1

Introduction

In reliability theory a key problem is to find out how the reliability of a complex system can be determined from knowledge of the reliabilities of its components. One inherent weakness of traditional binary reliability theory is that the system and the components are always described just as functioning or failed. This approach represents an oversimplification in many real-life situations where the system and their components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. The first attempts to replace this by a theory for multistate systems of multistate components were done in the late 1970s in Barlow and Wu (1978), El-Neweihi *et al.* (1978) and Ross (1979). This was followed up by independent work in Griffith (1980), Natvig (1982a), Block and Savits (1982) and Butler (1982) leading to proper definitions of a multistate monotone system and of multistate coherent systems and also of minimal path and cut vectors. Furthermore, in Funnemark and Natvig (1985) upper and lower bounds for the availabilities and unavailabilities, to any level, in a fixed time interval were arrived at for multistate monotone systems based on corresponding information on the multistate components. These were assumed to be maintained and interdependent. Such bounds are of great interest when trying to predict the performance process of the system, noting that exactly correct expressions are obtainable just for trivial systems. Hence, by the mid 1980s the basic multistate reliability theory was established. A review of the early development in this area is given in Natvig (1985a). Rather recently, probabilistic modeling of partial monitoring of components with

Multistate Systems Reliability Theory with Applications Bent Natvig

applications to preventive system maintenance has been extended by Gåsemyr and Natvig (2005) to multistate monotone systems of multistate components. A newer review of the area is given in Natvig (2007).

The theory was applied in Natvig *et al*. (1986) to an offshore electrical power generation system for two nearby oilrigs, where the amounts of power that may possibly be supplied to the two oilrigs are considered as system states. This application is also used to illustrate the theory in Gåsemyr and Natvig (2005). In Natvig and Mørch (2003) the theory was applied to the Norwegian offshore gas pipeline network in the North Sea, as of the end of the 1980s, transporting gas to Emden in Germany. The system state depends on the amount of gas actually delivered, but also to some extent on the amount of gas compressed, mainly by the compressor component closest to Emden. Rather recently the first book (Lisnianski and Levitin, 2003) on multistate system reliability analysis and optimization appeared. The book also contains many examples of the application of reliability assessment and optimization methods to real engineering problems. This has been followed up by Lisnianski *et al*. (2010).

Working on the present book a series of new results have been developed. Some generalizations of bounds for the availabilities and unavailabilities, to any level, in a fixed time interval given in Funnemark and Natvig (1985) have been established. Furthermore, the theory for Bayesian assessment of system reliability, as presented in Natvig and Eide (1987) for binary systems, has been extended to multistate systems. Finally, a theory for measures of component importance in nonrepairable and repairable multistate strongly coherent systems has been developed, and published in Natvig (2011), with accompanying advanced discrete simulation methods and an application to a West African production site for oil and gas.

1.1 Basic notation and two simple examples

Let $S = \{0, 1, \ldots, M\}$ be the set of states of the system; the $M + 1$ states representing successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Furthermore, let $C = \{1, \ldots, n\}$ be the set of components and S_i, $i = 1, \ldots, n$ the set of states of the ith component. We claim $\{0, M\} \subseteq S_i \subseteq S$. Hence, the states 0 and M are chosen to represent the endpoints of a performance scale that might be used for both the system and its components. Note that in most applications there is no need for the same detailed description of the components as for the system.

Let x_i, $i = 1, \ldots, n$ denote the state or performance level of the ith component at a fixed point of time and $\boldsymbol{x} = (x_1, \ldots, x_n)$. It is assumed that the state, ϕ, of the system at the fixed point of time is a deterministic function of $\boldsymbol{x}$, i.e. $\phi = \phi(\boldsymbol{x})$. Here $\boldsymbol{x}$ takes values in $S_1 \times S_2 \times \cdots \times S_n$ and ϕ takes values in S. The function ϕ is called the structure function of the system. We often denote a multistate system by (C, ϕ). Consider, for instance, a system of n components in parallel where $S_i = \{0, M\}, i = 1, \ldots, n$. Hence, we have a binary description of component states. In binary theory, i.e. when $M = 1$, the system state is 1 iff at least one component is functioning. In multistate theory we may let the state of the system be the number of components functioning, which is far more informative. In this case, for $M = n$,

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{n} x_i / n. \tag{1.1}$$

As another simple example consider the network depicted in Figure 1.1. Here component 1 is the parallel module of the branches a_1 and b_1 and component 2 the parallel module of the branches a_2 and b_2. For $i = 1, 2$ let $x_i = 0$ if neither of the branches work, 1 if one branch works and 3 if two branches work. The states of the system are given in Table 1.1.

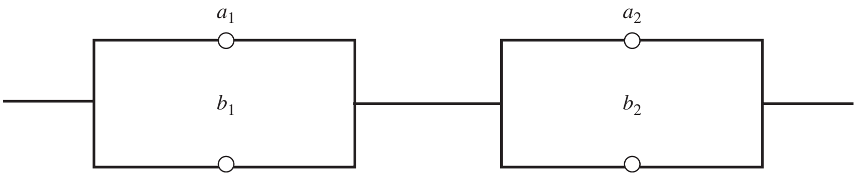

Figure 1.1 A simple network.

Note, for instance, that the state 1 is critical both for each component and the system as a whole in the sense that the failing of a branch leads

Table 1.1 States of the simple network system of Figure 1.1.

Component 2	3	0	2	3
	1	0	1	2
	0	0	0	0
		0	1	3
		Component 1		

to the 0 state. In binary theory the functioning state comprises the states $\{1, 2, 3\}$ and hence only a rough description of the system's performance is possible. It is not hard to see that the structure function is given by

$$\phi(\boldsymbol{x}) = x_1x_2 - I(x_1x_2 = 3) - 6I(x_1x_2 = 9), \tag{1.2}$$

where $I(\cdot)$ is the indicator function.

The following notation is needed throughout the book.

$(\cdot_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$.

$\boldsymbol{y} < \boldsymbol{x}$ means $y_i \le x_i$ for $i = 1, \ldots, n$, and $y_i < x_i$ for some i.

Let $A \subset C$. Then

$\boldsymbol{x}^A$ = vector with elements $x_i, i \in A$,

A^c = subset of C complementary to A.

1.2 An offshore electrical power generation system

In Figure 1.2 an outline of an offshore electrical power generation system, considered in Natvig *et al.* (1986), is given. The purpose of this system is to supply two nearby oilrigs with electrical power. Both oilrigs have their own main generation, represented by equivalent generators A_1 and A_3 each having a capacity of 50 MW. In addition, oilrig 1 has a standby generator A_2 that is switched into the network in case of outage of A_1 or A_3, or may be used in extreme load situations in either of the two oilrigs. The latter situation is, for simplicity, not treated in this book. A_2 is in cold standby, which means that a short startup time is needed before it is switched into the network. This time is neglected in the following model. A_2 also has a capacity of 50 MW. The control unit, U, continuously supervises the supply from each of the generators with automatic control of the switches. If, for instance, the supply from A_3 to oilrig 2 is not sufficient, whereas the supply from A_1 to oilrig 1 is sufficient, U can activate A_2 to supply oilrig 2 with electrical power through the standby subsea cables L.

The components to be considered here are A_1, A_2, A_3, U and L. We let the perfect functioning level M equal 4 and let the set of states of all components be $\{0, 2, 4\}$. For A_1, A_2 and A_3 these states are interpreted as

0: The generator cannot supply any power;

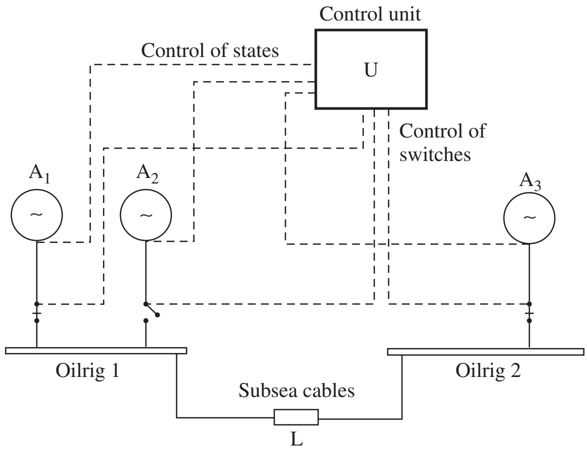

Figure 1.2 Outline of an offshore electrical power generation system.

2: The generator can supply a maximum of 25 MW;

4: The generator can supply a maximum of 50 MW.

Note that as an approximation we have, for these generators, chosen to describe their supply capacity on a discrete scale of three points. The supply capacity is not a measure of the actual amount of power delivered at a fixed point of time. There is continuous power-frequency control to match generation to actual load, keeping electrical frequency within prescribed limits.

The control unit U has the states

0: U will, by mistake, switch the main generators A_1 and A_3 off without switching A_2 on;

2: U will not switch A_2 on when needed;

4: U is functioning perfectly.

The subsea cables L are actually assumed to be constructed as double cables transferring half of the power through each simple cable. This leads to the following states of L

0: No power can be transferred;

2: 50% of the power can be transferred;

4: 100% of the power can be transferred.

Let us now, for simplicity, assume that the mechanism that distributes the power from A_2 to platform 1 or 2 is working perfectly. Furthermore, as a start, assume that this mechanism is a simple one either transferring no power from A_2 to platform 2, if A_2 is needed at platform 1, or transferring all power from A_2 needed at platform 2. Now let $\phi_1(A_1, A_2, U) =$ the amount of power that can be supplied to platform 1, and $\phi_2(A_1, A_2, A_3, U, L) =$ the amount of power that can be supplied to platform 2. ϕ_1 will now just take the same states as the generators whereas ϕ_2 can also take the following states

1: The amount of power that can be supplied is a maximum of 12.5 MW;

3: The amount of power that can be supplied is a maximum of 37.5 MW.

Number the components A_1, A_2, A_3, U, L successively 1, 2, 3, 4, 5. Then it is not too hard to be convinced that ϕ_1 and ϕ_2 are given respectively by

$$\phi_1(\boldsymbol{x}) = I(x_4 > 0)\min(x_1 + x_2 I(x_4 = 4), 4) \tag{1.3}$$

$$\phi_2(\boldsymbol{x}) = I(x_4 > 0)\min(x_3 + x_2 I(x_4 = 4) I(x_1 = 4) x_5/4, 4). \tag{1.4}$$

Let us still assume that the mechanism that distributes the power from A_2 to platform 1 or 2 is working perfectly. However, let it now be more advanced, transferring excess power from A_2 to platform 2 if platform 1 is ensured a delivery corresponding to state 4. Of course in a more refined model this mechanism should be treated as a component. The structure functions are now given by

$$\phi_1^*(\boldsymbol{x}) = \phi_1(\boldsymbol{x}) \tag{1.5}$$

$$\phi_2^*(\boldsymbol{x}) = I(x_4 > 0)\min(x_3 + \max(x_1 + x_2 I(x_4 = 4) - 4, 0) x_5/4, 4), \tag{1.6}$$

noting that $\max(x_1 + x_2 I(x_4 = 4) - 4, 0)$ is just the excess power from A_2 which one tries to transfer to platform 2.

1.3 Basic definitions from binary theory

Before going into the specific restrictions that we find natural to claim on the structure function ϕ, it is convenient first to recall some basic definitions from the traditional binary theory. This theory is nicely introduced in Barlow and Proschan (1975a).

Definition 1.1: A system is a binary monotone system (BMS) iff its structure function ϕ satisfies:

(i) $\phi(\boldsymbol{x})$ is nondecreasing in each argument

(ii) $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1 \quad \mathbf{0} = (0, \ldots, 0), \mathbf{1} = (1, \ldots, 1)$.

The first assumption roughly says that improving one of the components cannot harm the system, whereas the second says that if all components are in the failure state, then the system is in the failure state and, correspondingly, that if all components are in the functioning state, then the system is in the functioning state.

We now impose some further restrictions on the structure function ϕ.

Definition 1.2: A binary coherent system (BCS) is a BMS where each component is relevant, i.e. the structure function ϕ satisfies: $\forall i \in \{1, \ldots, n\}, \exists(\cdot_i, \boldsymbol{x})$ such that $\phi(1_i, \boldsymbol{x}) = 1, \phi(0_i, \boldsymbol{x}) = 0$.

A component which is not relevant is said to be irrelevant. We note that an irrelevant component can never directly cause the failure of the system. As an example of such a component consider a condenser in parallel with an electrical device in a large engine. The task of the condenser is to cut off high voltages which might destroy the electrical device. Hence, although irrelevant, the condenser can be very important in increasing the lifetime of the device and hence the lifetime of the whole engine. The limitation of Definition 1.2 claiming relevance of the components, is inherited by the various definitions of a multistate coherent system considered in this book.

Let $C_0(\boldsymbol{x}) = \{i | x_i = 0\}$ and $C_1(\boldsymbol{x}) = \{i | x_i = 1\}$.

Definition 1.3: Let ϕ be the structure function of a BMS. A vector $\boldsymbol{x}$ is said to be a path vector iff $\phi(\boldsymbol{x}) = 1$. The corresponding path set is $C_1(\boldsymbol{x})$. A minimal path vector is a path vector $\boldsymbol{x}$ such that $\phi(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} < \boldsymbol{x}$. The corresponding minimal path set is $C_1(\boldsymbol{x})$.

Definition 1.4: Let ϕ be the structure function of a BMS. A vector $\boldsymbol{x}$ is said to be a cut vector iff $\phi(\boldsymbol{x}) = 0$. The corresponding cut set is $C_0(\boldsymbol{x})$. A minimal cut vector is a cut vector $\boldsymbol{x}$ such that $\phi(\boldsymbol{y}) = 1$ for all $\boldsymbol{y} > \boldsymbol{x}$. The corresponding minimal cut set is $C_0(\boldsymbol{x})$.

We also need the following notation

$$\coprod_{i \in A} x_i = 1 - \prod_{i \in A}(1 - x_i) \quad x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2).$$

We then have the following representations for the series and parallel systems respectively

$$\min_{1\le i\le n} x_i = \prod_{i=1}^{n} x_i \qquad \max_{1\le i\le n} x_i = \coprod_{i=1}^{n} x_i. \tag{1.7}$$

Consider a BCS with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$. Since the system is functioning iff for at least one minimal path set all the components are functioning, or alternatively, iff for all minimal cut sets at least one component is functioning, we have the two following representations for the structure function

$$\phi(\boldsymbol{x}) = \coprod_{j=1}^{p} \prod_{i\in P_j} x_i = \max_{1\le j\le p} \min_{i\in P_j} x_i, \tag{1.8}$$

$$\phi(\boldsymbol{x}) = \prod_{j=1}^{k} \coprod_{i\in K_j} x_i = \min_{1\le j\le k} \max_{i\in K_j} x_i. \tag{1.9}$$

Definition 1.5: The monotone system (A, χ) is a module of the monotone system (C, ϕ) iff

$$\phi(\boldsymbol{x}) = \psi[\chi(\boldsymbol{x}^A), \boldsymbol{x}^{A^c}],$$

where ψ is a monotone structure function and $A \subseteq C$.

Intuitively, a module is a monotone subsystem that acts as if it were just a supercomponent. Consider again the example where a condenser is in parallel with an electrical device in a large engine. The parallel system of the condenser and the electrical device is a module, which is relevant.

Definition 1.6: A modular decomposition of a monotone system (C, ϕ) is a set of disjoint modules $\{(A_k, \chi_k)\}_{k=1}^{r}$ together with an organizing monotone structure function ψ, i.e.

(i) $C = \cup_{i=1}^{r} A_i$ where $A_i \cap A_j = \emptyset \quad i \neq j$,

(ii) $\phi(\boldsymbol{x}) = \psi[\chi_1(\boldsymbol{x}^{A_1}), \ldots, \chi_r(\boldsymbol{x}^{A_r})] = \psi[\boldsymbol{\chi}(\boldsymbol{x})]$.

Making a modular decomposition of a system is a way of breaking it into a collection of subsystems which can be dealt with more easily.

Definition 1.7: Given a BMS structure function ϕ, its dual structure function ϕ^D is given by

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}),$$

where $\boldsymbol{x}^D = (x_1^D, \ldots, x_n^D) = \mathbf{1} - \boldsymbol{x} = (1 - x_1, \ldots, 1 - x_n)$.

Definition 1.8: The random variables $T_1, \ldots, T_n$ are associated iff $\text{Cov}[\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})] \geq 0$ for all pairs of nondecreasing binary functions Γ, Δ. $\boldsymbol{T} = (T_1, \ldots, T_n)$.

We list some basic properties of associated random variables.

- P_1 Any subset of a set of associated random variables is a set of associated random variables.
- P_2 The set consisting of a single random variable is a set of associated random variables.
- P_3 nondecreasing and nonincreasing functions of associated random variables are associated.
- P_4 If two sets of associated random variables are independent of each other, then their union is a set of associated random variables.
- P_5 Independent random variables are associated.

1.4 Early attempts to define multistate coherent systems

We now return to multistate reliability theory and begin by discussing the structure function considered by Barlow and Wu (1978).

Definition 1.9: Let $P_1, \ldots, P_p$ be nonempty subsets of $C = \{1, \ldots, n\}$ such that $\cup_{i=1}^n P_i = C$ and $P_j \not\subseteq P_i, i \neq j$. Then

$$\phi(\boldsymbol{x}) = \max_{1 \leq j \leq p} \min_{i \in P_j} x_i. \tag{1.10}$$

If the sets $\{P_1, \ldots, P_p\}$ are considered minimal path sets in a binary system, they uniquely determine a BCS (C, ϕ_0) where ϕ_0 is defined by Equation (1.8). On the other hand, starting out with a BCS ϕ_0, its minimal path sets $\{P_1, \ldots, P_p\}$ are uniquely determined. Hence, what Barlow and Wu (1978) essentially do when defining their structure function is just to extend the domain and range of Equation (1.8) from $\{0, 1\}$ to $\{0, 1, \ldots, M\}$. It is hence a one-to-one correspondence between the binary structure function ϕ_0 and the multistate structure function ϕ. Furthermore, if $\{K_1, \ldots, K_k\}$ are the minimal cut sets of (C, ϕ_0) it follows from Theorem 3.5, page 12 of Barlow and Proschan (1975a) that for $\phi(\boldsymbol{x})$ of Equation (1.10) we have

$$\phi(\boldsymbol{x}) = \min_{1 \leq j \leq k} \max_{i \in K_j} x_i. \tag{1.11}$$

Setting $p = 1$ in Equation (1.10) and $k = 1$ in Equation (1.11), noting that $P_1 = K_1 = C$, we respectively get what are naturally called the multistate series and parallel systems.

El-Neweihi *et al.* (1978) suggest the following definition of a multistate coherent system.

Definition 1.10: Let $S_i = S, i = 1, \ldots, n$. A system is a multistate coherent system iff its structure function ϕ satisfies:

(i) $\phi(\boldsymbol{x})$ is nondecreasing in each argument

(ii) $\forall i \in \{1, \ldots, n\}, \forall j \in \{0, 1, \ldots, M\}, \exists(\cdot_i, \boldsymbol{x})$ such that, $\phi(j_i, \boldsymbol{x}) = j$ and $\phi(\ell_i, \boldsymbol{x}) \neq j \quad \ell \neq j$

(iii) $\forall j \in \{0, 1, \ldots, M\} \quad \phi(\boldsymbol{j}) = j \quad \boldsymbol{j} = (j, \ldots, j)$.

It is easy to see that the structure function of Definition 1.9 is just a special case of the one in Definition 1.10. Furthermore, note that (i) and (ii) of Definition 1.10 are generalizations of the claims of Definition 1.2. In the binary case, (iii) of Definition 1.10 is implied by the corresponding (i) and (ii). This is not true in the multistate case.

1.5 Exercises

1.1 Verify Equation (1.1).

1.2 Verify Equation (1.2).

1.3 Verify Equations (1.3) and (1.4).

1.4 Verify Equations (1.5) and (1.6).

1.5 Prove Property P_5 of associated random variables by applying Properties P_2 and P_4.

1.6 Show that the structure function of Definition 1.9 is just a special case of the one in Definition 1.10.