Part I

Background Material

1

Vibration of Single Degree of Freedom Systems

In this chapter, some of the basic concepts of vibration analysis for single degree of freedom (SDoF) discrete parameter systems will be introduced. The term ‘discrete (or sometimes lumped) parameter’ implies that the system is a combination of discrete rigid masses (or components) interconnected by flexible/elastic stiffness elements. Later it will be seen that a single DoF representation may be employed to describe the behaviour of a particular characteristic (or mode) shape of the system via what are known as modal coordinates. Multiple degree of freedom (MDoF) discrete parameter systems will be considered in Chapter 2. The alternative approach to modelling multiple DoF systems, as so-called ‘continuous’ systems, where components of the system are flexible and deform in some manner, is considered later in Chapters 3 and 4.

Much of the material in this introductory part of the book on vibrations is covered in detail in many other texts, such as Tse *et al.* (1978), Newland (1987), Rao (1995), Thomson (1997) and Inman (2006) and it is assumed that the reader has some engineering background so should have met many of the ideas before. Therefore, the treatment here will be as brief as is consistent with the reader being reminded, if necessary, of various concepts used later in the book. Such introductory texts on mechanical vibration should be referenced if more detail is required or if the reader’s background understanding is limited.

There are a number of ways of setting up the equations of motion for an SDoF system, e.g. Newton’s laws and D’Alembert’s principle. However, consistently throughout the book, Lagrange’s energy equations will be employed, although in one or two cases other methods are adopted as they offer certain advantages. In this chapter, the determination of the free and forced vibration response of an SDoF system to various forms of excitation relevant to aircraft loads will be considered. The idea is to introduce some of the core dynamic analysis methods (or tools) to be used later in aircraft aeroelasticity and loads calculations.

1.1 SETTING UP EQUATIONS OF MOTION FOR SINGLE DoF SYSTEMS

A single DoF system is one whose motion may be described by a single coordinate, i.e. a displacement or rotation. All systems that may be described by a single degree of freedom may be shown to have an identical form of governing equation, albeit with different symbols employed in each case. Two examples will be considered, a classical mass/spring/damper system and an aircraft control surface able to rotate about its hinge line but restrained by an actuator. These examples will illustrate translational and rotational motions.

Introduction to Aircraft Aeroelasticity and Loads J. R. Wright and J. E. Cooper

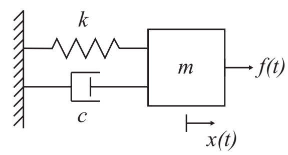

Figure 1.1 SDoF mass/spring/damper system.

1.1.1 Example: Classical Single DoF System

The classical form of an SDoF system is shown in Figure 1.1, and comprises a mass m, a spring of stiffness k and a viscous damper whose coefficient is c; a viscous damper is an idealized energy dissipation device where the force developed is linearly proportional to the relative velocity between its ends (note that the alternative approach of using hysteretic/structural damping will be considered later). The motion of the system is a function of time t and is defined by the displacement $x(t)$. A time-varying force $f(t)$ is applied to the mass.

Lagrange's energy equations are differential equations of the system expressed in what are sometimes termed 'generalized coordinates' but written in terms of energy and work quantities (Wells, 1967; Tse *et al.*, 1978). These equations will be suitable for a specific physical coordinate or a coordinate associated with a shape (see Chapter 3). Now, Lagrange's equation for an SDoF system with a displacement coordinate x may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial \Im}{\partial \dot{x}} + \frac{\partial U}{\partial x} = Q_x = \frac{\partial(\delta W)}{\partial(\delta x)}, \tag{1.1}$$

where T is the kinetic energy, U is the potential (or strain) energy, $\Im$ is the dissipative function, Q_x is the so-called generalized force and W is a work quantity.

For the SDoF example, the kinetic energy is given by

$$T = \frac{1}{2}m\dot{x}^2, \tag{1.2}$$

where the overdot indicates the derivative with respect to time, namely d/dt. The strain energy in the spring is

$$U = \frac{1}{2}kx^2. \tag{1.3}$$

The damper contribution may be treated as an external force, or else may be defined by the dissipative function

$$\Im = \frac{1}{2}c\dot{x}^2. \tag{1.4}$$

Finally, the effect of the force is included in Lagrange's equation by considering the incremental work done δW obtained when the force moves through an incremental displacement δx, namely

$$\delta W = f\,\delta x. \tag{1.5}$$

Substituting Equations (1.2) to (1.5) into Equation (1.1) yields the ordinary second-order differential equation

$$m\ddot{x} + c\dot{x} + kx = f(t). \tag{1.6}$$

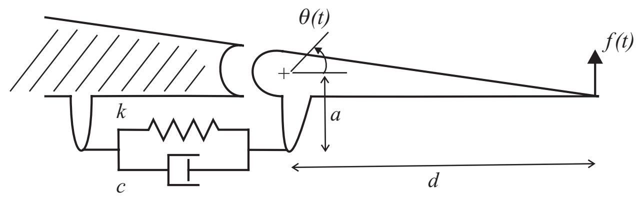

Figure 1.2 Single degree of freedom control surface/actuator system.

1.1.2 Example: Aircraft Control Surface

As an example of a completely different SDoF system that involves a rotational coordinate system, consider the control surface/actuator model shown in Figure 1.2. The control surface has a moment of inertia J about the hinge, the effective actuator stiffness and damping values are k and c respectively and the rotation of the control surface is θ rad. The actuator lever arm has length a. A force $f(t)$ is applied to the control surface at a distance d from the hinge. The main surface of the wing is assumed to be fixed rigidly as shown.

The energy, dissipation and work done functions corresponding to Equations (1.2) to (1.5) may be shown to be

$$T = \frac{1}{2}J\dot{\theta}^2, \qquad U = \frac{1}{2}k(a\theta)^2, \qquad \Im = \frac{1}{2}c(a\dot{\theta})^2, \qquad \delta W = (f\,d)\,\delta\theta, \tag{1.7}$$

where the angle of rotation is assumed to be small, so that, for example, $\sin\theta = \theta$. The work done term is a torque multiplied by a rotation. Then, applying the Lagrange equation with coordinate θ, it may be shown that

$$J\ddot{\theta} + ca^2\dot{\theta} + ka^2\theta = d\,f(t). \tag{1.8}$$

Clearly, this equation is of the same form as that in Equation (1.6). All SDoF systems have equations of a similar form, albeit with different symbols and units.

1.2 FREE VIBRATION OF SINGLE DoF SYSTEMS

In free vibration, an initial condition is imposed and motion then occurs in the absence of any external force. The motion takes the form of a nonoscillatory or oscillatory decay; the latter corresponds to the low values of damping normally encountered in aircraft, so only this case will be considered. The solution method is to assume a form of motion given by

$$x(t) = X\mathrm{e}^{\lambda t}, \tag{1.9}$$

where X is the amplitude and λ is the characteristic exponent defining the decay. Substituting Equation (1.9) into Equation (1.6), setting the applied force to zero and simplifying, yields the quadratic equation

$$\lambda^2 m + \lambda c + k = 0 \tag{1.10}$$

The solution of this ‘characteristic equation’ for the oscillatory motion case produces two complex roots, namely

$$\lambda_{1,2} = -\frac{c}{2m} \pm \mathrm{i}\sqrt{\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2}, \tag{1.11}$$

where the complex value $\mathrm{i} = \sqrt{-1}$. Equation (1.11) may be rewritten in the nondimensional form

$$\lambda_{1,2} = -\zeta\omega_n \pm \mathrm{i}\omega_\mathrm{n}\sqrt{1-\zeta^2} = -\zeta\omega_\mathrm{n} \pm \mathrm{i}\omega_\mathrm{d}, \tag{1.12}$$

where

$$\omega_\mathrm{n} = \sqrt{\frac{k}{m}}, \qquad \omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}, \qquad \zeta = \frac{c}{2m\omega_\mathrm{n}}. \tag{1.13}$$

Here ω_n is the (undamped) natural frequency (frequency in rad/s of free vibration in the absence of damping), ω_d is the damped natural frequency (frequency of free vibration in the presence of damping) and ζ is the *damping ratio* (i.e. c expressed as a proportion of the critical damping c_crit, the value at which motion becomes nonoscillatory); these parameters are basic and unique properties of the system.

Because there are two roots to Equation (1.10), the solution for the free vibration motion is given by the sum

$$x(t) = X_1 \mathrm{e}^{\lambda_1 t} + X_2 \mathrm{e}^{\lambda_2 t}. \tag{1.14}$$

After substitution of Equation (1.12) into Equation (1.14), the motion may be expressed in the form

$$x(t) = \mathrm{e}^{-\zeta\omega_\mathrm{n} t}\left[(X_1 + X_2)\cos\omega_\mathrm{d} t + \mathrm{i}(X_1 - X_2)\sin\omega_\mathrm{d} t\right]. \tag{1.15}$$

Since the displacement must be a real quantity, then X_1, X_2 must be complex conjugate pairs and Equation (1.15) simplifies to one of the classical forms

$$x(t) = \mathrm{e}^{-\zeta\omega_\mathrm{n} t}\left[A_1 \sin\omega_\mathrm{d} t + A_2 \cos\omega_\mathrm{d} t\right] \qquad \text{or} \qquad x(t) = A\mathrm{e}^{-\zeta\omega_\mathrm{n} t}\sin(\omega_\mathrm{d} t + \psi), \tag{1.16}$$

where the amplitude A and phase ψ (or amplitudes A_1, A_2) are unknown values, to be determined from the initial conditions for displacement and velocity. Thus this 'underdamped' motion is sinusoidal with an exponentially decaying envelope, as shown in Figure 1.3 for a case with general initial conditions.

1.2.1 Example: Aircraft Control Surface

Using Equation (1.8) for the control surface actuator system and comparing the expressions with those for the simple system, the undamped natural frequency and damping ratio may be shown by inspection to be

$$\omega_\mathrm{n} = \sqrt{\frac{ka^2}{J}} \qquad \text{and} \qquad \zeta = \frac{\mathrm{ca}}{2\sqrt{\mathrm{kJ}}}. \tag{1.17}$$

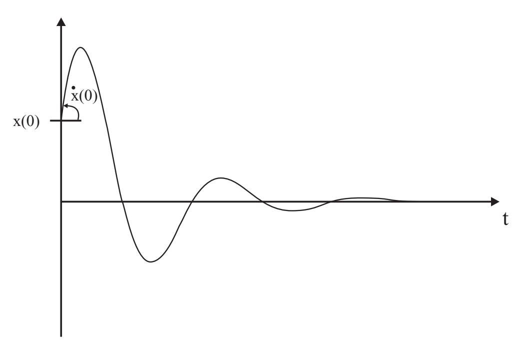

Figure 1.3 Free vibration response for an underdamped single degree of freedom system.

1.3 FORCED VIBRATION OF SINGLE DoF SYSTEMS

In determining aircraft loads for gusts and manoeuvres (see Chapters 13 to 17), the aircraft response to a number of different types of forcing functions needs to be considered. These tend to divide into three categories:

1. *Harmonic excitation* is primarily concerned with excitation at a single frequency (for engine or rotor out-of-balance and as a constituent part of continuous turbulence analysis).
2. *Nonharmonic deterministic excitation* includes the '1-cosine' input (for discrete gusts or runway bumps) and various shaped inputs (for flight manoeuvres); such forcing functions often have clearly defined analytical forms and tend to be of short duration, often called transient.
3. *Random excitation* includes continuous turbulence and runway profiles, the latter required for taxiing. Random excitation can be specified using a time or frequency domain description (see later).

The aircraft dynamics are sometimes nonlinear (e.g. doubling the input does not double the response), which complicates the solution process, but in this chapter only the linear case will be considered. The treatment of nonlinearity will be covered in later chapters, albeit only fairly briefly. In the following sections, the determination of the response to harmonic, transient and random excitation will be considered, using both time and frequency domain approaches. The extension to MDoF systems will be covered later in Chapter 2.

1.4 HARMONIC FORCED VIBRATION – FREQUENCY RESPONSE FUNCTIONS

The most important building block for forced vibration requires determination of the response to a harmonic (i.e. sinusoidal) force with frequency ω rad/s (or $\omega/(2\pi)$ Hz). The relevance to aircraft loads is primarily in helping to lay important foundations for behaviour of dynamic systems, e.g. continuous turbulence analysis. However, the real-life cases of engines or rotors and propellers can introduce harmonic excitation to the aircraft.

1.4.1 Response to Harmonic Excitation

When a harmonic force is applied, there is an initial transient response, followed by a steady-state phase where the response will also be sinusoidal at the same frequency as the excitation but lagging it in phase; only the steady-state response will be considered here, though the transient response may often be important.

The excitation input is defined by

$$f(t) = F \sin \omega t \tag{1.18}$$

and the steady-state response is given by

$$x(t) = X \sin(\omega t - \phi), \tag{1.19}$$

where F, X are the amplitudes and ϕ is the amount by which the response 'lags' the excitation in phase (so-called 'phase lag'). In one approach, the steady-state response may be determined by substituting these expressions into the equation of motion and then equating sine and cosine terms using trigonometric expansion.

However, an alternative approach uses complex algebra and will be adopted since it is more powerful and commonly used. In this approach, the force and response are rewritten in a complex notation as follows:

$$\begin{aligned} f(t) &= F\mathrm{e}^{\mathrm{i}\omega t} = F\cos\omega t + \mathrm{i}F\sin\omega t, \\ x(t) &= X\mathrm{e}^{\mathrm{i}(\omega t-\phi)} = (X\mathrm{e}^{-\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}\omega t} = \tilde{X}\mathrm{e}^{\mathrm{i}\omega t} = \tilde{X}\cos\omega t + \mathrm{i}\tilde{X}\sin\omega t. \end{aligned} \tag{1.20}$$

Here the phase lag is embedded in a new complex amplitude quantity $\tilde{X}$. Only the imaginary part of the excitation and response will be used for sine excitation; an alternative way of viewing this is that the solutions for both the sine and cosine excitation will be found simultaneously. The solution process is straightforward once the concepts have been grasped. The complex expressions in Equations (1.20) are now substituted into Equation (1.6). Noting that $\dot{x} = \mathrm{i}\omega\tilde{X}\mathrm{e}^{\mathrm{i}\omega t}$ and $\ddot{x} = -\omega^2\tilde{X}\mathrm{e}^{\mathrm{i}\omega t}$ and cancelling the exponential term, then

$$(-\omega^2 m + \mathrm{i}\omega c + k)\tilde{X} = F. \tag{1.21}$$

Thus the complex response amplitude may be solved algebraically so that

$$\tilde{X} = X\mathrm{e}^{-\mathrm{i}\phi} = \frac{F}{k - \omega^2 m + \mathrm{i}\omega c} \tag{1.22}$$

and equating real and imaginary parts from the two sides of the equation yields the amplitude and phase as

$$X = \frac{F}{\sqrt{\left(k-\omega^2 m\right)^2 + (\omega c)^2}} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{\omega c}{k - \omega^2 m}\right). \tag{1.23}$$

Hence, the time response may be calculated using X, ϕ from this equation.

1.4.2 Frequency Response Functions (FRFs)

An alternative way of writing Equation (1.22) is

$$H_{\mathrm{D}}(\omega) = \frac{\tilde{X}}{F} = \frac{1}{k - \omega^2 m + \mathrm{i}\omega c} \tag{1.24}$$

or in nondimensional form

$$H_{\mathrm{D}}(\omega) = \frac{1/k}{1 - (\omega/\omega_{\mathrm{n}})^2 + \mathrm{i}2\zeta\,(\omega/\omega_{\mathrm{n}})} = \frac{1/k}{1 - r^2 + \mathrm{i}2\zeta r} \quad \text{where} \quad r = \frac{\omega}{\omega_{\mathrm{n}}}. \tag{1.25}$$

Here $H_{\mathrm{D}}(\omega)$ is known as the displacement (or receptance, (Ewins, 1995)) frequency response function (FRF) of the system and is a system property. It dictates how the system behaves under harmonic excitation at any frequency. The equivalent velocity and acceleration FRFs are given by

$$H_{\mathrm{V}} = \mathrm{i}\omega H_{\mathrm{D}}, \qquad H_{\mathrm{A}} = -\omega^2 H_{\mathrm{D}} \tag{1.26}$$

since multiplication by $\mathrm{i}\omega$ in the frequency domain is equivalent to differentiation in the time domain ($\mathrm{i}^2 = -1$).

The quantity $kH_{\mathrm{D}}(\omega)$ is a nondimensional expression, or dynamic magnification, relating the dynamic amplitude to the static deformation for several damping values. The well-known ‘resonance’

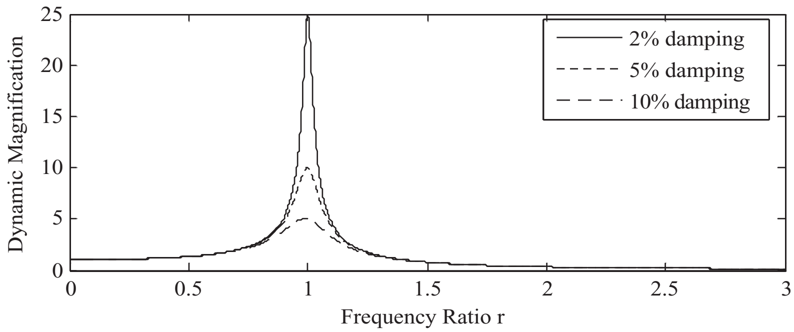

Figure 1.4 Displacement frequency response function for a single degree of freedom system.

phenomenon is shown in Figure 1.4 by the amplitude peak that occurs when the excitation frequency ω is at the 'resonance' frequency, close in value to the undamped natural frequency ω_n; the phase changes rapidly in this region, passing through 90° at resonance. Note that the resonant peak increases in amplitude as the damping ratio reduces and that the dynamic magnification (approximately $1/2\zeta$) can be extremely large.

1.4.3 Hysteretic (or Structural) Damping

So far, a viscous damping representation has been employed, based on the assumption that the damping force is proportional to velocity (and therefore to frequency). However, in practice, damping measurements in structures and materials have sometimes shown that damping is independent of frequency but acts in quadrature (i.e. is at 90° phase) to the displacement of the system. Such an internal damping mechanism is known as hysteretic (or sometimes structural) damping (Rao, 1995). It is common practice to combine the damping and stiffness properties of a system having hysteretic damping into a so-called complex stiffness, namely

$$k^* = k(1 + \mathrm{i}g), \tag{1.27}$$

where g is the structural damping coefficient or loss factor (not to be confused with the same symbol used for acceleration due to gravity) and the complex number indicates that the damping force is in quadrature with the stiffness force. The SDoF equation of motion amended to employ hysteretic damping may then be written as

$$m\ddot{x} + k(1 + \mathrm{i}g)x = f(t). \tag{1.28}$$

This is a rather peculiar equation, being expressed in the time domain but including the complex number; it is not possible to solve this equation in this form. However, it is feasible to write the equation in the time domain as

$$m\ddot{x} + c_{\mathrm{eq}}\dot{x} + kx = f(t), \tag{1.29}$$

where $c_{\mathrm{eq}} = gk/\omega$ is the equivalent viscous damping. This equation of motion may be used if the motion is dominantly at a single frequency. The equivalent damping ratio expression may be shown to be

$$\zeta_{\mathrm{eq}} = \frac{g}{2}\left(\frac{\omega_n}{\omega}\right) \tag{1.30}$$

or, if the system is actually vibrating at the natural frequency, then

$$\zeta_{\mathrm{eq}} = \frac{g}{2} \tag{1.31}$$

Thus the equivalent viscous damping ratio is half of the loss factor, and this factor of 2 is often seen when comparing flutter damping plots from the US and Europe (see Chapter 11).

An alternative way of considering hysteretic damping is to convert Equation (1.28) into the frequency domain, using the methodology employed earlier in Section 1.4.1, so yielding the FRF in the form

$$H_{\mathrm{D}}(\omega) = \frac{\tilde{X}}{F} = \frac{1}{k(1+\mathrm{i}g) - \omega^2 m} \tag{1.32}$$

and now the complex stiffness takes a more suitable form. Thus, a frequency domain solution of a system with hysteretic damping is acceptable, but a time domain solution assumes motion at essentially a single frequency. The viscous damping model, despite its drawbacks, does lend itself to more simple analysis, though both viscous and hysteretic damping models are widely used.

1.5 TRANSIENT/RANDOM FORCED VIBRATION – TIME DOMAIN SOLUTION

When a transient/random excitation is present, the time response may be calculated in one of three ways.

1.5.1 Analytical Approach

If the excitation is deterministic, having a relatively simple mathematical form (e.g. step, ramp), then an analytical method suitable for ordinary differential equations may be used (i.e. combination of complementary function and particular integral). Such an approach is impractical for more general forms of excitation. For example, a *unit step* force applied to the system initially at rest may be shown to give rise to the response (or so-called 'step response function') $s(t)$.

$$s(t) = x_{\mathrm{SRF}}(t) = \frac{1}{k}\left[1 - \frac{\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t}}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathrm{d}}t + \psi)\right] \qquad \text{with } \tan\psi = \frac{\sqrt{1-\zeta^2}}{\zeta}. \tag{1.33}$$

Note that the term in square brackets is the ratio of the dynamic-to-static response and this ratio is shown in Figure 1.5 for different dampings. Note that there is a tendency of the transient response to 'overshoot' the steady-state value, but this initial peak response is hardly affected by damping; this behaviour will be referred to later as 'dynamic overswing' when considering manoeuvres in Chapters 13 and 24.

Another important excitation is the *unit impulse* of force. This may be idealized crudely as a very narrow rectangular force–time pulse of unit area (i.e. strength) of 1 N s (the ideal impulse is the so-called Dirac-δ function, having zero width and infinite height). Because this impulse imparts an instantaneous change in momentum, the velocity changes by an amount equal to the impulse strength/mass), so the case is equivalent to free vibration with a finite initial velocity and zero initial displacement. Thus it may be shown that the response to a unit impulse (or the so-called 'impulse response function') $h(t)$ is

$$h(t) = x_{\mathrm{IRF}}(t) = \frac{1}{m\omega_{\mathrm{d}}}\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t}\sin\omega_{\mathrm{d}}t. \tag{1.34}$$

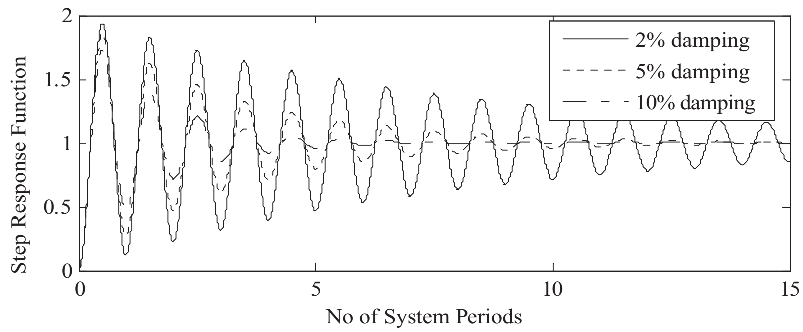

Figure 1.5 Dynamic-to-static ratio of step response for a single degree of freedom system.

The impulse response function (IRF) is shown plotted against nondimensional time for several dampings in Figure 1.6; the response starts and ends at zero. The y axis values depend upon the mass and natural frequency. The IRF may be used in the convolution approach described in Section 1.5.3.

1.5.2 Principle of Superposition

The principle of superposition, only valid for linear systems, states that if the responses to separate forces $f_1(t)$ and $f_2(t)$ are $x_1(t)$ and $x_2(t)$ respectively, then the response $x(t)$ to the sum of the forces $f(t) = f_1(t) + f_2(t)$ will be the sum of their individual responses, namely $x(t) = x_1(t) + x_2(t)$.

1.5.3 Example: Single Cycle of Square Wave Excitation – Response Determined by Superposition

Consider an SDoF system with an effective mass of 1000 kg, natural frequency 2 Hz and damping 5 % excited by a transient excitation consisting of a single cycle of a square wave with amplitude A and period τ_{square}. The response may be found by superposition of a step input of amplitude 1000 N at $t = 0$, a negative step input of amplitude 2000 N at $t = \tau_{\text{square}}/2$ and a single positive step input of amplitude 1000 N at $t = \tau_{\text{square}}$, as illustrated in Figure 1.7. The response may be calculated using the MATLAB program in appendix G in the companion website.

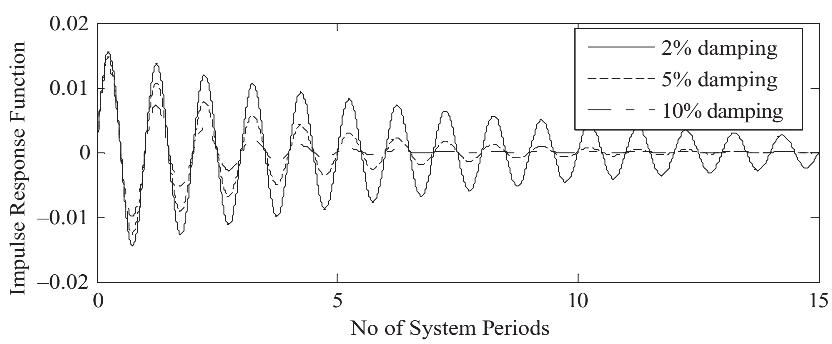

Figure 1.6 Impulse response function for a single degree of freedom system.

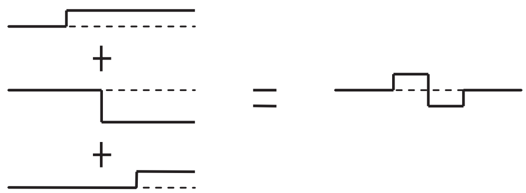

Figure 1.7 Single cycle of a square wave described by the principle of superposition.

Figure 1.8 shows the response when $\tau_{\mathrm{square}} = 0.5$ s, the period of the system; the dashed line shows the time scale of the input. In this case, the square wave pulse is nearly 'tuned' to the system (i.e. near to the resonance frequency) and so the response is significantly larger (by almost a factor of 2) than for a single on/off pulse. This is the reason why the number of allowable pilot control input reversals in a manoeuvre is strictly limited.

1.5.4 Convolution Approach

The principle of superposition illustrated above may be employed in the solution of the response to general transient/random excitation. The idea here is that a general excitation input may be represented by a sequence of very narrow (ideal) impulses of different heights (and therefore strengths), as shown in Figure 1.9. A typical impulse occurring at time $t = \tau$ is of height $f(\tau)$ and width $\mathrm{d}\tau$. Thus the corresponding impulse strength is $f(\tau)\,d\tau$ and the response to this impulse, using the unit impulse response function in Equation (1.34), is

$$x_\tau(t) = \{f(\tau)\,\mathrm{d}\tau\}\ h(t-\tau) = \frac{\{f(\tau)\,\mathrm{d}\tau\}}{m\omega_{\mathrm{n}}}\mathrm{e}^{-\zeta\omega_{\mathrm{n}}(t-\tau)}\sin\omega_{\mathrm{d}}(t-\tau) \quad \text{for} \quad t \geq \tau,$$
$$x_\tau(t) = 0 \quad \text{for} \quad t < \tau. \tag{1.35}$$

Note that the response is only nonzero *after* the impulse at $t = \tau$. The response to the entire excitation time history is equal to the summation of the responses to all the constituent impulses. Given that each impulse is $\mathrm{d}\tau$ wide, and allowing $\mathrm{d}\tau \to 0$, then the summation effectively becomes an integral, given by

$$x(t) = \int_{\tau=0}^{t} f(\tau)\,h(t-\tau)\,\mathrm{d}\tau. \tag{1.36}$$

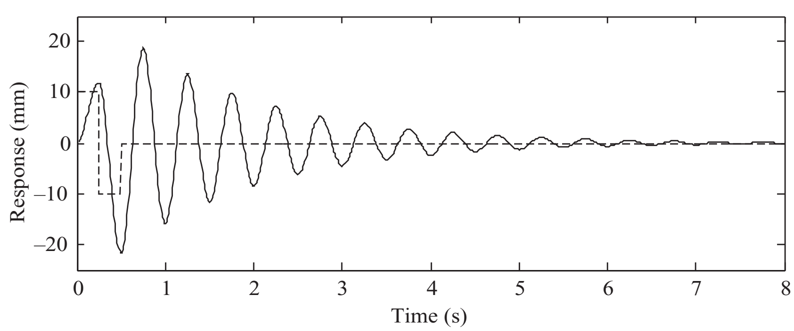

Figure 1.8 Response to a single cycle of square wave, using superposition.

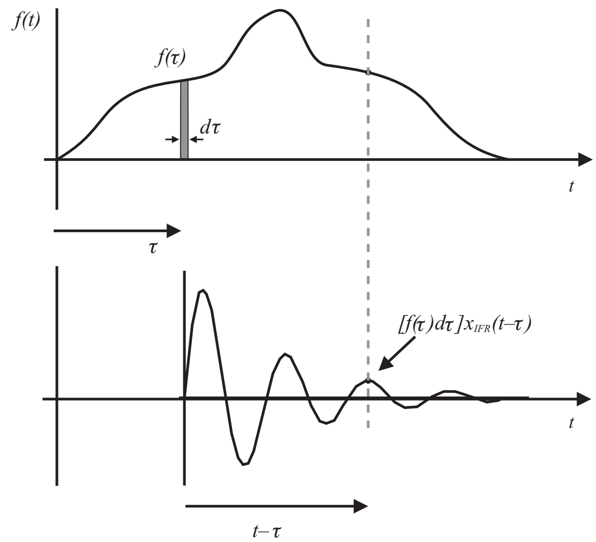

Figure 1.9 Convolution process.

This is known as the convolution integral (Newland, 1989; Rao, 1995) or, alternatively, the Duhamel integral (Fung, 1969). A shorthand way of writing this integral, where $*$ denotes convolution, is

$$x(t) = h(t) * f(t). \tag{1.37}$$

An alternative form of the convolution process may be written by treating the excitation as a combination of on/off steps and using the step response function $s(t)$, thus yielding a similar convolution expression (Fung, 1969)

$$x(t) = f(t)s(0) + \int_{\tau=0}^{t} f(\tau)\,\frac{\mathrm{d}s}{\mathrm{d}t}(t-\tau)\,\mathrm{d}\tau. \tag{1.38}$$

This form of convolution will be encountered in Chapters 10 and 16 for unsteady aerodynamics and gusts.

In practice, the convolution integrations would be performed numerically and not analytically. Thus the force input and impulse (or step) response function would need to be obtained in discrete, and not continuous, time form. The impulse response function may in fact be obtained numerically via the inverse Fourier transform of the frequency response function (see later).

1.5.5 Direct Solution of ODEs

An alternative approach for solving the ordinary differential equation, not requiring a closed form solution or performing a convolution, is to employ a numerical integration approach such as the Runge–Kutta or Newmark-β algorithms (Rao, 1995). To present one or both of these algorithms in detail is beyond the scope of this book. Suffice it to say that, knowing the response at the jth time value, the differential equation expressed at the $(j+1)$th time value is used, together with some assumption for the variation of the response within the step length, to predict the response at the $(j+1)$th time value.

In this book, time responses are sometimes calculated using numerical integration in the SIMULINK package called from a MATLAB program. The idea is illustrated using the earlier superposition example.

1.5.6 Example: Single Cycle of Square Wave Excitation – Response Determined by Numerical Integration

Consider again the SDoF system excited by the single square wave cycle as used in Section 1.5.3. The response may be found using numerical integration and may be seen to overlay the exact result in Figure 1.8 provided an adequately small step size is used (typically at least 30 points per cycle). The response is calculated using a Runge–Kutta algorithm in a MATLAB/SIMULINK program (see companion website).

1.6 TRANSIENT FORCED VIBRATION – FREQUENCY DOMAIN SOLUTION

The analysis leading up to the definition of the frequency response function in Section 1.4 considered only the response to an excitation input comprising a single sine wave at frequency ω rad/s. However, if the excitation was made up of several sine waves with different amplitudes and frequencies, the total steady-state response could be found by superposition of the responses to each individual sine wave, using the appropriate value of the FRF at each frequency. Again, because superposition is used, the approach only applies for linear systems.

1.6.1 Analytical Fourier Transform

In practice the definition of the FRF may be extended to cover a more general excitation by employing the Fourier transform (FT), so that

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{\text{Fourier transform of } x(t)}{\text{Fourier transform of } f(t)}, \tag{1.39}$$

where, for example, $X(\omega)$, the Fourier transform of $x(t)$, is given for a continuous signal by

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t. \tag{1.40}$$

The Fourier transform $X(\omega)$ is a complex function of frequency (i.e. spectrum) whose real and imaginary parts define the magnitude of the components of $\cos\omega t$ and $-\sin\omega t$ in the signal $x(t)$. The units of $X(\omega)$, $F(\omega)$ in this definition are typically m s and N s and the units of $H(\omega)$ are m/N. The inverse Fourier transform (IFT), not defined here, allows the frequency function to be transformed back into the time domain.

Although the Fourier transform is initially defined for an infinite continuous signal, and this would appear to challenge its usefulness, in practice inputs of finite length T may be used with the definition

$$X(\omega) = \frac{1}{T}\int_{0}^{T} x(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t. \tag{1.41}$$

In this case, the units of $X(\omega)$, $F(\omega)$ become m and N respectively, while units of $H(\omega)$ remain m/N. What is being assumed by using this expression is that $x(t)$ is in effect periodic with period T; i.e. the signal keeps repeating itself in a cyclic manner. Provided there is no discontinuity between the start and end of $x(t)$, then the analysis may be applied for a finite length excitation such as a pulse. If a discontinuity does exist, then a phenomenon known as 'leakage' occurs and additional incorrect Fourier amplitude components are introduced to represent the discontinuity; in practice, window functions (e.g.

Hanning, Hamming, etc.) are often applied to minimize this effect (Newland, 1987). The choice of the parameters in the analysis must be made carefully to minimize this error.

1.6.2 Discrete Fourier Transform

When using the Fourier transform in solving real problems, the discrete version (as opposed to the analytical version above) must be used. A detailed discussion of this is beyond the scope of the book, and other references should be studied, but some of the ideas will be seen in the example to follow Section 1.6.3 and in the MATLAB program (see companion website). In summary, the data record of length T $(= N\Delta t)$ is represented by a sequence $\{x(j\Delta t),\ j = 0, 1, 2, \ldots, (N-1)\Delta t\}$, with N (usually a power of 2) values at equal time intervals Δt s.

The resulting discrete Fourier transform (or DFT) is a sequence of discrete frequency domain values $X((j-1)\Delta f),\ \ j = 1, 2, \ldots, N/2+1$, i.e. from DC (zero frequency) to the so-called Nyquist frequency $(f_{\mathrm{Nyq}} = 1/(2\Delta t))$ at frequency intervals of $\Delta f = 1/T$. The values at the DC and Nyquist frequency are real but all the remaining values are complex, catering for the cosine and sine components.

It should be emphasized that it is important to understand the way in which the data are handled when performing forward and inverse transforms and this is well worth checking, e.g. using a simple case with a limited number of sine or cosine components and data points. Typically, the Fourier transform in the frequency domain is stored in a vector of N numbers (mostly complex), namely

$$\{X(0)X(\Delta f)X(2\Delta f)\cdots X(f_{\mathrm{Nyq}}-\Delta f)X(f_{\mathrm{Nyq}})X^*(f_{\mathrm{Nyq}}-\Delta f)\cdots X^*(2\Delta f)X^*(\Delta f)\}. \qquad (1.42)$$

It can be seen that the so-called ‘negative’ frequency values are conjugates of the positive frequency values (i.e. have the opposite sign for the imaginary parts, shown by $*$, which is not to be confused with convolution). They are stored further along the transform vector in the reverse direction. Thus the additional complex numbers provide no extra information, but when using numerical functions or subroutines to carry out the inverse transform to return to the time domain, it is essential to retain the data in this form. Again, a simple check may prevent considerable difficulty and possibly error later on.

1.6.3 Frequency Domain Response – Excitation Relationship

It may be seen that rearranging Equation (1.39) leads to

$$X(\omega) = H(\omega)\ F(\omega) \qquad (1.43)$$

and it is interesting to relate this to the time domain convolution Equation (1.37). The FRF and IRF are in fact Fourier transform pairs, e.g. the FRF is the Fourier transform of the IRF. Further, it may also be shown that by taking the Fourier transform of both sides of Equation (1.37), then Equation (1.43) results, i.e. convolution in the ‘time domain’ is equivalent to multiplication in the ‘frequency domain’. The extension of this approach for an MDoF system will be considered in Chapter 2.

A useful feature of Equation (1.43) is that it may be used to determine the response of a system, given the excitation time history, by going via a frequency domain route. Thus the response $x(t)$ of a linear system to a finite length transient excitation input $f(t)$ may be found by the following procedure (taking care over data storage):

1. Fourier transform $f(t)$ to find $F(\omega)$.
2. Determine the FRF $H(\omega)$ for the system.
3. Multiply the FRF and $F(\omega)$ using Equation (1.43) to determine $X(\omega)$.
4. Inverse Fourier transform $X(\omega)$ to find $x(t)$.

1.6.4 Example: Single Cycle of Square Wave Excitation – Response Determined via Fourier Transform

Consider again the SDoF system excited by a single square wave cycle as used in Section 1.5.3. The response is calculated using a MATLAB program (see companion website). Note that only a limited number of data points are used in order to allow the discrete values in the frequency and time domains to be seen; only discrete data points are plotted in the frequency domain functions. The results agree well with those in Figure 1.8 but the accuracy would improve as more data points were used to represent the signals.

1.7 RANDOM FORCED VIBRATION – FREQUENCY DOMAIN SOLUTION

There are two cases in aircraft loads where response to a random-type excitation is required: flying through continuous turbulence and taxiing on a runway with a nonsmooth profile. For continuous turbulence, it is normal practice to use a spectral approach based on a linearized model of the aircraft (see Chapter 16). When the effect of significant nonlinearity is to be explored, a time domain computation would need to be used. However, for taxiing (see Chapter 17), the solution would be carried out in the time domain using numerical integration of the equations of motion, as they are highly nonlinear due to the presence of the landing gear.

When a random excitation is considered, then a statistical approach is normally employed by defining the so-called power spectral density (PSD) of the excitation and response (Newland, 1987; Rao, 1995). For example, the PSD of $x(t)$ is defined by

$$S_{xx}(\omega) = \frac{T}{2\pi} X(\omega)^* X(\omega) = \frac{T}{2\pi} \left| X(\omega)^2 \right|, \tag{1.44}$$

where * denotes the complex conjugate (not to be confused with convolution). Thus the PSD is essentially proportional to the modulus squared of the Fourier amplitude at each frequency and would have units of density (m^2/rad s if $x(t)$ were a displacement). It is a measure of how the 'power' in $x(t)$ is distributed over the frequency range of interest. In practice, the PSD of a time signal could be computed from a long data record by employing some form of averaging of finite length segments of the data.

If Equation (1.43) is multiplied on both sides by its complex conjugate then

$$X(\omega)\, X^*(\omega) = H(\omega)F(\omega)\;\; H^*(\omega)F^*(\omega) = |H(\omega)|^2\;\; F(\omega)\, F^*(\omega) \tag{1.45}$$

and if the relevant scalar factors present in Equation (1.44) are accounted for, then Equation (1.45) becomes

$$S_{xx}(\omega) = |H(\omega)|^2\;\; S_{FF}(\omega). \tag{1.46}$$

Thus, knowing the definition of the excitation PSD $S_{FF}(\omega)$ (units N^2/rad s for force), the response PSD may be determined given the FRF for the system (m/N for displacement per force). It may be seen from Equation (1.46) that the spectral shape of the excitation is carried through to the response, but is filtered by the system dynamic characteristics. The extension of this approach for an MDoF system will be considered in Chapter 2. This relationship between the response and excitation PSDs is useful but phase information is lost.

In the analysis shown so far, the PSD $S_{xx}(\omega)$, for example, has been 'two-sided' in that values exist at both positive and negative frequencies; the latter are somewhat artificial but derive from the mathematics in that a positive frequency corresponds to a vector rotating anticlockwise at ω, whereas a negative frequency corresponds to rotation in the opposite direction. However, in practice the 'two-sided' (or double-sided) PSD is often converted into a 'one-sided' (or single-sided) function $\Phi_{xx}(\omega)$, existing only at nonnegative frequencies and calculated using

$$\Phi_{xx}(\omega) = 2S_{xx}(\omega), \qquad 0 \le \omega < \infty. \tag{1.47}$$

Single-sided spectra are in fact used in determining the response to continuous turbulence considered in Chapter 16, since the continuous turbulence PSD is defined in this way.

The mean square value is the corresponding area under the one-sided or two-sided PSD, so

$$\overline{x^2} = \int_0^{+\infty} \Phi_{xx}(\omega)\,\mathrm{d}\omega \qquad \text{or} \qquad \overline{x^2} = \int_{-\infty}^{+\infty} S_{xx}(\omega)\,\mathrm{d}\omega, \tag{1.48}$$

where clearly only a finite, not infinite, frequency range is used in practice. The root-mean-square value is the square root of the mean-square value.

1.8 EXAMPLES

Note that these examples may be useful preparation when carrying out the examples in later chapters.

1. An avionics box may be idealized as an SDoF system comprising a mass m supported on a mounting base via a spring k and damper c. The system displacement is $y(t)$ and the base displacement is $x(t)$. The base is subject to acceleration $\ddot{x}(t)$ from motion of the aircraft. Show that the equation of motion for the system may be written in the form $m\ddot{z} + c\dot{z} + kz = -m\ddot{x}(t)$ where $z = y - x$ is the relative displacement between the mass and the base (i.e. spring extension).
2. In a flutter test, the acceleration of an aircraft control surface following an explosive impact decays to a quarter of its amplitude after 5 cycles, which corresponds to an elapsed time of 0.5 s. Estimate the undamped natural frequency and the percentage of critical damping. [10 Hz, 4.4 %]
3. Determine an expression for the response of a single degree of freedom undamped system undergoing free vibration following an initial condition of zero velocity and displacement x_0.
4. Determine an expression for the time t_p at which the response of a damped SDoF system to excitation by a step force F_0 reaches a maximum [$\omega_n t_p = \pi/\sqrt{1-\zeta^2}$]. Show that the maximum response is given by the expression $xk/F_0 = 1 + \exp(-\zeta\pi/\sqrt{1-\zeta^2})$, noting the insensitivity to damping at low values.
5. Using the complex algebra approach for harmonic excitation and response, determine an expression for the transmissibility (i.e. system acceleration per base acceleration) for the base excited system in Example 1.
6. A motor mounted in an aircraft on four antivibration mounts may be idealized as an SDoF system of effective mass 20 kg. Each mount has a stiffness of 5000 N/m and a damping coefficient of 200 N s/m. Determine the natural frequency and damping ratio of the system. Also, estimate the displacement and acceleration response of the motor when it runs with a degree of imbalance equivalent to a sinusoidal force of ±40 N at 1200 rpm (20 Hz). Compare this displacement value to the static deflection of the motor on its mounts. [5.03 Hz, 63.2%, 0.128 mm, 2.02 m/s^2, 9.8 mm]
7. A machine of mass 1000 kg is supported on a spring/damper arrangement. In operation, the machine is subjected to a force of 750 cos ωt, where ω (rad/s) is the operating frequency. In an experiment, the operating frequency is varied and it is noted that resonance occurs at 75 Hz and that the magnitude of the FRF is 2.5. However, at its normal operating frequency this value is found to be 0.7. Find the normal

operating frequency and the support stiffness and damping coefficient. [118.3 Hz, 2.43×10^8 N/m, 1.97×10^5 N s/m]

8. An aircraft fin may be idealized in bending as an SDoF system with an effective mass of 200 kg, undamped natural frequency of 5 Hz and damping 3 % critical. The fin is excited via the control surface by an 'on/off' force pulse of magnitude 500 N. Using MATLAB and one or more of the (a) superposition, (b) simulation and (c) Fourier transform approaches, determine the pulse duration that will maximize the resulting response and the value of the response itself.
9. Using MATLAB, generate a time history of 16 data points with a time interval Δt of 0.05 s and composed of a DC value of 1, a sine wave of amplitude 3 at 4 Hz and cosine waves of amplitude -2 at 2 Hz and 1 at 6 Hz. Perform the Fourier transform and examine the form of the complex output sequence as a function of frequency to understand how the data are stored and how the frequency components are represented. Then perform the inverse FT and examine the resulting sequence, comparing it to the original signal.
10. Generate other time histories with a larger number of data values, such as (a) single (1-cosine) pulse, (b) multiple cycles of a sawtooth waveform and (c) multiple cycles of a square wave. Calculate the FT of each and examine the amplitude of the frequency components to see how the power is distributed.