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Background: On Georg Duffing and the Duffing equation

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1.1 Introduction

It is possibly the dream of many researchers to have an equation named after them. One person who achieved this was Georg Duffing, and this book is devoted to various aspects of his equation. This equation is enigmatic. In its original form, it essentially has only one extra nonlinear stiffness term compared to the linear second-order differential equation, which is the bedrock of vibrations theory, and this opens the door to a whole new world of interesting phenomena. Much of this was not known at the time of Georg Duffing, and is described in this book. The story behind the equation is also very interesting, because Georg Duffing was not an academic; he was an engineer, who carried out academic work in his spare time, as will be described later. In the present day when academics are being constantly reminded about the impact of their research work, and are constantly being judged by their output, in terms of publications, it is also interesting to look at the academic output from Georg Duffing and the impact of his work. Rarely is a paper or textbook written on nonlinear dynamics today without some reference to the Duffing equation, such is the impact of his work, yet he wrote less than ten publications in his life.

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The aim of this book is twofold. The first is to give a historical background to Duffing's work, and to track the evolution of his work to the present day. This is done in this chapter. The second aim is to provide a thorough treatment of the different forms of his equation through the various chapters written by the contributing authors. This will involve qualitative and quantitative analysis coupled with descriptions of the many physical phenomena that are described by the various forms of his equation.

Nowadays, the term 'Duffing equation' is used for any equation that describes an oscillator that has a cubic stiffness term, regardless of the type of damping or excitation. This, however, was not the case in Duffing's original work, in which he restricted his attention to the *free* and *forced harmonic vibration* of an oscillator in which the stiffness force had quadratic and cubic terms, and the damping considered was of the linear viscous type. In this book the contemporary view is taken and many forms of the Duffing equation are studied, with the notable exceptions of a randomly or parametrically excited oscillator.

1.2 Historical perspective

In any historical perspective, the authors undoubtedly provide their own interpretation of events, and this is also the case here. The history of nonlinear dynamics is vast and has many different threads to it, from the highly mathematical to the physical. It is not the intention of the authors to give a detailed history here – for this, the reader is referred to a review paper written by Holmes that covers the period 1885–1975 [1] and a slightly more recent paper by Shaw and Balachandran [2]. The authors restrict their attention to the historical perspective with respect to Duffing's work.

The concept of nonlinear vibrations was known long before Duffing wrote his book on oscillations [3], in which his famous equation is given. However, Duffing was the one to tackle the problem of a nonlinear oscillator in a systematic way starting with the linear oscillator, and examining the effects of quadratic and cubic stiffness nonlinearities. He emphasised the differences between the linear and the nonlinear oscillators for both free and forced vibration, also considering the effects of damping. Prior to Duffing, there had been some work on the mathematical analysis of nonlinear oscillators, for example by Hermann von Helmholtz [4] and Baron Rayleigh [5]. Two contemporaries of Duffing, Henri Poincaré (1854–1912) and Aleksandr Lyapunov (1857–1918), who were both giants in the history of nonlinear dynamics, did not appear to influence Duffing's work – at least they were not cited in his book.

In the story of nonlinear dynamics, as well as in Duffing's book, the pendulum plays a dominant role, and so it is appropriate to start the story with Galileo.

Galileo Galilei: 1564–1642. Galileo studied the pendulum and noticed that the natural frequency of oscillation was roughly independent of the amplitude of oscillation, i.e., they are isochronous. For it to be used in a time-keeping instrument, it needed to be forced because the oscillations diminished with time due to damping. He invented a mechanism to do this called an escapement [6]. This work was quickly followed by that of Huygens, who realised that the pendulum was inherently nonlinear.

Christiaan Huygens: 1629–1695. Huygens patented the pendulum clock in 1657. The early clocks had wide pendulum swings of up to 100°. Huygens discovered that wide swings made the pendulum inaccurate because he observed that the natural period was dependent upon the amplitude of motion, i.e., it was a nonlinear system. Subsequently the clocks were modified with a new escapement so that the pendulum swing was reduced to about 4–6°. Huygens also discovered that if the pendulum had a length that varied during the oscillation, according to an isochronous curve, then the frequency of oscillation became independent of the amplitude (effectively he linearised a nonlinear system) [7].

In many vibrating systems, it is the interaction between stiffness and mass that causes the 'interesting' dynamic behaviour. The first person to introduce the concept of stiffness theoretically was Hooke.

Robert Hooke: 1635–1703. Hooke is famous for his law [8], which gives the linear relationship between the applied force and resulting displacement of a linear spring. At the same time that Hooke was formulating the constitutive law for a spring, Newton was formulating his laws of motion, the most important of which for dynamical systems, is his second law.

Isaac Newton: 1643–1727. Newton, of course, is famous for his three laws of motion [9]. According to Truesdell [10], at the time of Newton and Hooke, simple harmonic motion (SHM) was not understood in the context of elastic bodies. However, Galileo was well aware of SHM in his study of the pendulum. Although vibration is often studied using rigid-body, lumped parameter systems (especially the study of nonlinear vibrations), a key area of practical interest is the vibrations of elastic bodies, such as beams, plates and shells. The first person to extend Hooke's law to such a system (a beam) was Liebnitz.

Gottfried Wilhelm Leibniz: 1646–1716. Leibniz is attributed with applying Hooke's law to a system containing moments; i.e., the bending moment is proportional to the second moment of area of a beam. This is thought to be the first application of calculus to a continuous system [10].

Although Hooke and Newton introduced some very important fundamental building blocks for mechanics, a general framework for the study of mechanics was lacking. The first person to provide some rudimentary tools for analysis was James Bernoulli.

James Bernoulli: 1654–1716. James Bernoulli developed the following approaches to solving problems in mechanics: balance of forces resolved in two fixed orthogonal directions; balance of forces normal and tangential to the line; virtual work; balance of moments. Truesdell [10] also attributes the first nonlinear law of elasticity to James Bernoulli. Around the same time, James Bernoulli's bother, John, was studying the vibration of a catenary, and then the vibration of a weighted string. During this study he formulated the equation for the natural frequency of a system.

John Bernoulli: 1667–1748. John Bernoulli studied the case of a string in tension loaded with weights. In this work he determined that the natural frequency of a system is equal to the square root of its stiffness divided by its mass, $\omega_{\mathrm{n}} = \sqrt{k/m}$, [11]. This is believed to be the first publication to state this relationship.

Some seventy years or so after Newton and Hooke formulated their laws for stiffness and mass, Euler connected them together in the form of a harmonically excited differential equation. This equation is the one that is taught to all students of vibration as a mathematical description of an undamped forced single-degree-of-freedom system.

Leonhard Euler: 1707–1783. Euler was the first person to write down the equation of motion of a harmonically forced, undamped linear oscillator, $m\ddot{x} + kx = F \sin \omega t$. He formally introduced the nondimensional driving frequency $\Omega = \omega/\omega_\mathrm{n}$ and noted that the response becomes infinite when $\Omega = 1$. Hence, he was the first person to explain the phenomenon of resonance [12].

More than 100 years later, Helmholtz was the first person to add a nonlinear stiffness term to Euler's equation of motion.

Hermann Von Helmholtz: 1821–1894. Helmholtz was the first person to include nonlinearity into the equation of motion for a harmonically forced undamped single degree-of-freedom oscillator. He postulated that the eardrum behaved as an asymmetric oscillator, such that the restoring force was $f = k_1 x + k_2 x^2$, which gave rise to additional harmonics in the response for a tonal input [4]. In the context of nonlinear dynamics, the equation $m\ddot{x} + k_1 x + k_2 x^2 = F \sin \omega t$ is now commonly known as the Helmholtz equation.

Around the same time that Helmholtz published his work, Rayleigh published his classic book on acoustics and vibration – The Theory of Sound [5]. This book had two volumes and covered an enormous amount of fundamental material in acoustics and vibration. In one small part of the first volume he considered a nonlinear oscillator.

John William Strutt, Third Baron Rayleigh: 1842–1919. Rayleigh considered the *free vibration* of a nonlinear single-degree-of-freedom oscillator. He studied the same system as Helmholtz, in which the force–deflection characteristic was quadratic, and he also investigated a system in which the force–deflection characteristic was symmetrical, given by $f = k_1 x + k_3 x^3$ [5]. In the latter case the equation of motion for this was given as $m\ddot{x} + k_1 x + k_2 x^3 = 0$. This is very close to Duffing's equation, but does not have a forcing term, and Rayleigh only provided a small amount of analysis, showing that nonlinear systems will vibrate at a fundamental frequency and harmonics of this frequency depending on the amplitude of vibration and the type of nonlinear stiffness force.

Also, around the time that Helmholtz and Rayleigh published their books [4,5], concerning vibrations and acoustics, Routh published his book on the dynamics of rigid bodies [13]. Among other things, he considered the *free vibration* of a system with a linear-plus cubic-stiffness force. For an undamped system he showed that the frequency of oscillation is affected by the amplitude.

Apart from the great pioneers mentioned above, who, motivated by acoustics, laid down the foundations for vibration theory, two other authors deserve a mention, because they directly inspired Duffing in his work. They are Von. O. Martienssen [14] and J. Biermanns [15]. In both of these papers an electrical system was studied in which included an inductor. For high current levels, the relationship between the current, i and the flux, ϕ is nonlinear. Biermanns showed that the nonlinear relationship between the current and the flux could be written as a power series, and if this is truncated at the third power as shown above, then the resulting equation for current is very similar to that given by Rayleigh, i.e., it can be modelled as $i = A_1\phi + A_3\phi^3$. This results in a 'hardening' characteristic, i.e., the current and the flux have the nonlinear relationship in the same way that force and displacement have in the mechanical system when the nonlinear term is positive. Martienssen observed this behaviour experimentally and reported the existence of the *jump-down phenomenon* as frequency was increased and the *jump-up phenomenon* as frequency was decreased. He also modelled the system and showed that between the jump-up and jump-down frequencies, three steady-state conditions could occur.

1.3 A brief biography of Georg Duffing

In 1994, F.P.J. Rimrott published a brief biography in Technische Mechanik [16] and part of this is translated in this chapter. The photograph of Georg Duffing is taken from this article and is shown in Figure 1.1.

Georg Wilhelm Christian Caspar Duffing was born on 11 April 1861 in Waldshut in Baden, Germany. He was the oldest of six children of the merchant Christian

Figure 1.1 George Duffing . Reprinted from [16], Copyright 1977, with permission from Technische Mechanik.

Duffing and his wife Julie, whose maiden name was Spies. A year after he was born the family moved to Mannheim, where the grandfather Spies, a carpenter, had a large woodyard on the shore of the river Neckar.

Georg Duffing had a gift for mathematics and a natural musical talent. He studied the violin and performed in public as a child, and played in a band in his youth.

From 1878 to 1883, Duffing embarked on his formal higher education. He spent one year at mathematical school, one year at engineering school and three years at the Mechanical Engineering school at the Polytechnic, which is the University Fridericiana in Karlsruhe, today [17]. Although he had a heart problem, which subsequently prevented him from doing military service, Duffing was among the best runners in Baden.

After his graduation, Duffing went to Köln to work for *Deutzer Motorenwerken*, where he developed steam engines, which were produced in 1905.

At age 46, he married Elizabeth Lofde from Berlin. They had four children.

In 1910 Duffing was invited to Westinghouse in the USA. He stayed there for several months and came home with enough money to work as a self-employed inventor and scientist.

The Duffing family moved to Berlin in 1913, mainly because he wanted to listen to the lectures of Max Planck on quantum theory. This was typical behaviour – he always wanted to gain more knowledge.

When the First World War broke out and money lost its value, Duffing was working on vibrations, brakes, gears and engines, by trial and error. On Sundays he would go to the laboratory of the Royal Technical Faculty with his oldest daughter, where Professor Eugen Meyer allowed him to conduct experiments. He patented his inventions; however, it did not improve his financial situation. During that time, he was studying vibrations described by particular differential equations. In 1917 he completed his 134-page monograph numbered 41/42, with the title "Forced oscillations with variable natural frequency and their technical significance" [3]. It was published in 1918 by Vieweg & Sohn and cost five Deutsch Marks. This is the work for which he is famous.

In 1921, when the Duffing family encountered financial difficulties, he received an invitation of work from Ölgesellschaft Stern & Sonneborn A.G., which became the famous *Shell* company. He was offered the position as head of a laboratory where he invented a viscosimeter for lubricants.

The family moved to Hamburg, where Duffing suffered from severe flu and thrombosis of his leg, the consequences of which remained with him for the rest of his life.

A tragical part of his life came in 1927 when the ship 'Cap Arcona', had technical problems during a voyage. Stern & Sonneborn, had provided the oil that was made in accordance with Duffing's instructions. During the voyage, an engine failed. Duffing checked the oil and found out that many types of oil had been mixed, probably to save money. There was a trial, where Duffing presented facts clearly and honestly. He was resolute as he had been throughout his life. However, because he had testified against Stern & Sonneborn, he lost his job.

The Duffing family moved back to Berlin in 1931. Although he was 70 years old he carried on his research and inventing activities. During the Second World War, he had particular difficulties during the bombing attacks, as he could not easily take shelter in the cellar because of the problems with his leg. They subsequently moved to a small peaceful town called Schwedt on the river Oder.

Georg Duffing died there on 5 April 1944 aged 83 years. He is buried in the Jerusalem Graveyard in Halleschen Tor in Berlin.

1.4 The work of Georg Duffing

Written records of Georg Duffing's work comprise his patents and publications. His patents were registered both in the USA and Germany. The very first patent seems to have been registered as a 'Speed regulator for explosion engines' in the USA in 1905, and has the number 799459 [18], the illustration of which is shown in Figure 1.2. In the first

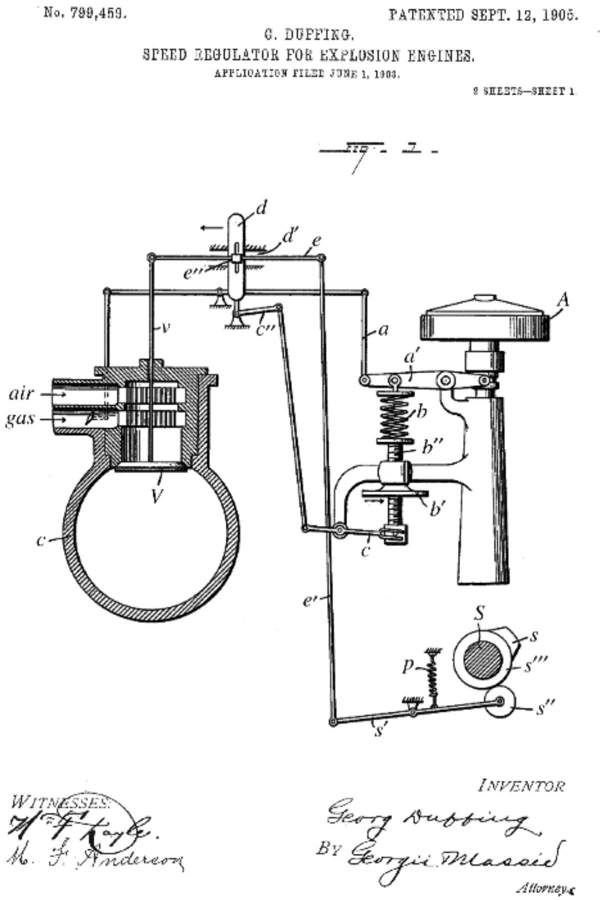

Figure 1.2 Illustration of Duffing's patent 'Speed regulator for explosion engines' [18].

paragraph of the written part of his application [18], Duffing wrote: “Be it known that I, Georg Duffing, engineer, a subject of the German Emperor, residing at 93 Deutzerstrasse, Mülheim-on-the-Rhine, Germany, have invented certain new and useful Improvements in Speed-Regulators for Explosion-Engine; and I do hereby declare the following to be a full, clear, and exact description of the invention, such as will enable others skilled in the art to which it apertains to make and use the same.” During the following decades Duffing invented many ‘new and useful improvements’, fewer of which were registered in the USA than in Germany. Some of the USA patents can be seen, for example in [19–21], while Rimrott [16] gave an extensive list of German patents.

In terms of publictions, it is hard to qualify Georg Duffing’s work as prolific, as he was the author of only nine publications. On the other hand, the fact that he was not an academic and that he was active only for about 25 years in the 20th century, make this number respectable. His publications includes books, book chapters and journal articles. They are listed in chronological order in Table 1.1.

Table 1.1 List of Duffing’s publications.

No	Publications
1.	G. Duffing, *Beitrag zur Bestimmung der Formveränderung gekröpfter Kurbelwellen*. Verlag von Julius, Berlin, 1906.
2.	G. Duffing, *Erzwungene schwingungen bei veränderlicher eigenfrequenz und ihre technische bedeutung*, Series: Sammlung Vieweg, No 41/42. Vieweg & Sohn, Braunschweig, 1918.
3.	L. Gümbel; G. Duffing, *Der heutige Stand der Schmierungsfrage. Zur numerischen Integration gewöhnlicher Differentialgleichungen I. und II,* Series: Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, No 224. Verlag des Vereines deutschen Ingenieure, Berlin 1920.
4.	G. Duffing, Beitrag zur Theorie der Flüssigkeitsbewegung zwischen Zapfen und Lager. *ZAMM Zeitschrift für Angewandte Mathematik und Mechanik*, 4, 296 Fig. 314, 1924.
5.	G. Duffing, Reibungsversuche am Gleitlager. *VDI - Zeitschrift*, 72, 495–499, 1928.
6.	G. Duffing, *Elastizität und Reibung beim Riementrieb*, Series: Sonderheft des Verbandes der Ledertreibriemen-Fabrikanten Deutschlands E. V., No 12, Berlin W 35, Kurfürstenstr. 148: Ledertreibriemen u. Techn. Lederartikel, 1930.
7.	G. Duffing, Die Schmiermittelreibung bei Gleitflächen von endlicher Breite; in *Handbuch der Physikalischen und Technischen Mechanik* Edited by F, Auerbach, W. Hort. Barth, Leipzig 1931.
8.	G. Duffing, Elastizität und Reibung beim Riementrieb. *Forschung im Ingenieurwesen*, 2, 99 Fig. 104, 1931.
9.	G. Duffing, Messung der Zähigkeit durch gleichförmige koachsiale Bewegung einer Kugel in einem Kreiszylinder, *ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik*, 13, 366 Fig. 373, 1933.

The motivation for this book and the publication for which Duffing is recognised, is the monograph listed as number 2, mentioned in the previous section and listed as [3] in the referencess of this chapter. The next section is devoted to the monograph and contains the description of its content. It should be noted that, although it is not the only book that Duffing wrote, the phrase ‘Duffing’s book’ will be used in relation to this particular publication only.

1.5 Contents of Duffing’s book

The title page of Duffing’s famous book is shown in Figure 1.3. It can be seen that the book was written in German, which was Duffing’s native tongue. To help the reader understand what was written in the book some key pages have been translated by Keith and Heather Worden, and these are shown in the Appendix of this book. A brief summary of the contents of Duffing’s book is given below.

1.5.1 Description of Duffing’s book

George Duffing was not an academic, but an engineer, as was clearly written on the title page of his book ‘Forced oscillations with variable natural frequency and their technical significance’. His motivation for the research reported in the book stemmed from his personal practical experience and observations of engineering systems. However, he was hoping that “the work would raise some interest in mathematical circles, because it requires some additional tools/knowledge and more time than one technician has.” Duffing repeated this wish several times through the book, wanting to “be timely” and admitting that it was the reason for him to deliver the results despite the fact that “they had not been completed”.

The book comprises seven chapters and five Appendices. It contains results on the response of both linear and nonlinear oscillatory systems obtained analytically, graphically, numerically and experimentally. The majority of the Appendices cover some necessary mathematical background work, which Duffing included to help the nonmathematician understand the content without having to search the literature.

In Chapter I a linear single-degree-of-freedom system excited by an arbitrary time-varying external force is considered using the convolution integral. An undamped system is analysed first, and this is followed by a damped system for periodic excitation only. As a special case, the response of the system under harmonic excitation is determined for resonance and offresonance conditions. This chapter serves as a reference, and describes simple systems for which results for nonlinear systems given in the subsequent chapters can be compared.

Chapter II is the most comprehensive – it is where free and forced undamped oscillations of the systems with a nonlinear restoring force are treated, and where the first significant results for these types of problems are given. Some of these systems are subsequently named after Duffing. The restoring force is assumed to

Erzwungene Schwingungen
bei veränderlicher Eigenfrequenz
und ihre technische Bedeutung

Von

Georg Duffing
Ingenieur

Mit 23 Abbildungen

Braunschweig
Druck und Verlag von Friedr. Vieweg & Sohn
1918

Figure 1.3 The title page of Duffing's book.

contain small quadratic and/or cubic nonlinearity. Duffing first considered free vibrations of a system with such a restoring force and obtained the first integral of motion corresponding to the energy conservation law. He then expressed the motion using Weierstrass elliptic functions. Separately, the case with softening cubic nonlinearity, corresponding to a symmetrical *potential well*, and the case with quadratic nonlinearity, corresponding to an asymmetrical *potential well*, are treated in this way. Further, Duffing studied forced vibrations for a system with cubic softening nonlinearity. Using previously obtained results, he applied the method of variation of constants to derive a fifth-order polynomial expression in one specifically defined parameter. He showed graphically that depending on the value of the forcing frequency compared to the natural frequency, the number of its roots can vary from one to three. The multivaluedness of the steady-state response is also confirmed by developing an iterative method – the method of successive approximation, which was subsequently called Duffing's method [22]. This technique is first validated on the linear system by demonstrating that its solution is equivalent to the sum of the complementary function and particular integral of the equation of motion. So that he could apply it to the forced vibrations of a softening system, he showed that the first approximation could be assumed to be harmonic at the frequency of excitation. As a result, he derived the frequency–amplitude equation, which is a cubic function of the amplitude. Graphical interpretation of the result shows that the multivaluedness of the response is dependent on how the excitation frequency compares with the natural frequency of the linearised system. Although Duffing was aware that to predict the response, one must examine the history of the response, i.e., of the hysteretic behaviour of nonlinear systems, surprisingly there is not a single frequency-response diagram in his book. To illustrate his findings, he provided the example of a forced pendulum, approximating the equation of motion to a system with a softening cubic restoring force. Duffing also analysed forced vibration for a system with quadratic nonlinearity by applying the method of successive approximation, assuming the first approximation to be the sum of a bias (DC) term and a harmonic term. After the derivation of the amplitude–frequency equation, it is solved graphically, and it is demonstrated that there can be multiple values of the amplitude for each frequency, where the number and the values of solutions are dependent upon whether or not the natural frequency is greater or less than the excitation frequency. In addition, the response of a forced system with negative quadratic and cubic nonlinearity is determined by means of Duffing's iterative method. The results are illustrated by investigating a pendulum that is excited by a constant plus harmonic force, whose equation of motion is transformed appropriately. The chapter is concluded with a summary of the main findings, and includes a table in which the differences between the responses of linear and nonlinear externally excited systems are highlighted.

Chapter III is devoted entirely to the experimental illustration and analysis of a system whose general equation of motion covers all the cases of *forced vibrations* considered in the previous chapter. The rig consisted of a pendulum which could be adjusted so that it corresponded to either a symmetrical or asymmetrical system. Duffing compared his theoretical results with those obtained experimentally and found satisfactory numerical agreement.

Chapter IV contains only one section, which is concerned with the influence of damping on the response of a softening cubic system with harmonic forced excitation. Again, the possibility of a multivalued response is shown graphically. Comparing the case of weak damping with the corresponding diagram for the undamped case, Duffing remarked that there is no qualitative difference below and above the natural frequency.

Stability analysis of the periodic motion of the harmonically excited oscillator with cubic nonlinearity is investigated in Chapter V. With this aim, a linearised variational equation of the perturbed solution is considered, but with regard to the pendulum.

In Chapter VI some real systems are considered that are of interest from a practical point of view and whose governing equations correspond to those considered earlier: first, an electrical circuit analysed in Martienssen's paper [14], which is related to a free oscillating cubic system; then, a synchronous generator whose equation of motion corresponds to the asymmetric pendulum equation; and finally, a three-phase generator whose equation of motion corresponds to the symmetric pendulum equation and for which the multivaluedness of the response is shown analytically and graphically.

Chapter VII, entitled 'Generalisations', is concerned with the application of Duffing's method to the study of the systems excited by a sum of several harmonic forces. The cases of a quadratic and cubic restoring force are dealt with separately. Duffing also pointed out the necessity to study nonlinear systems with many degrees of freedom, due to their technical significance.

In Appendix 1 some details about the Weierstrass elliptic functions are given, while the integration of elliptic differential equations is commented on in Appendix 2. Appendix 3 contains the algorithm on how to transform a certain differential form to the Weierstrass normal form. Free vibrations of a pendulum are studied in Appendix 4 by means of elliptic functions. In Appendix 5 the Ritz method is applied to the forced vibrations with either cubic or quadratic nonlinearity with the aim of obtaining the amplitude-frequency expression.

1.5.2 Reviews of Duffing's book

The appearance of Duffing's book was announced and its contribution recognised soon after it had been published. Two reviews appeared in scientific journals, both written by Professor Georg Hamel from Berlin.

The first review was in the Annual Bulletin of Mathematics ('Jahbrbuch der Mathematik' 1916–1918) [23]. According to Professor Hamel, the main aim of the book was to explain several significant phenomena that appear during oscillatory motion of an externally excited asymmetric pendulum. The reviewer highlighted the difference between the number and stability properties of the steady-state solutions of its approximate equation, in which the restoring force contains quadratic and cubic nonlinearity, and the linearised equation.

Another review was submitted in 1920 to the ZAMM-Journal of Applied Mathematics and Mechanics ('Zeitschrift fur Angewandte Mathematik und Mechanik') and published in its very first issue in 1921 [24]. At the beginning of the review, the main characteristics of the resonance response of a harmonically excited linear oscillator are listed. Then, stating that "the equations that describe numerous vibration problems are more complex", the example of a forced pendulum was given as an illustration. It was emphasised that its solution was obtained for free vibrations by using elliptic functions, but in case of forced vibrations this appeared to be unattainable. In order to overcome this problem, Duffing approximated the equation of the pendulum to an equation with softening cubic nonlinearity, assumed the solution of motion in the form of the first harmonic and applied three methods (the method of the variation of parameters, the method of successive approximation and the Ritz method), showing the possibility of a multivalued response. It is also noted that Duffing succeeded in confirming some results experimentally as well as discussing the equation of motion with both quadratic and cubic nonlinearity, and damped vibrations.

It is worth mentioning that the reviewer recognised and supported Duffing's wish and intention, writing [24]: "Strange vibration phenomena in relatively simple cases are enlightened in this study, as a reward for an engineer and as an inspiration for a mathematician to gain deeper insight."

1.6 Research inspired by Duffing's work

1.6.1 1918–1952

Following Duffing's book, it took some time for his work to become known. This could have been due to the fact that it was published in German. In what follows, a potted history of the research work that followed on from Duffing's book is given. It will be seen that it took about ten years for the book to be cited in a publication written in English, and this was in Timoshenko's book. It is possible that Timoshenko got to know of Duffing's work when he was at Westinghouse in the United States. He went there in 1922, more than a decade after Duffing's visit which was in 1910.

Possibly the earliest paper that cites Duffing's work was written by Hamel [25]. Hamel also wrote reviews of Duffing's book [23,24]. In this paper, Hamel studied the pendulum, but did not approximate the restoring force as a linear plus a cubic term as in Duffing's book. Rather, using the variational approach, he minimised the action integral, deriving the amplitude–frequency equation, obtaining a more accurate result.

Rüdenberg considered both mechanical and electrical systems with nonlinear restoring-force characteristics [26], continuing the work of Martienssen [14], Biermanns [15] and Duffing [3]. He considered both *free* and *forced oscillations*. For free oscillations, he considered undamped systems and for forced vibrations he considered both undamped and damped systems using a combination of analytical and geometrical approaches similar to that taken by Duffing. He assumed a harmonic response, but considered a generalised restoring force instead of than one of polynomial form, which permits graphical rather than closed-form solutions.

In 1924, Appleton studied the softening nonlinear behaviour of a galvanometer used in the laboratory in Cambridge University [27]. He observed that the output from the galvanometer could have two different values for certain current inputs. He modelled the system as Duffing had done for the pendulum and produced frequency-response curves that were similar to those observed in the laboratory. He also considered the stability of his solutions. Remarkably, Appleton did not refer to any literature, except to note that a paper by Waibal in *Annal der Physik* had observed hysteresis behaviour in a galvanometer.

Lachman wrote a paper concerning the solution of the exact equation describing the forced vibration of a pendulum in 1928 [28]. He used Duffing's name in the title of the paper, demonstrating that he was directly inspired by this equation appearing in Duffing's book.

Timoshenko's classic textbook was published in 1928 [29]. In it he considered simple mechanical systems with geometric nonlinearity and cites Duffing's book. This appears to be the first time that it was cited in a publication written in English, and is possibly the beginning of international acknowledgement of the importance of Duffing's pioneering work.

Five years later, Den Hartog, who was also employed by Westinghouse (1924–1932), developed a graphical method for solving the forced vibration of a system with a nonlinear spring, and compares his results directly with the method developed by Martienssen, the work that inspired Duffing. Duffing's book is cited in this paper [30].

In 1938, Rauscher developed an iterative method to determine the response of a forced nonlinear oscillator with a general nonlinear restoring force characteristic using the amplitude of free vibration of the oscillator as an initial guess [31]. He cited Duffing's book as being the long-established text on the subject.

Von Kármán published a paper in the Bulletin of the American Mathematical Society in 1940 [32], based on the fifteenth Josiah Willard Gibbs lecture that he gave in 1939. In this paper he described several nonlinear engineering problems, one of which involved *subharmonic resonance* due to nonlinear restoring forces. Duffing's book was listed in the bibliography.

In the late 1930s and 1940s a group of applied mathematicians worked on nonlinear problems in New York University. These were led by Richard Courant, who left Germany in the mid-1930s, where he had been an assistant to Hilbert at Göttingen. His group included Kurt Friedrichs, his former student who left Germany in 1937 to join him, and James Stoker, who subsequently wrote the seminal book on nonlinear vibrations [22]. In 1942 in a series of lectures given at Brown University [33] based on a course given by Friedrichs and Stoker at New York University, the equation of an oscillator in which the restoring force consists of a linear and a cubic term was described as Duffing's equation. This was 24 years after Duffing's book, and the authors believe this was the first occasion in which the equation was named in such a way.

In 1949, Levenson published a paper in the Journal of Applied Physics based on his doctoral work at New York University in which the Duffing equation and his name appears in the title [34]. In this paper, Levenson considered the harmonic and subharmonic response of the system. It appears that by 1949, some 31 years after Duffing's book, the equation describing the oscillator with a linear and cubic nonlinear restoring force had become known as the Duffing equation.

Following Stoker's book in 1950 [22], the Duffing equation sat alongside van der Pol's, equation as one of the classic equations in nonlinear vibrations and was being cited in a wide range of literature from physics, for example [35], to mathematics, for example [36]. This last paper was a review paper published in 1952; it appears to signal the end of the work in this area, apart from one paper on the transient behaviour of a ferroresonance circuit in 1956, for about a decade.

1.6.2 1962 to the present day

Since the 1960s, many journal papers have been published related to the Duffing equation. A survey has been carried out via SCOPUS to track the journal papers that used the word 'Duffing' in the title, abstract or keywords. The number of such papers published per year is shown in Figure 1.4. It can be seen that until the 1970s, only a few papers appeared per year, concerned mainly with finding an approximation for the displacement of the oscillator. Then, this number dramatically increased, which was because people started to recognise the Duffing equation as a model for different systems. Also, digital computers started to be used to solve analytically nonlinear

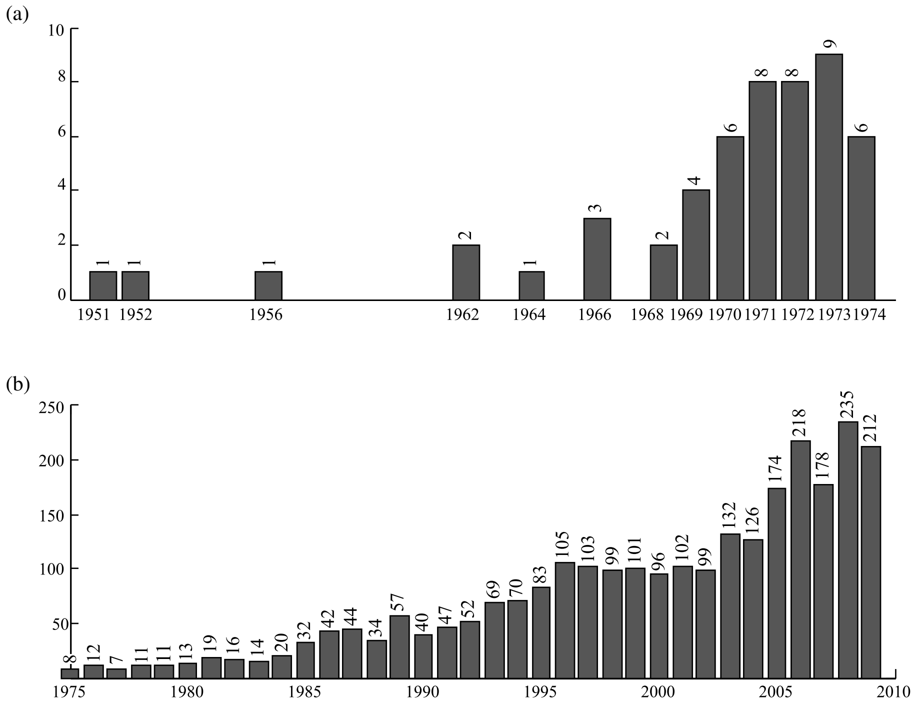

Figure 1.4 Number of publications referring to the word 'Duffing' in the title, abstract or keywords per year; a) for the period 1950–1974; b) for the period 1975–2009 (Source: Elsevier Scopus™, accessed 9 August 2008 and updated 30 March 2010).

ordinary differential equations. This increasing trend continued even further, when in 1976 Holmes and Rand published their paper on bifurcations of Duffing's equation and the application of catastrophe theory [37].

In the 1980s Ueda published his work on chaos, initially named 'randomly transitional phenomena' [38] and 'random phenomena' [39]. Reference [39] was a translated version of an earlier paper published in Japanese in the *Transactions of the Institute of Electrical Engineers of Japan*, Vol. A98, March 1978. The postscript in reference [39] sheds some interesting light on the discovery of chaotic behaviour in the purely cubic Duffing oscillator. Because this is such an important milestone in the study of the Duffing equation, it is copied in full below.

POSTSCRIPT by YOSHISUKE UEDA

I deem it a great honour to be given the opportunity to translate my article into English and I would like to express my thanks to the members of the editorial board. In the following I am writing down some comments and fond memories of days past when I was preparing the manuscript with tremendous difficulty.

It was on November 1961 when I met with chaotic motions in an analogue computer simulating a forced self-oscillatory system. Since then my interest has been held by the phenomenon, and I have been fascinated by the problem "what are steady states in non-linear systems?" After nearly ten years I understood "randomly transitional phenomenon", I published my findings in the *Transactions of the Institute of Electronics and Communication Engineers of Japan*, Vol. 56, April 1973 [10]. My paper then received a number of unfavorable criticisms from some of my colleagues: such as, "Your results are of no importance because you have not examined the effects of simulation and/or calculation errors at all", "Your paper is of little importance because it is merely an experimental result", "Your result is no more than a periodic oscillation. Don't form a selfish concept of steady states", and so forth. Professor Hiromu Momota of the Institute of Plasma Physics was the first to appreciate the worth of my work. He said "Your results give an important feature relating to stochastic phenomena" on 3 March 1974. Through his good offices I joined the Collaborating Research Program at the Institute of Plasma Physics at Nagoya University. These events gave me such unforgettable impressions that I continued the research with tenacity. At this moment I yearn for those days with great appreciation for their criticisms and encouragements.

By the middle of the 1970s, I had obtained many data of strange attractors for some systems of differential equations; but I had no idea to what journals and/or conferences I might submit these results. I was then lucky enough to meet with Professor David Ruelle who was visiting Japan in the early summer of 1978. He advised me to submit my results to the *Journal of Statistical Physics* [P1]. Further, he named the strange attractor of Fig. 3 "Japanese Attractor" and introduced it to the whole world [P2–P5]. At the same time chaotic behavior in deterministic systems began to come under the spotlight in various fields of natural sciences. I fortunately had several opportunities to present my accumulated results [P6–P11]. It is worthwhile mentioning that, due to the efforts of Professor David Ruelle and Professor Jean-Michel Kantor, the Japanese Attractor will be displayed at the National Museum of Sciences, Techniques and Industries which will open in Paris, 1986. In these circumstances this paper is a commemorative for me and I sincerely appreciate their kindness on these matters.

As the reader will notice in this translation and also in ref. [P1] I was rather nervous of using the term "strange attractor", because I had no understanding of its mathematical definition in those days. Although I do not think I fully understand the definition of it even today, I begin to use the term "strange attractor" without hesitation because it seems to agree with reality. However, it seems to me that the term "chaos", although it is short and simple, is a little bit exaggerated. In the universe one does have a lot more complicated, mysterious and incomprehensible phenomena! I should be interested in readers' views of my opinion.

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P11. Y Ueda and H. Ohta, *Chaos and Statistical Methods*, p. 161. Springer, Berlin (1984).

In 1979 Holmes's article 'A nonlinear oscillator with a strange attractor' appeared [40]. This is also a highly cited paper. The continuous growing trend of the published articles has included the development of analytical and numerical methods to find different solutions for motion and to study the phenomena associated with the equations of motion. Investigation of the transition between different regimes has also been widely researched as has identification of the systems, and different problems of synchronisation and control, etc.

What is also apparent is the diversity of disciplines in which Duffing's equation appears. As illustrated in Figure 1.5, the majority of publications in the SCOPUS survey belongs to Engineering. Around 25% of them are from Physics and Astronomy and 19% from Mathematics. Computer Science encompasses 6%, Material Science and Multidisciplinary studies 2%, while Chemical Engineering has only 1% of the publications. The rest of the disciplines, such as Earth Science, Biochemistry or Biology each have less than 1% of the total, so are given in a cumulative way.

It should be emphasised that it is not the equation of motion with a positive linear term and cubic nonlinearity that was named after Duffing, but many other homogeneous or inhomogeneous second-order ordinary differential equations were also

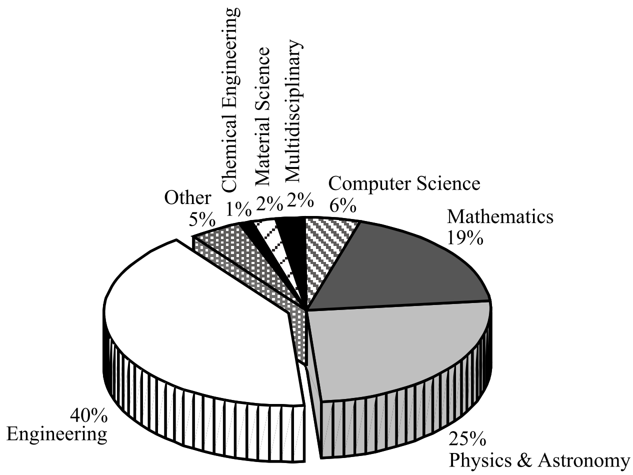

Figure 1.5 Percentage of publications referring to the word 'Duffing' in the title, abstract or keywords for the period 1950–2009 per disciplines (Source: Elsevier Scopus™, accessed 9 August 2008 and updated 30 March 2010).

called the Duffing equation(s) and formed the basis of many mathematical models of different systems. Some of the practical examples of the systems whose dynamic behaviour is described by these equations are given in Chapter 2 of this book. Subsequent chapters are concerned with different forms of the Duffing equation(s) each contains some references that can help the reader to track the most important and the most influential publications associated with each form of the equation studied.

1.7 Some other books on nonlinear dynamics

As mentioned previously, Stoker's book [22] was a key publication in the field of nonlinear vibrations in the 1950s. From the 1960s there have been many books published in this area, demonstrating the rapid development of the topic and growing interest by various communities. Several books are listed in this chapter, so that the reader can probe more deeply into the topics of their choice. The books are grouped together as follows; mathematical treatment of equations of motion [41–48], nonlinear phenomena, with a focus on chaos [49–54], and finally those which have more of an engineering bias [55–59]. The books by Hayashi [42] and Nayfeh and Mook [43] are considered to be of particular importance as they give the concepts and analytical methods for the study of nonlinear oscillators. In addition, the former provides experimental results and the latter includes an extensive bibliography.

1.8 Overview of this book

This book has been created with the aim of enabling the reader to gradually gain insight into the equations associated with Duffing's name, the oscillators that they describe, methods that are used to study their response and related phenomena. Following the historical background given in Chapter 1, Chapter 2 shows how some real dynamic systems can be modelled approximately by the Duffing equation. Chapters 3 and 4 are concerned primarily with qualitative and quantitative analysis of free vibration problems (Chapter 4 does, however, contain some quantitative analysis of forced vibrations). These two chapters set the scene for the remaining four chapters, which are concerned with different forced vibration problems. More details about each book chapter, as well as the Appendix and the Glossary, are given below.

As mentioned above, Chapter 2 gives practical examples of systems whose dynamic behaviour is described by different forms of the Duffing equation. Various physical systems are chosen to illustrate the physical phenomena that result in different forms of this equation. These equations are subsequently nondimensionalised to link with the other chapters in the book. In addition, several basic types of geometric nonlinearity are described: hardening (with positive linear stiffness and positive cubic stiffness), softening (with positive linear stiffness and negative cubic stiffness), systems with negative linear-positive nonlinear stiffness (i.e., with a double/two/twin-well potential) and, finally, a system that is purely nonlinear (no linear term). The equations that describe the systems with these types of nonlinearity are subsequently investigated in more detail in later chapters for the case of free and forced vibrations and different damping mechanisms.

Chapter 3 is concerned with free vibration*s* of a system with viscous damping. Qualitative analysis is conducted to demonstrate that the system undergoes *local bifurcations* when the linear stiffness and damping are changed. It is shown that negative linear stiffness and negative linear damping can produce buckling and *self-excited oscillation*, respectively. It is also shown that nonlinear stiffness characterises the postbuckling behaviour, i.e., the existence of nontrivial *fixed points* and their stability. The effect of nonlinear damping on the existence and magnitude of the *steady-state* response for the self-excited system is demonstrated. Furthermore, more global aspects of the *bifurcation* are investigated. By using a *Hamiltonian* structure, some of the qualitative characteristics of nonlinear dynamics are also studied.

Some quantitative methods for obtaining the solutions of various forms of the Duffing equation with hardening, softening, negative linear-positive nonlinear stiffness and pure cubic nonlinearity are presented in Chapter 4. Two groups of analytical methods are shown: nonperturbation and perturbation techniques. The following asymptotic methods are considered: the straightforward expansion method, the parameter-expanding method (the elliptic Lindstedt–Poincaré method), the generalised averaging method, the parameter perturbation method (elliptic Krylov–Bogolubov method), the elliptic harmonic balance method, the elliptic Galerkin method (the weighted residual method), the homotopy perturbation method and the homotopy analysis method. For all the methods discussed, the common factor is the generating solutions of the differential equations that describe the free or harmonically forced oscillations of the Duffing oscillator. These are based on *Jacobi elliptic functions*. To illustrate the use of these methods, several examples are given. To assess the accuracy of the approximate analytical solutions, they are compared with numerical solutions. It is shown that the analytical results obtained are in good agreement with the solutions from numerical integration even for the cases when the nonlinearity and/or the excitation force are not small.

In Chapter 5, forced harmonic oscillations of the Duffing oscillator with linear viscous damping are explored. For weak nonlinearities and weak damping, the perturbation method is used to obtain an analytical approximation for the primary resonance response. In order to study the stability of periodic responses of the forced Duffing oscillator, local stability analysis is carried out on the equations describing the slow timescale evolution. In addition, secondary resonance corresponding to strong (hard) excitation is also discussed. The combination of analytical and numerical investigations presented in this chapter is used to illustrate the *jump phenomenon* and the rich variety of nonlinear phenomena possible in the system with a hardening, softening and pure cubic nonlinearity.

Chapter 6 contains the study of a harmonically excited Duffing oscillator with different damping mechanisms, focusing on the effects of these damping mechanisms on the response of a system with a hardening, softening, negative linear-positive nonlinear stiffness and pure cubic nonlinearity. All velocity-dependent damping mechanisms are treated by using the concept of *equivalent viscous damping*. The *break-loose frequency* is introduced in the case of *Coulomb damping*. The stability analysis of the harmonic solution, *period-doubling bifurcation* and *Melnikov criterion* are obtained for linear and cubic damping. Some experimental and numerical results are included to investigate some typical trends in the response.

Forced harmonic vibration in a Duffing oscillator with negative linear stiffness and linear viscous damping are examined in Chapter 7. Various aspects of the dynamical behaviour of the Duffing oscillator with a *twin-well potential* are investigated by the combined use of analytical and numerical tools. Nonlinear periodic oscillations are discussed first, and the classical nonlinear *resonance* is studied in detail by the method of multiple scales. Then, transition to a complex response is investigated by using *bifurcation diagrams*, basins of attractions, and stable and unstable *invariant manifolds*, by summarising the regions of different dynamical response in a comprehensive behaviour chart. Analytical prediction of the transition to *chaos* via the *Melnikov criterion* is then presented. Finally, nonclassical issues such as control of *homoclinic bifurcation* and *chaos*, and *dynamical integrity* are discussed in detail with the aim of highlighting the most important ideas and objectives.

In the last chapter, the forced harmonic vibrations of an asymmetric nonlinear system are investigated. Two nonlinear asymmetric systems are described. The first is a pure cubic nonlinear oscillator with a constant and a harmonic force acting on it, and the second is a harmonically excited oscillator with both quadratic and cubic nonlinearity. Both of these systems have a *single-well potential*. Different analytical and numerical approaches are used to study and illustrate the rich dynamics of the systems, which include multiple *jumps* in the *hysteretic behaviour* and different routes to chaos.

This book also has an Appendix, which contains various sections of Duffing's book that have been tranlsated into English. His book has been cited many times since 1918, the year it was published, but to the editors' knowledge, it has never been translated into English. The sections have been chosen to give a flavour of the book, reflecting aspects of Duffing's work closely related to the content of this book.

This book ends with a Glossary, containing a list of some definitions and terms used. The aim of providing such a list is to enable the reader to go through the book smoothly, without any need to look elsewhere for background information, and to make this book appropriate for a wide-range of readers interested in the content. The terms in bold in the Glossary are written in italics in the main text, when they appear for the first time in each chapter.

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