delta} = \mu_0 A$. According to Equation (1.111), the tangential stress in an air gap is given by

$$\sigma_{F\mathrm{tan}} = \mu_0 H_\mathrm{n} H_\mathrm{tan} = \mu_0 H_\mathrm{n} A = B_\mathrm{n} A. \tag{1.115}$$

This equation gives a local time-dependent value for the tangential stress when local instantaneous values for the normal flux density B_n and the linear current density A are given. Air-gap flux density and linear current density thereby determine the tangential stress occurring in electrical machines. If we want to emphasize the place and time dependence of the stress, we may write the expression in the form

$$\sigma_{F\mathrm{tan}}(x,t) = \mu_0 H_\mathrm{n}(x,t)\, H_\mathrm{tan}(x,t) = \mu_0 H_\mathrm{n}(x,t)\, A(x,t) = B_\mathrm{n}(x,t)\, A(x,t). \tag{1.116}$$

This expression is a very important starting point for the dimensioning of an electrical machine. The torque of the machine may be directly determined by this equation when the rotor dimensions are selected.

Example 1.7: Assume a sinusoidal air-gap flux density distribution having a maximum value of 0.9 T and a sinusoidal linear current density with a maximum value of 40 kA/m in the air gap. To simplify the case, also assume that the distributions are overlapping; in other words, there is no phase shift. This condition may occur on the stator surface of a synchronous machine; however, in the case of an induction machine, the condition never takes place in the steady state, since the stator also has to carry the magnetizing current. In our example, both the diameter and the length of the rotor are 200 mm. What is the power output, if the rotation speed is 1450 min^{-1}?

Solution: Because $\sigma_{F\mathrm{tan}}(x) = \hat{B}_\mathrm{n} \sin(x)\, \hat{A} \sin(x)$, the average tangential stress becomes $\overline{\sigma}_{F\mathrm{tan}}(x) = 0.5 \hat{B}_\mathrm{n} \hat{A} = 18$ kPa. The active surface area of the rotor is $\pi D l = 0.126\,\mathrm{m}^2$. When we multiply the rotor surface area by the average tangential stress, we obtain 2270 N. This tangential force occurs everywhere at a radial distance of 0.1 m from the centre of the axis, the torque being thus 227 N m. The angular velocity is 151 rad/s, which produces a power of approximately 34 kW. These values are quite close to the values of a real, totally enclosed 30 kW induction machine.

In electrical machines, the tangential stresses typically vary between 10 and 50 kPa depending on the machine construction, operating principle and especially on the cooling. For instance, the values for totally enclosed, permanent magnet synchronous machines vary typically between 20 and 30 kPa. For asynchronous machines, the values are somewhat lower. In induction machines with open-circuit cooling, the value of 50 kPa is approached. Using direct cooling methods may give notably higher tangential stresses.

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Example 1.8: Calculate the force between two iron bodies when the area of the air gap between the bodies is $10\,\mathrm{cm}^2$, and the flux density is 1.5 T. The relative permeability of iron is assumed to be 700. It is also assumed that the tangential component of the field strength is zero.

Solution:

$$\sigma_{F\mathrm{n}} = \frac{1}{2}\mu_0\left(H_\mathrm{n}^2\right) = \frac{1}{2}\mu_0\left(\frac{B_\mathrm{n}}{\mu_0}\right)^2 = 8.95\cdot 10^5\,\frac{\mathrm{V\,s}}{\mathrm{A\,m}}\left(\frac{\mathrm{A}}{\mathrm{m}}\right)^2$$
$$= 8.95\cdot 10^5\,\frac{\mathrm{V\,A\,s}}{\mathrm{m}^3} = 8.95\cdot 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2}.$$

This is the stress in the air gap. The force acting on the iron can be approximated by multiplying the stress by the area of the air gap. Strictly speaking, we should investigate the permeability difference of the iron and air, the force acting on the iron therefore being

$$F_{F\mathrm{n}} = S\sigma_{F\mathrm{n}}\left(1-\frac{1}{\mu_{\mathrm{rFe}}}\right) = \left(1-\frac{1}{700}\right)0.001\ \mathrm{m}^2\cdot 8.95\cdot 10^5\,\frac{\mathrm{N}}{\mathrm{m}^2} = 894\,\mathrm{N}.$$

No magnetic force is exerted on the air (a nonmagnetic material ($\mu_\mathrm{r} = 1$)), although some stress occurs in the air because of the field strength. Only the part of air-gap flux that is caused by the magnetic susceptibility of the iron circuit creates a force. By applying the stress tensor, we may now write for a normal force

$$F_{F\mathrm{n}} = \frac{B_\delta^2 S_\delta}{2\,\mu_0}\left(1-\frac{1}{\mu_\mathrm{r}}\right). \tag{1.117}$$

For iron, $1/\mu_\mathrm{r} \ll 1$, and thus, in practice, the latter term in Equation (1.117) is of no significance, unless the iron is heavily saturated.

From this example, we may conclude that the normal stress is usually notably higher than the tangential stress. In these examples, the normal stress was 895 000 Pa, and the tangential stress 18 000 Pa. Some cases have been reported in which attempts have been made to apply normal stress in rotating machines.

1.6 Self-Inductance and Mutual Inductance

Self-inductances and mutual inductances are the core parameters of electrical machines. Permeance is generally determined by

$$\Lambda = \frac{\Phi}{\Theta} = \frac{\Phi}{Ni} \tag{1.118}$$

and inductance by

$$L = N\frac{\Phi}{i} = \frac{\Psi}{i} = N^2 \Lambda. \tag{1.119}$$

Inductance describes a coil's ability to produce flux linkage. Therefore, also its unit H (henry) is equal to V s/A. Correspondingly, the mutual inductance L_{12} is determined from the flux linkage Ψ_{12}, created in winding 1 by the current i_2 that flows in winding 2,

$$L_{12} = \frac{\Psi_{12}}{i_2}. \tag{1.120}$$

In the special case where the flux Φ_{12}, created by the current of winding 2, penetrates all the turns of windings 1 and 2, the mutual permeance between the windings is written as

$$\Lambda_{12} = \frac{\Phi_{12}}{N_2 i_2}, \tag{1.121}$$

and the mutual inductance as

$$L_{12} = N_1 N_2 \Lambda_{12}. \tag{1.122}$$

Here, N_1 is the number of turns of the winding in which the voltage is induced, and N_2 is the number of turns of the winding that produces the flux.

The energy equation for a magnetic circuit can be written with the flux linkage as

$$W_\Phi = \int_0^t iL\frac{\mathrm{d}i}{\mathrm{d}t}\mathrm{d}t = \int_0^t i\frac{\mathrm{d}\Psi}{\mathrm{d}t}\mathrm{d}t = \int_0^\Psi i\,\mathrm{d}\Psi. \tag{1.123}$$

If an integral has to be calculated, the volume under observation can be divided into flux tubes. A flux flowing in such a flux tube is created by the influence of N turns of the winding. By taking into account the fact that the field strength $\boldsymbol{H}$ is created by the current i according to the equation $\oint \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = k_\mathrm{w} N i$, the equation for the sum of the energies of all flux tubes in the volume observed, that is the total energy of the magnetic circuit that was previously given by the current and flux linkage, may be written as

$$W_\Phi = \int_0^\Psi i\,\mathrm{d}\Psi = \int_0^\Phi k_\mathrm{w} N i\,\mathrm{d}\Phi = \int_0^B \oint \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l}\cdot S\,\mathrm{d}B = \int_0^B \int_V H\,\mathrm{d}V\,\mathrm{d}B = \int_V \int_0^B H\,\mathrm{d}B\,\mathrm{d}V. \tag{1.124}$$

The volume integration has to be performed over the volume V in which the flux in question is passing. Energy per volume is thus written in the familiar form

$$\frac{\mathrm{d}W_\Phi}{\mathrm{d}V} = \int_0^B H\mathrm{d}B, \tag{1.125}$$

the energy stored in the complete magnetic circuit being

$$W_\Phi = \int_V \int_0^B H \, \mathrm{d}B \, \mathrm{d}V. \tag{1.126}$$

As the flux linkage is proportional to the current i, $\Psi = Li$, the energy can be given also as

$$W_\Phi = L \int_0^i i \, \mathrm{d}i = {}^1\!/\!_2 Li^2. \tag{1.127}$$

Equation (1.126) yields

$$\frac{\mathrm{d}W_\Phi}{\mathrm{d}V} = \frac{1}{2}HB, \tag{1.128}$$

$$W_\Phi = \frac{1}{2}\int_V HB \, \mathrm{d}V = \frac{1}{2}\int_V \mu H^2 \, \mathrm{d}V. \tag{1.129}$$

From Equations (1.119), (1.127) and (1.129), we can calculate an ideal overall magnetic permeance for a magnetic circuit of volume V

$$\Lambda = \frac{1}{N^2 i^2}\int_V HB \, \mathrm{d}V = \frac{1}{N^2 i^2}\int_V \mu H^2 \, \mathrm{d}V. \tag{1.130}$$

Let us now investigate two electric circuits with a common magnetic energy of

$$W_\Phi = \int_0^{\Psi_1} i_1 \, \mathrm{d}\Psi_1 + \int_0^{\Psi_2} i_2 \, \mathrm{d}\Psi_2. \tag{1.131}$$

Also in this case, the magnetic energy can be calculated from Equations (1.125) and (1.126). We can see that the common flux flowing through the flux tube n is

$$\Phi_\mathrm{n} = \frac{\Psi_\mathrm{n1}}{N_1} = \frac{\Psi_\mathrm{n2}}{N_2} = BS_\mathrm{n}. \tag{1.132}$$

This flux tube is magnetized by the sum current linkage of two windings N_1 and N_2

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = i_1 N_1 + i_2 N_2. \tag{1.133}$$

In a linear system, the fluxes are directly proportional to the sum magnetizing current linkage $i_1 N_1 + i_2 N_2$, and thus we obtain an energy

$$W_\Phi = \frac{1}{2}(i_1 \Psi_1) + \frac{1}{2}(i_2 \Psi_2). \tag{1.134}$$

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Because the flux linkages are created with two windings together, they can be divided into parts

$$\Psi_1 = \Psi_{11} + \Psi_{12} \quad \text{and} \quad \Psi_2 = \Psi_{22} + \Psi_{21}. \tag{1.135}$$

Now, the flux linkages and inductances can be linked

$$\Psi_{11} = L_{11} i_1, \quad \Psi_{12} = L_{12} i_2, \quad \Psi_{22} = L_{22} i_2, \quad \Psi_{21} = L_{21} i_1. \tag{1.136}$$

In Equation (1.136), there are the self-inductances L_{11} and L_{22}, and the mutual inductances L_{12} and L_{21}. The magnetic energy can now be rewritten as

$$\begin{aligned} W_\Phi &= \tfrac{1}{2}\left(L_{11} i_1^2 + L_{12} i_1 i_2 + L_{22} i_2^2 + L_{21} i_2 i_1\right) \\ &= W_{11} + W_{12} + W_{22} + W_{21}. \end{aligned} \tag{1.137}$$

The magnetic energy of the magnetic field created by the two current circuits can thus be divided into four parts, two parts representing the energy of the self-inductances and two parts representing the energy of the mutual inductances. Correspondingly, the magnetic energy density in a certain volume can be written according to Equation (1.128), after the substitution

$$H = H_1 + H_2 \quad \text{and} \quad B = B_1 + B_2, \tag{1.138}$$

in the form

$$\frac{\mathrm{d}W_\Phi}{\mathrm{d}V} = \frac{1}{2}\left(H_1 B_1 + H_1 B_2 + H_2 B_2 + H_2 B_1\right). \tag{1.139}$$

Since in this equation $H_1 B_2 = H_2 B_1$, the energies and inductances have to behave as $W_{12} = W_{21}$ and $L_{12} = L_{21}$. This gives

$$W_\Phi = W_{11} + 2W_{12} + W_{22} = \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2. \tag{1.140}$$

Equations (1.137) and (1.139) yield

$$W_{12} = \frac{1}{2} L_{12} i_1 i_2 = \frac{1}{2} \int_V H_1 B_2 \,\mathrm{d}V. \tag{1.141}$$

Now, we obtain for the permeance between the windings $\Lambda_{12} = \Lambda_{21}$, which corresponds to the mutual inductance, by comparing Equation (1.122),

$$\Lambda_{12} = \frac{1}{N_1 i_1 N_2 i_2} \int_V \mu H_1 H_2 \,\mathrm{d}V. \tag{1.142}$$

If the field strengths are created by sinusoidal currents with a phase difference γ

$$i_1 = \hat{i}_1 \sin \omega t \quad \text{and} \quad i_2 = \hat{i}_2 \sin(\omega t + \gamma), \tag{1.143}$$

the mutual average energy of the fields created by these currents is obtained from

$$W_{12\mathrm{av}} = \frac{1}{2}L_{12}\frac{1}{2\pi}\int_{0}^{2\pi}\hat{i}_1\hat{i}_2\,\mathrm{d}\omega t = \frac{1}{2}L_{12}\hat{i}_1\hat{i}_2\cos\gamma. \tag{1.144}$$

Correspondingly, the permeance between the windings is

$$\Lambda_{12} = \frac{2}{N_1\hat{i}_1N_2\hat{i}_2}\int_V \mu H_1H_2\cos\gamma\,\mathrm{d}V. \tag{1.145}$$

In these equations, γ is the time-dependent phase angle either between the currents in two windings or between the partial field strengths created by these currents.

Mutual inductances are important in rotating electrical machines. A machine is, however, usually treated using an equivalent electric circuit in which all the machine windings are presented at the same voltage level. In such cases the mutual inductances are replaced by the magnetizing inductance L_m, which can be calculated using the transformation ratio K as $L_m = KL_{12}$. Further discussion of the magnetic circuit properties and inductances, such as magnetizing inductance L_m, will be given in Chapter 3.

1.7 Per Unit Values

When analysing electrical machines, especially in electric drives, per unit values are often employed. This brings certain advantages to the analysis, since they show directly the relative magnitude of a certain parameter. For instance, if the relative magnetizing inductance of an asynchronous machine is $l_m = 3$, it is quite high. On the other hand, if it is $l_m = 1$, it is rather low. Now it is possible to compare machines, the rated values of which differ from each other.

Relative values can be obtained by dividing each dimension by a base value. When considering electric motors and electric drives, the base values are selected accordingly:

- Peak value for rated stator phase current $\hat{i}_N$. (It is, of course, also possible to select the root mean square (RMS) stator current as a base value, instead. In such a case the voltage also has to be selected accordingly.)
- Peak value for rated stator phase voltage $\hat{u}_N$.
- Rated angular frequency $\omega_N = 2\pi f_{sN}$.
- Rated flux linkage, corresponding also to the rated angular velocity $\hat{\Psi}_N$.
- Rated impedance Z_N.
- Time in which 1 radian in electrical degrees, $t_N = 1\ \mathrm{rad}/\omega_N$, is travelled at a rated angular frequency. Relative time τ is thus measured as an angle $\tau = \omega_N t$.
- Apparent power S_N corresponding to rated current and voltage.
- Rated torque T_N corresponding to rated power and frequency.

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When operating with sinusoidal quantities, the rated current of the machine is I_N and the line-to-line voltage is U_N:

$$\text{The base value for current } I_\mathrm{b} = \hat{i}_\mathrm{N} = \sqrt{2} I_\mathrm{N}. \tag{1.146}$$

$$\text{The base value for voltage } U_\mathrm{b} = \hat{u}_\mathrm{N} = \sqrt{2}\frac{U_\mathrm{N}}{\sqrt{3}}. \tag{1.147}$$

$$\text{Angular frequency } \omega_\mathrm{N} = 2\pi f_\mathrm{sN}. \tag{1.148}$$

$$\text{The base value for flux linkage } \Psi_\mathrm{b} = \Psi_\mathrm{N} = \frac{\hat{u}_\mathrm{N}}{\omega_\mathrm{N}}. \tag{1.149}$$

$$\text{The base value for impedance } Z_\mathrm{b} = Z_\mathrm{N} = \frac{\hat{u}_\mathrm{N}}{\hat{i}_\mathrm{N}}. \tag{1.150}$$

$$\text{The base value for inductance } L_\mathrm{b} = L_\mathrm{N} = \frac{\hat{u}_\mathrm{N}}{\omega_\mathrm{N}\hat{i}_\mathrm{N}}. \tag{1.151}$$

$$\text{The base value for capacitance } C_\mathrm{b} = C_\mathrm{N} = \frac{\hat{i}_\mathrm{N}}{\omega_\mathrm{N}\hat{u}_\mathrm{N}}. \tag{1.152}$$

$$\text{The base value for apparent power } S_\mathrm{b} = S_\mathrm{N} = \frac{3}{2}\hat{i}_\mathrm{N}\hat{u}_\mathrm{N} = \sqrt{3} U_\mathrm{N} I_\mathrm{N}. \tag{1.153}$$

$$\text{The base value for torque } T_\mathrm{b} = T_\mathrm{N} = \frac{3}{2\omega_\mathrm{N}}\hat{i}_\mathrm{N}\hat{u}_\mathrm{N}\cos\varphi_\mathrm{N} = \frac{\sqrt{3} U_\mathrm{N} I_\mathrm{N}}{\omega_\mathrm{N}}\cos\varphi_\mathrm{N}. \tag{1.154}$$

The relative values to be used are

$$u_\mathrm{s,pu} = \frac{u_\mathrm{s}}{\hat{u}_\mathrm{N}}, \tag{1.155}$$

$$i_\mathrm{s,pu} = \frac{i_\mathrm{s}}{\hat{i}_\mathrm{N}}, \tag{1.156}$$

$$r_\mathrm{s,pu} = \frac{R_\mathrm{s}\hat{i}_\mathrm{N}}{\hat{u}_\mathrm{N}}, \tag{1.157}$$

$$\Psi_\mathrm{s,pu} = \frac{\omega_\mathrm{N}\Psi_\mathrm{s}}{\hat{u}_\mathrm{N}}, \tag{1.158}$$

$$\omega_\mathrm{pu} = \frac{\omega}{\omega_\mathrm{N}} = \frac{n}{f_\mathrm{N}} = n_\mathrm{pu}, \tag{1.159}$$

where n is the rotational speed per second, and

$$\tau = \omega_\mathrm{N} t. \tag{1.160}$$

The relative values of inductances are the same as the relative values of reactances. Thus, we obtain for instance

$$l_\mathrm{m.pu} = \frac{L_\mathrm{m}}{L_\mathrm{b}} = \frac{L_\mathrm{m}}{\dfrac{\hat{u}_\mathrm{N}}{\omega_\mathrm{N}\hat{i}_\mathrm{N}}} = \frac{\hat{i}_\mathrm{N}}{\hat{u}_\mathrm{N}} X_\mathrm{m} = x_\mathrm{m,pu}, \tag{1.161}$$

where X_m is the magnetizing reactance.

We also have a mechanical time constant

$$T_J = \omega_N \left(\frac{\omega_N}{p}\right)^2 \frac{2J}{3\hat{u}_N \hat{i}_N \cos\varphi_N}, \tag{1.162}$$

where J is the moment of inertia. According to (1.162), the mechanical time constant is the ratio of the kinetic energy of a rotor rotating at synchronous speed to the power of the machine.

Example 1.9: A 50 Hz star-connected, four-pole, 400 V induction motor has the following nameplate values: $P_N = 200\,\mathrm{kW}$, $\eta_N = 0.95$, $\cos\varphi_N = 0.89$, $I_N = 343\,\mathrm{A}$, $I_S/I_N = 6.9$, $T_{max}/T_N = 3$, and rated speed $1485\,\mathrm{min}^{-1}$. The no-load current of the motor is 121 A. Give an expression for the per unit inductance parameters of the motor.

Solution:

The base angular frequency $\omega_N = 2\pi f_{sN} = 314/\mathrm{s}$.

The base value for flux linkage $\Psi_b = \hat{\Psi}_N = \dfrac{\hat{u}_N}{\omega_N} = \dfrac{\sqrt{2}\cdot 230\,\mathrm{V}}{314/\mathrm{s}} = 1.036\,\mathrm{V\,s}$.

The base value for inductance $L_b = L_N = \dfrac{\hat{\Psi}_N}{\hat{i}_N} = \dfrac{1.036\,\mathrm{V\,s}}{\sqrt{2}\cdot 343\,\mathrm{A}} = 2.14\,\mathrm{mH}$.

The no-load current of the machine is 121 A, and the stator inductance of the machine is thereby about $L_s = 230\,\mathrm{V}/(121\,\mathrm{A}\cdot 314/\mathrm{s}) = 6.06\,\mathrm{mH}$. We guess that 97% of this belongs to the magnetizing inductance. $L_m = 0.97 \cdot 6.06\,\mathrm{mH} = 5.88\,\mathrm{mH}$.

The per unit magnetizing inductance is now $L_m/L_b = 5.88/2.14 = 2.74 = l_{m,pu}$ and the stator leakage $l_{s\sigma} = 0.03 \cdot 6.06\,\mathrm{mH} = 0.18\,\mathrm{mH}$. $l_{s\sigma,pu} = 0.18/2.14 = 0.084$.

We may roughly state that the per unit short-circuit inductance of the motor is, according to the starting current ratio, $l_k \approx 1/(I_S/I_N) = 0.145$. Without better knowledge, we divide the short-circuit inductance 50:50 for the stator and rotor per unit leakages: $l_{s\sigma,pu} = l_{r\sigma,pu} = 0.0725$. This differs somewhat from the above-calculated $l_{s\sigma,pu} = 0.18/2.14 = 0.084$. However, the guess that 97% of the stator inductance $l_{s,pu} = l_{s\sigma,pu} + l_{m,pu}$ seems to be correct enough.

The motor per unit slip is $s = (n_{syn} - n)n_{syn} = (1500 - 1485)/1500 = 0.006\,73$. The motor per unit slip at low slip values is directly proportional to the per unit rotor resistance. Thus, we may assume that the rotor per unit resistance is of the same order, $r_{r,pu} \approx 0.0067$.

The rated efficiency of the motor is 95%, which gives 5% per unit losses to the system. If we assume 1% stator resistance $r_{s,pu} \approx 0.01$, and 0.5% excess losses, we have 2.8% $(5 - 1 - 0.5 - 0.67 = 2.8\%)$ per unit iron losses in the motor. Hence, the losses in the machine are roughly proportional to the per unit values of the stator and rotor resistances. For more detailed information, the reader is referred to Chapter 7.

1.8 Phasor Diagrams

When investigating the operation of electrical machines, sinusoidally alternating currents, voltages and flux linkages are often illustrated with phasor diagrams. These diagrams are based either on generator logic or on motor logic; the principle of the generator logic is that for instance the flux linkage created by the rotor magnetization of a synchronous machine induces an electromotive force in the armature winding of the machine. Here, Faraday's induction law is applied in the form

$$e = -\frac{\mathrm{d}\Psi}{\mathrm{d}t}. \tag{1.163}$$

The flux linkage for a rotating-field machine can be presented as

$$\Psi\,(t) = \hat{\Psi} e^{\mathrm{j}\omega t}. \tag{1.164}$$

The flux linkage is derived with respect to time

$$e = -\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\mathrm{j}\omega\hat{\Psi} e^{\mathrm{j}\omega t} = e^{-\mathrm{j}\frac{\pi}{2}}\omega\hat{\Psi} e^{\mathrm{j}\omega t} = \omega\hat{\Psi} e^{\mathrm{j}\left(\omega t - \frac{\pi}{2}\right)}. \tag{1.165}$$

The emf is thus of magnitude $\omega\hat{\Psi}$ and its phase angle is 90 electrical degrees behind the phasor of the flux linkage. Figure 1.18 illustrates the basic phasor diagrams according to generator and motor logic.

As illustrated in Figure 1.18 for generator logic, the flux linkage Ψ_f generated by the rotor of a synchronous machine induces a voltage E_f to the armature winding of the machine when the machine is rotating. The stator voltage of the machine is obtained by reducing the proportion of the armature reaction and the resistive voltage loss from the induced voltage. If the machine is running at no load, the induced voltage E_f equals the stator voltage U_s.

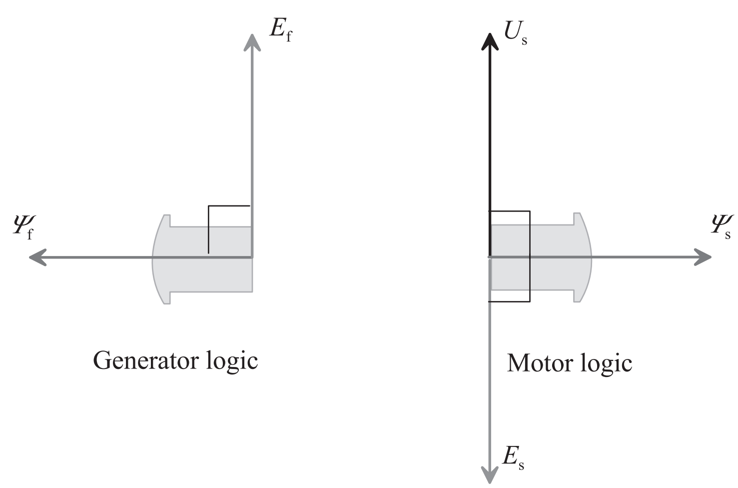

Figure 1.18 Basic phasor diagrams for generator and motor logic

Motor logic represents an opposite case. According to the induction law, the flux linkage can be interpreted as an integral of voltage. If the stator resistive losses are neglected we can write

$$\Psi_\mathrm{s} \approx \int u_\mathrm{s}\mathrm{d}t = \int \hat{u}_\mathrm{s} e^{\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\mathrm{j}\omega}\hat{u}_\mathrm{s} e^{\mathrm{j}\omega t} = \frac{1}{\omega}\hat{u}_\mathrm{s} e^{\mathrm{j}\left(\omega t - \frac{\pi}{2}\right)}. \tag{1.166}$$

The phasor of the flux linkage is 90 electrical degrees behind the voltage phasor. Again, differentiating the flux linkage with respect to time produces an emf. In the case of Figure 1.18, a flux linkage is integrated from the voltage, which further leads to the derivation of a back emf now cancelling the supply voltage. As is known, this is the case with inductive components. In the case of a coil, a major part of the supply voltage is required to overcome the self-inductance of the coil. Only an insignificant voltage drop takes place in the winding resistances. The resistive losses have therefore been neglected in the above discussion.

Example 1.10: A 50 Hz synchronous generator field winding current linkage creates at no load a stator winding flux linkage of $\hat{\Psi}_\mathrm{s} = 15.6\,\mathrm{V\,s}$. What is the internal induced phase voltage (and also the stator voltage) of the machine?

Solution: The induced voltage is calculated as

$$e_\mathrm{F} = -\frac{\mathrm{d}\Psi_\mathrm{s}}{\mathrm{d}t} = -\mathrm{j}\omega\hat{\Psi}_\mathrm{s} e^{\mathrm{j}\omega t} = -\mathrm{j}314/\mathrm{s}\cdot 15.6\,\mathrm{V\,s}\cdot e^{\mathrm{j}\omega t} = 4900\,\mathrm{V}\cdot e^{\mathrm{j}\omega t}.$$

Thus 4900 V is the peak value of the stator phase voltage, which gives an effective value of the line-to-line voltage:

$$U_\mathrm{ll} = \frac{4900\,\mathrm{V}}{\sqrt{2}}\sqrt{3} = 6000\,\mathrm{V}.$$

Hence, we have a 6 kV machine at no load ready to be synchronized to the network.

Example 1.11: A rotating-field motor is supplied by a frequency converter at 25 Hz and 200 V fundamental effective line-to-line voltage. The motor is initially a 400 V star-connected motor. What is the stator flux linkage in the inverter supply?

Solution: The stator flux linkage is found by integrating the phase voltage supplied to the stator

$$\Psi_\mathrm{s} \approx \int u_\mathrm{s}\mathrm{d}t = \int \hat{u}_\mathrm{s} e^{\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\mathrm{j}25\cdot 2\pi}\frac{\sqrt{2}\cdot 200\,\mathrm{V}}{\sqrt{3}} e^{\mathrm{j}\omega t} = 1.04\,\mathrm{V\,s}\cdot e^{\mathrm{j}\left(\omega t - \frac{\pi}{2}\right)}.$$

Consequently, the flux linkage amplitude is 1.04 V s and it is lagging in a 90° phase shift the voltage that creates the flux linkage.

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