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Preliminary notions and definitions

Before we set out on our exploration of lower previsions, we start with a brief overview of the basic notation used throughout this book. More specific notation will be introduced in further chapters, as we go along. Refer to Appendix E₃₉₆ for a comprehensive list of symbols and notations used in the book.

We also recall and prove a few basic definitions and results from elementary measure and functional theory, as well as topology. We need to rely on these quite often throughout the book, so it seems convenient to herd them all into this easily accessible corner. Of course, the reader may wish to skip this chapter and simply refer back to it when necessary.

Somewhat more detailed introductions to specific topics, such as linear spaces, topology, Choquet integration, and extended real numbers, can be found in the appendices.

We assume that the reader is familiar with the basics of set theory and logic (proposition, set, union, intersection, complementation, inclusion, relation, function, map), as well as calculus (derivative, Riemann integral, sequence).

1.1 Sets of numbers

$\mathbb{N}$ denotes the set of natural numbers including zero. Sometimes, we need infinity as well, or we do not need zero; in these cases, we write

$$\mathbb{N}^* := \mathbb{N} \cup \{\infty\} \text{ and } \mathbb{N}_{>0} := \mathbb{N} \setminus \{0\} \text{ and } \mathbb{N}_{>0}^* := \mathbb{N}^* \setminus \{0\}.$$

As usual, $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers and $\mathbb{R}$ is the set of real numbers. The set of extended real numbers is denoted by

$$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}.$$

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We say that a real number a is **positive** when $a > 0$, **negative** when $a < 0$, **non-positive** when $a \leq 0$, and **non-negative** when $a \geq 0$. The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$, and the set of strictly positive real numbers is denoted by $\mathbb{R}_{> 0}$:

$$\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\} \text{ and } \mathbb{R}_{> 0} := \{x \in \mathbb{R} : x > 0\} = \mathbb{R}_{\geq 0} \setminus \{0\}.$$

For any subset B of $\mathbb{R}^*$, we define its **supremum** $\sup B$ and **infimum** $\inf B$ as its smallest upper and greatest lower bound, respectively:

$$\begin{aligned} \sup B &:= \min \{a \in \mathbb{R}^* : a \geq b \text{ for all } b \in B\} \\ \inf B &:= \max \{a \in \mathbb{R}^* : a \leq b \text{ for all } b \in B\}. \end{aligned}$$

Addition and multiplication can be extended from the real to the extended real numbers in a straightforward manner; refer to the extensive discussion in Appendix D₃₉₁ for more details. The only case that deserves some care is the sum of $-\infty$ and $+\infty$, which is undefined. We call a sum $a + b$ of two extended real numbers **well defined** whenever it cannot be reduced to the cases $-\infty + (+\infty)$ or $+\infty + (-\infty)$.

1.2 Gambles

In this section, we are mainly concerned with real-valued maps defined on a non-empty set, or **space**, $\mathcal{X}$. Because such maps play such an essential role in the theory of lower previsions, they are given a special name: following Walley (1991), a real map is called a **gamble** – however, in contrast to Walley’s (1991) use of the term gamble, in this book, gambles need not be bounded. When we consider bounded gambles, we always explicitly say so. We denote the set of all gambles by $\mathbb{G}(\mathcal{X})$.

In the course of this book, we shall come across many other sets of this type that contain special types of gambles, defined on sets such as $\mathcal{X}$. When no confusion can arise as to what underlying set $\mathcal{X}$, we are talking about, we often omit explicit reference to $\mathcal{X}$. For instance, we simply write $\mathbb{G}$ instead of $\mathbb{G}(\mathcal{X})$, when the underlying space $\mathcal{X}$ is clear from the context.

If $*$ is a binary operation on the set of real numbers $\mathbb{R}$, then we extend $*$ point-wise to a binary operation on gambles as follows: for any two gambles f and g on $\mathcal{X}$, $f * g$ is the gamble on $\mathcal{X}$ defined by

$$(f * g)(x) := f(x) * g(x) \text{ for all } x \in \mathcal{X}.$$

This allows us, for instance, to consider the gambles $f + g$, $f - g$ and fg .

Gambles can be ordered point-wise: for any pair of gambles f and g on $\mathcal{X}$, we say that

$$f \leq g \text{ whenever } (\forall x \in \mathcal{X})(f(x) \leq g(x)), \quad (1.1)$$

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where of course the ‘ $\leq$ ’ on the right-hand side is the usual ordering of real numbers. We also use the notation ‘ $f < g$ ’ for ‘ $f \leq g$ and not $f = g$ ’. The binary relation $\leq$ on $\mathbb{G}$ is a **partial order**: it is reflexive, antisymmetrical and transitive.¹ The **supremum**, or smallest upper bound, $f \vee g$ of two gambles f and g on $\mathcal{X}$ with respect to this partial order is given by their point-wise maximum, and their **infimum**, or greatest lower bound; $f \wedge g$ is given by their point-wise minimum, so

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\} \text{ for all } x \text{ in } \mathcal{X}.$$

These operators should not be confused with the **supremum** $\sup f$ and **infimum** $\inf f$ of a gamble f on $\mathcal{X}$, which are defined as the extended real numbers:

$$\begin{aligned} \sup f &:= \sup \{f(x) : x \in \mathcal{X}\} = \min \{a \in \mathbb{R}^* : a \geq f\} \\ \inf f &:= \inf \{f(x) : x \in \mathcal{X}\} = \max \{a \in \mathbb{R}^* : a \leq f\}. \end{aligned}$$

If $\sup f$ is finite, then we say that the gamble f is **bounded above**, and similarly, f is **bounded below** if $\inf f$ is finite. We say that f is **bounded** when it is bounded above and below. We denote the set of all bounded gambles on $\mathcal{X}$ by $\mathbb{B}(\mathcal{X})$, or more simply by $\mathbb{B}$.

We also define the **absolute value** $|f|$ of a gamble f by $|f|(x) := |f(x)|$ for all x in $\mathcal{X}$, and the **negation** $-f$ of f as $(-f)(x) := -f(x)$ for all x in $\mathcal{X}$. Clearly, $f + (-g)$ is the same thing as $f - g$.

It is convenient to identify a real number $a \in \mathbb{R}$ with the constant gamble $a(x) := a$ for all $x \in \mathcal{X}$.² For instance, the expression $a \geq f$, where f is a gamble on $\mathcal{X}$ and a is a real number, means that $a \geq f(x)$ for all x in $\mathcal{X}$; we have already used this earlier in our definition of the supremum and infimum of a gamble f . As another example, $f \geq 0$ means that the gamble f is nowhere negative (or equivalently, everywhere non-negative): we simply say that f is **non-negative**. Similarly, we say that f is **non-positive** when $f \leq 0$. For another example, we define the **scalar product** λf of a real number λ and a gamble f as the point-wise product to the constant gamble λ and f :

$$(\lambda f)(x) := \lambda f(x) \text{ for all } x \text{ in } \mathcal{X}.$$

As indicated earlier, we make one important exception to the rule of interpreting binary operators in a point-wise manner: we take $f < 0$ to mean that the gamble f is nowhere positive (or equivalently, everywhere non-positive), and actually negative in at least one element of its domain:

$$f < 0 \text{ whenever } f \leq 0 \text{ and } f \neq 0; \quad (1.2)$$

we then say that f is **negative**. Similarly, we call f **positive** and write $f > 0$ when $-f < 0$, or equivalently, $f \geq 0$ and $f \neq 0$.

¹ For a detailed discussion of partially ordered sets, refer to Davey and Priestley (1990) or Birkhoff (1995). Basic definitions can also be found in Appendix A₃₆₈.

² As it would be silly and even pedantic to distinguish notationally between constant gambles and real numbers, we also use the notation $\mathbb{R}$ for the set of all constant gambles on $\mathcal{X}$.

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Given a gamble f , we can consider its **non-negative part** $f^+ := 0 \vee f$ and its **non-positive part** $f^- := -(0 \wedge f)$. These names are inspired by the fact that $f^+ \geq 0$, $-f^- \leq 0$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Observe that also $f^+ \wedge f^- = 0$ and $f^+ \vee f^- = |f|$.

Definition 1.1 (Special sets of gambles) We call a subset $\mathcal{K}$ of $\mathbb{G}$,

- (a) **negation-invariant** if it is closed under negation, meaning that $-\mathcal{K} := \{-f : f \in \mathcal{K}\} \subseteq \mathcal{K}$;³
- (b) a **cone** if it is closed under non-negative scalar multiplication,⁴ meaning that $\lambda\mathcal{K} := \{\lambda f : f \in \mathcal{K}\} \subseteq \mathcal{K}$ for all $\lambda \in \mathbb{R}_{\geq 0}$;
- (c) **convex** if it is closed under convex combinations, which means that $\lambda\mathcal{K} + (1 - \lambda)\mathcal{K} := \{\lambda f + (1 - \lambda)g : f, g \in \mathcal{K}\} \subseteq \mathcal{K}$ for all $\lambda \in (0, 1)$;
- (d) a **convex cone** if it is closed under addition and non-negative scalar multiplication, or, in other words, under non-negative linear combinations: $\text{nonneg}(\mathcal{K}) \subseteq \mathcal{K}$, where

$$\text{nonneg}(\mathcal{K}) := \left\{ \sum_{k=1}^n \lambda_k f_k : n \in \mathbb{N}, f_k \in \mathcal{K}, \lambda_k \in \mathbb{R}_{\geq 0} \right\};$$

- (e) a **linear space** if it is closed under addition and scalar multiplication, or, in other words, under linear combinations: $\text{span}(\mathcal{K}) \subseteq \mathcal{K}$, where

$$\text{span}(\mathcal{K}) := \left\{ \sum_{k=1}^n \lambda_k f_k : n \in \mathbb{N}, f_k \in \mathcal{K}, \lambda_k \in \mathbb{R} \right\};$$

- (f) a **$\wedge$ -semilattice** if it is closed under point-wise minimum and a **$\vee$ -semilattice** if it is closed under point-wise maximum;
- (g) a **lattice** if it is closed under point-wise minimum and maximum, that is, at the same time a $\wedge$ -semilattice and a $\vee$ -semilattice;
- (h) a **linear lattice** if it is both a linear space and a lattice.⁵

A linear space can also be defined as a negation-invariant convex cone.

The set $\{f \in \mathbb{G}(\mathcal{X}) : f \geq 0\}$ of all non-negative gambles on $\mathcal{X}$ is denoted by $\mathbb{G}_{\geq 0}(\mathcal{X})$, or simply by $\mathbb{G}_{\geq 0}$. Similarly, the set $\{f \in \mathbb{B}(\mathcal{X}) : f \geq 0\}$ of all non-negative bounded gambles on $\mathcal{X}$ is denoted by $\mathbb{B}_{\geq 0}(\mathcal{X})$, or simply by $\mathbb{B}_{\geq 0}$. Both $\mathbb{G}_{\geq 0}$ and $\mathbb{B}_{\geq 0}$ are convex cones.

It is clear that $\mathbb{G}$, $\mathbb{B}$ and $\mathbb{R}$ are linear lattices. In the course of this book, we shall come across quite a few other interesting subsets of $\mathbb{G}$ that are linear lattices too.

³ In this and some of the subsequent conditions, the inclusion can actually be replaced by an equality.

⁴ The mathematical literature is particularly messy when it comes to defining cones. A cone is sometimes defined as being closed under *positive* scalar multiplication (see, for instance, Rockafellar, 1970, p. 13), or as what we call a *convex cone* (Wong and Ng, 1973, p. 1), which in its turn is called a *wedge* by others (Holmes, 1975, p. 17). We follow Boyd and Vandenberghe (2004, Section 2.1.5).

⁵ In Appendix A₃₆₈ we give a more general definition of linear spaces and linear lattices. A linear lattice in the present sense is also a linear lattice in the sense of Definition A.6₃₇₀, with respect to the point-wise order $\leq$ on gambles defined in Equation (1.1)₂.

1.3 Subsets and their indicators

There are important special gambles that correspond to subsets of $\mathcal{X}$. With a subset A of $\mathcal{X}$, also called an **event**, we can associate a $\{0, 1\}$ -valued gamble I_A given by

$$I_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \text{ for all } x \text{ in } \mathcal{X}.$$

This bounded gamble I_A is called the **indicator** of A . For a collection $\mathcal{A}$ of subsets of $\mathcal{X}$, we sometimes look at the corresponding collection of indicators:

$$I_{\mathcal{A}} := \{I_A : A \in \mathcal{A}\}.$$

The set of all subsets of $\mathcal{X}$ is the **power set** of $\mathcal{X}$, and it is denoted by $\mathcal{P}(\mathcal{X})$, or simply $\mathcal{P}$, when it is clear from the context which space $\mathcal{X}$ we are talking about. We call a subset (or event) **proper** if it is neither empty nor equal to $\mathcal{X}$.

Union, intersection and difference of sets are denoted as usual as $A \cup B$, $A \cap B$ and $A \setminus B$, respectively. For the complement $\mathcal{X} \setminus A$ of a set A , we also write A^c .

With a gamble f on $\mathcal{X}$ and a real number α , we associate the following **level sets**:

$$\{f \geq \alpha\} := \{x \in \mathcal{X} : f(x) \geq \alpha\}$$

$$\{f > \alpha\} := \{x \in \mathcal{X} : f(x) > \alpha\}$$

$$\{f = \alpha\} := \{x \in \mathcal{X} : f(x) = \alpha\}.$$

We emphasise that, say, the ‘>’ in $\{f > \alpha\}$ is interpreted here in a point-wise manner; the exception discussed near Equation (1.2)₃ does not apply here.

1.4 Collections of events

We now turn to special collections of subsets of $\mathcal{X}$.

Definition 1.2 (Fixed vs free) A subset $\mathcal{C}$ of $\mathcal{P}$ is called **fixed** if its intersection is non-empty: $\bigcap \mathcal{C} \neq \emptyset$, so some element of $\mathcal{X}$ belongs to all elements of $\mathcal{C}$. Otherwise, $\mathcal{C}$ is called **free**.

Definition 1.3 (Filters and proper filters) A subset $\mathcal{F}$ of $\mathcal{P}$ is called a **filter** if

(i) $\mathcal{F}$ is **increasing**: if $A \in \mathcal{F}$ and $A \subseteq B$ then also $B \in \mathcal{F}$;

(ii) $\mathcal{F}$ is **closed under finite intersections**: if $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called a **proper filter** if in addition $\mathcal{X} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$, or equivalently, if $\mathcal{F}$ is a proper subset of $\mathcal{P}$. We denote set of all proper filters by $\mathbb{F}(\mathcal{X})$, or simply by $\mathbb{F}$ if it is clear from the context which space $\mathcal{X}$ we are talking about.

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Definition 1.4 (Filter bases and sub-bases) A set $\mathcal{C}$ is called a **filter base** for a filter $\mathcal{F}$ if $\mathcal{F} = \{A \subseteq \mathcal{X} : (\exists C \in \mathcal{C})(C \subseteq A)\}$. A set $\mathcal{C}'$ is called a **filter sub-base** for a filter $\mathcal{F}$ if the collection of finite intersections of elements of $\mathcal{C}'$ constitutes a filter base for $\mathcal{F}$.

Definition 1.5 (Ultrafilters) A proper filter $\mathcal{U}$ is called an **ultrafilter** if additionally either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$ for all A in $\mathcal{P}$, or equivalently, if there is no other proper filter that includes $\mathcal{U}$. We denote set of all ultrafilters by $\mathbb{U}(\mathcal{X})$, or simply by $\mathbb{U}$ if it is clear from the context which space $\mathcal{X}$ we are talking about.

An ultrafilter $\mathcal{U}$ is *fixed* if and only if it contains a singleton, that is, if $\mathcal{U} = \{A \subseteq \mathcal{X} : x \in A\}$ for some $x \in \mathcal{X}$.

Sometimes, instead of filters, the dual notion of **ideals** is used: a decreasing collection $\mathcal{I}$ of events that is closed under finite unions. A **proper ideal** contains $\emptyset$ but not $\mathcal{X}$. In fact, from an ideal $\mathcal{I}$, we can always construct a filter $\{A^c : A \in \mathcal{I}\}$ by element-wise complementation, and vice versa.

If a lattice of gambles contains only (indicators of) events, we call it a **lattice of events**. A lattice of events can also be seen as a collection of subsets of $\mathcal{X}$ that is closed under (finite) intersection and union. If it is also closed under set complementation and contains the empty set $\emptyset$, it is a *field*.

Definition 1.6 (Fields and σ -fields) A **field** $\mathcal{F}$ on $\mathcal{X}$ is a collection of subsets of $\mathcal{X}$ that contains the empty set $\emptyset$ and is closed under finite unions and complementation. A **σ -field** $\mathcal{F}$ on $\mathcal{X}$ is a field on $\mathcal{X}$ that is also closed under countable unions.

Next, we discuss two standard ways to provide topological spaces⁶ with σ -fields. So, assume that $\mathcal{X}$ is a topological space.

Definition 1.7 (Borel σ -field) The smallest σ -field on $\mathcal{X}$ that contains all open sets is called the **Borel σ -field** on $\mathcal{X}$ and is denoted by $\mathcal{B}(\mathcal{X})$, or simply by $\mathcal{B}$. Its members are called **Borel sets**.

For instance, $\mathcal{B}(\mathbb{R})$ is the smallest σ -field that contains all open intervals (a, b) (where $a, b \in \mathbb{R}$ and $a < b$). In measure theory, this is the standard way to equip $\mathbb{R}$ with a σ -field.

Another very useful way to equip a topological space $\mathcal{X}$ with a σ -field goes via the so-called G_δ sets. A set is G_δ if it is a countable intersection of open sets – remember that, generally, only finite intersections of open sets are open, so a G_δ set need not be open.

Definition 1.8 (Baire σ -field) The smallest σ -field that contains all compact G_δ sets is called the **Baire σ -field** and is denoted by $\mathcal{B}_0(\mathcal{X})$, or simply by $\mathcal{B}_0$. Its members are called **Baire sets**.

A gamble on $\mathcal{X}$ has **compact support** if it is zero outside some compact subset of $\mathcal{X}$. A very useful characterisation of $\mathcal{B}_0(\mathcal{X})$ goes as follows (Schechter, 1997, Section 20.34).

⁶ For a brief overview of relevant notions from topology, refer to Appendix B₃₇₁.

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Theorem 1.9 *If $\mathcal{X}$ is a locally compact Hausdorff space, then $\mathcal{B}_0(\mathcal{X})$ is the smallest σ -field that makes all continuous bounded gambles on $\mathcal{X}$ with compact support measurable.*

In fact, all continuous gambles with compact support are bounded – we emphasise here that they are bounded because of our focus on bounded gambles when we need to apply this result. Measurability is introduced a bit further on in Definition 1.17₁₂. For measurability with respect to σ -fields, such as the Baire σ -field, see in particular Proposition 1.19₁₅.

It can be shown that every Baire set is also a Borel set: $\mathcal{B}_0(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{X})$. Equality holds for compact metric spaces $\mathcal{X}$ such as compact subsets of $\mathbb{R}$ but not for arbitrary locally compact Hausdorff spaces (Schechter, 1997, Section 20.35).

1.5 Directed sets and Moore–Smith limits

Consider a non-empty set $\mathcal{A}$ provided with a binary relation $\leq$ that satisfies the following properties:

- (i) $\leq$ is **reflexive**: $\alpha \leq \alpha$ for all α in $\mathcal{A}$;
- (ii) $\leq$ is **transitive**: if $\alpha \leq \beta$ and $\beta \leq \gamma$, then also $\alpha \leq \gamma$ for all α, β , and γ in $\mathcal{A}$;
- (iii) $\leq$ satisfies the **composition property**: for all α and β in $\mathcal{A}$, there is some γ in $\mathcal{A}$ such that both $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Any such set equipped with a relation that is transitive, reflexive, and that satisfies the composition property, is called a **directed set** and is said to have the **Moore–Smith property**.

A **net** f in a set $\mathcal{Y}$ on a directed set $\mathcal{A}$ is a map $f : \mathcal{A} \rightarrow \mathcal{Y}$. If the set $\mathcal{Y}$ is provided with a topology⁷ (of open sets) $\mathcal{T}$, then we say that the net $f : \mathcal{A} \rightarrow \mathcal{Y}$ **converges** to an element y of $\mathcal{Y}$ if for any open set $O \in \mathcal{T}$ containing y – or, in other words, for any open neighbourhood O of y – there is some $\alpha_O \in \mathcal{A}$ such that $f(\alpha) \in O$ for all $\alpha_O \leq \alpha$. Instead of $f(\alpha)$, we also write f_α .⁸

Alternatively, if the topology on $\mathcal{Y}$ is determined by a (semi)norm $\|\bullet\|$, then the net f converges to y if, for all real $\varepsilon > 0$, there is some $\alpha_\varepsilon \in \mathcal{A}$ such that $\|f(\alpha) - y\| < \varepsilon$ for all $\alpha_\varepsilon \leq \alpha$.

A net of propositions p is said to hold **eventually** if there is some $\alpha^* \in \mathcal{A}$ such that $p(\alpha)$ holds for all $\alpha^* \leq \alpha$ (see, for instance, Schechter (1997, Section 7.7) for the terminology). This allows us to say that a net $f : \mathcal{A} \rightarrow \mathcal{Y}$ converges to y if f eventually belongs to every open neighbourhood of y .

In particular, an **extended real net** f on a directed set $\mathcal{A}$ is a map $f : \mathcal{A} \rightarrow \mathbb{R}^*$. If f assumes only values in $\mathbb{R}$, it is called a **real net**.

⁷ See Appendix B₃₇₁ for more details about the various topological notions discussed in this section.

⁸ We usually write f_α when the elements f_α of $\mathcal{Y}$ are themselves functions defined on some set $\mathcal{X}$, whose values $f_\alpha(x)$ in $x \in \mathcal{X}$ we then have to consider. This allows us to avoid writing $f(\alpha)(x)$...

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An extended real net f is said to be **non-increasing** if, for all $\alpha \leq \beta$, we have that $f(\alpha) \geq f(\beta)$. Similarly, f is said to be **non-decreasing** if, for all $\alpha \leq \beta$, we have that $f(\alpha) \leq f(\beta)$.

An extended real net f converges to a real number if there is some real number a such that, for every $\varepsilon > 0$, there is some $\alpha_\varepsilon \in \mathcal{A}$ such that $|f(\alpha) - a| < \varepsilon$ for all $\alpha_\varepsilon \leq \alpha$ (see Moore and Smith, 1922, Section I, p. 103). The net f converges to $+\infty$ if it is eventually greater than any real number: for all $R \in \mathbb{R}$, there is some $\alpha_R \in \mathcal{A}$ such that $f(\alpha) > R$ for all $\alpha_R \leq \alpha$. Similarly, f converges to $-\infty$ if it is eventually smaller than any real number: for all $R \in \mathbb{R}$, there is some $\alpha_R \in \mathcal{A}$ such that $f(\alpha) < R$ for all $\alpha_R \leq \alpha$.

If an extended real net f converges to some extended real number a , then this a is unique, and it is called the **Moore–Smith limit**, or simply the **limit**, of f . We denote this limit by $\lim_{\alpha \in \mathcal{A}} f(\alpha)$ or $\lim_{\alpha} f(\alpha)$ or $\lim_{\alpha} f_{\alpha}$, or simply $\lim f$. If $a = \lim f$, then we also write $f_{\alpha} \rightarrow a$ and say that f_{α} **converges to** a . If the net f has a limit $\lim f$, we say that it **converges**.

The Moore–Smith limit is a natural generalisation of the limit of sequences, $\mathbb{N}$, provided with the natural order of natural numbers is a directed set. The following examples show that nets appear in other interesting situations as well. All of them will be used at some point in this book. For a discussion of nets and their fundamental rôle in topology, refer, for instance, to Willard (1970).

Example 1.10 Consider the set $\mathbb{R}^2$, partially ordered by the component-wise ordering: $(a, b) \leq (c, d) \Leftrightarrow a \leq c$ and $b \leq d$. This ordering is clearly reflexive and transitive and also satisfies the composition property: for any (a, b) and (c, d) in $\mathbb{R}^2$, both $(a, b) \leq (\max\{a, c\}, \max\{b, d\})$ and $(c, d) \leq (\max\{a, c\}, \max\{b, d\})$. An extended real net f on $\mathbb{R}^2$ is then a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^*$. It follows from the definition given earlier that this net converges to a real number f^* if and only if, for all $\varepsilon > 0$, there is some real number R such that $|f(a, b) - f^*| < \varepsilon$ for all $a, b \geq R$. We then write that $\lim_{a, b \rightarrow +\infty} f(a, b) = f^*$.

We can of course also impose other partial orders on $\mathbb{R}^2$, which lead to other limits. For instance, if we now let $(a, b) \leq (c, d) \Leftrightarrow a \geq c$ and $b \leq d$, we again get a directed set, and the net $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ converges to a real number f^* if and only if, for all $\varepsilon > 0$, there is some real number R such that $|f(a, b) - f^*| < \varepsilon$ for all $-a, b \geq R$. We then write that $\lim_{(a, b) \rightarrow (-\infty, +\infty)} f(a, b) = f^*$.

Nets on $\mathbb{N}^2$ and their convergence are similarly defined, and in this case, we also use the notation $f_{m, n}$ instead of $f(m, n)$. ♦

Example 1.11 If we partially order a proper filter $\mathcal{F}$ with the ‘includes’ relation $\supseteq$, then we see that $\mathcal{F}$ is a *directed set*: $\supseteq$ is clearly reflexive and transitive, and for any A and B in $\mathcal{F}$, we know that $A \cap B \in \mathcal{F}$, $A \supseteq A \cap B$ and $B \supseteq A \cap B$, so $\supseteq$ satisfies the composition property on $\mathcal{F}$. This implies that we can use proper filters to define nets, and we can associate limits with proper filters. A net p of propositions defined on $\mathcal{F}$ is said to hold **eventually** if there is some $F^* \in \mathcal{F}$ such that $p(F)$ holds for all $F^* \supseteq F$. ♦

Example 1.12 Let $\mathfrak{P}_{\mathcal{F}}$ denote the set of all finite partitions of $\mathcal{X}$ whose elements belong to some given field $\mathcal{F}$ on $\mathcal{X}$. We define a binary relation $\leq$ on $\mathfrak{P}_{\mathcal{F}}$: say that $\mathcal{A} \leq \mathcal{B}$ whenever $\mathcal{B}$ is a **refinement** of $\mathcal{A}$, that is, whenever every element of $\mathcal{B}$ is a

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subset of some element of $\mathcal{A}$. Then $\leq$ is *reflexive*: every finite partition is a refinement of itself. $\leq$ is also *transitive*: if a finite partition $\mathcal{A}$ refines a finite partition $\mathcal{B}$, and $\mathcal{B}$ refines a finite partition $\mathcal{C}$, then $\mathcal{A}$ also refines $\mathcal{C}$. And $\leq$ satisfies the *composition property*: because $\mathcal{F}$ is a field, every two finite partitions in $\mathcal{F}$ have a common finite refinement in $\mathcal{F}$, that is, for every $\mathcal{A}$ and $\mathcal{B}$ in $\mathfrak{P}_{\mathcal{F}}$, there is a $\mathcal{C}$ in $\mathfrak{P}_{\mathcal{F}}$ such that $\mathcal{A} \leq \mathcal{C}$ and $\mathcal{B} \leq \mathcal{C}$. So $\mathfrak{P}_{\mathcal{F}}$ is a directed set with respect to $\leq$, and as a consequence, we can take the Moore–Smith limit of nets on $\mathfrak{P}_{\mathcal{F}}$. $\blacklozenge$

Moore–Smith limits of nets have more or less the same important properties as limits of sequences. We list a few that will be important for the discussion in this book. Their proofs are obvious.

Proposition 1.13 *Consider extended real nets f and g , and a real number $\lambda \neq 0$.*

- (i) *If f converges, then $\inf f \leq \lim f \leq \sup f$.*
- (ii) *If f is non-increasing, then f converges and $\lim f = \inf f$; similarly, if f is non-decreasing, then f converges and $\lim f = \sup f$.*
- (iii) *If any two of the nets f , g and $f + g$ converge, then the third also converges and $\lim(f + g) = \lim f + \lim g$, provided that the sum on the right-hand side is well defined.*
- (iv) *If any one of the nets f and λf converges, then the other also converges, and $\lim(\lambda f) = \lambda \lim f$.*
- (v) *If $f \geq g$ and both f and g converge, then $\lim f \geq \lim g$.*
- (vi) *If the nets f and g converge, then so do the nets $f \wedge g$ and $f \vee g$, and $\lim(f \wedge g) = \lim f \wedge \lim g$ and $\lim(f \vee g) = \lim f \vee \lim g$.*

We can also define the **limit inferior** $\liminf f$ and the **limit superior** $\limsup f$ of an extended real net f as follows:

$$\liminf f = \lim_{\alpha} \inf f_{\alpha} := \sup_{\alpha \in \mathcal{A}} \inf_{\beta \geq \alpha} f_{\beta} \quad \text{and} \quad \limsup f = \lim_{\alpha} \sup f_{\alpha} := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \geq \alpha} f_{\beta}. \quad (1.3)$$

Clearly, the net $g_{\alpha} := \inf_{\beta \geq \alpha} f_{\beta}$ is non-decreasing and therefore converges, with $\liminf f = \lim g = \lim_{\alpha} \inf_{\beta \geq \alpha} f_{\beta}$. Similarly, $\limsup f = \lim_{\alpha} \sup_{\beta \geq \alpha} f_{\beta}$.

1.6 Uniform convergence of bounded gambles

A nice illustration of the convergence of nets is provided by the notion of uniform convergence of bounded gambles, which we shall need several times in the rest of this book.

We can provide the set $\mathbb{B}$ of bounded gambles on $\mathcal{X}$ with the so-called **supremum norm** $\|\bullet\|_{\text{inf}}$ given by⁹

$$\|f\|_{\text{inf}} := \sup |f| \quad \text{for all bounded gambles } f \text{ on } \mathcal{X}.$$

It is clear that $\|f\|_{\text{inf}}$ is a (finite) real number for any bounded gamble f .

⁹ The reason for this perhaps surprising notation will become clear further on, in Definition 4.25₆₄.

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With the notations used in Section 1.5, we now let $\mathcal{Y} := \mathbb{B}$ and consider a net $f : \mathcal{A} \rightarrow \mathbb{B}$ in $\mathbb{B}$, whose values f_α are bounded gambles on $\mathcal{X}$. This net converges to a bounded gamble $g \in \mathbb{B}$ if

$$(\forall \varepsilon > 0)(\exists \alpha_\varepsilon)(\forall \alpha \leq \alpha_\varepsilon) \sup |g - f_\alpha| < \varepsilon, \quad (1.4)$$

and then we say that the net f_α converges to g **in supremum norm** or that it converges to g **uniformly**. The net f_α is then also called **uniformly convergent**. Interestingly, we can consider $\sup |g - f_\alpha|$ as a real net, and we see that the net of bounded gambles f_α converges uniformly to g if and only if this real net converges to zero: $\lim_\alpha \sup |g - f_\alpha| = 0$, or in other words, $\sup |g - f_\alpha| \rightarrow 0$.

For any set $\mathcal{K} \subseteq \mathbb{B}$ of bounded gambles on $\mathcal{X}$, its **uniform closure**¹⁰ $\text{cl}(\mathcal{K})$ is defined as the set of limits of all uniformly convergent nets (or, what is equivalent in this case, sequences, see, for instance, Willard (1970, Chapter 4 and Section 11.7)) of elements of $\mathcal{K}$:

$$\begin{aligned} \text{cl}(\mathcal{K}) &:= \{g \in \mathbb{B} : f_\alpha \rightarrow g \text{ uniformly for some net } f_\alpha \text{ in } \mathcal{K}\} \\ &= \{g \in \mathbb{B} : f_n \rightarrow g \text{ uniformly for some sequence } f_n \text{ in } \mathcal{K}\}. \end{aligned}$$

The set $\mathcal{K}$ is **uniformly closed** if it contains the limits of all uniformly convergent nets (or sequences) in $\mathcal{K}$, or, in other words, if $\text{cl}(\mathcal{K}) = \mathcal{K}$.

1.7 Set functions, charges and measures

The following definition describes well-known extensions for set functions. By a **set function** μ , we mean a map from a collection of events $\mathcal{A}$ to the set $\mathbb{R}_{\geq 0}$ of non-negative real numbers, such that (i) $\emptyset \in \mathcal{A}$ and $\mu(\emptyset) = 0$ and (ii) μ is **monotone**: if $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$ for all A and B in $\mathcal{A}$.

Definition 1.14 Let μ be a set function defined on a lattice $\mathcal{A}$ of subsets of $\mathcal{X}$ containing the empty set. The **inner set function** μ_* and the **outer set function** μ^* of μ , or induced by μ , are the maps defined for all $B \subseteq \mathcal{X}$ by

$$\begin{aligned} \mu_*(B) &:= \sup \{\mu(A) : A \in \mathcal{A} \text{ and } A \subseteq B\}, \\ \mu^*(B) &:= \inf \{\mu(A) : A \in \mathcal{A} \text{ and } A \supseteq B\}. \end{aligned}$$

Clearly, μ_* and μ^* are monotone as well, and they coincide with μ on its domain $\mathcal{A}$, implying that $\mu_*(\emptyset) = \mu^*(\emptyset) = 0$. But μ_* and μ^* are not necessarily real valued; however, they are real valued when $(\emptyset \text{ and } \mathcal{X} \text{ belongs to } \mathcal{A})$. In that case, they really deserve the name ‘(inner and outer) *set function*’.

To define probability charges, we first identify a sufficiently large collection of events of interest. For convenience, we assume that this collection has the minimal structure of a field, see Definition 1.6₆.

¹⁰ For a brief discussion of the notion of topological closure, refer to Appendix B₃₇₁.

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Next, to each event A in a field $\mathcal{F}$, we attach a(n extended) real number $\mu(A)$ that measures something related to A : belief in the occurrence of A , mass, charge, capacity of A and so on. We borrow the following definition from Bhaskara Rao and Bhaskara Rao (1983).

Definition 1.15 *Let $\mathcal{F}$ be a field on $\mathcal{X}$. A **charge** μ on $\mathcal{F}$, also called **finitely additive measure**, is an $\mathbb{R}^*$ -valued map on $\mathcal{F}$ that assumes at most one of the values $+\infty$ and $-\infty$ and satisfies*

- (i) $\mu(\emptyset) = 0$, and
- (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$.

A charge μ on $\mathcal{F}$ is called a **probability charge** if additionally it is **positive**,

- (iii) $\mu(A) \geq 0$ for any $A \in \mathcal{F}$,

and **normalised**,

- (iv) $\mu(\mathcal{X}) = 1$.

The set of all probability charges on $\mathcal{F}$ is denoted by $\mathbb{P}(\mathcal{F})$. Finally, a charge μ is said to be **σ -additive** if additionally $\mathcal{F}$ is a σ -field and

- (v) for any sequence A_n of pairwise disjoint sets in $\mathcal{F}$, the limit $\lim_{n \rightarrow +\infty} \sum_{k=0}^n \mu(A_k)$ exists in $\mathbb{R}^*$ and is equal to $\mu(\bigcup_{n \in \mathbb{N}} A_n)$ (and hence, the limit is independent of the order of the sequence).

A σ -additive charge is simply called a **measure** and a σ -additive probability charge a **probability measure**.

A probability charge is a special set function. Perhaps the most commonly used charge is the **Lebesgue measure** λ on $\mathcal{B}(\mathbb{R})$, which is defined as the unique σ -additive measure on $\mathcal{B}(\mathbb{R})$ such that

$$\lambda((a, b)) = \lambda([a, b)) = \lambda((a, b]) = \lambda([a, b]) = b - a$$

for any $a, b \in \mathbb{R}$ ($a \leq b$): it measures the length of intervals of $\mathbb{R}$. Vitali (1905) proved that there is no σ -additive measure on all of $\mathcal{P}(\mathbb{R})$ that has this property – if we accept the Axiom of Choice (see, for instance, Schechter (1997, Section 21.22)). This is one of the reasons for introducing the Borel σ -field $\mathcal{B}(\mathbb{R})$. Sometimes the Lebesgue measure is defined on larger σ -fields (see, for instance, Halmos (1974, Section 15)). The Lebesgue measure can of course also be defined, by taking appropriate restrictions, on any real interval, rather than on the space $\mathbb{R}$ itself.

As another example, consider the **total variation** $|\mu|$ of a charge μ on a field $\mathcal{F}$. It can be defined as follows, using the set $\mathfrak{P}_{\mathcal{F}}$ of all finite partitions of $\mathcal{X}$ whose elements belong to the field $\mathcal{F}$, discussed in Example 1.12_g. This set is directed by the refinement relation $\leq$. With any $A \in \mathcal{F}$, we can consider the extended real net v^A

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on $\mathfrak{P}_{\mathcal{F}}$ that maps any finite partition $\mathcal{C}$ in $\mathfrak{P}_{\mathcal{F}}$ to the extended real number

$$v^A(\mathcal{C}) := \sum_{C \in \mathcal{C}} |\mu(C \cap A)|.$$

Using the properties of the charge μ , it is not difficult to show that the net v^A is non-decreasing, and therefore converges, by Proposition 1.13₉, to some extended real number

$$|\mu|(A) := \lim_{\mathcal{C}} \sum_{C \in \mathcal{C}} v^A(C) = \sup_{\mathcal{C} \in \mathfrak{P}_{\mathcal{F}}} \sum_{C \in \mathcal{C}} |\mu(C \cap A)| \text{ for any } A \in \mathcal{F}.$$

It is easy to check that $|\mu|$ is a charge as well. When μ is positive – meaning that $\mu(A) \geq 0$ for all $A \in \mathcal{F}$, which holds, for instance, for probability charges and also for the Lebesgue measure – the total variation $|\mu|$ is equal to μ .

1.8 Measurability and simple gambles

In this book, we use charges and fields much more often than we do measures and σ -fields: we seem to have little need for σ -additivity. Measurability of functions is a notion that is commonly associated with σ -fields in the context of measures (see, for instance, Kallenberg, 2002), so it will be useful to come up with a definition for measurability that works for fields $\mathcal{F}$ in the context of charges as well.

In the context of this book, we shall only have to rely on a notion of measurability for bounded gambles, so that is what we concentrate on here. We begin with a definition of $\mathcal{F}$ -simple gambles, which are necessarily bounded.

Definition 1.16 (Simple gambles) *Let $\mathcal{F}$ be a field on $\mathcal{X}$. A gamble f on $\mathcal{X}$ is called $\mathcal{F}$ -simple if it belongs to the linear span of $I_{\mathcal{F}}$, or, in other words, if $f = \sum_{i=1}^n a_i I_{A_i}$ for some $n \in \mathbb{N}$, $a_1, \dots, a_n$ in $\mathbb{R}$ and $A_1, \dots, A_n$ in $\mathcal{F}$. The sum $\sum_{i=1}^n a_i I_{A_i}$ is called a **representation** of f . The set of $\mathcal{F}$ -simple gambles is denoted by $\text{span}(\mathcal{F})$.*

Note that $\text{span}(\mathcal{F})$ is a simplified notation for $\text{span}(I_{\mathcal{F}})$, the linear span of all indicators of elements of $\mathcal{F}$.

Next, we define $\mathcal{F}$ -measurability. The characterisation (A) in Definition 1.17 is due to Greco (1981), and the characterisation (B) is due to Janssen (1999), who also established equivalence with Greco's definition. We give a proof that is shorter than the one by Janssen.

Definition 1.17 *Let $\mathcal{F}$ be a field on $\mathcal{X}$ and let f be a bounded gamble on $\mathcal{X}$. Then the following conditions are equivalent; if any (and hence all) of them are satisfied, we say that f is $\mathcal{F}$ -measurable.*

(A) *For any $a \in \mathbb{R}$ and any $\epsilon > 0$, there is an $A \in \mathcal{F}$ such that*

$$\{f \geq a\} \supseteq A \supseteq \{f \geq a + \epsilon\}.$$

(B) *There is a sequence f_n of $\mathcal{F}$ -simple gambles that converges uniformly to f , that is, $\lim_{n \rightarrow +\infty} \sup |f - f_n| = 0$, or, in other words, $f \in \text{cl}(\text{span}(\mathcal{F}))$.*

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The set of all $\mathcal{F}$ -measurable bounded gambles is denoted by $\mathbb{B}_{\mathcal{F}}(\mathcal{X})$ and given by

$$\mathbb{B}_{\mathcal{F}}(\mathcal{X}) := \text{cl}(\text{span}(\mathcal{F})).$$

If it is clear from the context which space $\mathcal{X}$ we are talking about, we also use the simpler notation $\mathbb{B}_{\mathcal{F}}$.

Obviously, $\mathbb{B}_{\mathcal{F}}$ is a uniformly closed linear lattice that contains all constant gambles. Let us prove that the conditions are indeed equivalent.

Proof. (A) $\Rightarrow$ (B). Assume that condition (A) is satisfied. Let $\varepsilon > 0$. Then (B) is established if we can find an $\mathcal{F}$ -simple gamble g such that $\sup |f - g| \leq \varepsilon$.

Let $a_0, \dots, a_n$ be any finite sequence of real numbers such that $a_0 < \inf f$, $0 < a_{i+1} - a_i < \frac{\varepsilon}{3}$ for $i \in \{0, \dots, n-1\}$ and $\sup f < a_n$. We can always find such a sequence by making n large enough. Define $A_i := \{f \geq a_i\}$ for $i \in \{0, \dots, n\}$. We infer from (A) that there is some sequence $B_0, \dots, B_{n-1}$ of members of $\mathcal{F}$ such that

$$A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \cdots \supseteq A_{n-1} \supseteq B_{n-1} \supseteq A_n.$$

With $a_{-1} := 0$, define the $\mathcal{F}$ -simple gamble $g := \sum_{i=0}^{n-1} (a_i - a_{i-1}) I_{B_i}$. This g has the desired property $\sup |f - g| \leq \varepsilon$: choose any $x \in \mathcal{X}$, then we show that $|f(x) - g(x)| < \varepsilon$. First, observe that, by construction of the $a_0, \dots, a_n$, there is a unique $j_x \in \{1, \dots, n\}$ such that $a_{j_x-1} \leq f(x) < a_{j_x}$. So $x \in A_i$ for $i \leq j_x - 1$ and $x \notin A_i$ for $i \geq j_x$. We then infer from the construction of the sequence $B_0, \dots, B_{n-1}$ that $I_{B_i}(x) = 1$ for $i \leq j_x - 2$ and $I_{B_i}(x) = 0$ for $i \geq j_x$; for $i = j_x - 1$, both the values 0 and 1 are possible. Hence, indeed

$$\begin{aligned} |f(x) - g(x)| &= \left| f(x) - \sum_{i=0}^{n-1} (a_i - a_{i-1}) I_{B_i}(x) \right| \\ &\leq \left| f(x) - \sum_{i=0}^{j_x-2} (a_i - a_{i-1}) \right| + |a_{j_x-1} - a_{j_x-2}| \\ &= |f(x) - a_{j_x-2}| + |a_{j_x-1} - a_{j_x-2}| \\ &\leq |f(x) - a_{j_x-1}| + 2|a_{j_x-1} - a_{j_x-2}| \\ &< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which establishes the first part of the proof.

(B) $\Rightarrow$ (A). Consider any $a \in \mathbb{R}$ and $\varepsilon > 0$, and suppose there is a sequence of $\mathcal{F}$ -simple gambles such that $\sup |f - f_n|$ converges to zero. Then there is some $n_\varepsilon \in \mathbb{N}$ such that $\sup |f - f_{n_\varepsilon}| < \frac{\varepsilon}{2}$ and therefore $f(x) - \varepsilon < f_{n_\varepsilon}(x) - \varepsilon/2 < f(x)$ for all $x \in \mathcal{X}$. This guarantees that

$$\{f \geq a\} \supseteq \left\{ f_{n_\varepsilon} \geq a + \frac{\varepsilon}{2} \right\} \supseteq \{f \geq a + \varepsilon\}.$$

Now simply observe that $\left\{ f_{n_\varepsilon} \geq a + \varepsilon/2 \right\}$ belongs to $\mathcal{F}$, because f_{n_ε} is $\mathcal{F}$ -simple. $\square$

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Either of these equivalent approaches to defining measurability with respect to a field has been taken by a number of authors (see, for instance, Greco (1981), Denneberg (1994), Janssen (1999) and Bhaskara Rao and Bhaskara Rao (1983); Bhaskara Rao and Bhaskara Rao speak of $\mathcal{F}$ -continuity rather than $\mathcal{F}$ -measurability).

Hildebrandt (1934, Section 1(f)) and Walley (1991, Section 3.2.1) use definitions for $\mathcal{F}$ -measurability that are stronger than ours, unless $\mathcal{F}$ is a σ -field. However, when $\mathcal{F}$ is a field but not a σ -field, then Hildebrandt's (1934) and Walley's (1991) sets of $\mathcal{F}$ -measurable bounded gambles are not even linear spaces, which makes us prefer our weaker definition given earlier.

As a special case, a $\mathcal{P}$ -simple gamble is any gamble with a finite number of values, and it is simply called a **simple gamble**. For simple gambles, $\mathcal{F}$ -measurability is easier to characterise than for bounded gambles.

Proposition 1.18 *Let $\mathcal{F}$ be a field, and let f be a simple gamble. Then the following statements are equivalent:*

- (i) f is $\mathcal{F}$ -simple;
- (ii) f is $\mathcal{F}$ -measurable;
- (iii) for any $x \in \mathbb{R}$, the set $\{f \geq x\}$ belongs to $\mathcal{F}$;
- (iv) for any $x \in \mathbb{R}$, the set $\{f > x\}$ belongs to $\mathcal{F}$.

Proof. A bounded gamble f is clearly simple if and only if it assumes a finite number of values, say $f_1 < f_2 < \dots < f_n$, and then (see also Corollary C.4₃₈₁):

$$f = \sum_{k=1}^n f_k I_{\{f=f_k\}} = f_1 + \sum_{k=2}^n (f_k - f_{k-1}) I_{\{f \geq f_k\}}. \quad (1.5)$$

We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Proving (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) is similar.

(i) $\Rightarrow$ (ii). If f is $\mathcal{F}$ -simple, then there are $m \geq 1$, $a_\ell \in \mathbb{R}$ and $F_\ell \in \mathcal{F}$ such that $f = \sum_{\ell=1}^m a_\ell I_{F_\ell}$. This implies that the f_k are of the form $\sum_{\ell \in L_k} a_\ell$ and the corresponding $\{f = f_k\}$ of the form $\bigcap_{\ell \in L_k} F_\ell \in \mathcal{F}$ for appropriately chosen $L_k \subseteq \{1, \dots, m\}$. Hence, each $\{f \geq f_k\} \in \mathcal{F}$, and this implies that f is $\mathcal{F}$ -measurable: simply take $A := \{f \geq f_k\}$ in Definition 1.17(A)₁₂.

(ii) $\Rightarrow$ (iii). Consider any $x \in \mathbb{R}$. It is clear that $\{f \geq x\}$ is either empty or of the form $\{f \geq f_k\}$. We may therefore assume without loss of generality that $\{f \geq x\}$ has the form $\{f \geq f_k\}$. Moreover, we can always choose a and ε in Definition 1.17(A)₁₂ in such a way that $\{f \geq a\} = \{f \geq f_k\} = \{f \geq a + \varepsilon\}$, so $\{f \geq f_k\} \in \mathcal{F}$.

(iii) $\Rightarrow$ (i). Immediately from Equation (1.5) and the fact that by assumption all $\{f \geq f_k\} \in \mathcal{F}$. $\square$

When $\mathcal{F}$ is a σ -field, this simple characterisation carries over to all $\mathcal{F}$ -measurable bounded gambles, which means that our measurability definition reduces to the classical definition of $\mathcal{F}$ -measurability for bounded gambles (see, for instance, also

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Kallenberg, 2002, Lemma 1.11). As the proof is short, we repeat it here.

Proposition 1.19 *Let $\mathcal{F}$ be a σ -field. A bounded gamble f is $\mathcal{F}$ -measurable if and only if, for any $a \in \mathbb{R}$, the set $\{f > a\}$ belongs to $\mathcal{F}$, or equivalently, if and only if, for any $a \in \mathbb{R}$, the set $\{f \geq a\}$ belongs to $\mathcal{F}$.*

Proof. ‘if’. Simply take $A := \{f > a\}$ or $A := \{f \geq a\}$ in Definition 1.17(A)₁₂.

‘only if’. Suppose f is $\mathcal{F}$ -measurable. By Definition 1.17(A)₁₂, there is some sequence A_n in $\mathcal{F}$ such that

$$\left\{f \geq a + \frac{1}{n+1}\right\} \supseteq A_n \supseteq \left\{f \geq a + \frac{2}{n+1}\right\}.$$

Taking the countable union over $n \in \mathbb{N}$ on all sides, this leads to

$$\{f > a\} \supseteq \bigcup_{n \in \mathbb{N}} A_n \supseteq \{f > a\},$$

which means that $\{f > a\} = \bigcup_{n \in \mathbb{N}} A_n$. As $\mathcal{F}$ is a σ -field, it is closed under countable unions, and hence, $\bigcup_{n \in \mathbb{N}} A_n$ belongs to $\mathcal{F}$. This establishes the proposition.

For the other part, construct the sequence A_n in $\mathcal{F}$ such that

$$\left\{f \geq a - \frac{2}{n+1}\right\} \supseteq A_n \supseteq \left\{f \geq a - \frac{1}{n+1}\right\}$$

and take countable intersections to arrive at the desired result. □

When $\mathcal{F} = \mathcal{P}$, measurability is no longer an issue, as all bounded gambles are trivially $\mathcal{P}$ -measurable.

Proposition 1.20 *The set of simple gambles $\text{span}(\mathcal{P})$ is uniformly dense in $\mathbb{B}$, meaning that any bounded gamble is a uniform limit of a sequence of simple gambles: $\mathbb{B} = \text{cl}(\text{span}(\mathcal{P}))$.*

Proof. Immediately from the equivalence proof for Definition 1.17₁₂. □

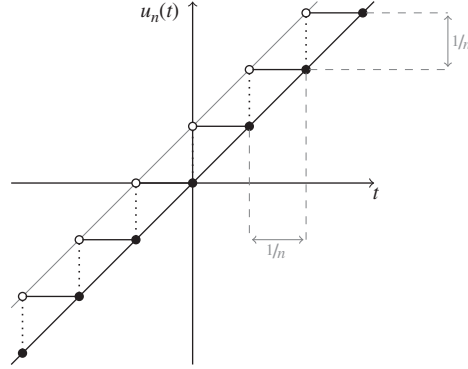
The following surprisingly simple lemma, due to Denneberg (1994, Lemma 6.2, pp. 73–74) – but we believe part (iv)_∧ is new – shows that *all* gambles f , and not just the bounded ones, can be uniformly approximated in a very convenient and systematic manner by a sequence of ‘primitive gambles’ $u_n \circ f$, which turn out to be simple whenever f is bounded. Here $u_n : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$u_n(t) := \inf \left\{ \frac{k}{n} : k \in \mathbb{Z} \text{ and } \frac{k}{n} \geq t \right\} \text{ for all } t \in \mathbb{R}, \quad (1.6)$$

where $n \in \mathbb{N}_{>0}$. It is an increasing step function with equidistant steps of width and height $1/n$ such that (see the following graph, which is also helpful in understanding a number of steps in the proof of Lemma 1.21_∧ further on)

$$t \leq u_n(t) \leq t + \frac{1}{n} \text{ for all } t \in \mathbb{R}. \quad (1.7)$$

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Lemma 1.21 For any gamble f on $\mathcal{X}$ and all $n \in \mathbb{N}$, we have that $f \leq u_n \circ f \leq f + 1/n$, so the sequence of gambles $u_n \circ f$ converges uniformly to f . Moreover, the following statements hold for all gambles f and g on $\mathcal{X}$:

- (i) The subsequence $u_{2^n} \circ f$ of $u_n \circ f$ is non-increasing.
- (ii) If f is bounded, then $u_n \circ f$ is simple for all $n \in \mathbb{N}$.
- (iii) $u_n \circ f + u_n \circ g - 2/n \leq u_n \circ (f + g) \leq u_n \circ f + u_n \circ g$ for all $n \in \mathbb{N}$, so the sequence $u_n \circ f + u_n \circ g$ converges uniformly to $f + g$.
- (iv) if f and g are comonotone in the sense of Definition C.2₃₇₈, then so are $u_n \circ f$ and $u_n \circ g$.

Proof. That $f \leq u_n \circ f \leq f + 1/n$ follows immediately from Equation (1.7)₁.

(i). This follows at once from $u_{2^n}(t) \leq u_n(t)$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$ (see also the earlier graph for more geometrical insight).

(ii). If f is bounded, then we see that $\inf f \leq f \leq u_n \circ f \leq f + 1/n \leq \sup f + 1/n$, and therefore $u_n \circ f$ is bounded as well. As it then necessarily only assumes a finite number of values, it is a simple gamble.

(iii). It suffices to prove that $u_n(t_1) + u_n(t_2) - 2/n < u_n(t_1 + t_2) \leq u_n(t_1) + u_n(t_2)$ for all real t_1 and t_2 . On the one hand, we see that

$$\begin{aligned}
 u_n(t_1 + t_2) &= \inf \left\{ \frac{\ell}{n} : \ell \in \mathbb{Z} \text{ and } \frac{\ell}{n} \geq t_1 + t_2 \right\} \\
 &= \inf \left\{ \frac{\ell_1}{n} + \frac{\ell_2}{n} : \ell_1, \ell_2 \in \mathbb{Z} \text{ and } \frac{\ell_1}{n} + \frac{\ell_2}{n} \geq t_1 + t_2 \right\} \\
 &\leq \inf \left\{ \frac{\ell_1}{n} + \frac{\ell_2}{n} : \ell_1, \ell_2 \in \mathbb{Z} \text{ and } \frac{\ell_1}{n} \geq t_1 \text{ and } \frac{\ell_2}{n} \geq t_2 \right\} \\
 &= u_n(t_1) + u_n(t_2);
 \end{aligned}$$

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on the other hand, we infer from Equation (1.7)₁₅ that

$$\begin{aligned} u_n(t_1) + u_n(t_2) - u_n(t_1 + t_2) &= [u_n(t_1) - t_1] + [u_n(t_2) - t_2] - [u_n(t_1 + t_2) - (t_1 + t_2)] \\ &< \frac{1}{n} + \frac{1}{n} + 0. \end{aligned}$$

(iv). Consider arbitrary x_1 and x_2 in $\mathcal{X}$ and assume that $u_n(f(x_1)) < u_n(f(x_2))$. Then $f(x_1) < f(x_2)$ because u_n is non-decreasing, and therefore $g(x_1) \leq g(x_2)$, because f and g are comonotone. We conclude that $u_n(g(x_1)) \leq u_n(g(x_2))$, again because u_n is non-decreasing. $\square$

1.9 Real functionals

A real-valued function Γ defined on some subset $\mathcal{K}$ of $\mathbb{G}$ will be called a (real) **functional** on $\mathcal{K}$. It is called **monotone** if

$$f \leq g \Rightarrow \Gamma(f) \leq \Gamma(g) \text{ for all } f \text{ and } g \text{ in } \mathcal{K}.$$

In very much the same way as monotone set functions give rise to inner set functions, monotone real functionals may lend themselves to inner extension.

Definition 1.22 (Inner extension) *If Γ is a monotone real functional defined on a lattice of bounded gambles¹¹ $\mathcal{K}$ that contains all constant gambles, then its **inner extension** Γ_* is the real functional defined on all bounded gambles f on $\mathcal{X}$ by*

$$\Gamma_*(f) := \sup \{ \Gamma(g) : g \in \mathcal{K} \text{ and } g \leq f \}. \quad (1.8)$$

This inner extension Γ_* is obviously monotone as well, and it coincides with Γ on its domain $\mathcal{K}$.

We now turn to a number of other possible properties of real functionals, for which it is easiest to assume that they are defined on a convex cone $\mathcal{K}$ of gambles.

Definition 1.23 *A real functional Γ defined on a convex cone of gambles $\mathcal{K}$ is called*

- (a) **positively homogeneous** if $\Gamma(\lambda f) = \lambda \Gamma(f)$ for all $f \in \mathcal{K}$ and all real $\lambda > 0$;
- (b) **non-negatively homogeneous** if $\Gamma(\lambda f) = \lambda \Gamma(f)$ for all $f \in \mathcal{K}$ and all real $\lambda \geq 0$;
- (c) **homogeneous** if $\Gamma(\lambda f) = \lambda \Gamma(f)$ for all $f \in \mathcal{K}$ and all real λ ;¹²

¹¹ Of course, we could consider inner extensions for functionals defined on gambles that are not bounded; our sole reason for not doing so here is because we do not need to in the context of this book.

¹² Sometimes, in the literature (see, for instance, Schechter, 1997, Definition 12.24), a real functional Γ is called *homogeneous* if $\Gamma(\lambda f) = |\lambda| \Gamma(f)$ for all $f \in \mathcal{K}$ and all real λ .

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- (d) **convex** if $\Gamma(\lambda f + (1 - \lambda)g) \leq \lambda\Gamma(f) + (1 - \lambda)\Gamma(g)$ for all $f, g \in \mathcal{K}$ and all real $\lambda \in (0, 1)$;
- (e) **concave** if $\Gamma(\lambda f + (1 - \lambda)g) \geq \lambda\Gamma(f) + (1 - \lambda)\Gamma(g)$ for all $f, g \in \mathcal{K}$ and all real $\lambda \in (0, 1)$.

If in addition $\mathcal{K}$ is a linear space, then we call Γ

- (a) **super-additive** if $\Gamma(f + g) \geq \Gamma(f) + \Gamma(g)$ for all $f, g \in \mathcal{K}$;
- (b) **sub-additive** if $\Gamma(f + g) \leq \Gamma(f) + \Gamma(g)$ for all $f, g \in \mathcal{K}$;
- (c) **additive** if $\Gamma(f + g) = \Gamma(f) + \Gamma(g)$ for all $f, g \in \mathcal{K}$;
- (d) **linear** if $\Gamma(\lambda f + \mu g) = \lambda\Gamma(f) + \mu\Gamma(g)$ for all $f, g \in \mathcal{K}$ and all real λ, μ .

There is a very straightforward characterisation of real functionals that can be extended linearly, whose formulation and simple proof we borrow from Schechter (1997, Proposition 11.10).

Proposition 1.24 *A real functional Γ defined on a set $\mathcal{K}$ of gambles on $\mathcal{X}$ can be extended to a linear functional on $\text{span}(\mathcal{K})$ if and only if*

$$\sum_{k=1}^n \lambda_k f_k = 0 \Rightarrow \sum_{k=1}^n \lambda_k \Gamma(f_k) = 0 \text{ for all } n \in \mathbb{N}_{>0}, \lambda_k \in \mathbb{R} \text{ and } f_k \in \mathcal{K} \quad (1.9)$$

and in that case, this linear extension is unique.

Proof. If there is a linear extension Γ' to $\text{span} \mathcal{K}$, it must be unique, because linearity requires that it should satisfy

$$\Gamma' \left(\sum_{k=1}^n \lambda_k f_k \right) = \sum_{k=1}^n \lambda_k \Gamma(f_k) \text{ for all } n \in \mathbb{N}_{>0}, \lambda_k \in \mathbb{R} \text{ and } f_k \in \mathcal{K}.$$

To show that there is a linear extension, we only need to show that this definition of Γ' is consistent. Assume that $\sum_{k=1}^n \lambda_k f_k = \sum_{\ell=1}^m \mu_\ell g_\ell$, where $m, n \in \mathbb{N}_{>0}$ and $\lambda_k, \mu_\ell \in \mathbb{R}$ and $f_k, g_\ell \in \mathcal{K}$. Then it follows from Equation (1.9) that $\sum_{k=1}^n \lambda_k \Gamma(f_k) = \sum_{\ell=1}^m \mu_\ell \Gamma(g_\ell)$. $\square$

To conclude, we mention a convenient continuity property.

Definition 1.25 (Monotone convergence) *A real functional Γ defined on a set $\mathcal{K}$ of gambles on $\mathcal{X}$ satisfies **downward monotone convergence** if for every non-increasing sequence f_n in $\mathcal{K}$ such that*

- (i) $a \geq f_1 \geq f_2 \geq \dots$ for some $a \in \mathbb{R}$, and
- (ii) the point-wise limit $\lim_{n \rightarrow +\infty} f_n$ of the sequence f_n also belongs to $\mathcal{K}$,

*it holds that $\Gamma(\lim_{n \rightarrow +\infty} f_n) = \lim_{n \rightarrow +\infty} \Gamma(f_n)$. It satisfies **upward monotone convergence** if for every non-decreasing sequence g_n in $\mathcal{K}$ such that*

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- (i) $b \leq g_1 \leq g_2 \leq \dots$ for some $b \in \mathbb{R}$, and
- (ii) the point-wise limit $\lim_{n \rightarrow +\infty} g_n$ of the sequence g_n also belongs to $\mathcal{K}$,

it holds that $\Gamma(\lim_{n \rightarrow +\infty} g_n) = \lim_{n \rightarrow +\infty} \Gamma(g_n)$. We say that Γ satisfies **monotone convergence** if it satisfies both upward and downward monotone convergence.

In the literature, monotone convergence is more usually defined in terms of *non-negative non-decreasing* sequences (see, for instance, Schechter, 1997, Sections 21.38 and 24.29). For real linear functionals that satisfy constant additivity (meaning that $\Gamma(f + a) = a\Gamma(1) + \Gamma(f)$ for all f in $\mathcal{K}$ and $a \in \mathbb{R}$), such as the usual types of integrals, our definition is evidently equivalent with the usual formulation.

Proposition 1.26 *If Γ is a real linear functional defined on a linear space $\mathcal{K}$, then the following statements are equivalent:*

- (i) Γ satisfies upward monotone convergence;
- (ii) Γ satisfies downward monotone convergence;
- (iii) Γ satisfies monotone convergence.

Proof. Immediate, once we realise that $\mathcal{K}$ is negation-invariant, $\Gamma(-f) = -\Gamma(f)$ for all $f \in \mathcal{K}$, and $\lim_{n \rightarrow +\infty} -f_n = -\lim_{n \rightarrow +\infty} f_n$. $\square$

We differentiate between upward and downward monotone convergence here because we will mainly be working outside the ambit of linear functionals and will come across examples of both types (see, for instance, Proposition 5.12₉₁ and its proof and Proposition 5.21₁₀₀).

1.10 A useful lemma

We shall need the following simple lemma a few times.

Lemma 1.27 *Let a, b and ϵ be real numbers and assume that $\epsilon \geq 0$. If $a \leq b \leq a + \epsilon$, then, for every real number c , we have that $ac - \epsilon|c| \leq bc \leq ac + \epsilon|c|$.*

Proof. If $c \geq 0$, then we have that $ac \leq bc \leq (a + \epsilon)c$, which implies that

$$ac - \epsilon|c| \leq ac \leq bc \leq (a + \epsilon)c \leq ac + \epsilon|c|.$$

On the other hand, if $c < 0$, then we have that $ac \geq bc \geq (a + \epsilon)c$, which implies that

$$ac + \epsilon|c| \geq ac \geq bc \geq (a + \epsilon)c \geq ac - \epsilon|c|.$$

In both cases, we find that $ac - \epsilon|c| \leq bc \leq ac + \epsilon|c|$. $\square$

