1 An Introduction to Ordinary Differential Equations

1.1 INTRODUCTION AND OBJECTIVES

Part I of this book is devoted to an overview of ordinary and partial differential equations. We discuss the mathematical theory of these equations and their relevance to quantitative finance. After having read the chapters in Part I you will have gained an appreciation of one-factor and multi-factor partial differential equations.

In this chapter we introduce a class of *second-order ordinary differential equations* as they contain derivatives up to order 2 in one independent variable. Furthermore, the (unknown) function appearing in the differential equation is a function of a single variable. A simple example is the *linear* equation

$$Lu \equiv a(x)u'' + b(x)u' + c(x)u = f(x) \tag{1.1}$$

In general we seek a solution u of (1.1) in conjunction with some auxiliary conditions. The coefficients a, b, c and f are known functions of the variable x. Equation (1.1) is called linear because all coefficients are independent of the unknown variable u. Furthermore, we have used the following shorthand for the first- and second-order derivatives with respect to x:

$$u' = \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{and} \quad u'' = \frac{\mathrm{d}^2u}{\mathrm{d}x^2} \tag{1.2}$$

We examine (1.1) in some detail in this chapter because it is part of the Black–Scholes equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{1.3}$$

where the asset price S plays the role of the independent variable x and t plays the role of time. We replace the unknown function u by C (the option price). Furthermore, in this case, the coefficients in (1.1) have the special form

$$\begin{aligned} a(S) &= \tfrac{1}{2}\sigma^2 S^2 \\ b(S) &= rS \\ c(S) &= -r \\ f(S) &= 0 \end{aligned} \tag{1.4}$$

In the following chapters our intention is to solve problems of the form (1.1) and we then apply our results to the specialised equations in quantitative finance.

1.2 TWO-POINT BOUNDARY VALUE PROBLEM

Let us examine a general second-order ordinary differential equation given in the form

$$u'' = f(x; u, u') \tag{1.5}$$

where the function f depends on three variables. The reader may like to check that (1.1) is a special case of (1.5). In general, there will be many solutions of (1.5) but our interest is in defining extra conditions to ensure that it will have a unique solution. Intuitively, we might correctly expect that two conditions are sufficient, considering the fact that you could integrate (1.5) twice and this will deliver two constants of integration. To this end, we determine these extra conditions by examining (1.5) on a *bounded* interval (a, b). In general, we discuss linear combinations of the unknown solution u and its first derivative at these end-points:

$$\begin{aligned} a_0 u(a) - a_1 u'(a) &= \alpha\ , \ |a_0| + |a_1| \neq 0 \\ b_0 u(b) + b_1 u'(b) &= \beta\ , \ |b_0| + |b_1| \neq 0 \end{aligned} \tag{1.6}$$

We wish to know the conditions under which problem (1.5), (1.6) has a unique solution. The full treatment is given in Keller (1992), but we discuss the main results in this section. First, we need to place some restrictions on the function f that appears on the right-hand side of equation (1.5).

Definition 1.1. The function $f(x, u, v)$ is called uniformly Lipschitz continuous if

$$|f(x; u,v) - f(x; w, z)| \leq K \max(|u - w|, |v - z|) \tag{1.7}$$

where K is some constant, and x, ut, and v are real numbers.

We now state the main result (taken from Keller, 1992).

Theorem 1.1. *Consider the function $f(x; u, v)$ in (1.5) and suppose that it is uniformly Lipschitz continuous in the region R, defined by:*

$$R : a \leq x \leq b,\ u^2 + v^2 < \infty \tag{1.8}$$

Suppose, furthermore, that f has continuous derivatives in R satisfying, for some constant M,

$$\frac{\partial f}{\partial u} > 0, \qquad \left|\frac{\partial f}{\partial v}\right| \leq M \tag{1.9}$$

and, that

$$a_0 a_1 \geq 0, \qquad b_0 b_1 \geq 0, \qquad |a_0| + |b_0| \neq 0 \tag{1.10}$$

Then the boundary-value problem (1.5), (1.6) *has a unique solution.*

This is a general result and we can use it in new problems to assure us that they have a unique solution.

1.2.1 Special kinds of boundary condition

The linear boundary conditions in (1.6) are quite general and they subsume a number of special cases. In particular, we shall encounter these cases when we discuss boundary conditions for the Black–Scholes equation. The main categories are:

- Robin boundary conditions
- Dirichlet boundary conditions
- Neumann boundary conditions.

The most general of those is the Robin condition, which is, in fact, (1.6). Special cases of (1.6) at the boundaries $x = a$ or $x = b$ are formed by setting some of the coefficients to zero. For example, the boundary conditions at the end-point $x = a$:

$$\begin{aligned} u(a) &= \alpha \\ u'(a) &= \beta \end{aligned} \tag{1.11}$$

are called Dirichlet and Neumann boundary conditions at $x = a$ and at $x = b$, respectively.

Thus, in the first case the value of the unknown function u is known at $x = a$ while, in the second case, its derivative is known at $x = b$ (but not u itself). We shall encounter the above three types of boundary condition in this book, not only in a one-dimensional setting but also in multiple dimensions. Furthermore, we shall discuss other kinds of boundary condition that are needed in financial engineering applications.

1.3 LINEAR BOUNDARY VALUE PROBLEMS

We now consider a special case of (1.5), namely (1.1). This is called a **linear equation** and is important in many kinds of applications. A special case of Theorem 1.1 occurs when the function $f(x; u, v)$ is linear in both u and v. For convenience, we write (1.1) in the canonical form

$$-u'' + p(x)u' + q(x)u = r(x) \tag{1.12}$$

and the result is:

Theorem 1.2. *Let the functions $p(x)$, $q(x)$ and $r(x)$ be continuous in the closed interval $[a, b]$ with*

$$\begin{aligned} &q(x) > 0, \quad a \le x \le b, \\ &a_0 a_1 \ge 0, \quad |a_0| + |a_1| \ne 0, \\ &b_0 b_1 \ge 0, \quad |b_0| + |b_1| \ne 0, \end{aligned} \tag{1.13}$$

Assume that

$$|a_0| + |b_0| \ne 0$$

then the two-point boundary value problem (BVP)

$$\begin{aligned} &Lu \equiv -u'' + p(x)u' + q(x)u = r(x), \quad a < x < b \\ &a_0 u(a) - a_1 u'(a) = \alpha, \qquad b_0 u(b) + b_1 u'(b) = \beta \end{aligned} \tag{1.14}$$

has a unique solution.

Remark. The condition $|a_0| + |b_0| \ne 0$ excludes boundary value problems with Neumann boundary conditions at both ends.

1.4 INITIAL VALUE PROBLEMS

In the previous section we examined a differential equation on a bounded interval. In this case we assumed that the solution was defined in this interval and that certain boundary conditions were defined at the interval's end-points. We now consider a different problem where we wish to find the solution on a semi-infinite interval, let's say (a, ∞). In this case we define the initial value problem (IVP)

$$\begin{aligned} u'' &= f(x; u, u') \\ a_0 u(a) - a_1 u'(a) &= \alpha \\ b_0 u(a) - b_1 u'(a) &= \beta \end{aligned} \tag{1.15}$$

where we assume that the two conditions at $x = a$ are independent, that is

$$a_1 b_0 - a_0 b_1 \neq 0 \tag{1.16}$$

It is possible to write (1.15) as a first-order system by a change of variables:

$$\begin{aligned} &u' = v, \qquad v' = f(x; u, v) \\ &a_0 u(a) - a_1 v(a) = \alpha \\ &b_0 u(a) - b_1 v(a) = \beta \end{aligned} \tag{1.17}$$

This is now a first-order system containing no explicit derivatives at $x = a$. System (1.17) is in a form that can be solved numerically by standard schemes (Keller, 1992). In fact, we can apply the same transformation technique to the boundary value problem (1.14) to get

$$\begin{aligned} &-v' + p(x)v + q(x)u = r(x) \\ &u' = v \\ &a_0 u(a) - a_1 v(a) = \alpha, \\ &b_0 u(b) + b_1 v(b) = \beta \end{aligned} \tag{1.18}$$

This approach has a number of advantages when we apply finite difference schemes to approximate the solution of problem (1.18). First, we do not need to worry about approximating derivatives at the boundaries and, second, we are able to approximate v with the same accuracy as u itself. This is important in financial engineering applications because the first derivative represents an option's delta function.

1.5 SOME SPECIAL CASES

There are a number of common specialisations of equation (1.5), and each has its own special name, depending on its form:

$$\begin{aligned} &\text{Reaction–diffusion:} && u'' = q(x)u \\ &\text{Convection–diffusion:} && u'' = p(x)u' \\ &\text{Diffusion:} && u'' = 0 \end{aligned} \tag{1.19}$$

Each of these equations is a model for more complex equations in multiple dimensions, and, we shall discuss the time-dependent versions of the equations in (1.19). For example, the convection–diffusion equation has been studied extensively in science and engineering and has applications to fluid dynamics, semiconductor modelling and groundwater flow, to name just a few (Morton, 1996). It is also an essential part of the Black–Scholes equation (1.3).

We can transform equation (1.1) into a more convenient form (the so-called *normal form*) by a change of variables under the constraint that the coefficient of the second derivative $a(x)$ is always positive. For convenience we assume that the right-hand side term f is zero. To this end, define

$$\begin{aligned} p(x) &= \exp \int \frac{b(x)}{a(x)}\, \mathrm{d}x \\ q(x) &= \frac{c(x)p(x)}{a(x)} \end{aligned} \tag{1.20}$$

If we multiply equation (1.1) (note $f = 0$) by $p(x)/a(x)$ we then get:

$$\frac{\mathrm{d}}{\mathrm{d}x} p(x) \frac{\mathrm{d}u}{\mathrm{d}x} + q(x)u = 0 \tag{1.21}$$

This is sometimes known as the *self-adjoint form*. A further change of variables

$$\zeta = \int \frac{\mathrm{d}x}{p(x)} \tag{1.22}$$

allows us to write (1.21) to an even simpler form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\zeta^2} + p(x)q(x)u = 0 \tag{1.23}$$

Equation (1.23) is simpler to solve than equation (1.1).

1.6 SUMMARY AND CONCLUSIONS

We have given an introduction to second-order ordinary differential equations and the associated two-point boundary value problems. We have discussed various kinds of boundary conditions and a number of sufficient conditions for uniqueness of the solutions of these problems. Finally, we have introduced a number of topics that will be required in later chapters.