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A synthetic view

Mathematical finance has been an exponentially growing field of research in the last decades and is still impressively active. There are also many directions and subfields under the hat of 'finance' and researchers from very different fields, such as economics (of course), engineering, mathematics, numerical analysis and recently statistics, have been involved in this area.

This chapter is intended to give a guidance on the reading of the book and to provide a better focus on the topics discussed herein. The book is intended to be self-contained in its exposition, introducing all the concepts, including very preliminary ones, which are required to better understand more complex topics and to appreciate the details and the beauty of some of the results.

This book is also very computer-oriented and it often moves from theory to applications and examples. The R statistical environment has been chosen as a basis. All the code presented in this book is free and available as an R statistical package called **opefimor** on CRAN.[1]

There are many good publications on mathematical finance on the market. Some of them consider only mathematical aspects of the matter at different level of complexity. Other books that mix theoretical results and software applications are usually based on copyright protected software. These publications touch upon the problem of model calibration only incidentally and in most cases the focus is on discrete time models mainly (ARCH, GARCH, etc.) with notable exceptions.

The main topics of this book are the description of models for asset dynamics and interest rates along with their statistical calibration. In particular, the attention is on continuous time models observed at discrete times and calibration techniques for them in terms of statistical estimation. Then pricing of derivative contracts on a single underlining asset in the Black and Scholes-Merton framework (Black and Scholes 1973; Merton 1973), pricing of basket options, volatility, covariation and regime switching analysis are considered. At the same

[1] CRAN, Comprehensive R Archive Network, http:://cran.r-project.org.

Option Pricing and Estimation of Financial Models with R, First Edition. Stefano M. Iacus.

time, the book considers jump diffusions and telegraph process models and pricing under these dynamics.

1.1 The world of derivatives

There are many kinds of financial markets characterized by the nature of the financial products exchanged rather than their geographical or physical location. Examples of these markets are:

- **stock markets**: this is the familiar notion of stock exchange markets, like New York, London, Tokyo, Milan, etc.;
- **bond markets**: for fixed return financial products, usually issued by central banks, etc.;
- **currency markets or foreign exchange markets**: where currencies are exchanged and their prices are determined;
- **commodity markets**: where prices of commodities like oil, gold, etc. are fixed;
- **futures and options markets**: for derivative products based on one or more other underlying products typical of the previous markets.

The book is divided into two parts (although some natural overlapping occurs). In the first part the modelling and analysis of dynamics of assets prices and interest rates are presented (Chapters 3, 4 and 5). In the second part, the theory and practice on derivatives pricing are presented (Chapters 6 and 7). Chapter 2 and part of Chapter 3 contain basic probabilistic and statistical infrastructure for the subsequent chapters. Chapter 4 introduces the numerical basic tools which, usually in finance, complement the analytical results presented in the other parts. Chapter 8 presents an introduction to recently introduced models which go beyond the standard model of Black and Scholes and the Chapter 9 presents accessory results for the analysis of financial time series which are useful in risk analysis and portfolio choices.

1.1.1 Different kinds of contracts

Derivatives are simply contracts applied to financial products. The most traded and also the object of our interest are the *options*. An option is a contract that gives the right to sell or buy a particular financial product at a given price on a predetermined date. They are clearly asymmetric contracts and what is really sold is the 'option' of exercise of a given right. Other asymmetric contracts are so-called *futures* or *forwards*. Forwards and futures are contracts which oblige one to sell or buy a financial product at a given price on a certain date to another party. Options and futures are similar in that, e.g., prices and dates are prescribed but clearly in one case what is traded is an opportunity of trade and in the other an obligation. We mainly focus on option pricing and we start with an example.

1.1.2 Vanilla options

Vanilla options[2] is a term that indicates the most common form of options. An option is a contract with several ingredients:

- the *holder*: who subscribes the financial contract;
- the *writer*: the seller of the contract;
- the *underlying asset*: the financial product, usually but not necessarily a stock asset, on which the contract is based;
- the *expiry date*: the date on which the right (to sell or buy) the underlying asset can be exercised by the holder;
- the *exercise or strike price*: the predetermined price for the underlying asset at the given date.

Hence, the holder buys a right and not an obligation (to sell or buy), conversely the writer is obliged to honor the contract (sell or buy at a given price) at the expiry date.

The right of this choice has an economical value which has to be paid in advance. At the same time, the writer has to be compensated from the obligation. Hence the problem of fixing a *fair* price for an option contract arises. So, option pricing should answer the following two questions:

- how much should one pay for his right of choice? i.e. how to fix the price of an option in order to be accepted by the holder?
- how to minimize the risk associated with the obligation of the writer? i.e. to which (economical) extent can the writer reasonably support the cost of the contract?

Example 1.1.1 (From Wilmott *et al.* (1995)) *Suppose that there exists an asset on the market which is sold at \$25 and assume we want to fix the price of an option on this asset with an expiry date of 8 months and exercise price of buying this asset at \$25. Assume there are only two possible scenarios: (i) in 8 months the price of the asset rises to \$27 or (ii) in 8 months the price of the asset falls to*

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[2] From *Free On-Line Dictionary of Computing*, http://foldoc.doc.ic.ac.uk. Vanilla : *(Default flavour of ice cream in the US) Ordinary flavour, standard. When used of food, very often does not mean that the food is flavoured with vanilla extract! For example, 'vanilla wonton soup' means ordinary wonton soup, as opposed to hot-and-sour wonton soup. This word differs from the canonical in that the latter means 'default', whereas vanilla simply means 'ordinary'. For example, when hackers go to a Chinese restaurant, hot-and-sour wonton soup is the canonical wonton soup to get (because that is what most of them usually order) even though it isn't the vanilla wonton soup*.

\$23. In case (i) the potential holder of the option can exercise the right, pay \$25 to the writer to get the asset, sell it on the market at \$27 to get a return of \$2, i.e.

$$\$27 - \$25 = \$2.$$

In scenario (ii), the option will not be exercised, hence the expected return is \$0. If both scenarios are likely to happen with the same probability of $\frac{1}{2}$*, the expected return for the potential holder of this option will be*

$$\frac{1}{2} \times \$0 + \frac{1}{2} \times \$2 = \$1.$$

So, if we assume no transaction costs, no interest rates, etc., the fair value of this option should be \$1. If this is the fair price, a holder investing \$1 in this contract could gain $-\$1 + \$2 = \$1$*, which means 100% of the invested money in scenario (i) and in scenario (ii)* $-\$1 + \$0 = -\$1$*, i.e. 100% of total loss. Which means that derivatives are extremely risky financial contracts that, even in this simple example, may lead to 100% of gain or 100% of loss.*

Now, assume that the potential holder, instead of buying the option, just buys the asset. In case (i) the return from this investment would be $-\$25 + \$27 = \$2$ *which means* $+2/25 = 0.08$ *(+8%) and in scenario (ii)* $-\$25 + \$23 = -\$2$ *which equates to a loss of value of* $-2/25 = -0.08$ *(−8%).*

From the previous example we learn different things:

- the value of an option reacts quickly (instantaneously) to the variation of the underlying asset;
- to fix the fair price of an option we need to know the price of the underlying asset at the expiry date: either we have a crystal ball or a good predictive model. We try the second approach in Chapters 3 and 5;
- the higher the final price of the underlying asset the larger will be the profit; hence the price depends on the present and future values of the asset;
- the value of the option also depends on the strike price: the lower the strike price, the less the loss for the writer;
- clearly, the expiry date of the contract is another key ingredient: the closer the expiry date, the less the uncertainty on future values of the asset and vice versa;
- if the underlying asset has high volatility (variability) this is reflected by the risk (and price) of the contract, because it is less certainty about future values of the asset. The study of volatility and Greeks will be the subject of Chapters 5, 6 and 9.

It is also worth remarking that, in pricing an option (as any other risky contract) there is a need to compare the potential revenue of the investment against fixed return contracts, like bonds, or, at least, interest rates. We will discuss models for the description of interest rates in the second part of Chapter 5. To summarize, the value of an option is a function of roughly the following quantities:

$$\begin{aligned}\text{option value} = f(&\text{current asset price, strike price,}\\ &\text{final asset price, expiry date,}\\ &\text{interest rate)}\end{aligned}$$

Although we can observe the current price of the asset and predict interest rates, and we can fix the strike price and the expiry date, there is still the need to build predictive models for the final price of the asset. In particular, we will not be faced with two simple scenarios as in the previous example, but with a complete range of values with some variability which is different from asset to asset. So not only do we need good predictive models but also some statistical assessment and calibration of the proposed models. In particular we will be interested in calculating the expected value of f mainly as a function of the final value of the asset price, i.e.

$$\text{payoff} = \mathbb{E}\{f(\cdots)\}$$

this is the *payoff* of the contract which will be used to determine the fair value of an option. This payoff is rarely available in closed analytical form and hence simulation and Monte Carlo techniques are needed to estimate or approximate it. The bases of this numerical approach are set in Chapter 4.

The option presented in Example 1.1.1 was in fact a *call* option, where call means the 'right to buy'. An option that gives a right to sell at some price is called a *put* option. In a put option, the writer is obliged to buy from the holder an asset to some given price (clearly, when the underlying asset has a lower value on the market). We will see that the structure of the payoff of a put option is very similar to that of a call, although specular considerations on its value are appropriate, e.g. while the holder of a call option hopes for the rise of the price of the assets, the owner of the put hopes for the decrease of this price, etc. Table 1.1 reports put and call prices for the Roll Royce asset. When the strike price is 130, the cost of a call is higher than the cost of the put. This is because the current price is 134 and even a small increase in the value produces a gain of at least \$4. In the case of the put, the price should fall more than \$4 in order

Table 1.1 *Financial Times*, 4 Feb. 1993. (134): asset price at closing on 3 Feb. 1993. Mar., June, Sep.: expiry date, third Wednesday of each month.

		Calls			Puts		
Option	Ex. Price	Mar	Jun	Sep	Mar	Jun	Sep
R.Royce	130	11	15	19	9	14	17
(134)	140	6	11	16	16	20	23

to exercise the option. Of course all the prices are functions of the expiry dates. This is a similar situation but with smaller prices for options with a higher strike price (140).

1.1.3 Why options?

Usually options are not primary financial contracts in one's portfolio, but they are often used along with assets on which the derivative is based. A traditional investor may decide to buy stocks of a company if he or she believes that the company will increase its value. If right, the investor can sell at a proper time and obtain some gain, if wrong the investor can sell the shares before the price falls too much. If instead of real stocks the investor buys options on that stock, her fall or gain can go up to 100% of the investment as shown in the trivial example. But if one is risk adverse and wants to add a small risk to the portfolio, a good way to do this is to buy regular stocks and some options on the same stock. Also, in a long-term strategy, if one owns shares and options of the same asset and some temporary decrease of value occurs, one can decide to use or buy options to compensate this temporary loss of value instead of selling the stocks. For one reason or another, options are more liquid than the underlying assets, i.e. there are more options on an asset than available shares of that asset.

So options imply high risk for the holder which, in turn, implies complete loss of investment up to doubling. Symmetrically, the writer exposes himself to this obligation for a small initial payment of the contract (see e.g. Table 1.1). So, who on earth may decide to be a writer of one of these contracts of small revenue and high risk? Because an option exists on the market, their price should be fixed in a way that is considered convenient (or fair) for both the holder and the writer. Surely, if writers have more information on the market than a casual holder, then transition costs and other technical aspects may give enough profit to afford the role of writers. The strategy that allows the writer to cover the risk of selling an option to a holder at a given price is called *hedging*. More precisely, the hedging strategy is part of the way option pricing is realized (along with the notion of *non-arbitrage* which will be discussed in details in Chapter 6). Suppose we have an asset with decreasing value. If a portfolio contains only assets of this type, its value will decrease accordingly. If the portfolio contains only put options on that asset, the value of the portfolio will increase. A portfolio which includes both assets and put options in appropriate proportion may reduce the risk to the extent of eliminating the risk (*risk free strategy*). Hedging is a portfolio strategy which balances options and assets in order to reduce the risk. If the writer is able to sell an option at some price slightly higher than its real value, he may construct a hedging strategy which covers the risk of selling the option and eventually gain some money, i.e. obtain a risk-free profit. Risk-free strategies (as defined in Chapter 6) are excluded in the theory of Black and Scholes.

1.1.4 A variety of options

Options like the ones introduced so far are called *European* options. The name European has nothing to do with the market on which they are exchanged but on the typology of the contract itself. European options are contracts for which the right to sell (European call option) or buy (European put option) can be exercised only at a fixed expiry date. These options are the object of Chapter 6.

Options which can be exercised during the whole period of existence of the contract are called *American options*. Surely, the pricing of American options is more complicated than the pricing of European options because instead of a single fixed horizon, the complete evolution of the underlying asset has to be predicted in the most accurate way. In particular, the main point in possessing an American option is to find the optimal time on which exercise the option. This is the object of Chapter 7.

In both cases, options have not only an initial value (the initial fair price) but their value changes with time and options can be exchanged on the market before expiry date. So, the knowledge of the price of an option over the whole life of the contract is interesting in both situations.

Another classification of options is based on the way the payoff is determined. Even in the case of European options, it might happen that the final payoff of the option is determined not only by the last value of the underlying asset but also on the complete evolution of the price of the same asset, for example, via some kind of averaging. These are called *exotic options* (or *path-dependent options*). This is typical of options based on underlying products like commodities, where usually the payoff depends on the distance between the strike price and the average price during the whole life of the contract, the maximal or minimal value, etc.) or interest rates, where some geometric average is considered.

Average is a concept that applies to discrete values as well as to continuous values (think about the expected value of random variables). Observations always come in discrete form as a sequence of numbers, but analytical calculations are made on continuous time processes. The errors due to discretization of continuous time models affect both calibration and estimation of the payoffs. We will discuss this issue throughout the text.

Path-dependent options can be of European or American type and can be further subclassified according to the following categories which actually reflect analytical ways to treat them:

- barrier options: exercised only if the underlying asset reaches (or not) a prescribed value during the whole period (for example, in a simple European option with a given strike price, the option may be exercised by the holder only if the asset does not grow too much in order to contain the risk);

- Asian options: the final payoff is a function of some average of the price of the underlying asset;
- lookback options: the payoff depends on the maximal or minimal value of the asset during the whole period.

Notice that in this brief outlook on option pricing we mention only options on a single asset. Although very pedagogical to explain basic concepts of option pricing, many real options are based on more than one underlying asset. We will refer to these options as *basket options* and consider them in Chapter 6, Section 6.7. As for any portfolio strategy, correlation of financial products is something to take into account and not just the volatility of each single asset included in the portfolio. We will discuss the monitoring of volatility and covariance estimation of multidimensional financial time series in Chapter 9.

1.1.5 How to model asset prices

Modern mathematical finance originated from the doctoral thesis of Bachelier (1900) but was formally proposed in a complete financial perspective by Black and Scholes (1973) and Merton (1973). The basic model to describe asset prices is the *geometric Brownian motion*. Let us denote by $\{S(t), t \geq 0\}$ the price of an asset at time t, for $t \geq 0$. Consider the small time interval $\mathrm{d}t$ and the variation of the asset price in the interval $[t, t + \mathrm{d}t)$ which we denote by $\mathrm{d}S(t) = S(t + \mathrm{d}t) - S(t)$. The return for this asset is the ratio between $\mathrm{d}S(t)$ and $S(t)$. We can model the returns as the result of two components:

$$\frac{\mathrm{d}S(t)}{S(t)} = \text{deterministic contribution} + \text{stochastic contribution}$$

the deterministic contribution is related to interest rates or bonds and is a risk free trend of this model (usually called the *drift*). If we assume a constant return μ, after $\mathrm{d}t$ times, the deterministic contribution to the returns of S will be $\mu \mathrm{d}t$:

$$\text{deterministic contribution} = \mu \mathrm{d}t.$$

The stochastic contribution is instead related to exogenous and nonpredictable shocks on the assets or on the whole market. For simplicity, these shocks are assumed to be symmetric, zero mean etc., i.e. typical Gaussian shocks. To separate the contribution of the natural volatility of the asset from the stochastic shocks, we assume further that the stochastic part is the product of $\sigma > 0$ (the volatility) and the variation of stochastic Gaussian noise $\mathrm{d}W(t)$:

$$\text{stochastic contribution} = \sigma \mathrm{d}W(t).$$

It is further assumed that the stochastic variation $\mathrm{d}W(t)$ has a variance proportional to the time increment, i.e. $\mathrm{d}W(t) \sim N(0, \mathrm{d}t)$. The process $W(t)$, which is such that $\mathrm{d}W(t) = W(t + \mathrm{d}t) - W(t) \sim N(0, \mathrm{d}t)$, is called the Wiener process or Brownian motion. Putting all together, we obtain the following equation:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W(t),$$

which we can rewrite in differential form as follows:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t). \tag{1.1}$$

This is a difference equation, i.e. $S(t + \mathrm{d}t) - S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)(W(t + \mathrm{d}t) - W(t))$ and if we take the limit as $\mathrm{d}t \to 0$, the above is a formal writing of what is called a *stochastic differential equation*, which is intuitively very simple but mathematically not well defined as is. Indeed, taking the limit as $\mathrm{d}t \to 0$ we obtain the following differential equation:

$$S'(t) = \mu S(t) + \sigma S(t) W'(t)$$

but the $W'(t)$, the derivative of the Wiener process with respect to time, is not well defined in the mathematical sense. But if we rewrite (1.1) in integral form as follows:

$$S(t) = S(0) + \mu \int_0^t S(u)\mathrm{d}u + \sigma \int_0^t S(u)\mathrm{d}W(u)$$

it is well defined. The last integral is called *stochastic integral* or *Itô integral* and will be defined in Chapter 3. The geometric Brownian motion is the process $S(t)$ which solves the stochastic differential equation (1.1) and is at the basis of the Black and Scholes and Merton theory of option pricing. Chapters 2 and 3 contain the necessary building blocks to understand the rest of the book.

1.1.6 One step beyond

Unfortunately, the statistical analysis of financial time series, as described by the geometric Brownian motion, is not always satisfactory in that financial data do not fit very well the hypotheses of this theory (for example the Gaussian assumption on the returns). In Chapter 8 we present other recently introduced models which account for several discrepancies noticed on the analysis of real data and theoretical results where the stochastic noise $W(t)$ is replaced by other stochastic processes. Chapter 9 treats some advanced applied topics like monitoring of the volatility, estimation of covariation for asynchronous time series, model selection for sparse diffusion models and explorative data analysis of financial time series using cluster analysis.

1.2 Bibliographical notes

The principal text on mathematical finance is surely Hull (2000). This is so far the most complete overview of modern finance. Other text may be good companion to enter the arguments because they use different level of formalism. Just to mention a few, we can signal the two books Wilmott (2001) and Wilmott *et al*. (1995). The more mathematically oriented reader may prefer books like Shreve (2004a,b), Dineen (2005), Williams (2006), Mikosch (1998), Cerný (2004), Grossinho *et al*. (2006), Korn and Korn (2001) and Musiela and Rutkowski (2005). More numerically oriented publications are Fries (2007), Jäckel (2002), Glasserman (2004), Benth (2004), Ross (2003), Rachev (2004) and Scherer and Martin (2005). Other books more oriented to the statistical analysis of financial times series are Tsay (2005), Carmona (2004), Ruppert (2006) and Franke *et al*. (2004).

References

Bachelier, L. (1900). Théorie de la spéculation. *Analles Scientifique de l'École Normale Superieure* **17**, 21–86.

Benth, F. (2004). *Option Theory with Stochastic Analysis. An introduction to Mathematical Finance*. Springer-Verlag Berlin, Heidelberg.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637–654.

Carmona, R. (2004). *Statistical Analysis of Financial Data in S-Plus*. Springer, New York.

Cerný, A. (2004). *Mathematical Techniques in Finance. Tools for Incomplete Markets*. Princeton University Press, Princeton, N.J.

Dineen, S. (2005). *Probability Theory in Finance. A Mathematical Guide to the Black-Scholes Formula*. American Mathematical Society, Providence, R.I.

Franke, J., Härdle, W., and Hafner, C. (2004). *Statistics of Financial Markets: An Introduction*. Springer, New York.

Fries, C. (2007). *Mathematical Finance. Theory, Modeling, Implementation*. John Wiley & Sons, Inc., Hoboken, N.J.

Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering*. Springer, New York.

Grossinho, M., Shiryaev, A., Esqível, M., and Oliveira, P. (2006). *Stochastic Finance*. Springer Science Business Media, Inc., New York.

Hull, J. (2000). *Options, Futures and Other Derivatives*. Prentice-Hall, Englewood Cliffs, N.J.

Jäckel, P. (2002). *Monte Carlo Methods in Finance*. John Wiley & Sons, Ltd, Chichester, England.

Korn, R. and Korn, E. (2001). *Option Pricing and Portfolio Optimization*. American Mathematical Society, Providence, R.I.

Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4**, 1, 141–183.

Mikosch, T. (1998). *Elementary Stochastic Calculus with Finance in View*. World Scientific, Singapore.

Musiela, M. and Rutkowski, M. (2005). *Martingale Methods in Financial Modeling. Second Edition*. Springer-Verlag Berlin, Heidelberg.

Rachev, S. (2004). *Handbook of Computational and Numerical Methods in Finance*. Birkhäuser, Boston.

Ross, S. (2003). *An Elementary Introduction to Mathematical Finance. Options and Other Topics*. Cambridge University Press, New York.

Ruppert, D. (2006). *Statistics and Finance: An Introduction*. Springer, New York.

Scherer, B. and Martin, D. (2005). *Introduction to Modern Portfolio Optimization with NuOPT, S-Plus and S+Bayes*. Springer Science Media Business, Inc., New York.

Shreve, S. (2004a). *Stochastic Calculus for Finance I. The Binomial Asset Pricing Model*. Springer, New York.

Shreve, S. (2004b). *Stochastic Calculus for Finance II. Continuous-Time Models*. Springer, New York.

Tsay, R. S. (2005). *Analysis of Financial Time Series. Second Edition*. John Wiley & Sons, Inc., Hoboken, N.J.

Williams, R. (2006). *Introduction to the Mathematics of Finance*. American Mathematical Society, Providence, R.I.

Wilmott, P. (2001). *Paul Wilmott Introduces Quantitative Finance*. John Wiley & Sons, Ltd, Chichester, England.

Wilmott, P., Howison, S., and Dewynne, J. (1995). *The Mathematics of Financial Derivatives. A Student Introduction*. Cambridge University Press, New York.