Chapter 1

Deformation and Kinematics. Mass Balance

The aim of this chapter is to describe the deformation and the kinematics of a porous medium formed from a deformable skeleton and a fluid saturating the porous space. The underlying idea consists in approaching the porous medium as the superimposition of two continua, the skeleton continuum and the fluid continuum. The description of the deformation and of the kinematics of each continuum considered separately differs in no way from that of a monophasic continuous medium. Nevertheless, the skeleton deformation is eventually the one that can actually be observed so it is the one discussed in the following.

The laws of physics governing the evolution of a porous continuum involve the time rate of the physical quantities attached either to the skeleton or to the fluid whatever their further distinct movement. Accordingly the particle derivative is therefore introduced, allowing us to follow separately the motions of the skeleton and the fluid. A first illustration of its use is given at the end of this chapter by expressing the mass balance for the two superimposed continua.

1.1 The Porous Medium and the Continuum Approach

1.1.1 Connected and Occluded Porosity. The Matrix

A saturated porous medium is composed of a matrix and a porous space, the latter being filled by a fluid. The connected porous space is the space through which the fluid actually flows and whose two points can be joined by a path lying entirely within it so that the fluid phase remains continuous there. The matrix is composed of both a solid part and a possible occluded porosity, whether saturated or not, but through which no filtration occurs. The connected porosity is the ratio of the volume of the connected porous space to the total volume. In what follows the term 'porosity', used without further specification, refers to the entire connected porosity.

Poromechanics O. Coussy
 ISBN 0-470-84920-7

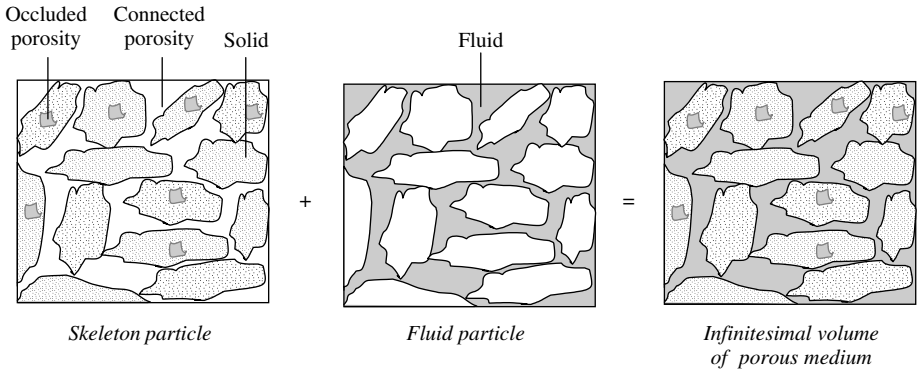

Figure 1.1: The porous medium as the superimposition of two continuous media: a skeleton particle and a fluid particle coincide with the same geometrical infinitesimal volume.

1.1.2 Skeleton and Fluid Particles. Continuity Hypothesis

A porous medium can be treated as the superimposition of two continua, the skeleton continuum and the fluid continuum. Accordingly, as illustrated in Fig. 1.1, any infinitesimal volume can be treated as the superimposition of two material particles. The first is the skeleton particle formed from the matrix and the connected porous space emptied of fluid. The second is the fluid particle formed from the fluid saturating the connected porous space and from the remaining space without the matrix.

A continuous description of a medium, which is heterogeneous at the microscopic scale, requires the choice of a macroscopic scale at which the inner constitution of matter is ignored in the analysis of the macroscopic physical phenomena. For instance, the porosity is associated with an elementary volume including sufficient material to be representative of the filtration process. More generally the hypothesis of continuity assumes the existence of a representative elementary volume which is relevant at the macroscopic scale for all the physical phenomena involved in the intended application. The physics is supposed to vary continuously from one to another of those juxtaposed infinitesimal volumes whose junction constitutes the porous medium. In addition, continuous deformation of the skeleton assumes that two skeleton particles, juxtaposed at a given time, were always so and will remain so.

1.2 The Skeleton Deformation

When subjected to external forces and to variations in pressure of the saturating fluid, the skeleton deforms. The description of this deformation differs in no way from that of a standard solid continuum and is succinctly developed below.

1.2.1 Deformation Gradient and Transport Formulae

At time $t = 0$ consider an initial configuration for the skeleton. In this configuration a skeleton particle is located by its position vector $\mathbf{X}$ of components X_i, in a Cartesian

coordinate frame of orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. At time t the skeleton has deformed and lies in the current configuration. In this configuration the particle whose initial position vector was $\mathbf{X}$ is now located by its current position vector $\mathbf{x}$ of components $x_i(X_j, t)$. We write:

$$\mathbf{X} = X_i \mathbf{e}_i; \quad \mathbf{x} = x_i(X_j, t)\mathbf{e}_i \tag{1.1}$$

with a summation on the repeated subscript i. In what follows this convention is adopted and, provided that no further indication is given, the index notation refers to a Cartesian coordinate system.

Deformation gradient and transport of a vector. In the initial configuration consider an infinitesimal material vector $d\mathbf{X}$ joining the skeleton particle located at $\mathbf{X}$ to the juxtaposed particle located at $\mathbf{X} + d\mathbf{X}$. After deformation $d\mathbf{X}$ becomes $d\mathbf{x}$ joining the same skeleton particles in their new positions, $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$ (see Fig. 1.2). The vector $d\mathbf{x}$ can be obtained from $d\mathbf{X}$ by differentiating (1.1):

$$d\mathbf{x} = \frac{\partial x_i}{\partial X_j} dX_j \mathbf{e}_i \tag{1.2}$$

or, equivalently:

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \tag{1.3}$$

where:

$$\mathbf{F} = \nabla_X \mathbf{x}; \quad F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{1.4}$$

In (1.4) ∇_X stands for the nabla operator relative to the initial configuration. $\mathbf{F}$ is called the *deformation gradient*. It transports any material vector $d\mathbf{X}$ onto its deformed $d\mathbf{x}$. Its inverse $\mathbf{F}^{-1}$ and its transpose ${}^t\mathbf{F}$ are respectively defined by:

$$d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x}; \quad d\mathbf{x} = d\mathbf{X} \cdot {}^t\mathbf{F} \tag{1.5}$$

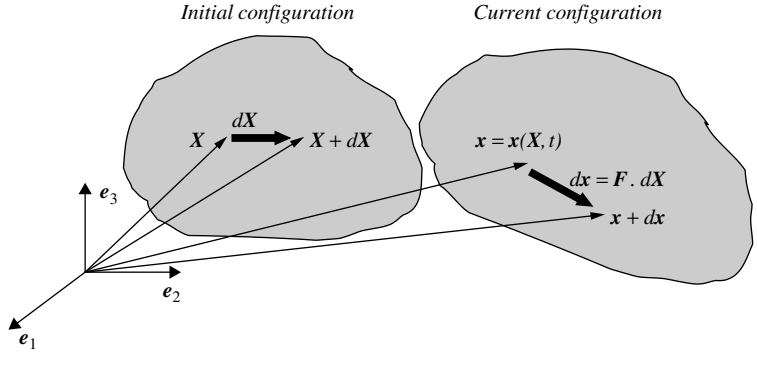

Figure 1.2: Deformation gradient $\mathbf{F}$ and transport of a material vector $d\mathbf{X}$.

and satisfy:

$$({}^t\mathbf{F})_{ij} = F_{ji}; \quad (\mathbf{F}^{-1})_{ij} = \frac{\partial X_i}{\partial x_j} \tag{1.6}$$

Deformation gradient and displacement. Let $\boldsymbol{\xi}(\mathbf{X}, t)$ be the displacement vector of the particle whose initial and current positions are $\mathbf{X}$ and $\mathbf{x}$. We write:

$$\mathbf{x} = \mathbf{X} + \boldsymbol{\xi} \tag{1.7}$$

From definitions (1.4) and (1.7) deformation gradient $\mathbf{F}$ can be expressed as a function of displacement vector $\boldsymbol{\xi}$ according to:

$$\mathbf{F} = \mathbf{1} + \nabla_X \boldsymbol{\xi}; \quad F_{ij} = \delta_{ij} + \frac{\partial \xi_i}{\partial X_j} \tag{1.8}$$

where δ_{ij} is the Kronecker delta, that is $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Volume transport. The current infinitesimal volume $d\Omega_t = dx_1\, dx_2\, dx_3$ is equal to the composed product:

$$d\Omega_t = (d\mathbf{x}_1,\, d\mathbf{x}_2,\, d\mathbf{x}_3) = d\mathbf{x}_1 \cdot (d\mathbf{x}_2 \times d\mathbf{x}_3) \tag{1.9}$$

where $d\mathbf{x}_i = dx_i\, \mathbf{e}_i$ (with no summation). The linearity of the composed product with respect to the vectors it combines allows us to write:

$$d\Omega_t = (\mathbf{F} \cdot d\mathbf{X}_1, \mathbf{F} \cdot d\mathbf{X}_2, \mathbf{F} \cdot d\mathbf{X}_3) = \det \mathbf{F}\, (d\mathbf{X}_1,\, d\mathbf{X}_2,\, d\mathbf{X}_3) \tag{1.10}$$

As a consequence any initial material volume $d\Omega_0$ transforms into the material volume $d\Omega_t$ through the relation:

$$d\Omega_t = J\, d\Omega_0 \tag{1.11}$$

where $J = \det \mathbf{F}$ is the Jacobian of the deformation.

Surface transport. Consider a material surface dA, oriented by the unit normal $\mathbf{N}$. Throughout the deformation dA transforms into the material surface da, oriented by the unit normal $\mathbf{n}$. Since vectors $\mathbf{N}$ and $\mathbf{n}$ are not material vectors, they do not match in the deformation. Let $\mathbf{U}$ be any material vector in the initial configuration. The material cylinder of initial volume $\mathbf{N} \cdot \mathbf{U}\, dA$ transforms into the material cylinder of volume $\mathbf{n} \cdot \mathbf{F} \cdot \mathbf{U}\, da$ (see Fig. 1.3). According to (1.11) we write:

$$\mathbf{n} \cdot \mathbf{F} \cdot U\, da = J\, \mathbf{N} \cdot \mathbf{U}\, dA \tag{1.12}$$

Since (1.12) holds whatever the vector $\mathbf{U}$, we derive:

$$\mathbf{n}\, da = J\, {}^t\mathbf{F}^{-1} \cdot \mathbf{N}\, dA; \qquad n_i\, da = J \frac{\partial X_j}{\partial x_i} N_j\, dA \tag{1.13}$$

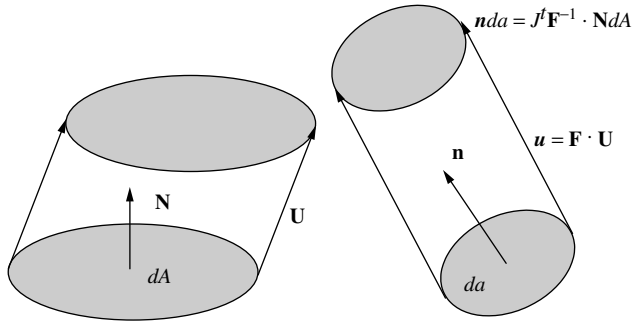

Figure 1.3: Transport of the oriented material surface $\mathbf{N}\,dA$ onto its deformed $\mathbf{n}\,da$.

Let $\mathbf{v}$ be any vector attached to the current configuration and let $\mathbf{V}$ be the associated vector attached to the initial configuration and defined in such a way that the flow of $\mathbf{v}$ through da coincides with the flow of $\mathbf{V}$ through dA. We write:

$$\mathbf{v}\cdot\mathbf{n}\,da = \mathbf{V}\cdot\mathbf{N}\,dA; \qquad v_i n_i\,da = J V_i \frac{\partial X_j}{\partial x_i} N_j\,dA \tag{1.14}$$

From (1.13) and (1.14) we derive:

$$\mathbf{V} = J\mathbf{F}^{-1}\cdot\mathbf{v}; \qquad V_i = J\frac{\partial X_i}{\partial x_j} v_j \tag{1.15}$$

Integration of (1.14) over the volumes Ω_0 and Ω_t matching in the deformation, followed by use of the divergence theorem and relation (1.11), gives the useful identity:

$$\nabla_x\cdot\mathbf{v}\,d\Omega_t = \nabla_X\cdot\mathbf{V}\,d\Omega_0; \qquad J\frac{\partial v_i}{\partial x_i} = \frac{\partial V_i}{\partial X_i} \tag{1.16}$$

1.2.2 Eulerian and Lagrangian Porosities. Void Ratio

Let n be the Eulerian porosity, so that the fluid occupies the volume $n\,d\Omega_t$ in the current configuration. Since the skeleton material volume $d\Omega_t$ changes throughout the deformation, porosity n does not properly quantify the volume change undergone by the porous space attached to the initial material volume $d\Omega_0$. In contrast to the Eulerian porosity n, which refers to the current volume $d\Omega_t$, the change in the porous space is eventually better captured by the Lagrangian porosity ϕ, which refers the current porous volume to the initial volume $d\Omega_0$ according to:

$$\phi\,d\Omega_0 = n\,d\Omega_t; \qquad \phi = Jn \tag{1.17}$$

For its part the current degree of compactness of a porous material is well captured by the void ratio e defined as the ratio of the current porous volume to the current volume of the matrix. Owing to its definition the void ratio e is a Eulerian variable, with no Lagrangian counterpart, and is expressed as a function of n in the form:

$$e = \frac{n}{1-n} \tag{1.18}$$

1.2.3 Strain Tensor

Deformation induces changes in both the lengths of the material vectors and the angles between them. The Green–Lagrange strain tensor $\mathbf{\Delta}$ measures these changes by quantifying the variation of the scalar product of two material vectors $d\mathbf{X}$ and $d\mathbf{Y}$ transforming the deformation throughout into $d\mathbf{x}$ and $d\mathbf{y}$. We write:

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}; \quad d\mathbf{y} = \mathbf{F} \cdot d\mathbf{Y} : \quad d\mathbf{x} \cdot d\mathbf{y} - d\mathbf{X} \cdot d\mathbf{Y} = 2d\mathbf{X} \cdot \mathbf{\Delta} \cdot d\mathbf{Y} \tag{1.19}$$

With the help of (1.5) $\mathbf{\Delta}$ can be written as a function of deformation gradient $\mathbf{F}$ according to:

$$\mathbf{\Delta} = \frac{1}{2}({}^t\mathbf{F} \cdot \mathbf{F} - \mathbf{1}) \tag{1.20}$$

Being symmetric, tensor $\mathbf{\Delta}$ admits three real eigenvalues Δ_J ($J = I, II, III$). The latter are the principal strains and are associated with the eigenvectors $\mathbf{e}_J$ ($J = I, II, III$), which are the principal directions of the deformation such as $\mathbf{\Delta} \cdot \mathbf{e}_J = \Delta_J \mathbf{e}_J$. The orthogonality of the principal directions, writing $\mathbf{e}_I \cdot \mathbf{e}_J = 0$, is preserved in the deformation. Indeed:

$$\left(\mathbf{e}_I \cdot {}^t\mathbf{F}\right) \cdot (\mathbf{F} \cdot \mathbf{e}_J) = 2\mathbf{e}_I \cdot \mathbf{\Delta} \cdot \mathbf{e}_J = 2\Delta_{I \text{ or } J}\mathbf{e}_I \cdot \mathbf{e}_J = 0 \tag{1.21}$$

The gradient $\mathbf{R}$ of the rotation that rigidly transports the set of orthogonal principal directions $\mathbf{e}_J$ to their final directions is an isometry so that the related strain tensor is zero, resulting in ${}^t\mathbf{R} = \mathbf{R}^{-1}$. Therefore the gradient $\mathbf{F}$ of any transformation decomposes as:

$$\mathbf{F} = \mathbf{D} \cdot \mathbf{R} \tag{1.22}$$

that is the rotation $\mathbf{R}$ followed by the actual deformation $\mathbf{D}$, the latter involving no rotation and matching the dilation of the principal directions of deformation. Equivalently, the gradient $\mathbf{F}$ can also decompose as the actual deformation $\mathbf{D}'$ matching the dilation, followed by the rotation $\mathbf{R}'$ relative to that of the eigenvectors. Accordingly, the strain tensor $\mathbf{\Delta}$ accounts entirely for the actual deformation since:

$$\mathbf{\Delta} = \frac{1}{2}(\mathbf{D}^2 - \mathbf{1}) \tag{1.23}$$

By means of (1.8) $\mathbf{\Delta}$ can finally be expressed as a function of the displacement vector $\boldsymbol{\xi}$ according to:

$$\mathbf{\Delta} = \frac{1}{2}\left(\nabla_X \boldsymbol{\xi} + {}^t\nabla_X \boldsymbol{\xi} + {}^t\nabla_X \boldsymbol{\xi} \cdot \nabla_X \boldsymbol{\xi}\right) \tag{1.24a}$$

$$\Delta_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial X_j} + \frac{\partial \xi_j}{\partial X_i} + \frac{\partial \xi_k}{\partial X_i}\frac{\partial \xi_k}{\partial X_j}\right) \tag{1.24b}$$

1.2.4 Infinitesimal Transformation and the Linearized Strain Tensor

In many problems a first-order approximation to the finite theory can be carried out under the condition of infinitesimal transformation, that is

$$\|\nabla \boldsymbol{\xi}\| \ll 1 \tag{1.25}$$

where the norm $\|(\cdot)\|$ of $(\cdot)$ has not been specified because of the equivalence of all the norms in a vectorial space of finite three dimensions. Moreover, as far as only spatial derivations are concerned, in the limit of infinitesimal transformation the current and the initial configurations merge, so the nabla operator ∇ can be used with no need for a subscript referring to a particular configuration, that is $\nabla = \nabla_X \equiv \nabla_x$.

Under condition (1.25) the Green–Lagrange strain tensor $\boldsymbol{\Delta}$ reduces to the linearized strain tensor $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\Delta} \simeq \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{\xi} + {}^t\nabla \boldsymbol{\xi}); \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \tag{1.26}$$

Since $\boldsymbol{\Delta}$ has the same order of magnitude as $\nabla_X \boldsymbol{\xi}$, infinitesimal transformation implies infinitesimal deformation, that is $\|\boldsymbol{\Delta}\| \ll 1$. In contrast, the deformation may be infinitesimal whereas the transformation is not. For instance, in a rigid body motion $\boldsymbol{\Delta}$ is zero whereas $\nabla_X \boldsymbol{\xi}$ can have any order of magnitude.

Under the approximation of infinitesimal transformation, (1.8) gives:

$$(J = \det \mathbf{F}) \simeq \left(1 + \nabla \cdot \boldsymbol{\xi} = 1 + \frac{\partial \xi_i}{\partial x_i} = 1 + \varepsilon_{ii}\right) \tag{1.27}$$

From now on let ϵ be the linearized volume dilation of the skeleton, that is:

$$\epsilon = \varepsilon_{ii} = \nabla \cdot \boldsymbol{\xi} \tag{1.28}$$

so that (1.11) takes the form:

$$d\Omega_t \simeq (1 + \epsilon)\ d\Omega_0 \tag{1.29}$$

The observable macroscopic volume dilation undergone by the skeleton is due both to the change in porosity and to the volume dilation ϵ_s undergone by the solid matrix, although the latter is not accessible from purely macroscopic experiments. Analogously to (1.29) the definition of ϵ_s allows us to write:

$$d\Omega_t^s = (1 + \epsilon_s)\ d\Omega_0^s \tag{1.30}$$

Owing to the respective definition of Eulerian and Lagrangian porosities, n and ϕ (see §1.2.2), the volume occupied by the matrix is linked to the overall volume through the relations:

$$d\Omega_t^s = (1 - n)\ d\Omega_t = d\Omega_t - \phi\, d\Omega_0; \quad d\Omega_0^s = (1 - \phi_0)\, d\Omega_0 \tag{1.31}$$

Combining the above equations, we finally derive the volume balance:

$$\epsilon = (1 - \phi_0)\epsilon_s + \phi - \phi_0 \tag{1.32}$$

In the absence of any occluded porosity, the solid grains forming the matrix generally undergo negligible volume changes so that the matrix can be considered as incompressible. Accordingly we let $\epsilon_s = 0$ in (1.32), giving:

$$\epsilon = \phi - \phi_0 \tag{1.33}$$

As is usually done in soil mechanics, it can be more convenient to use the void ratio e instead of the volumetric dilation ϵ. Combining (1.17), (1.18) and (1.33), we obtain:

$$\epsilon = \frac{e - e_0}{1 + e_0} \tag{1.34}$$

Under the approximation of infinitesimal transformation, the diagonal term ε_{ii} (with no summation) is equal to the linear dilation in the $\mathbf{e}_i$ direction, while twice the non-diagonal term, $\gamma_{ij} = 2\varepsilon_{ij}$ $(i \neq j)$, is equal to the distortion related to directions $\mathbf{e}_i$ and $\mathbf{e}_j$, that is the change undergone by the angle made between the material vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ that were normal prior to the deformation.

1.3 Kinematics

The description of the skeleton deformation by means of the deformation gradient $\mathbf{F}$ is by nature a Lagrangian description. The fields are functions of time t and of position vector $\mathbf{X}$ locating the skeleton particle in the initial configuration. The latter does not vary with time and the kinematics of the skeleton results from a simple time derivation.

In contrast to the Lagrangian approach, the Eulerian approach involves only the current configuration, with no reference to any initial configuration. The approach is carried out by using the velocity field $\mathbf{V}^\pi(\mathbf{x}, t)$ of the particle coinciding at time t with the geometrical point located at $\mathbf{x}$. The particle can be either a skeleton particle, $\pi = s$, or a fluid particle, $\pi = f$.[1] At time t, the same Eulerian approach applies to both particles since the skeleton continuum and the fluid continuum merge in the same current configuration.

1.3.1 Particle Derivative

Definition

The particle derivative $d^\pi\mathcal{G}/dt$ with respect to particle π $(= s$ or $f)$ of some field $\mathcal{G}$ is the time derivative of $\mathcal{G}$ that an observer attached to the particle would derive. This observer records the variation $d^\pi\mathcal{G}$ of quantity $\mathcal{G}$ between times t and $t + dt$. For instance, the

[1] It would have been more rigorous to make a distinction between the index referring to the matter at the macroscopic scale (for instance, sk for the skeleton particle and fl for the fluid particle) and the index referring to the matter at the mesoscopic scale (for instance, s for the solid matrix, as in (1.32), and f for the fluid). However, for the sake of simplicity of notation, we chose not to make this distinction.

origin of the coordinate being fixed, the velocity field $\mathbf{V}^{\pi}(\mathbf{x}, t)$ of particle π located at $\mathbf{x}$ reads:

$$\frac{d^{\pi}\mathbf{x}}{dt} = \mathbf{V}^{\pi}(\mathbf{x}, t); \quad \pi = s \text{ or } f \tag{1.35}$$

Particle derivative of a material vector

Definition (1.35) allows us to write the particle derivative of the material vector $d\mathbf{x}$ in the form:

$$\frac{d^{\pi}}{dt}(d\mathbf{x}) = \frac{d^{\pi}}{dt}[(\mathbf{x}+d\mathbf{x}) - \mathbf{x}] = \mathbf{V}^{\pi}(\mathbf{x}+d\mathbf{x}, t) - \mathbf{V}^{\pi}(\mathbf{x}, t) \tag{1.36}$$

so that:

$$\frac{d^{\pi}}{dt}(d\mathbf{x}) = \nabla_x \mathbf{V}^{\pi} \cdot d\mathbf{x}; \quad \left(\nabla_x \mathbf{V}^{\pi}\right)_{ij} = \frac{\partial V_i^{\pi}}{\partial x_j}; \quad \pi = s \text{ or } f \tag{1.37}$$

Particle derivative of a material volume

Starting from (1.9), we express the particle derivative of the material volume $d\Omega_t$ in the form:

$$\frac{d^{\pi}}{dt}(d\Omega_t) = \frac{d^{\pi}}{dt}(d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3) = \frac{d^{\pi}}{dt}(d\mathbf{x}_1 \cdot (d\mathbf{x}_2 \times d\mathbf{x}_3)) \tag{1.38}$$

The linearity of the composed product $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with regard to vectors $\mathbf{v}_{i=1,2,3}$ allows us to write:

$$\begin{aligned}\frac{d^{\pi}}{dt}(d\Omega_t) = &\left(\frac{d^{\pi}}{dt}(d\mathbf{x}_1),\ d\mathbf{x}_2,\ d\mathbf{x}_3\right) \\ &+ \left(d\mathbf{x}_1,\ \frac{d^{\pi}}{dt}(d\mathbf{x}_2),\ d\mathbf{x}_3\right) + \left(d\mathbf{x}_1,\ d\mathbf{x}_2,\ \frac{d^{\pi}}{dt}(d\mathbf{x}_3)\right)\end{aligned} \tag{1.39}$$

Use of (1.37) gives:

$$\begin{aligned}\frac{d^{\pi}}{dt}(d\Omega_t) = &\left(\frac{\partial V_i^{\pi}}{\partial x_1}\,\mathbf{e}_i\,dx_1,\ d\mathbf{x}_2,\ d\mathbf{x}_3\right) \\ &+ \left(d\mathbf{x}_1,\ \frac{\partial V_i^{\pi}}{\partial x_2}\,\mathbf{e}_i\,dx_2,\ d\mathbf{x}_3\right) + \left(d\mathbf{x}_1,\ d\mathbf{x}_2,\ \frac{\partial V_i^{\pi}}{\partial x_3}\,\mathbf{e}_i\,dx_3\right)\end{aligned} \tag{1.40}$$

The product $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is zero as soon as two vectors among vectors $\mathbf{v}_{i=1,2,3}$ are colinear. Thus:

$$\frac{d^{\pi}}{dt}(d\Omega_t) = \left(\frac{\partial V_1^{\pi}}{\partial x_1} + \frac{\partial V_2^{\pi}}{\partial x_2} + \frac{\partial V_3^{\pi}}{\partial x_3}\right)(d\mathbf{x}_1,\ d\mathbf{x}_2,\ d\mathbf{x}_3) \tag{1.41}$$

or equivalently:

$$\frac{d^{\pi}}{dt}(d\Omega_t) = \left(\nabla_x \cdot \mathbf{V}^{\pi}\right) d\Omega_t; \quad \pi = s \text{ or } f \tag{1.42}$$

Particle derivative of a field

The particle derivative $d^\pi \mathcal{G}/dt$ with respect to particle π $(= s$ or $f)$ of field $\mathcal{G}(\mathbf{x}, t)$ turns out to be the time derivative of $\mathcal{G}$ when letting $\mathbf{x}$ match the successive positions $\mathbf{x}^\pi(t)$ occupied by the particle. We write:

$$\frac{d^\pi \mathcal{G}}{dt} = \frac{\partial \mathcal{G}}{\partial t} + (\nabla_x \mathcal{G}) \cdot \mathbf{V}^\pi \tag{1.43}$$

For instance, the acceleration $\boldsymbol{\gamma}^\pi$ of particle π is the particle derivative of velocity $\mathbf{V}^\pi(\mathbf{x}, t)$:

$$\boldsymbol{\gamma}^\pi = \frac{d^\pi \mathbf{V}^\pi}{dt} = \frac{\partial \mathbf{V}^\pi}{\partial t} + \left(\nabla_x \mathbf{V}^\pi\right) \cdot \mathbf{V}^\pi; \quad \gamma_i^\pi = \frac{\partial V_i^\pi}{\partial t} + \frac{\partial V_i^\pi}{\partial x_j} V_j^\pi \tag{1.44}$$

Particle derivative of a volume integral

The particle derivative applies to the volume integral of any quantity $\mathcal{G}$ according to:

$$\frac{d^\pi}{dt} \int_{\Omega_t} \mathcal{G} d\Omega_t = \int_{\Omega_t} \frac{d^\pi}{dt} (\mathcal{G} d\Omega_t) \tag{1.45}$$

Use of (1.42) and (1.43) allows us to rewrite (1.45) in the form:

$$\frac{d^\pi}{dt} \int_{\Omega_t} \mathcal{G} d\Omega_t = \int_{\Omega_t} \left(\frac{d^\pi \mathcal{G}}{dt} + \mathcal{G} \nabla_x \cdot \mathbf{V}^\pi \right) d\Omega_t \tag{1.46}$$

or, equivalently:

$$\frac{d^\pi}{dt} \int_{\Omega_t} \mathcal{G} d\Omega_t = \int_{\Omega_t} \left(\frac{\partial \mathcal{G}}{\partial t} + \nabla_x \cdot (\mathcal{G} \mathbf{V}^\pi) \right) d\Omega_t \tag{1.47}$$

Use of the divergence theorem finally provides the alternative expression:

$$\frac{d^\pi}{dt} \int_{\Omega_t} \mathcal{G} d\Omega_t = \int_{\Omega_t} \frac{\partial \mathcal{G}}{\partial t} d\Omega_t + \int_{\partial \Omega_t} \mathcal{G} \mathbf{V}^\pi \cdot \mathbf{n} \, da \tag{1.48}$$

where $\partial \Omega_t$ stands for the border of volume Ω_t, while $\mathbf{n}$ is the outward unit normal to surface da.

1.3.2 Strain Rates

The particle derivative allows the Eulerian description of the kinematics of the deformation that refers only to the current configuration. The Eulerian strain rate tensor $\mathbf{d}^\pi$ is defined by the relation:

$$\frac{d^\pi}{dt} (d\mathbf{x} \cdot d\mathbf{y}) = 2 d\mathbf{x} \cdot \mathbf{d}^\pi \cdot d\mathbf{y} \tag{1.49}$$

where $d\mathbf{x}$ and $d\mathbf{y}$ are any infinitesimal skeleton ($\pi = s$) or fluid ($\pi = f$) material vectors. Use of (1.37) into (1.49) gives:

$$\mathbf{d}^{\pi} = \frac{1}{2}\left(\nabla_x \mathbf{V}^{\pi} + {}^t\nabla_x \mathbf{V}^{\pi}\right); \quad d_{ij}^{\pi} = \frac{1}{2}\left(\frac{\partial V_i^{\pi}}{\partial x_j} + \frac{\partial V_j^{\pi}}{\partial x_i}\right) \tag{1.50}$$

Definition (1.50) of $\mathbf{d}^{\pi}$ allows us to decompose the deformation kinematics of material vector $d\mathbf{x}$ in the form:

$$\frac{d^{\pi}}{dt}(d\mathbf{x}) = \boldsymbol{\Omega}^{\pi} \cdot d\mathbf{x} + \mathbf{d}^{\pi} \cdot d\mathbf{x} \tag{1.51}$$

where $\boldsymbol{\Omega}^{\pi}$ is the rotation rate tensor attached to the antisymmetric part of $\nabla_x \mathbf{V}^{\pi}$:

$$\boldsymbol{\Omega}^{\pi} = \frac{1}{2}\left(\nabla_x \mathbf{V}^{\pi} - {}^t\nabla_x \mathbf{V}^{\pi}\right); \quad \Omega_{ij}^{\pi} = \frac{1}{2}\left(\frac{\partial V_i^{\pi}}{\partial x_j} - \frac{\partial V_j^{\pi}}{\partial x_i}\right) \tag{1.52}$$

The term $\boldsymbol{\Omega}^{\pi} \cdot d\mathbf{x}$ in (1.51) induces no strain rate since it accounts for the infinitesimal rotation of material vector $d\mathbf{x}$ according to:

$$\boldsymbol{\Omega}^{\pi} \cdot d\mathbf{x} = 2\boldsymbol{\omega}^{\pi} \times d\mathbf{x} \tag{1.53}$$

where $\boldsymbol{\omega}^{\pi}$ is the vorticity vector:

$$\boldsymbol{\omega}^{\pi} = \nabla_x \times \mathbf{V}^{\pi} \tag{1.54}$$

The Eulerian decomposition (1.51) of the kinematics of deformation is to be compared with the Lagrangian decomposition (1.22) of the deformation.

In contrast to the Eulerian approach to the kinematics of the skeleton deformation, the Lagrangian approach consists in deriving (1.19) with respect to time:[2]

$$\frac{d^s}{dt}(d\mathbf{x} \cdot d\mathbf{y}) = 2d\mathbf{X} \cdot \frac{d\boldsymbol{\Delta}}{dt} \cdot d\mathbf{Y} \tag{1.55}$$

Using transport formulae $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$ and $d\mathbf{y} = \mathbf{F} \cdot d\mathbf{Y}$, the comparison between (1.49) for $\pi = s$ and (1.55) leads to the transport formula:

$$\mathbf{d}^s = {}^t\mathbf{F}^{-1} \cdot \frac{d\boldsymbol{\Delta}}{dt} \cdot \mathbf{F}^{-1}; \quad d_{ij}^s = \frac{\partial X_k}{\partial x_i}\frac{d\Delta_{kl}}{dt}\frac{\partial X_l}{\partial x_j} \tag{1.56}$$

According to (1.26) and (1.56), the infinitesimal transformation approximation turns out to consider $\mathbf{d}^s \simeq d\boldsymbol{\varepsilon}/dt$.

[2]When taking the skeleton particle derivative of Lagrangian quantities, as for instance $d^s\boldsymbol{\Delta}/dt$, we will adopt a standard time derivative notation, such as for instance $d\boldsymbol{\Delta}/dt$ in (1.55). Indeed, there is no ambiguity since $\boldsymbol{\Delta} = \boldsymbol{\Delta}(\mathbf{X}, t)$. Furthermore, the particle derivative with respect to the fluid of a Lagrangian quantity does not generally present any physical interest.

1.4 Mass Balance

1.4.1 Equation of Continuity

Let ρ_s and ρ_f be the mesoscopic or intrinsic matrix and fluid mass densities so that $\rho_s\,(1-n)\,d\Omega_t$ and $\rho_f n\,d\Omega_t$ are respectively the skeleton mass and the fluid mass currently contained in the material volume $d\Omega_t$. Accordingly the macroscopic or apparent skeleton and fluid mass densities are respectively $\rho_s\,(1-n)$ and $\rho_f n$. When no mass change occurs, neither for the skeleton nor the fluid contained in the volume Ω_t, the mass balance can be expressed in the form:

$$\frac{d^s}{dt}\int_{\Omega_t}\rho_s\,(1-n)\;d\Omega_t = 0 \tag{1.57a}$$

$$\frac{d^f}{dt}\int_{\Omega_t}\rho_f n\,d\Omega_t = 0 \tag{1.57b}$$

Applying (1.45) and (1.47) to (1.57) we get:

$$\frac{d^s}{dt}(\rho_s\,(1-n)\,d\Omega_t) = 0 \tag{1.58a}$$

$$\frac{d^f}{dt}(\rho_f n\,d\Omega_t) = 0 \tag{1.58b}$$

and the Eulerian continuity equations:

$$\frac{\partial(\rho_s\,(1-n))}{\partial t} + \nabla_x\cdot\left(\rho_s\,(1-n)\,\mathbf{V}^s\right) = 0 \tag{1.59a}$$

$$\frac{\partial(\rho_f n)}{\partial t} + \nabla_x\cdot\left(\rho_f n\mathbf{V}^f\right) = 0 \tag{1.59b}$$

1.4.2 The Relative Flow Vector of a Fluid Mass. Filtration Vector. Fluid Mass Content

The appropriate formulation of the constitutive equations for the skeleton accounting for the skeleton–fluid couplings will require referring the motion of the fluid to the initial configuration of the skeleton. With that purpose in mind let $J_f\,da$ be the fluid mass flowing between time t and $t+dt$ through the infinitesimal skeleton material surface da oriented by the unit normal $\mathbf{n}$. We write:

$$J_f\,da = \mathbf{w}\cdot\mathbf{n}\,da \tag{1.60}$$

where $\mathbf{w}\,(\mathbf{x},t)$ is the Eulerian relative flow vector of fluid mass. Since the quantity $n(\mathbf{V}^f - \mathbf{V}^s)\cdot\mathbf{n}\,dadt$ is the infinitesimal fluid volume flowing through the skeleton surface da during the infinitesimal time dt (see Fig. 1.4), the relative vector of fluid mass $\mathbf{w}$ is consistently defined by:

$$\mathbf{w} = \rho_f\mathcal{V};\quad \mathcal{V} = n(\mathbf{V}^f - \mathbf{V}^s) \tag{1.61}$$

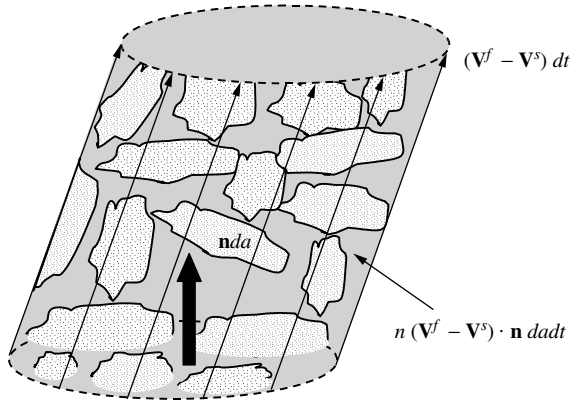

Figure 1.4: The infinitesimal fluid volume flowing through the skeleton surface da during the infinitesimal time dt. Eulerian porosity n is still to be associated with the infinitesimal fluid volume $\left(\mathbf{V}^f - \mathbf{V}^s\right)\cdot\mathbf{n}\,dadt$, and not with the skeleton infinitesimal surface da through which the fluid flows.

where $\mathcal{V}$ is the filtration vector. Use of definition (1.61) allows us to refer the fluid mass balance to the skeleton motion by rearranging the fluid continuity equation (1.59b) in the form:

$$\frac{d^s(\rho_f n)}{dt} + \rho_f n\,\nabla_x\cdot\mathbf{V}^s + \nabla_x\cdot\mathbf{w} = 0 \tag{1.62}$$

The Lagrangian approach to the fluid mass balance can be carried out by introducing the current Lagrangian fluid mass content m_f per unit of initial volume $d\Omega_0$. The latter relates to the current Eulerian fluid mass content $\rho_f n$ per unit of current volume $d\Omega_t$ according to:

$$\rho_f n\,d\Omega_t = m_f\,d\Omega_0 \tag{1.63}$$

Use of (1.17) and (1.63) gives the useful relation:

$$m_f = \rho_f\phi \tag{1.64}$$

where ϕ stands for the Lagrangian porosity (see §1.2.2). Furthermore, let $\mathbf{M}(\mathbf{X}, t)$ be the Lagrangian vector attached to the initial configuration and linked to vector $\mathbf{w}$ through the relation:

$$\mathbf{w}\cdot\mathbf{n}\,da = \mathbf{M}\cdot\mathbf{N}\,dA \tag{1.65}$$

where surfaces da and dA correspond in the skeleton deformation. Accordingly, letting $\mathbf{w} = \mathbf{v}$ and $\mathbf{M} = \mathbf{V}$ in (1.14), Eqs. (1.15) and (1.16) provide the transport formulae:

$$\mathbf{M} = J\mathbf{F}^{-1}\cdot\mathbf{w};\quad \nabla_x\cdot\mathbf{w}\,d\Omega_t = \nabla_X\cdot\mathbf{M}\,d\Omega_0 \tag{1.66a}$$

$$M_i = J\frac{\partial X_i}{\partial x_j}w_j;\quad J\frac{\partial w_i}{\partial x_i} = \frac{\partial M_i}{\partial X_i} \tag{1.66b}$$

Substitution of (1.63) and (1.66) into (1.62) premultiplied by $d\Omega_t$, and use of (1.42) with $\pi = s$, provide the Lagrangian fluid continuity equation in the form:

$$\frac{dm_f}{dt} + \nabla_X \cdot \mathbf{M} = 0; \qquad \frac{\partial m_f\ (\mathbf{X}, t)}{\partial t} + \frac{\partial M_i}{\partial X_i} = 0 \tag{1.67}$$

Analogously, the Lagrangian approach to the mass balance of the skeleton turns out to integrate (1.58a) in the form:

$$\rho_s\,(1-n)\,d\Omega_t = \rho_s^0(1-n_0)d\Omega_0 \tag{1.68}$$

where ρ_s^0 and $n_0 = \phi_0$ stand respectively for the initial matrix mass density and for the initial porosity. Use of (1.11) allows us to write:

$$m_s = m_s^0 = \rho_s^0(1-\phi_0) \tag{1.69}$$

where $m_s = J\rho_s\,(1-n)$ denotes the skeleton mass content per unit of initial volume $d\Omega_0$ and remains constantly equal to its initial value m_s^0 representing the skeleton mass density $\rho_s^0(1-\phi_0)$. Equations (1.67) and (1.69) constitute the skeleton Lagrangian alternative to the Eulerian continuity equations (1.59).

1.5 Advanced Analysis

1.5.1 Particle Derivative with a Surface of Discontinuity

Some applications involve propagation fronts across which discontinuities occur. In order to derive the particle derivative of an integral accounting for these discontinuities, let Σ be a surface of discontinuity travelling within the material volume Ω_t and subdividing the latter into two subvolumes Ω_1 and Ω_2. Let $\mathbf{n}$ be the unit normal to Σ oriented in the direction of travel towards the downstream subvolume Ω_2. Finally, let $[[\mathcal{G}]]$ denote the jump across Σ in the direction of $\mathbf{n}$ of the discontinuous quantity $\mathcal{G}$ (see Fig. 1.5):

$$[[\mathcal{G}]] = \mathcal{G}_2 - \mathcal{G}_1 \tag{1.70}$$

The normal speed of displacement $\mathbf{c} = c\,\mathbf{n}$ of the surface of discontinuity Σ is the velocity at which a geometrical point belonging to Σ moves along the normal $\mathbf{n}$. During the time duration dt the infinitesimal surface da belonging to Σ sweeps out the infinitesimal volume $\mathbf{c}\cdot\mathbf{n}\,da\,dt$. During the infinitesimal time dt the volumetric density $\mathcal{G}$ related to the latter undergoes the variation $-[[\mathcal{G}]]$. In order to account for this sudden variation, the particle derivative of a volume integral (1.48) has to be modified according to:

$$\frac{d^\pi}{dt}\int_{\Omega_\alpha} \mathcal{G}d\Omega_t = \int_{\Omega_t} \frac{\partial \mathcal{G}}{\partial t} d\Omega_t + \int_{\partial\Omega_t} \mathcal{G}\mathbf{V}^\pi \cdot \mathbf{n}\,da - \int_\Sigma [[\mathcal{G}]]\,\mathbf{c}\cdot\mathbf{n}\,da \tag{1.71}$$

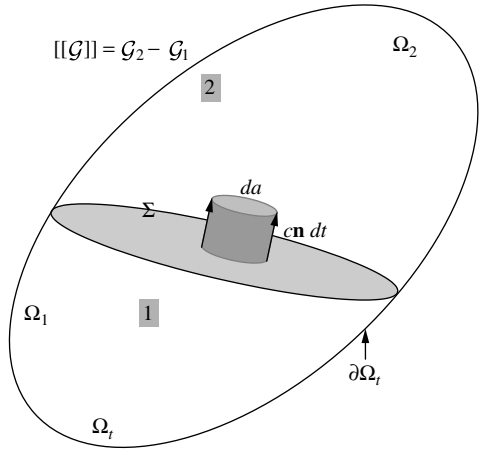

Figure 1.5: Infinitesimal volume swept out during the infinitesimal time dt by an infinitesimal surface da belonging to a surface of discontinuity Σ moving at the normal velocity of displacement $\mathbf{c} = c\,\mathbf{n}$.

Separate application of the divergence theorem to subdomains Ω_α, $\alpha = 1$ or 2, gives:

$$\int_{\Omega_\alpha} \nabla_x \cdot (\mathcal{G}\mathbf{V}^\pi) d\Omega_t = \int_{\partial\Omega_\alpha \cap \partial\Omega_t} \mathcal{G}\mathbf{V}^\pi \cdot \mathbf{n}\, da + \int_\Sigma \mathcal{G}\mathbf{V}^\pi \cdot \boldsymbol{v}_\alpha\, da; \qquad \boldsymbol{v}_1 = -\boldsymbol{v}_2 = \mathbf{n} \tag{1.72}$$

Since $\Omega_t = \Omega_1 \cup \Omega_2$ and $\partial\Omega_t = (\partial\Omega_1 \cap \partial\Omega_t) \cup (\partial\Omega_2 \cap \partial\Omega_t)$, the two previous equations lead to:

$$\frac{d^\pi}{dt}\int_{\Omega_t} \mathcal{G} d\Omega_t = \int_{\Omega_t} \left(\frac{\partial \mathcal{G}}{\partial t} + \nabla_x.(\mathcal{G}\mathbf{V}^\pi)\right) d\Omega_t + \int_\Sigma \left[\left[\mathcal{G}(\mathbf{V}^\pi - \mathbf{c})\right]\right] \cdot \mathbf{n}\, da \tag{1.73}$$

1.5.2 Mass Balance with a Surface of Discontinuity. The Rankine–Hugoniot Jump Condition

Use of (1.73) in (1.57) provides the mass balance equations accounting for a surface of discontinuity. We write:

$$\int_{\Omega_t} \left(\frac{\partial(\rho_s\,(1-n))}{\partial t} + \nabla_x \cdot \left(\rho_s\,(1-n)\,\mathbf{V}^s\right)\right) d\Omega_t + \int_\Sigma \left[\left[\rho_s\,(1-n)\,(\mathbf{V}^s - \mathbf{c})\right]\right] \cdot \mathbf{n}\, da = 0 \tag{1.74a}$$

$$\int_{\Omega_t} \left(\frac{\partial(\rho_f n)}{\partial t} + \nabla_x \cdot \left(\rho_f n \mathbf{V}^f\right)\right) d\Omega_t + \int_\Sigma \left[\left[\rho_f n(\mathbf{V}^f - \mathbf{c})\right]\right] \cdot \mathbf{n}\; da = 0 \tag{1.74b}$$

The first term in the above equations allows us to recover continuity equations (1.59), whereas the second term provides the jump or Rankine–Hugoniot conditions:

$$\left[\left[\rho_s\,(1-n)\,(\mathbf{V}^s-\mathbf{c})\right]\right]\cdot\mathbf{n}=0 \tag{1.75a}$$

$$\left[\left[\rho_f n(\mathbf{V}^f-\mathbf{c})\right]\right]\cdot\mathbf{n}=0 \tag{1.75b}$$

The jump condition means that the particles passing across the surface of discontinuity Σ do not undergo any mass change. Referring to the skeleton motion, the jump condition (1.75b) relative to the fluid is conveniently rewritten in the form:

$$\left[\left[\mathbf{w}-\rho_f n(\mathbf{c}-\mathbf{V}^s)\right]\right]\cdot\mathbf{n}=0 \tag{1.76}$$

Let us apply the jump condition to the special case of a surface of discontinuity Σ constituted by the interface between two different porous media. Assuming that the two media are perfectly bonded, $\mathbf{V}^s$ remains continuous across their interface so that the displacement speed $c=\mathbf{c}\cdot\mathbf{n}$ equals the normal skeleton velocity $\mathbf{V}^s\cdot\mathbf{n}$. Consequently, at the interface Σ of two bonded porous media, the jump condition (1.76), which eventually ensures that no liquid filtration occurs along the interface, reduces to:

$$[[\mathbf{w}]]\cdot\mathbf{n}=0 \tag{1.77}$$

During time dt the infinitesimal surface da belonging to the surface of discontinuity Σ sweeps out, in the current configuration, the skeleton material volume $(\mathbf{c}-\mathbf{V}^s)\cdot\mathbf{n}\,da\,dt$. In the meantime the infinitesimal surface dA associated with da through transport formula (1.13) sweeps out, in the initial configuration, the skeleton volume $\mathbf{C}\cdot\mathbf{N}\,dA\,dt$. Lagrangian normal speed $\mathbf{C}$ is the speed at which a geometrical point belonging to the surface of discontinuity moves between times t and $t+dt$ in the reference configuration along the normal $\mathbf{N}$. By using (1.64) and (1.65), the Eulerian jump condition (1.76) can be transported to the initial configuration to furnish the Lagrangian jump condition:

$$\left[\left[\mathbf{M}-m_f\,\mathbf{C}\right]\right]\cdot\mathbf{N}=0 \tag{1.78}$$

With the aim of giving a first illustration of the above Lagrangian approach to the Rankine–Hugoniot jump condition (see §5.4.4 for a second illustration), let us consider a porous material subjected to a dissolution process (e.g. of leaching type, see §7.1), in which the solid matrix (e.g. calcium) progressively dissolves into the interstitial solution filling the porous space. At the microscopic scale of the latter, the problem at hand involves the propagation of a surface of discontinuity in the initial configuration, even though the dissolution front representing the surface of discontinuity in the current configuration remains constantly identified with the current internal walls of the porous space. At the microscopic scale we apply (1.78) in the form:

$$\{(\mathbf{M}-m\mathbf{C})\cdot\mathbf{N}\}_{\text{intact}}=\{(\mathbf{M}-m\mathbf{C})\cdot\mathbf{N}\}_{\text{solute}} \tag{1.79}$$

where, for the sake of simplicity, we retain the same notation as that used at the macroscopic scale: $\mathbf{M}$ represents the Lagrangian relative flux vector with respect to the solid

matrix; m is the total mass per unit of initial volume of the species subjected to dissolution; $C = \mathbf{C} \cdot \mathbf{N}$ is the Lagrangian speed of propagation front Σ, which separates, in the initial configuration the still intact zone from the already dissolved one. The above equation eventually expresses the mass conservation related to the thin layer of solid matrix currently passing from the intact state, as part of the solid matrix, to the dissolved state, as contributing to the solute. On the intact side we write $\mathbf{M} = 0$ and $m = m_0$ since the still intact material, that is the current solid matrix, has no relative motion with respect to itself. Returning to the macroscopic scale, let $d\phi_{ch}/dt$ be the chemical contribution to the total rate $d\phi/dt$ of Lagrangian porosity, that is the one due only to the dissolution process irrespective of strain effects. According to the above analysis, $d\phi_{ch}/dt$ can be expressed in the form:

$$\frac{d\phi_{ch}}{dt} = \frac{1}{d\Omega_0} \int_{\Sigma} C dA \tag{1.80}$$

1.5.3 Mass Balance and the Double Porosity Network

Some materials such as rocks or concrete sometimes exhibit two very distinct porous networks. Roughly speaking, the first is formed of rounded pores, while the second is formed of penny-shaped cracks. Although these two networks can exchange fluid mass between them, their quite different geometries result in distinct evolution laws of their pore pressure and require a separate analysis. To this end let subscript 1 refer to the porous network formed of cracks and subscript 2 to the one formed of pores. While the continuity equation related to the skeleton remains unchanged, the fluid mass balance now reads:

$$\frac{d^{f_1}}{dt} \int_{\Omega_t} \rho_{f_1} n_1 \, d\Omega_t = \int_{\Omega_t} \overset{\circ}{r}_{2\to 1} \, d\Omega_t \tag{1.81a}$$

$$\frac{d^{f_2}}{dt} \int_{\Omega_t} \rho_{f_2} n_2 \, d\Omega_t = \int_{\Omega_t} \overset{\circ}{r}_{1\to 2} \, d\Omega_t \tag{1.81b}$$

where $\overset{\circ}{r}_{\alpha\to\beta}$ stands for the rate of fluid mass flowing from network α into network β, while n_α is the porosity associated with network α so that $n = n_1 + n_2$.

Mass conservation requires $\overset{\circ}{r}_{\alpha\to\beta} = -\overset{\circ}{r}_{\beta\to\alpha}$ and application of (1.45) to (1.81) gives:

$$\frac{d^{f_1}}{dt}(\rho_{f_1} n_1 \, d\Omega_t) = -\overset{\circ}{r}_{1\to 2} \, d\Omega_t; \quad \frac{d^{f_2}}{dt}(\rho_{f_2} n_2 \, d\Omega_t) = \overset{\circ}{r}_{1\to 2} \, d\Omega_t \tag{1.82}$$

Using (1.46), we obtain two separate continuity equations for the fluid flowing through the crack network and for the one flowing through the pore network:

$$\frac{\partial(\rho_{f_1} n_1)}{\partial t} + \nabla_x \cdot \left(\rho_{f_1} n_1 \mathbf{V}^{f_1}\right) = -\overset{\circ}{r}_{1\to 2} \tag{1.83a}$$

$$\frac{\partial(\rho_{f_2} n_2)}{\partial t} + \nabla_x \cdot \left(\rho_{f_2} n_2 \mathbf{V}^{f_2}\right) = \overset{\circ}{r}_{1\to 2} \tag{1.83b}$$

The Lagrangian alternative to the previous Eulerian fluid continuity equations can be written as:

$$\frac{d^s m_1}{dt} + \nabla_X \cdot \mathbf{M}^{(1)} = -\overset{\circ}{m}_{1\to 2} \tag{1.84a}$$

$$\frac{d^s m_2}{dt} + \nabla_X \cdot \mathbf{M}^{(2)} = \overset{\circ}{m}_{1\to 2} \tag{1.84b}$$

with:

$$m_\alpha = J\rho_{f_\alpha} n_\alpha = \rho_{f_\alpha}\phi_\alpha; \quad \overset{\circ}{m}_{1\to 2} = J\overset{\circ}{r}_{1\to 2} \tag{1.85}$$

The approach can be adapted in order to account for possible long-term exchanges of fluid mass between the occluded porosity, through which no fluid flow significantly occurs in the short-term range, and the connected porous network, through which the fluid flows at any time. This fluid exchange can be viewed as an exchange of mass between the skeleton, including the occluded porosity and its fluid, and still referred to by s, and the fluid saturating the porous network referred to by f. Analogously to (1.83) and (1.84) the Eulerian and Lagrangian approaches to the continuity equation respectively read:

$$\frac{\partial(\rho_s\,(1-n))}{\partial t} + \nabla_x \cdot \left(\rho_s\,(1-n)\,\mathbf{V}^s\right) = -\overset{\circ}{r}_{s\to f} \tag{1.86a}$$

$$\frac{\partial(\rho_f n)}{\partial t} + \nabla_x \cdot \left(\rho_f n \mathbf{V}^f\right) = \overset{\circ}{r}_{s\to f} \tag{1.86b}$$

and:

$$\frac{d^s m_s}{dt} = -\overset{\circ}{m}_{s\to f} \tag{1.87a}$$

$$\frac{d^s m_f}{dt} + \nabla_X \cdot \mathbf{M} = \overset{\circ}{m}_{s\to f} \tag{1.87b}$$

Similarly, consider a material subjected to a dissolution process. Let $\overset{\circ}{m}_{s\to sol}$ be the rate of solid mass (index s) which currently dissolves per unit of initial volume $d\Omega_0$ in solute form (index sol), so that the mass conservation of the solid matrix, the solute and the solvent (index w for liquid water) can be expressed in the form:

$$\frac{d^s m_s}{dt} = -\overset{\circ}{m}_{s\to sol} \tag{1.88a}$$

$$\frac{d^s m_{sol}}{dt} + \nabla_X \cdot \mathbf{M}^{sol} = \overset{\circ}{m}_{s\to sol} \tag{1.88b}$$

$$\frac{d^s m_w}{dt} + \nabla_X \cdot \mathbf{M}^w = 0 \tag{1.88c}$$

In addition, from the analysis of §1.5.2 and from (1.80) we derive:

$$\overset{\circ}{m}_{s\to sol} = \rho_s^0 \frac{d\phi_{ch}}{dt} \tag{1.89}$$