1

Basic Concepts

1.1 Introduction

First, a definition must be given for what constitutes a star. *A star can be defined as a self-gravitating celestial object in which there is, or there once was (in the case of dead stars), sustained thermonuclear fusion of hydrogen in their core.* For example, in the Sun, hydrogen, which is the most abundant element in the Universe, is fused into helium via the nuclear reaction $4\,^{1}H \rightarrow \,^{4}He$ + energy. Fusion is only present in the central regions of stars, because there exists a minimum threshold temperature at which this exothermic reaction can be ignited (which is of the order of ten million degrees for this particular reaction). For hydrogen nuclei (protons) to be fused, they must have a close approach on the order of distance at which the strong nuclear force comes into play.[1] The strong nuclear force is responsible for binding the nucleons (protons and neutrons) in the nucleus and contrary to gravity, for instance, its field of action is limited to a distance on the order of 10^{-15} m. At the high temperatures found in the centres of stars, the kinetic energy of the protons is sufficient to vanquish the repulsive Coulomb force between them and bring the protons within the distance where the attractive strong nuclear force becomes dominant. Protons can then fuse together while emitting energy.

The energy emitted by thermonuclear reactions is given by Einstein's famous $E = \Delta mc^2$ formula, where Δm is the difference in mass between the species on the left-hand and right-hand sides of the arrow found in the nuclear reaction given above and c is the speed of light in vacuum. However, the hydrogen burning reaction given above can be a bit misleading, since it suggests that four protons meet to form a helium nucleus. In reality, a series of nuclear reactions is needed to give this global reaction. On another note, even though only a small fraction of a star's mass will be transformed to energy during its lifetime, it will suffice to compensate for the energy irradiated at its surface.

[1] Here, a simple phenomenological explanation of nuclear fusion is given. In reality, quantum tunnelling intervenes. This will be discussed in more detail in Chapter 6.

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Details concerning various nuclear reactions of importance in stars will be discussed in Chapter 6.

Stars are formed following the gravitational collapse of cold molecular clouds found in the Universe. As the cloud or portions of it collapses, it can be shown (see Chapter 2) that approximately half of the gravitational energy gained is used to increase the internal temperature of the cloud and the remaining energy is irradiated as electromagnetic radiation in space. If the mass of the collapsed cloud is sufficient (i.e. more than approximately 8% of the mass of the Sun), the central temperatures will attain a value superior to the threshold temperature for sustained hydrogen fusion, which would by definition, lead to star birth. The solar mass is $M_\odot = 1.989 \times 10^{33}\,$g, where the symbol $\odot$ represents the Sun.[2] The physical properties of stars are often given in units of the corresponding value for the Sun. The gravitational collapse will continue until equilibrium is reached, where the nuclear energy generated per unit time (or its power) at the centre of the star equals the power output at its surface due to radiation emission. A star at this stage of its life is commonly called a main-sequence star. Since gravity has radial symmetry, a star will have a spherical shape (unless it has a high rotational speed). More details concerning stellar formation will be given in Chapter 2.

A star shines (or emits radiation) because of its high surface temperature. For example, the surface temperature of the Sun is approximately 5800 K, while its central temperature is approximately 16 million K. The decrease of the temperature as a function of distance from the centre is a natural occurrence that causes energy transport from the central regions to the surface of the Sun. Since the gas composing a star is characterized by an opacity to radiation, an observer looking at a star can only see its exterior regions, which is commonly called the photosphere or stellar atmosphere, having a geometrical depth of up to a few per cent of the stellar radius. This is similar to looking in a cloud of fog, being able to see only a certain distance before light signals are attenuated. The radiative field exiting a star depends on the temperature of these outer layers and is associated to their blackbody spectra. The physical properties of blackbodies will be discussed in Section 1.3 and will lead to an explanation why stars have different colours.

There are three modes of transportation of energy in stars. The most important is radiation. For this mode, the energy is transported when electromagnetic radiation diffuses from the central regions of stars towards its exterior. In regions where the radiative opacity becomes large, convection can dominate energy transport. Convection is the transport of energy by the vertical movements of cells of matter in the stars. Conduction is the third mode of transportation of energy in stars. However, this mode is rarely important. More details concerning energy transport will be discussed in Chapters 3 and 5.

As mentioned above, a star begins its life by transforming hydrogen to helium in its core. As time passes, the abundance of hydrogen gradually decreases in the star's core, and eventually, the fuel for this particular nuclear process, namely hydrogen, will all be spent. As hydrogen is transformed into helium, the structure of the star readjusts. The core contracts causing an increase of the central temperatures until possibly, depending on the initial mass of the star, helium fuses to produce carbon via the well-known triple-α reaction: $3\,{}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C}$ + energy. Meanwhile, the outer regions of the star expand. The star then becomes what is called a red giant. The final destiny of a star depends almost solely on

[2]Other physical properties of the Sun are given in Appendix C.

its initial mass; it will either become a white dwarf, a neutron star or a black hole. More details concerning stellar evolution will be given in Chapter 6.

For massive stars, a succession of nuclear reactions will occur during their different stages of evolution. The thermonuclear reactions in these stars are responsible for the synthesis of various elements, such as carbon, oxygen, silicon, etc. up to iron. This process is called nucleosynthesis. As known from the Big-Bang theory, at the beginning of the Universe, only hydrogen, helium and trace amounts of lithium were created. The formation of the other elements takes place in stars. Stars can therefore be seen as the Universe's production factories, generating all atoms heavier than helium, except for some lithium. In astronomy, elements heavier than helium are called metals and the fraction of the mass composed of metals is called the metallicity (Z). The metallicity of outer layers of the Sun is approximately $Z = 0.0169$. Meanwhile, the mass fraction of hydrogen (X) and helium (Y) at the surface of the Sun are, respectively, $X = 0.7346$ and $Y = 0.2485$ (and therefore $X + Y + Z = 1$). All of the atoms of these heavy elements found on Earth were created in stars, which then exploded in the form of supernovae ejecting this enriched matter into space. Some of this enriched matter was later found in the primordial cloud from which the Sun and the Earth were created. Life itself would be impossible without the creation of the elements in stars.

This is why stars are fundamental for our existence and can be considered as the main building blocks of the Universe. It is then crucial to understand them via the study of stellar astrophysics. This field of study is fascinating since it incorporates all major fields of physics (see Figure 1.1): nuclear, atomic, molecular and quantum physics, electromagnetism, relativity, thermodynamics, hydrodynamics, etc. This book aims to give the reader an introduction to this fundamental subject by emphasising the physical concepts involved and their specific importance in stars.

1.2 The Electromagnetic Spectrum

As is known from quantum mechanics, electromagnetic radiation has two personalities. It sometimes behaves like waves and at other times like particles. These particles are called photons. These two aspects of radiation are known as the wave–particle duality. For most radiative processes in stars, like an atomic absorption of a photon for example, radiation will act like a photon, rather than a wave. The wave–particle duality also applies to matter.

The energy (E) of photons is related to the frequency (ν) and wavelength (λ) of the associated electromagnetic wave via the following expression

$$E = h\nu = \frac{hc}{\lambda} \tag{1.1}$$

where h is the Planck constant and c is the speed of light in vacuum.

Even though a photon of wavelength λ has no mass, it possesses momentum p equal to

$$p = \frac{E}{c} = \frac{h}{\lambda} \tag{1.2}$$

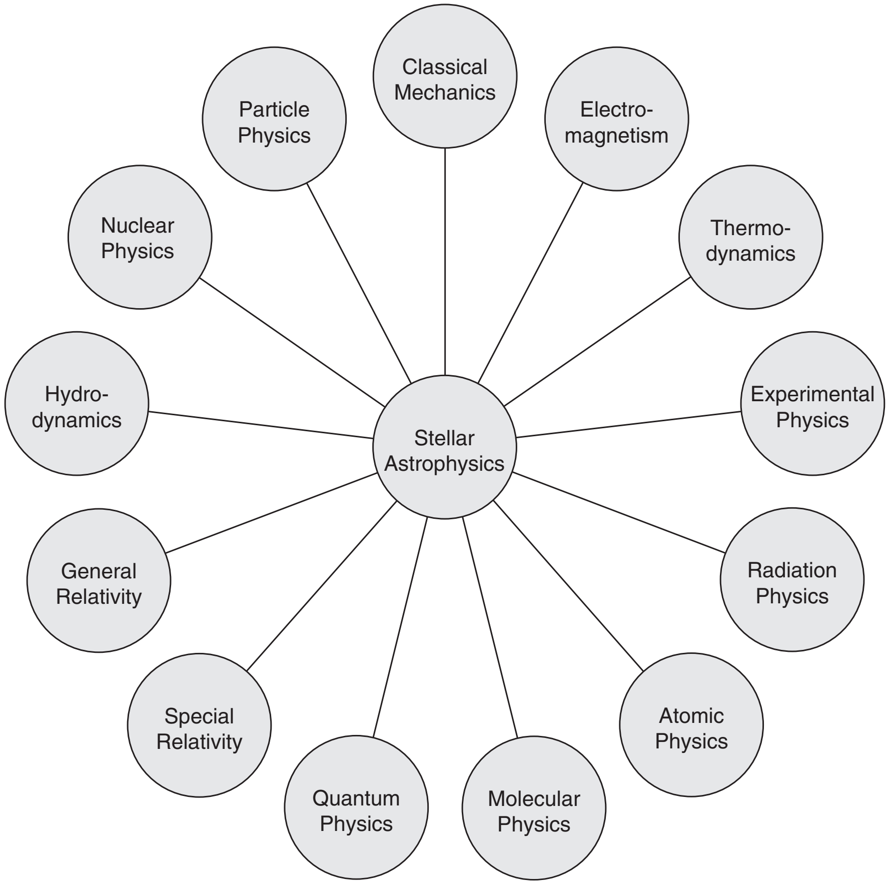

Figure 1.1 Figure illustrating the various fields of physics that intervene in stars.

As will be shown later, this physical quantity is of great importance in stars. Momentum transfer occurs from the radiation field to the stellar plasma following atomic absorption of photons, and this causes what is called radiation pressure.

The electromagnetic spectrum can be divided into a number of regions (see Table 1.1). It should be noted that the boundaries of these regions can vary from one source to another. For example, in astronomy the radio region often includes microwaves (0.1 cm $< \lambda <$ 100 cm). The visible part of the electromagnetic spectrum is in the range 4000 Å $< \lambda <$ 7000 Å where Å represents a unit of length called the angstrom and is equal to 10^{-8} cm. Within the visible part of the spectrum, several colours (blue, yellow, etc.) can be observed that are defined by wavelength. The approximate (or representative) wavelengths of these colours are given in Table 1.2. The most energetic photons in the visible spectrum are violet; whereas the least energetic are red.

Earth's atmosphere is opaque to most wavelengths except those in the visible part of the spectrum and in some parts of the radio. This is why Earth-based observatories detect either visible or radio waves, while ultraviolet or X-ray observatories are placed in orbit around the Earth. Since the vast majority of the information gathered from the Universe

Table 1.1 The electromagnetic spectrum.

Region	Wavelength range
Radio	>0.1 cm
Infrared	7000 Å to 0.1 cm
Visible	4000 to 7000 Å
Ultraviolet	100 to 4000 Å
X-ray	0.1 to 100 Å
Gamma-ray	<0.1 Å

Table 1.2 Approximate wavelength of colours.

Colour	Wavelength (Å)
Violet	4200
Blue	4700
Green	5300
Yellow	5800
Orange	6100
Red	6600

comes in the form of electromagnetic radiation, it is imperative to properly understand the interaction between radiation and matter.

1.3 Blackbody Radiation

In everyday life, when observing an object, what is detected is the light that it is reflecting. For instance, if when looking at a red object, the reason why it is red is that the object in question is absorbing most colours except red, which is being reflected. In sunlight or light emitted by most household bulbs, there exist all of the colours of visible part of the electromagnetic spectrum. That is why it is preferable to wear light clothing (optimally white) in hot weather, since it will reflect most of the light that falls upon it. Meanwhile, black objects absorb most of the visible light they receive.

A body will also emit radiation whose spectra will depend on its temperature. By definition, a blackbody is a physical entity that absorbs all radiation that falls upon it. Radiation emanating from a blackbody is due uniquely to its thermal energy.

The German physicist Max Planck (1858–1947) showed that a blackbody with temperature T emits a continuous spectrum of radiation characterized by a function $B_\nu(T)$, commonly called the Planck function. The units of this function are[3] erg/s/Hz/cm^2/sr and are those of the physical quantity called specific intensity (I_ν, see Section 3.3 for more details). In the field of astrophysics the cgs (standing for centimetre-gram-second) unit system is

[3]The unit erg is the unit of energy in the cgs system while sr is the unit of solid angle (see Chapter 3 for more details). One erg equals 10^{-7} J (see Appendix B).

the norm. The main physical constants in cgs used throughout this book can be found in Appendix A, while both cgs and S.I. (or the international system) units and conversion factors are given in Appendix B.

The monochromatic flux (F_ν) is defined as the quantity of energy in the spectral range between ν and $\nu + \mathrm{d}\nu$ emitted per unit surface, per unit time in units of erg/s/Hz/cm^2. In Chapter 3, it will be shown that for a blackbody, this quantity is given by the simple relation $F_\nu = \pi B_\nu$. It should be noted that in some physics textbooks, the Planck function given is the flux instead of the specific intensity and a factor π will then appear there.

The Planck function depends only on T and ν and is given by the following expression

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{1.3}$$

where k is the Boltzmann constant. This function is isotropic and thus independent of the direction.

The Planck distribution can also be written per unit wavelength (B_λ). Since, for a given blackbody, the integration over the entire spectra of B_ν and B_λ must be equal

$$B_\nu \mathrm{d}\nu = -B_\lambda \mathrm{d}\lambda \tag{1.4}$$

and

$$B_\lambda = -B_\nu \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{c}{\lambda^2} B_\nu = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \tag{1.5}$$

The cgs units of B_λ are erg/s/cm/cm^2/sr. Sometimes, units per unit wavelength in Å, erg/s/Å/cm^2/sr are used instead. Figure 1.2 illustrates Planck functions for several temperatures.

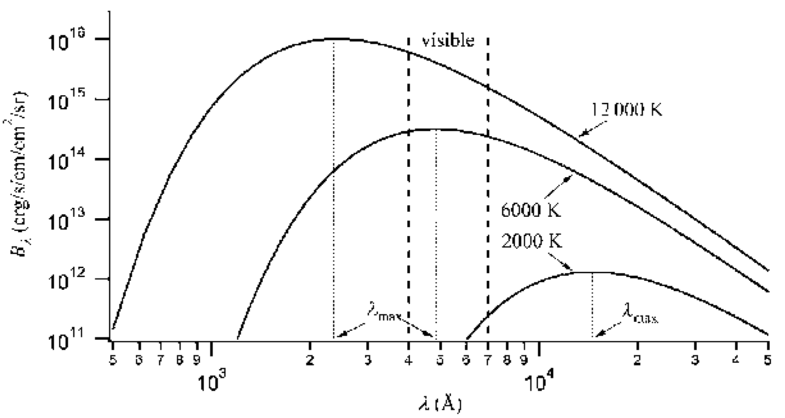

Figure 1.2 Planck distributions (B_λ) as a function of wavelength for $T = 2000$, 6000 and 12 000 K. The λ_{max} associated to each function and the visible part of the spectrum are also identified in this figure.

The energy distribution emitted by a blackbody leads to two laws. The first, the Stefan–Boltzmann law, gives the total power output per unit area F (or integrated flux in units of $\mathrm{erg/s/cm^2}$) of a blackbody with temperature T is

$$F = \int_0^\infty F_\nu \mathrm{d}\nu = \int_0^\infty \pi B_\nu \mathrm{d}\nu = \sigma T^4 \tag{1.6}$$

where σ is the Stefan–Boltzmann constant. To obtain this result, an integration of the monochromatic flux over the entire electromagnetic spectrum has been carried out (see Exercise 1.1). It shows that the energy output of a blackbody increases very rapidly with temperature. It should be noted that a blackbody with a higher temperature emits more energy at *all* wavelengths than a cooler one (see Figure 1.2). Since a star can be approximated by a blackbody (see Figure 1.8 in Section 1.6), a massive star having a high surface temperature, will emit much more power than a low-mass star that possesses a lower surface temperature. Massive stars will then have a shorter lifespan than smaller ones, since they burn their hydrogen at a much faster rate to compensate for their high brightness (this higher rate of nuclear burning is actually due to higher central temperatures). This topic will be discussed in more detail in Chapter 6.

A second law can also be derived from B_λ. It can be shown (see Exercise 1.2), that the wavelength $\lambda_{\max}$, at which the function B_λ is at its maximum, varies inversely with temperature (see Figure 1.2)

$$\lambda_{\max} = \frac{0.290\,\mathrm{K\,cm}}{T} \tag{1.7}$$

This equation is called Wien's law. It explains why hotter blackbodies (or stars) are blue and cooler ones are red. For example, when a blacksmith puts a piece of iron in the fire, it first starts glowing red. Then, as it gets hotter, it becomes white and even blue, hence the term *white hot*. When the piece of iron is at room temperature, it emits almost no visible light since the maximum of its energy distribution is found in the infrared. For that reason, when a person is lost in the forest, a search can be undertaken using infrared detectors. The body of a human being has a temperature of about 310 K (or 37 °C) and is hotter than the surrounding nature with a temperature of about 293 K (or 20 °C) depending on the season. A human body emits much more infrared radiation than these surroundings.

Figure 1.2 shows that a blackbody with a temperature of 2000 K has its $\lambda_{\max}$ in the infrared part of the electromagnetic spectrum, a 6000-K blackbody has its maximum emission in the visible region of the spectrum, while a 12 000-K blackbody has its $\lambda_{\max}$ in the ultraviolet. Since the human eye is more sensitive to photons with wavelengths in the blue part of the electromagnetic spectrum than those in the violet portion, the hottest stars in the sky seem blue, even though the maximum of the energy distribution of these stars is in the violet or even in the ultraviolet. They seem blue, because they emit more blue light than the other less energetic colours, due to the slope of the Planck distribution. The Sun is yellow, because its $\lambda_{\max}$ lies in the visible part of the electromagnetic spectrum (see Example 1.1).

Example 1.1: Calculate λ_{max} for the Sun.

Answer:

The surface temperature of the Sun is approximately 5800 K. If the radiation field of the Sun is approximated by that of a blackbody

$$\lambda_{max} = \frac{0.290\,\mathrm{K\,cm}}{T} = \frac{0.290\,\mathrm{K\,cm}}{5800\,\mathrm{K}} = 5\times10^{-5}\,\mathrm{cm} = 5000\,\text{Å} \tag{1.8}$$

This wavelength lies in the green part of the visible region of the spectrum. But since the Sun also emits a lot of blue, yellow and red light, the human eye, which is not equally sensitive to all wavelengths, incorporates all of these colours and sees the Sun as yellow.

Special Topic – The Greenhouse Effect

The average temperature on the Earth's surface is regulated by the amount of energy it receives from the Sun and the amount irradiated to space. The Earth's atmosphere is transparent to the visible part of the electromagnetic spectrum. Since the temperature at the Sun's surface is approximately 5800 K, its spectrum maximum is in the visible region and thus a lot of energy crosses the atmosphere and reaches the Earth's surface. Meanwhile the Earth's surface has an approximate temperature of 290 K and emits mostly infrared radiation. However, molecules such as H_2O and CO_2 can absorb infrared radiation and thus keep some heat in the terrestrial system. If it wasn't for the atmosphere, the temperature at our planet's surface would be more than 30 degrees cooler than it is now.

Unfortunately, human activity, such as the burning of fossil fuels, has increased the amount of pollutants (mostly CO_2) in our atmosphere. The increase of the abundances of these gases, called greenhouse gases, amplifies the opacity of the atmosphere to infrared radiation, which decreases the amount of energy lost to space. This process leads to a slight increase of the Earth's temperature and is called the greenhouse effect. Even the relatively small temperature increases expected are predicted to have important negative ecological impacts.

1.4 Luminosity, Effective Temperature, Flux and Magnitudes

The luminosity of a star is defined as the radiative power output emanating from its surface and is given in units of erg/s. The luminosity is an intrinsic value of a star and is not related to its distance from the observer. To obtain the luminosity, one must integrate the radiation field emitted over the entire electromagnetic spectrum and over the entire surface of the star. In the cases treated here, the flux will be assumed to be constant over the entire stellar surface. The luminosity is then obtained by simply multiplying the integrated flux (F) by the value of the star's surface area.

The effective temperature T_{eff} of a given star is defined as being the temperature needed for a blackbody with the same radius R_* as this star, to have the same luminosity L_* as this star. Since the integrated flux at the surface of this hypothetical blackbody is σT_{eff}^4, its luminosity is

$$L_* = 4\pi R_*^2 \sigma T_{eff}^4 \tag{1.9}$$

and the effective temperature of a star is

$$T_{eff} = \left(\frac{L_*}{4\pi R_*^2 \sigma} \right)^{1/4} \tag{1.10}$$

The integrated radiative flux at the surface of a star, in units of erg/s/cm^2, can also be written as a function of luminosity

$$F = \frac{L_*}{4\pi R_*^2} = \sigma T_{eff}^4 \tag{1.11}$$

At a distance r larger than R_* from the centre of the star, the integrated flux is

$$F(r) = \sigma T_{eff}^4 \left(\frac{R_*}{r} \right)^2 \tag{1.12}$$

Contrarily to the luminosity, the flux depends on the distance of the observer from the star. This equation shows the effect of the geometrical dilution of the flux as a function of distance from a star. This results from the fact that the luminosity is being distributed over a spherical surface of value $4\pi r^2$.

The human eye has a nonlinear response to light intensity. For example, a star that has an observed flux 10 times greater than a neighbouring star will not seem ten times brighter to the human eye. Thus, for practical and technological reasons, ancient astronomers divided the visible stars into a number of magnitude classes that better measures brightness with respect to the human eye than does flux. Unfortunately, these astronomers chose an unconventional scale such that the brighter stars have a lower magnitude. Magnitude is a relative scale that measures the logarithmic value of the radiative flux. A modern definition of magnitude is given by the formula

$$m_1 - m_2 = 2.5 \log \left(\frac{F_2}{F_1} \right) \tag{1.13}$$

which gives the difference of magnitudes of two stars as a function of their observed flux. This formula was chosen so that two stars with flux ratio of 100 will have a magnitude difference of 5 and, again for historical reasons, so that magnitude decreases when flux increases. Since the magnitude depends on the flux, it also depends on the distance separating the observer from the star. The magnitude m observed from Earth is called the apparent magnitude. An absolute magnitude M is then defined as the magnitude at a distance of 10 parsecs (1 pc = 3.26 light years[4]). Since the formula above is given on a relative scale, its usefulness is limited unless it is calibrated by fixing a magnitude for a given flux. Historically, the star Vega was chosen to have a magnitude of zero, so any object brighter than this standard star will have a negative magnitude.

It can be easily demonstrated (see Example 1.2) that the difference between the apparent and the absolute magnitude of a star is related to its distance d (in parsecs) to the observer via the equation

$$m - M = 5\log\left(\frac{d}{10}\right) \tag{1.14}$$

The value m–M is often called the distance modulus.

Example 1.2: Demonstrate the distance modulus equation given above.

Answer:

The definition of the magnitude is

$$m_1 - m_2 = 2.5\log\left(\frac{F_2}{F_1}\right) \tag{1.15}$$

For a given star with an apparent magnitude of m and an absolute magnitude of M, the magnitudes in the equation above may be defined as $m_1 = m$ and $m_2 = M$. Also, the flux at distance d from the star of luminosity L is $F_1 = L/(4\pi d^2)$. Finally, the flux at a distance $d_{10} = 10\,\mathrm{pc}$, $F_2 = L/(4\pi d_{10}{}^2)$. Therefore

$$m - M = 2.5\log\left(\frac{d}{d_{10}}\right)^2 \tag{1.16}$$

and if d is expressed in parsecs, this equation becomes

$$m - M = 5\log\left(\frac{d}{10}\right) \tag{1.17}$$

However, since it is impossible to observe the entire spectrum of a star, it is useful to define a magnitude for a given portion of the electromagnetic spectrum. The study of radiation inside a certain range of wavelength, commonly called a photometric band, is

[4]The parsec is a unit of distance defined in Section 6.9.5, while the light year is the distance travelled by light in vacuum during a one-year period.

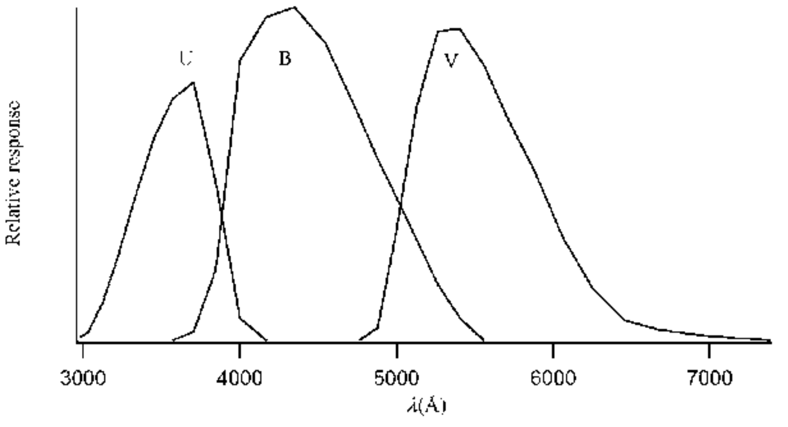

Figure 1.3 Response of U, B and V photometric indices (data from Arp, H.C.,*The Astrophysical Journal*, 133, 874 (1961)).

Table 1.3 Visual magnitudes of various astronomical objects.

Object name	m_V
Sun	–26.73
Full Moon	–12.7
Venus#	–4.5
Jupiter#	–2.5
Sirius	–1.44
Rigel	0.12
Saturn#	0.7
Deneb	1.23
Polaris	1.97

#At maximum brightness.

called photometry. To obtain the flux inside a given photometric band, a filter that is transparent to the radiation found inside this band and opaque to the photons outside of it, is placed in front of a photon detector.

Since radiation at different energies reacts with materials in different ways, telescopes and detectors must be adapted to the energy range of interest. Naturally, in the visible region of the spectrum, an optical telescope is used to accumulate the light on the detector. Figure 1.3 illustrates the transparency of such filters in the visible (V), blue (B) and ultraviolet (U) portions of the visible spectrum. These transparency functions must be taken into account when comparing observed magnitudes to theoretical values.

The brightest star in the sky is Sirius, while the faintest stars that are visible by the human eye have an apparent visual magnitude of approximately 6. Table 1.3 shows the apparent visual magnitudes of several well-known astronomical objects.

Example 1.3: Knowing that the apparent visual magnitude of the Sun is −26.73, calculate its absolute magnitude.

Answer:

The Sun is by definition at a distance of one astronomical unit (AU) from the Earth. Since 1 AU = 1.496×10^{13} cm = 4.848×10^{-6} pc, the distance modulus equation

$$m_V - M_V = 5\log\left(\frac{d}{10}\right) \tag{1.18}$$

may be used to find the solution.
Replacing the known values in the equation above

$$-26.73 - M_V = 5\log\left(\frac{4.848\times10^{-6}\ \text{pc}}{10\ \text{pc}}\right) \tag{1.19}$$

leads to $M_V = 4.84$.

Later, it will be shown that the absolute magnitude of a star can be determined by spectroscopy. Spectroscopy is defined as the study of radiation with respect to wavelength. Since the apparent magnitude can be obtained by photometric observations, the distance to stars can then be determined with the distance modulus equation (Eq. 1.14).

The definition of magnitude given above (Eq. 1.13) can also be applied to magnitudes of two photometric bands of a single star. If one obtains photometric measurements of two photometric bands for a star, the flux ratio of these bands can be used to obtain its effective temperature. To better illustrate this, an example is shown in Figure 1.4, where the flux of

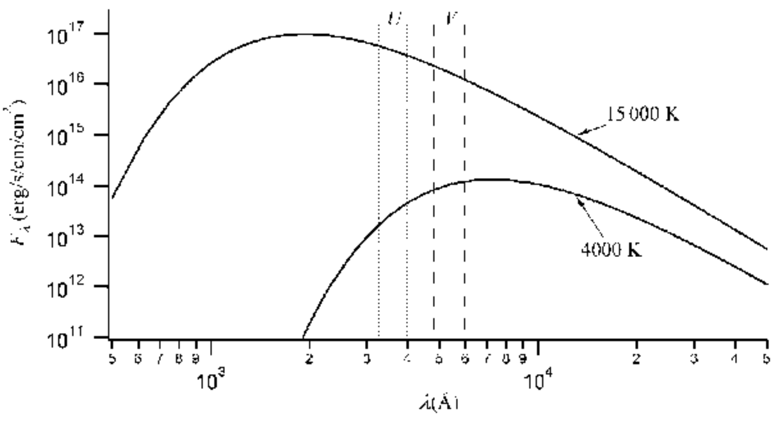

Figure 1.4 Monochromatic flux (F_λ) as a function of wavelength for two stars with T_{eff} = 4000 and 15 000 K approximated by blackbody radiation. The approximate positions of two photometric filters (U and V) are also shown.

a star is approximated by that of a blackbody with temperature T_{eff}. Two photometric bands for two blackbodies of different temperatures are shown. From this illustration, it is found that the ratio F_U/F_V (and thus $m_V - m_U$) increases with temperature. Since the blackbody flux is a well-known quantity, a value F_U/F_V is associated to each temperature. Assuming that the theoretical fluxes of stars with various effective temperatures can be calculated via the study of stellar atmospheres (see Chapter 4), the observed values of two apparent photometric magnitudes can be used to obtain T_{eff}. If nothing obstructs the light coming from the stars (interstellar clouds for example), $m_V - m_U$ is independent of distance to the observer. Typically, however, the presence of interstellar absorption or scattering necessitates certain corrections to be brought to the observed photometric magnitudes.

1.5 Boltzmann and Saha Equations

A star is composed of gaseous plasma containing both neutral and ionised atoms as well as free electrons. These free electrons come from ionisation. Ionisation is a process by which an atom loses one or more of its bound electrons. The atoms of a given element in various states of ionisation are called ions. In spectroscopy, ions are represented by the elemental nomenclature followed by a roman number. For example, CI is neutral carbon, CII is singly ionised carbon, and CVII is carbon ionised six times (i.e. a bare nucleus). Each ion of an element has its specific atomic energy levels. For reasons that will become clearer in later chapters, it is important to know the relative population of the various states of ionisation for each element present as a function of stellar depth, as well as the population among the various atomic energy levels for each of these ions. These quantities are critical for calculating the radiative opacity, which is the capacity of matter to absorb electromagnetic radiation. Opacity affects how radiation is transported from the inner to the outer portions of a star (see Chapter 3 for more details).

The field of statistical physics shows that the atomic energy levels of a given ion are populated inversely exponentially as a function of their energy: lower energy levels are naturally more populated than higher-lying energy levels. This being said, a bound electron can be excited to a higher energy level by two processes. Firstly, the energy needed for the bound electron to change levels can be obtained during a collision of the atom with another particle, for instance, a free electron. In this case, the kinetic energy of the free electron is used to excite the bound electron. The second process that can cause an excitation of an ion, is the absorption of a photon with energy equal to that of the electron transition (i.e. of energy equal to the difference between the two levels under consideration). These are called bound–bound transitions, since an electron goes from one bound state to another; whereas ionisation is a bound–free transition since the electron goes form a bound to a free state (see Figure 1.5). When collisions are the dominant processes that influence the energy-level populations (which is often the case in stars), the ratio of the population of two energy levels of a given ion in a gas at temperature T is given by the Boltzmann equation

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} e^{-\frac{(E_i - E_j)}{kT}} \tag{1.20}$$

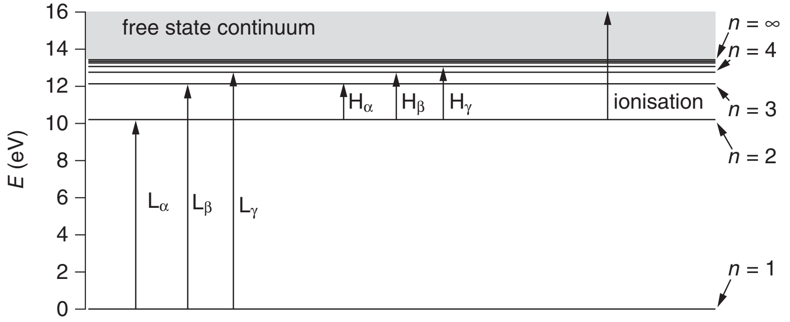

Figure 1.5 Energy levels of hydrogen in eV. Various bound-bound transitions are also shown, as well as a bound–free transition from level $n = 2$ (see Section 1.6 for more details).

where k is the Boltzmann constant, n_i is the number of atoms per unit volume (or population) in energy level i of the ion under consideration and g_i is the degeneracy of this level. The reader is reminded that the degeneracy of an energy level is the number of quantum states with the same energy. The quantity E_i is the energy of level i relative to the fundamental level, *which is set to zero*.

However, this form of the Boltzmann equation is not often useful. Instead, the ratio of the population of a given energy level to the total population of the ion under consideration is more useful. This quantity, which is useful for radiative opacity calculations (see Chapter 3), can be written (see Example 1.4)

$$\frac{n_i}{n_{\rm ion}} = \frac{g_i}{U_{\rm ion}} e^{-\frac{E_i}{kT}} \tag{1.21}$$

with

$$U_{\rm ion} = \sum_{n=1}^{\infty} g_n e^{-\frac{E_n}{kT}} \tag{1.22}$$

where $U_{\rm ion}$ is called the partition function of the ion under consideration, and $n_{\rm ion}$ is its total population. This form of the Boltzmann equation shows that the fraction of ions in a given energy level is equal to the portion of the partition function related to this level.

Example 1.4: Demonstrate the equation

$$\frac{n_i}{n_{\rm ion}} = \frac{g_i}{U_{\rm ion}} e^{-\frac{E_i}{kT}} \tag{1.23}$$

Answer:

From the equation

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} e^{-\frac{(E_i - E_j)}{kT}} \tag{1.24}$$

and since $E_1 = 0$, n_i with respect to the population of the fundamental level n_1 is written

$$\frac{n_i}{n_1} = \frac{g_i}{g_1} e^{-\frac{E_i}{kT}} \tag{1.25}$$

Meanwhile, the total population of the ion under consideration is

$$n_{\text{ion}} = \sum_{m=1}^{\infty} n_m = \frac{n_1}{g_1} \sum_{m=1}^{\infty} g_m e^{-\frac{E_m}{kT}} = \frac{n_1}{g_1} U_{\text{ion}} \tag{1.26}$$

The two equations above can be used to show that

$$\frac{n_i}{n_{\text{ion}}} = \frac{g_i}{U_{\text{ion}}} e^{-\frac{E_i}{kT}} \tag{1.27}$$

To better understand these concepts, it is instructive to apply them to hydrogen, which has well-known energy levels that can be calculated analytically via Bohr's atomic model. In units of electronvolts (eV),[5] E_n for the hydrogen atom is

$$E_n = 13.6\left[1 - \frac{1}{n^2}\right] \tag{1.28}$$

where n is the principal quantum number of the atomic energy level under consideration. Figure 1.5 shows the energy levels of hydrogen, and some transitions that can take place among them (see next section for more details). The degeneracy of a given level n is equal to $g_n = 2n^2$ for hydrogen.

To calculate the partition function, an infinite number of terms, related to the energy levels, must be summed. Unfortunately, for large values of n, the degeneracy (g_n) increases rapidly while the exponential found in the partition function equation ($e^{-\frac{E_n}{kT}}$) tends towards a constant value. The sum will then diverge for any temperature. Luckily, some simple physical considerations can alleviate this problem.

To better illustrate this problem, the case of hydrogen will be discussed. According to the Bohr model of the atom, the radius of the hydrogen atom in level n is $r = a_0 n^2$, where $a_0 = 0.529$ Å is the radius of the fundamental level of hydrogen (called the Bohr radius). The infinite sum needed to calculate the partition function is not physical, since for high-lying levels, the electron will eventually be closer to another nucleus than its own. An infinite sum for the partition function makes sense only if the atom in question is alone in

[5] 1 eV = 1.6×10^{-12} erg.

the Universe, which is obviously not the case! It should also be noted that in the analytical development leading to the Bohr radius equation, it is usually supposed that the only force on the electron is the attractive Coulomb force between the nucleus and the electron. So here again, the Universe is approximated to be composed only of the atom under consideration. A cut-off level of quantum number $n_{max,}$ where the levels superior to this energy level are no longer bound to the nucleus, can be defined and used to approximate the value of the partition function. This can also be interpreted as a lowering of the continuum shown in Figure 1.5. It can be shown that for a pure hydrogen gas, $n_{max} = (2a_0)^{-1/2}(N)^{-1/6}$ where N is the number density of hydrogen atoms (see Example 1.5). The partition function can then be approximated by a finite sum

$$U = \sum_{n=1}^{n_{max}} g_n e^{-\frac{E_n}{kT}} \tag{1.29}$$

Example 1.5: Show that for a pure hydrogen gas the cut-off value of the energy levels can be approximated by $n_{max} = (2a_0)^{-1/2}(N)^{-1/6}$ when calculating the partition function and where N is the number density of hydrogen atoms in the gas.

Answer:

By supposing that the average distance between two hydrogen atoms in the gas is $2d$, the number density is thus one atom per $(2d)^3$ volume

$$N = \frac{1}{(2d)^3} \tag{1.30}$$

The maximum value of n where the electron is still closer to the initial nucleus than a neighbouring one is $r_n \leq d$ where $r_n = a_0 n^2$. The variable n_{max} may be defined by the following

$$r_{max} = a_0 n_{max}^2 = d = \frac{1}{2N^{1/3}} \tag{1.31}$$

and thus

$$n_{max} = \frac{1}{\sqrt{2a_0}\, N^{1/6}} \tag{1.32}$$

Since ionised hydrogen has no atomic energy levels because it has lost its only electron, its partition function equals unity (i.e. it may be assumed that this ion has a single state of energy equal to 0 eV). This partition function is necessary to solve the equations describing ionisation of hydrogen shown below. At low temperatures, the partition function of neutral hydrogen can be approximated by the statistical weight of the fundamental energy level $g_1 = 2$ since the other terms in the sum (see Eq. 1.29) become small.

Example 1.6: Find the temperature at which the number density of hydrogen atoms in the fundamental state is equal to that of its second excited state (n = 3).

Answer:

From the Boltzmann equation

$$\frac{n_1}{n_3} = \frac{g_1}{g_3} e^{-\frac{(E_1 - E_3)}{kT}} = 1 \tag{1.33}$$

and since $g_1 = 2$, $g_3 = 18$, $E_1 = 0\,\text{eV}$ and $E_3 = 12.09\,\text{eV}$,

$$\frac{2}{18} e^{\frac{12.09\,\text{eV}}{kT}} = 1 \tag{1.34}$$

This becomes

$$\frac{12.09\,\text{eV}}{kT} = \ln(9) \tag{1.35}$$

and by using the value $k = 8.617 \times 10^{-5}\,\text{eV/K}$, the temperature is thus $T = 63\,900\,\text{K}$.

In stars, the local temperature increases as a function of depth. Moreover, deeper inside the stars, more energetic collisions will take place. This is due to the fact that according to statistical physics, the average thermal velocity of the particles in the stellar plasma is proportional to $T^{1/2}$. These collisions will cause excitations of atoms to higher energy levels (as described by the Boltzmann equation) and can also lead to ionisation of these atoms. Another process that can lead to an atom losing an electron is the absorption of a sufficiently energetic photon (see Figure 1.5). This process is called photoionisation. The freed electrons will contribute to the total gas pressure P. The reader is reminded that for an ideal gas, the equation of state is $P = n_{\text{tot}}kT$, where n_{tot} is the total number density of particles in the gas. This number density includes both the free electrons and the ions that are present in the plasma. A new physical quantity μ called the mean molecular weight of the particles in the gas may be defined by writing $n_{\text{tot}} = \frac{\rho}{\mu m_{\text{H}}}$, where ρ is the gas mass density (often simply called the density) and m_{H} is the mass of the hydrogen atom. Therefore, since density is given by the following equation

$$\rho = \sum_i n_i m_i \tag{1.36}$$

the mean molecular weight is

$$\mu = \frac{1}{m_{\text{H}} n_{\text{tot}}} \sum_i n_i m_i \tag{1.37}$$

where the sum over i runs over all types of particles present in the plasma including free electrons. The mean molecular weight gives the average mass of the particles in units of m_H. For instance, in a completely ionised hydrogen gas, $\mu = \frac{m_p + m_e}{2m_H} \approx \frac{1}{2}$, where m_p and m_e are respectively the proton and electron masses. The mean molecular weight is a useful concept that is used in stellar astrophysics and will be employed on several occasions in this book.

When collision processes dominate (which is often the case inside stars), the equation that regulates ionisation is called the Saha equation. It can be written

$$\frac{n_{i+1}}{n_i} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} \frac{2U_{i+1}}{U_i} e^{-\frac{E_{ion}}{kT}} \tag{1.38}$$

where n_i and n_{i+1} are the populations of neighbouring ions of a given element, n_e is the number density of free electrons in the gas (often called the electronic density), T the local temperature, U_i and U_{i+1} are the corresponding partition functions and E_{ion} is the ionising energy of ion *i from its fundamental energy level*. Here, ion $i + 1$ is the more highly ionised ion.

From this equation, it may be deduced that ionisation increases with temperature. This is related to the fact that more energetic collisions are possible in hotter plasma. Also, for a given temperature, ionisation decreases with increasing electronic density. An increase in n_e fills the phase space of free electrons and increases recombination of free electrons with ions (i.e. deionisation).

The equation shown above gives the relative populations of two neighbouring ionisation states. However, this quantity is not often useful in astrophysical applications. As will be discussed in Chapter 3, to calculate the radiative opacity for a given elemental species, the population of each energy level needs to be known, which necessitates the knowledge of the population of each ionisation state. A quantity that is critical for such calculations is the ionisation fraction. The ionisation fraction is the portion of atoms in a given ionisation state of the element under consideration. The ionisation fraction f_i of ionisation state i can be written

$$f_i = \frac{n_i}{n_1 + n_2 + n_3 + n_4 + \ldots} \tag{1.39}$$

and by dividing both the numerator and the denominator by the neutral state's population n_1

$$f_i = \frac{\left(\frac{n_i}{n_1}\right)}{1+\left(\frac{n_2}{n_1}\right)+\left(\frac{n_3}{n_1}\right)+\left(\frac{n_4}{n_1}\right)+\ldots} = \frac{\left(\frac{n_i}{n_{i-1}}\right)\left(\frac{n_{i-1}}{n_{i-2}}\right)\cdots\left(\frac{n_2}{n_1}\right)}{1+\left(\frac{n_2}{n_1}\right)+\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right)+\left(\frac{n_4}{n_3}\right)\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right)+\ldots} \tag{1.40}$$

A series of multiplications of Saha equations (Eq. 1.38) is thus obtained, that once calculated, will give the value of the ionisation fraction (assuming n_e and T are known).

Special Topic – Ionisation Energies

Ionisation energies for the first five ionisation stages for a large number of elements are given in Appendix D. Figure 1.6 shows the ionisation energy for neutral atoms as a function of atomic number. It is shown that there exists a local maximum of the ionisation energy for noble gases (He, Ne, Ar, etc …). These maxima are shifted to other elements for higher stages of ionisation. For example, for the singly ionised ion, maxima are found for LiII, NaII and KII (see Appendix D). These ions possess electronic configurations having respectively 2, 10 and 20 electrons and have filled electronic shells. They are also called noble gas electronic configurations.

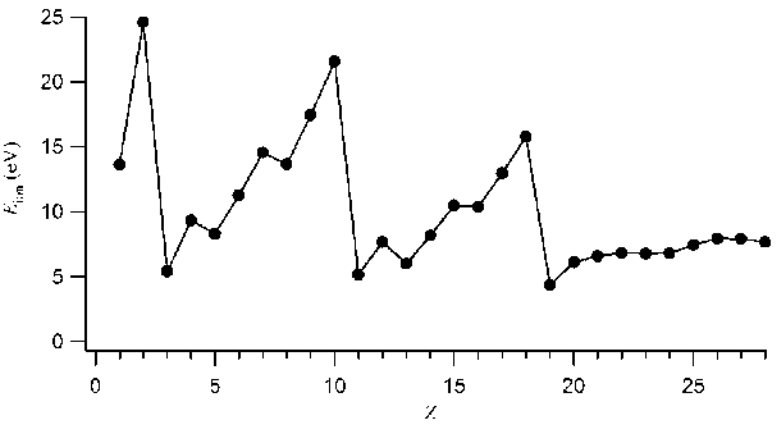

Figure 1.6 Ionisation energy (from the fundamental atomic energy state) as a function of atomic number for neutral atoms.

Of course, the ionisation fraction will vary with depth in stars, with more highly ionised ions appearing in deeper stellar layers. Figure 1.7 shows the ionisation fractions of the first 13 calcium ions in a star with $T_{\rm eff}$ = 7600 K. In this figure, there exists a large plateau for CaIII and CaXI ionisation fractions. These ions have large ionisation energies since they are in noble-gas configurations (respectively, those of Ar and Ne). These noble-gas configurations stay populated for a large domain of temperatures compared to other electronic configurations because of their large ionisation energy. Since the atomic energy levels (and therefore the absorption transitions) are different for each ionisation state, the radiative opacity of a given element will also vary with depth.

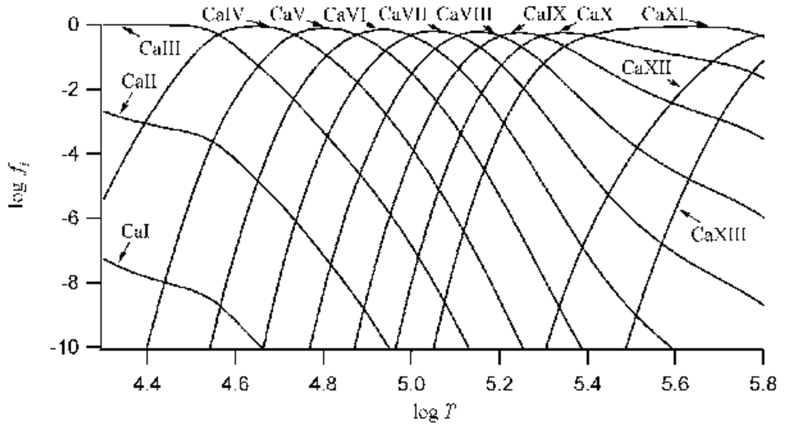

Figure 1.7 Ionisation fractions (f_i) of Ca ions as a function of temperature (or depth) in the interior of a star with $T_{\mathrm{eff}} = 7600\,\mathrm{K}$. The surface of the star is found at the left side of the horizontal axis.

Example 1.7: For a given star, calculate the fraction of neutral atoms in a gas composed of pure hydrogen at a depth where $T = 12\,000\,\mathrm{K}$ and $n_e = 2.0 \times 10^{15}\,\mathrm{cm}^{-3}$ (assume that the partition function of neutral hydrogen $U_{\mathrm{I}} = 2$).

Answer:

In a pure hydrogen gas, the free electrons come exclusively from hydrogen ionisation and therefore $n_e = n_{\mathrm{II}}$ where n_{II} represents the population of HII ions.
From the Saha equation

$$\frac{n_{\mathrm{II}}}{n_{\mathrm{I}}} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} \frac{2U_{\mathrm{II}}}{U_{\mathrm{I}}} e^{-\frac{E_{\mathrm{ion}}}{kT}} \tag{1.41}$$

where $E_{\mathrm{ion}} = 13.6\,\mathrm{eV}$ and $U_{\mathrm{II}} = 1$. By inserting the appropriate values into this equation, n_{I} is obtained

$$n_{\mathrm{I}} = \frac{n_e^2}{6.156 \times 10^{15}\ \mathrm{cm}^{-3}} = 6.498 \times 10^{14}\ \mathrm{cm}^{-3} \tag{1.42}$$

The ionisation fraction of neutral hydrogen is then

$$f_{\mathrm{I}} = \frac{n_{\mathrm{I}}}{n_{\mathrm{I}} + n_{\mathrm{II}}} = 0.245 \text{ or } 24.5\% \tag{1.43}$$

Example 1.8: Calculate the electronic density (n_e) in a gas at $T = 14\,000$ K composed of pure hydrogen where 70% of the atoms are ionised (assume $U_I = 2$).

Answer:

Since

$$\frac{n_{II}}{n_I + n_{II}} = 0.7 \tag{1.44}$$

therefore, $n_I = 0.428\ n_{II}$. Also, since the gas under consideration is made of pure hydrogen $n_{II} = n_e$.
From the Saha equation

$$\frac{n_{II} n_e}{n_I} = \left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} \frac{2U_{II}}{U_I} e^{-\frac{E_{ion}}{kT}} \tag{1.45}$$

where $E_{ion} = 13.6$ eV and $U_{II} = 1$. By inserting the appropriate values into this equation

$$\frac{n_{II} n_e}{n_I} = \frac{n_e^2}{0.428 n_e} = 2.33 n_e = 5.08 \times 10^{16}\ \text{cm}^{-3} \tag{1.46}$$

and $n_e = 2.18 \times 10^{16}\,\text{cm}^{-3}$.

It will be shown in Chapter 4 that the application of the Saha equation in real stars is more complex than the relatively simple examples shown above. In stellar models, since a large number of elements are present a large series of Saha equations has to be solved simultaneously. Atomic data included in the calculation of the partition functions and the Saha equations must then be known for all elements present. Such calculations therefore necessitate considerable computing resources.

Finally, it should be mentioned that the Boltzmann and Saha equations, respectively, give, statistically speaking, the portion of atoms in a given atomic level and in the various ionisation states. However, a single atom's state (atomic or ionisation) will constantly change as a function of time due to interactions with other particles. Generally, these interactions are induced by collisions, but radiative excitations and ionisations can sometimes be important. This will be discussed further in Chapter 3.

1.6 Spectral Classification of Stars

In astronomy, many objects, be it meteorites, galaxies or stars are classified. These classifications aim at a better understanding of the group of objects under consideration. In this section, one such classification will be discussed, namely the spectral classification of stars.

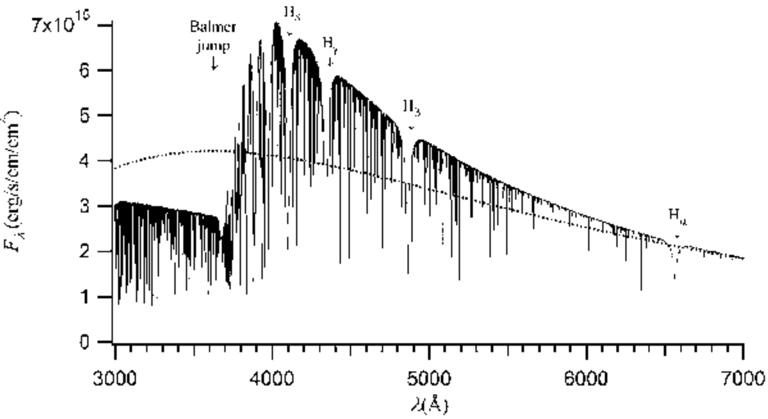

Figure 1.8 Theoretical monochromatic flux emerging form an A type star with $T_{\rm eff} = 8000\,\rm K$. The first four Balmer absorption lines, as well as the Balmer jump, are identified in this figure. Thousands of other absorption atomic lines can also be seen. This theoretical flux was obtained with the Phoenix stellar atmosphere code (Hauschildt, P.H., Allard, F. and Baron, E., *The Astrophysical Journal*, 512, 377 (1999)) while using the elemental abundances found in the Sun. The flux at the surface of a blackbody with $T = 8000\,\rm K$ (dotted curve) is also shown.

As photons diffuse towards the surface of a star, they can interact with the atoms present in the stellar plasma. A photon can, for example, be absorbed when its energy is used to excite an electron from a lower to an upper bound state of an atom. The absorption features seen in the spectrum from these transitions are called atomic lines (see Figure 1.8). If the atomic energy levels were precisely defined, only photons with a single value of λ could be absorbed by the transition under consideration. The value of λ is related to the energy difference between the upper and lower levels associated to the transition. The photon wavelength necessary for an electronic excitation from level n to level m is

$$\lambda_{n\rightarrow m} = \frac{hc}{E_m - E_n} \tag{1.47}$$

However, because of the uncertainty principle of the quantum theory, the energy levels cannot be precisely defined, thus giving an absorption profile with a certain width. Additionally, since the atoms in the star have a velocity distribution associated to the local temperature, called the Maxwell distribution, the Doppler effect as well as broadening by pressure (or collisions) will also play a role in the widening of the atomic lines (see Chapter 4 for more details).

For a given absorption line of an ion to be present in the spectra, the lower (or initial) level must be populated (i.e. Boltzmann equation) and of course, the ion must also be present (i.e. the Saha equation). Since a star's spectrum emerges from its photosphere, its effective temperature will play a pivotal role in determining which atomic lines are present in the spectrum.

Table 1.4 Lyman and Balmer series.

Lyman Series			Balmer Series		
Name	Transition ($n \to m$)	Wavelength	Name	Transition ($n \to m$)	Wavelength
L_α	$1 \to 2$	1216 Å	H_α	$2 \to 3$	6563 Å
L_β	$1 \to 3$	1025 Å	H_β	$2 \to 4$	4861 Å
L_γ	$1 \to 4$	972 Å	H_γ	$2 \to 5$	4341 Å
.			.		
.			.		
.			.		
Lyman jump	$1 \to \infty$	911 Å	Balmer jump	$2 \to \infty$	3646 Å

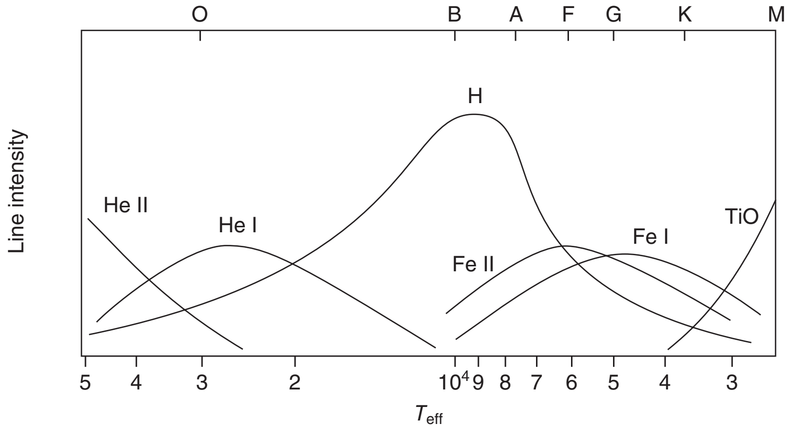

Figure 1.9 Approximate line intensity as a function of T_{eff} for several ions. The spectral types (these are positioned at the coolest temperature for each class) and the intensity of the TiO molecular bands are also shown.

Let's first discuss the behaviour of hydrogen lines in stellar spectra. Figure 1.5 shows the energy levels of hydrogen and some of the transitions that can occur. These transitions can be grouped as per their initial level. The lines emanating from the $n = 1$ level are called the Lyman lines (L_α, L_β, L_γ, etc.) and are found in the ultraviolet part of the spectrum. The Balmer series (H_α, H_β, H_γ, etc.) emanate from $n = 2$ and are in the visible part of the spectrum, while the Paschen lines (from $n = 3$) are found in the infrared. More details concerning the Lyman and Balmer series are given in Table 1.4. At the surface of cool stars, almost all of the hydrogen atoms are in the fundamental level and the Balmer lines (found in the visible spectrum) are very weak. The Lyman lines are also weak since relatively few ultraviolet photons exist in the spectrum of such a cool star. For hotter stars (say $T_{\text{eff}} = 8000\,\text{K}$ or so), the hydrogen atoms found in the $n = 2$ level begin to be significantly populated and the Balmer lines are then quite intense (see Figure 1.9). For even hotter stars, the intensity of the Balmer lines decreases, owing to the fact that the quantity of neutral hydrogen atoms contributing to the presence of the Balmer lines diminishes due to ionisation.

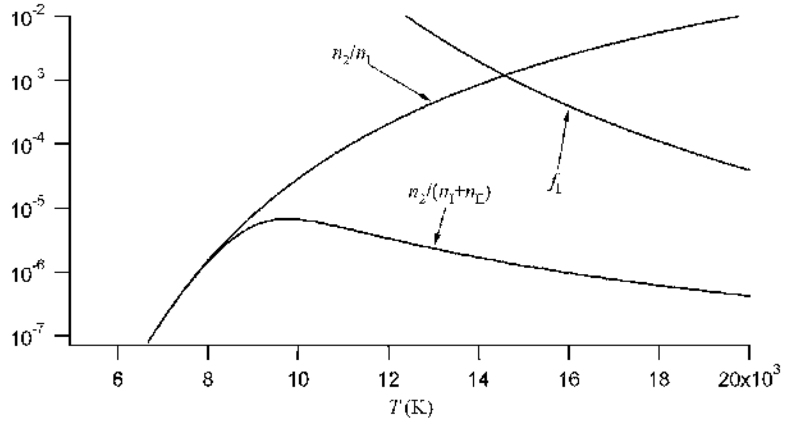

Figure 1.10 Illustration showing the portion of neutral hydrogen atoms found in the $n = 2$ level (n_2/n_I), the neutral ionisation fraction ($f_I = \frac{n_I}{n_I + n_{II}}$) and the product of these two factors that give the portion of all hydrogen atoms found in the $n = 2$ level (i.e. $\frac{n_2}{n_I + n_{II}}$).

Figure 1.10 illustrates the two contributing factor explaining why hydrogen Balmer lines are at their strongest for stars with surface temperatures around 10 000 K. The portion of neutral hydrogen atoms found in the $n = 2$ level increases with temperature, while the neutral ionisation fraction decreases. The line strength depends on the product of these two factors which has a maximum at $T \approx 10\,000$ K.

Similar tendencies are observed for the atomic lines of the other elements (see Figure 1.9). For example, FeI lines are strong in cool stars. But for hotter stars, FeII, FeIII, etc., eventually dominate. The position, with respect to $T_{\rm eff}$, of maximum strength of the atomic lines of various ions is related to their ionisation energy. For example, the ionisation energy of FeI is 7.9 eV, while it is 24.6 eV for HeI, the FeI atomic transitions are thus at lower energies than those of HeI. This explains why FeI lines are more prominent in cooler stars than those of HeI. The relative strength of atomic lines of different ions (either of the same or of a different element) can be used to estimate the surface temperature of stars. Such studies fall in the field of research called stellar spectroscopy.

Photons can also be absorbed during photoionisation. For hydrogen, the ionisation energy from its fundamental level is 13.6 eV, whereas it is 3.4 eV from its first excited state. The synthetic spectrum of Figure 1.8 shows a large flux decrease near $\lambda = 3646$ Å, due to the ionisation of hydrogen from level $n = 2$. This spectral feature is called the Balmer jump. As mentioned previously, the minimum energy of photons that can ionise hydrogen from this level is 3.4 eV. When more energetic photons are absorbed by this bound–free transition, the excess of energy is transformed to kinetic energy transferred to the ejected electron.

Stars are generally divided into seven spectral classes or types: O, B, A, F, G, K and M going from hotter (bluer) to cooler (redder) effective temperatures. This classical categorization of stellar spectra, based mainly on the strength of hydrogen Balmer lines, is called the Harvard classification. The A-type stars fall where the strongest (or deepest)

Table 1.5 Spectral classes.

Spectral class	T_{eff}	Spectral characteristics	Colour	Example
O	>30 000 K	HeII strong, H faint, multiply-ionised metals strong	blue	λ Ori
B	10 000–30 000 K	HeI strong, H moderate	blue-white	Rigel
A	7500–10 000 K	H lines at their maximum	white	Vega
F	6000–7500 K	Singly ionised metals strong, H moderate	white-yellow	Procyon
G	5000–6000 K	Singly ionised metals strong, H faint	yellow	Sun
K	3500–5000 K	Strong neutral and Singly ionised metals, H faint	orange	Arcturus
M	<3500 K	Strong molecule bands (i.e. TiO), strong neutral metals, H very faint	red	Betelgeuse

hydrogen lines are observed. As discussed above, two processes, excitation and ionisation, conspire to give the largest portion of hydrogen atoms in the $n = 2$ level in A-type stars (see Figure 1.10). A useful mnemonic to remember the order of the spectral classes is *'Oh Be A Fine Girl (or Guy, depending on the reader's preference), Kiss Me'*. The spectral features and T_{eff} of the different spectral classes are given in Table 1.5.

Simple molecules (TiO, CH, H_2O, etc.) can also exist in cooler stars and may absorb radiation not only through electronic transitions but also via rotational or vibrational transitions. These transitions are called bands instead of lines and are found in the infrared region of the spectrum. In hotter stars, the molecules are destroyed by photodissociation due to energetic photons, or by energetic collisions; hence, no molecular bands are observed in the spectra of such stars.

Hot stars are often called early-type stars, while cooler stars are called late-type stars. These terms came about when astronomers erroneously thought that stars began their lives as hot stars and cooled down during their lifespan.

The spectral classes can also be subdivided into 10 partitions. These subdivisions are identified by a single Arabic digit increasing from the hotter end to the cooler end of the spectral class (i.e. F0 stars are hotter than F9 stars). The spectral class of the Sun is G2.

All spectral types are not equally populated. There are fewer high-mass stars (i.e. type O and B) than less massive ones (i.e. type K and M). This is associated to the process of stellar formation that does not uniformly create stars with respect to their mass. This will be discussed in Chapter 2.

Several types of stars do not fit into the classical spectral classification given above. For instance, ApBp stars (p standing for peculiar) are A and B type stars with strong magnetic fields and large observed abundance anomalies. Abundance anomalies, are defined as when the abundances of some elements are very different from those expected (either those found in the Sun, or in the vicinity of the star under consideration). These abundance anomalies or peculiarities strongly modify their spectra which differentiate them from normal A-type stars. For example, in the case of an overabundance for a given element,

Table 1.6 Solar abundances of the most abundant elements.

Element	N_{elem}/N_{tot}
H	9.097×10^{-1}
He	8.890×10^{-2}
O	7.742×10^{-4}
C	3.303×10^{-4}
Ne	1.119×10^{-4}
N	1.021×10^{-4}
Mg	3.458×10^{-5}
Si	3.228×10^{-5}
Fe	3.154×10^{-5}
S	1.475×10^{-5}

its lines are much stronger. Abundances are often given relative to those of the Sun. Table 1.6 shows the abundances of the most abundant elements found in the Sun. A more complete set of solar-abundance data is given in Appendix E.

Abundance anomalies are believed to be caused by diffusion of the elements within the star, caused partly by the radiative force transferred to ions. The radiative force is due to momentum transfer from photons to atoms during line absorption for instance. The diffusion process can cause an accumulation or depreciation of certain species at different depths (see Chapter 7 for more details). Abundances observed at the surface of a star are not always indicative of the average abundances of the elements within the whole star.

Among other types of stars with peculiar spectra are Am (m standing for metallic) and HgMn stars (where Hg and Mn are generally overabundant by several orders of magnitudes at their surface as compared to their solar abundance). Another example of stars that can't be classified in the types shown in Table 1.5 are Be stars (e standing for emission). These stars are surrounded by gas, and emission lines are observed in their spectra. Emission lines are spectral features that resemble inverted absorption lines or spikes in the flux. Many other peculiar spectral types not mentioned here also exist.

Abundances found in stars are also used to define their population. There are three types of stellar populations. Population I stars are young stars with relatively large metallicity, while population II stars are older stars with a smaller value of metallicity. Population III stars are the oldest stars that, hypothetically, have zero metallicity. However, the stars of this population have never been directly observed. The Sun is a population I star. The relation between the age of a star and its metallicity can be explained by results from the Big-Bang theory and stellar evolution. As mentioned previously, at the beginning of the Universe, only hydrogen and helium were present, with the exception of a trace of lithium. Therefore, the first generation of stars (population III) did not contain any metals except for this trace element. As this generation of stars evolved, some become supernovae thereby enriching the interstellar medium with the newly synthesized heavy elements. Following generations of stars were then composed of this enriched matter, which translated into increasing metallicities. This process will be explained in more detail in Chapter 6.

1.7 The Hertzsprung–Russell Diagram

As discussed in Section 1.4, the luminosity of a star depends on both its radius and effective temperature. A famous diagram, called the Hertzsprung–Russell (hereafter H–R) diagram, shows the relation between the luminosity and the effective temperature of stars. In such diagrams, the direction of the abscissa (T_{eff}) is reversed (see Figures 1.11 and 1.12). This tool for studying stars was developed by the Danish astronomer Ejnar Hertzsprung

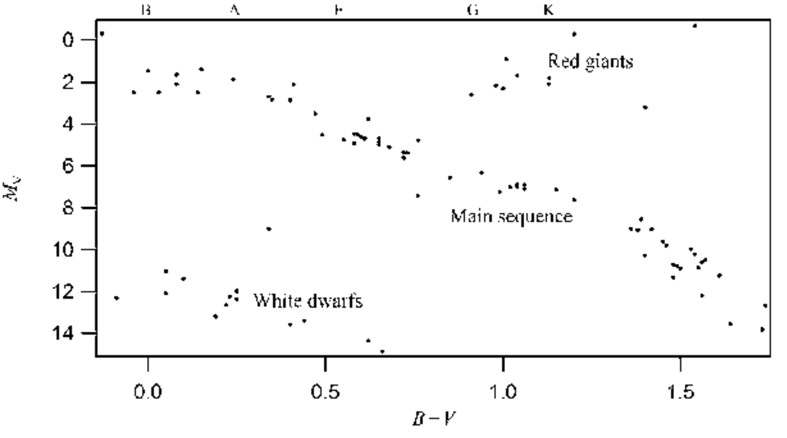

Figure 1.11 A sample taken among the 1000 nearest stars on a color-magnitude H–R diagram. The spectral types are also shown (these are positioned at the coolest temperature for each class).

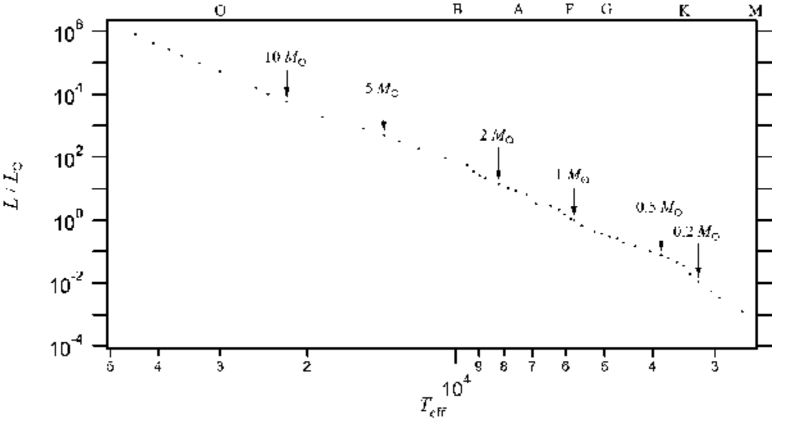

Figure 1.12 The main sequence on an H–R diagram. Several values of the mass are given. The spectral types are also shown (these are positioned at the coolest temperature for each class).

(1873–1967) and the American astronomer Henry Norris Russell (1877–1957) at the beginning of the twentieth century. The H–R diagram is extremely useful when studying the evolution of stars, since there are well-determined paths along which stars should travel as they evolve. These paths depend mostly on stellar mass (see Figure 6.10). During evolution, both the $T_{\rm eff}$ and the radius of a star change. Its spectral type will also be time dependent. Observational astronomers often use an absolute magnitude scale instead of luminosity, and $m_{\rm B} - m_{\rm V}$ instead of effective temperature. These are called colour-magnitude diagrams (see Figure 1.11). The colour index $m_{\rm B} - m_{\rm V}$ is usually written as $B - V$.

Figure 1.11 represents an observational H–R diagram containing a sample taken among the 1000 nearest stars, obtained from the Gleise star catalogue. A large portion of these stars are concentrated on a branch called the main sequence. This is where stars begin their lives and stay while burning hydrogen in their core. In this figure, the stars found above the main sequence are red giants; whereas those below are white dwarfs. These regions of the H–R diagram are often called branches. During its evolution, a star eventually leaves the main sequence, its radius increases and its $T_{\rm eff}$ at first decreases, giving a red giant star. It can then become a supergiant and possibly a white dwarf, depending on the value its initial mass (see Chapter 6 for more details).

For many reasons, a certain scatter is observed along each branch. For example, as time evolves, stars move in the H–R diagram. Even stars on the main sequence branch move slightly during their hydrogen-burning phase, their structure changes as more helium is produced in their core. Another factor that causes scatter is the varying metallicity among the stars. This leads to structural changes that modify their position on the H–R diagram. Observational errors can also add to the observed scatter.

When moving from the upper left to the lower right along the main sequence, the stars found there have lower masses and $T_{\rm eff}$ (see Figure 1.12). High-mass stars are more luminous because their central temperatures are higher and therefore they fuse hydrogen and produce nuclear energy at a higher rate. Their central temperatures are higher due to the large amount of gravitational energy that can be released during their formation (see Chapter 2). Figure 1.12 shows main-sequence stars of various masses within an H–R diagram. The range of masses for stars is approximately $0.08\,M_\odot \leq M \leq 120\,M_\odot$. The upper limit is related to the fact that high radiation pressure present at the surface of such massive stars pushes out any additional mass that would otherwise be gravitationally attracted to the star during its formation. However, the value of this upper limit is quite uncertain. The lower limit of this range exists because the central temperature of astronomical objects with $M \leq 0.08\,M_\odot$ does not attain the value needed for substantive and sustained hydrogen fusion. Objects with masses just below this limit are called brown dwarfs. These astronomical objects will be described in Chapter 6. Meanwhile, the range of effective temperature of main-sequence stars is approximately $2000\,{\rm K} \leq T_{\rm eff} \leq 60\,000\,{\rm K}$.

For main-sequence stars, the relation between the mass and radius is nearly linear (see Figure 1.13); whereas the luminosity increases much faster than mass (see Figure 1.12). This stems from the dependence of luminosity on R_* and $T_{\rm eff}$, $L_* = 4\pi R_*^2 \sigma T_{\rm eff}^4$. As mentioned previously, more fundamentally, the luminosity of a star is determined by the nuclear power generated in its core, which itself depends on the central temperature. The relation between the luminosity and stellar mass is critical for estimating the lifespan of main-sequence stars (see Chapter 6).

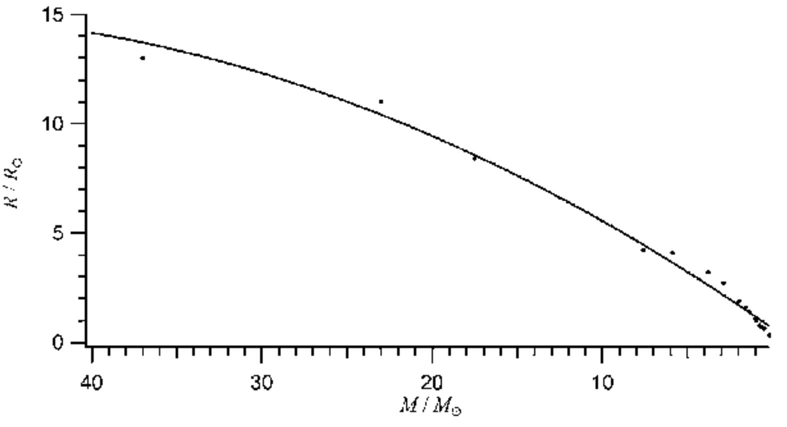

Figure 1.13 Relation between mass and radius for main-sequence stars (dots). Also shown is a curve fitted to the data.

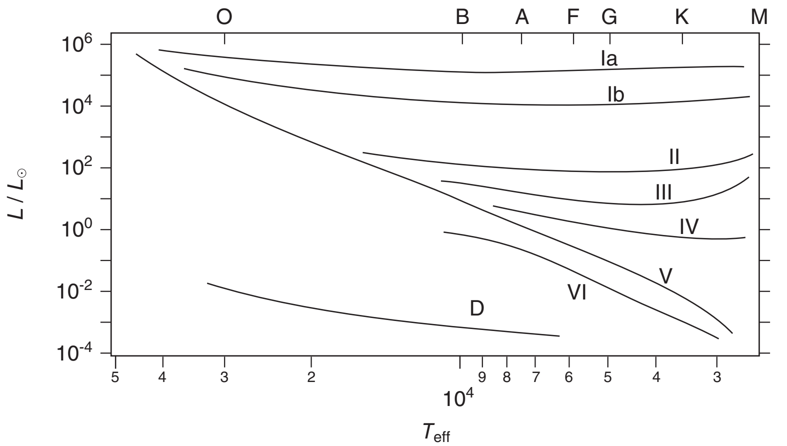

Figure 1.14 Luminosity classes of the H–R diagram. These are identified in Table 1.7. The spectral types are also shown (these are positioned at the coolest temperature for each class).

For a given T_{eff}, stars can have very different luminosities due to differing radii. A star of a given T_{eff} can, for instance, be a white dwarf, a main-sequence, or a supergiant star. A supergiant can have a radius up to the order of 1000 $R_\odot$ (where $R_\odot = 6.955 \times 10^{10}$ cm), while white dwarfs typically have $R \approx 0.01\ R_\odot$. This explains their position in the H–R diagram vis-à-vis the luminosity axis. The spectral class of a star is thus not sufficient to correctly specify its evolutionary status, since its spectral type depends solely on the physical properties of its photosphere. To solve this problem, luminosity classes (see Figure 1.14 and Table 1.7) are defined as a second parameter to the spectral classification of stars. These luminosity classes are related to differing evolutionary stages. For example, the Sun has a spectral type G2V, V being the luminosity class of a main-sequence star.

Table 1.7 Luminosity classes.

Ia	Bright supergiants
Ib	Supergiants
II	Bright giants
III	Giants
IV	Subgiants
V	Main-sequence stars (or dwarfs)
VI (or sd)	Subdwarfs
D (or VII)	White dwarfs

It should be noted that the T_{eff} range given in Table 1.5 for the spectral classes and shown in the figures found in this chapter are those of main-sequence stars. These T_{eff} ranges are slightly shifted for other luminosity classes (see Exercise 1.14).

Main-sequence stars are also called dwarfs. As shown in Figure 1.14, there exists a class of stars called subdwarfs found just below the main sequence. Subdwarf stars have low metallicities. This leads to a smaller radius and higher T_{eff} than a main-sequence star with the same mass. This larger T_{eff} can be explained by the fact that the outer layers are closer to the stellar core. In other words, the smaller radius leads to a higher flux, thus a larger T_{eff}.

In conclusion, the global properties of a star can be defined by three fundamental parameters: mass, radius and luminosity. With the luminosity and the radius, the effective temperature is defined by Eq. 1.10. A star found at a given point in the H–R diagram (i.e. with known luminosity and effective temperature) isn't completely defined since stars with different masses can pass at a same point in the H–R diagram during their lifetime. Its mass is needed to define it completely. Secondary parameters such as the abundances of the elements present in the star, the presence of magnetic fields, stellar rotation, etc. can also come into play. The fundamental parameters for main-sequence stars are given in Appendix G.

1.8 Summary

Modes of energy transport in stars: radiation, convection and conduction

$$\text{Planck distribution: } B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1} \tag{1.48}$$

$$\text{Stefan–Boltzmann law: } F = \sigma T^4 \tag{1.49}$$

$$\text{Wien's law: } \lambda_{\max} = \frac{0.290\,\text{K cm}}{T} \tag{1.50}$$

$$\text{Luminosity: } L_* = 4\pi R_*^2 \sigma T_{\text{eff}}^4 \tag{1.51}$$

$$\text{Integrated flux } \left(\text{for } r \geq R_*\right): F(r) = \sigma T_{\text{eff}}^4 \left(\frac{R_*}{r}\right)^2 \tag{1.52}$$

Magnitude: $m_1 - m_2 = 2.5\log\left(\frac{F_2}{F_1}\right)$ (1.53)

Distance modulus: $m - M = 5\log\left(\frac{d}{10}\right)$ (1.54)

Boltzmann equation: $\frac{n_i}{n_{\text{ion}}} = \frac{g_i}{U_{\text{ion}}}\,\mathrm{e}^{-\frac{E_i}{kT}}$ (1.55)

Saha equation: $\frac{n_{i+1}}{n_i} = \frac{1}{n_{\mathrm{e}}}\left(\frac{2\pi m_{\mathrm{e}} kT}{h^2}\right)^{\frac{3}{2}}\frac{2U_{i+1}}{U_i}\,\mathrm{e}^{-\frac{E_{\text{ion}}}{kT}}$ (1.56)

Density: $\rho = \sum_i n_i m_i$ (1.57)

Mean molecular weight: $\mu = \frac{\rho}{m_{\mathrm{H}} n_{\text{tot}}}$ (1.58)

Ionisation fraction: $f_i = \frac{\left(\frac{n_i}{n_{i-1}}\right)\left(\frac{n_{i-1}}{n_{i-2}}\right)\cdots\left(\frac{n_2}{n_1}\right)}{1+\left(\frac{n_2}{n_1}\right)+\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right)+\left(\frac{n_4}{n_3}\right)\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right)+\ldots}$ (1.59)

Spectral types (in order of decreasing T_{eff}): O, B, A, F, G, K and M

Three fundamental parameters of stars: mass, radius and luminosity

1.9 Exercises

1.1 Demonstrate the Stefan–Boltzmann law.

1.2 Demonstrate Wien's law (numerical problem).

1.3 A binary star system is observed, and since the separation between the two stars is much smaller that the distance of the system from the observer, it can be supposed that both stars are found at the same distance from Earth. The absolute magnitude in a given photometric band of the first star is determined to be –0.5, while its apparent magnitude is 3.5. If the apparent magnitude of the second star is 4.5, what is its absolute magnitude? At what distance (in light-years) is the binary system from the observer?

1.4 What is the numerical difference between the absolute magnitudes of two stars having the same T_{eff}, where one of these stars is in the giant phase and has a radius 15 times larger than the other star, which finds itself on the main sequence?

1.5 At what distance would the Sun have to be to have the same apparent magnitude as a 100-W light bulb found 100 m away? Express your answer in ly.

1.6 Assuming a flux equal that of a blackbody, calculate the percentage of the flux for stars with $T_{\mathrm{eff}} = 5000$, 10 000 and 20 000 K, capable of ionising hydrogen from level $n = 2$ (numerical problem)?

1.7 Calculate the temperature at which the number density of hydrogen atoms in the first excited state is ten times less than the number density of those in the fundamental level.

1.8 A hypothetical ion of an element has a degeneracy equal to $4n^2$, where n is the principal quantum number. At $T = 40\,000$ K, the ratio of the number density in level $n = 3$ to that of the fundamental ($n = 1$) is 0.25. Find the energy of level $n = 3$, assuming $E_1 = 0$.

1.9 What is the ionisation fraction of HI at a depth where $T = 9000$ K and $P = 140$ dyn/cm^2 in a star composed of pure hydrogen (assume $U_{\mathrm{I}} = 2$)?

1.10 Calculate the total number density (n_{tot}) and the density (ρ) at a depth in a star composed of pure hydrogen where $T = 9500$ K and 35 % of the atoms are ionised (assume $U_{\mathrm{I}} = 2$). What percentage of hydrogen atoms are in the energy level $n = 2$?

1.11 Calculate the pressure in a pure hydrogen gas at $T = 12\,000$ K that has 20 % of its atoms in the ionisation state HII (assume $U_{\mathrm{I}} = 2$).

1.12 At a certain depth in a star, three ions of a given element have the following ionisation fractions: $f_1 = 0.10$, $f_2 = 0.85$ and $f_3 = 0.05$. Their partition functions are: $U_1 = 1$, $U_2 = 2$ and $U_3 = 8$. The ionisation energy from the fundamental level for ion 1 is 30 eV and it is 55 eV for ion 2. Calculate n_e and T at this depth.

1.13 Figure 1.15 shows a portion of the spectra for two stars named A and B. The two curves shown in this figure are vertically shifted for visual effect. Using the relative inten-

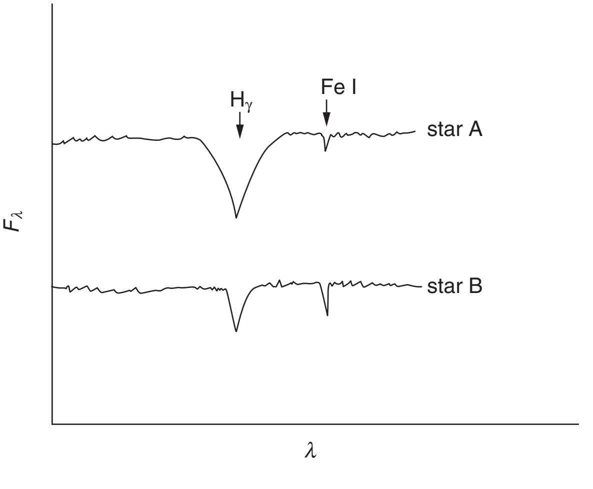

Figure 1.15 Illustration of the spectra of two stars showing the line H_γ and an atomic line from the ion FeI. These spectra are vertically shifted for visual effect (see Exercise 1.13).

sities of the hydrogen (H_γ) line and an atomic line from the ion FeI, which of these two stars is hotter? Why?

1.14 The effective temperature of a main-sequence star with spectral type B2 is approximately 22 000 K. Whereas, the effective temperature for a luminosity III class star of the same spectral type (i.e. with the same relative intensities of the various lines) possesses an effective temperature almost 2000 K lower than this value. Using the theoretical concepts seen in this chapter, explain the reason for the discrepancy.