CHAPTER 1

INTRODUCTION

In this chapter we will define the problem we are solving and give mathematical models of the problem, based on the physical laws of nature. Before we do this, let's jump in with an example.

Alice and Bob

Alice has just sent Bob a question in a game of Truth or Dare. The question is represented by two digital symbols (s_1 and s_2) as shown in Table 1.1. After sending an initial symbol s_0, the symbols are sent one at a time. Each is modified as it travels along a direct path to the receiver, so that it gets multiplied by -10. The symbols also travel along a second path, bouncing off a building, as shown in Fig. 1.1. The signal along this path gets multiplied by 9 and delayed so that it arrives at the same time as the next symbol arrives along the direct path. There is also noise which is added to the received signal.

At Bob's phone, the received values can be *modeled* as

$$\begin{aligned} r_1 &= -10s_1 + 9s_0 + n_1 \\ r_2 &= -10s_2 + 9s_1 + n_2. \end{aligned} \tag{1.1}$$

Suppose the actual received values are

$$r_1 = 1, \quad r_2 = -7. \tag{1.2}$$

Channel Equalization for Wireless Communications: From Concepts to Detailed Mathematics, First Edition. Gregory E. Bottomley.

Table 1.1 Possible messages

Index	Representation s_1 s_2	Message
1	+1 −1	"Do you like classical music?"
2	−1 −1	"Do you like soccer?"
3	+1 +1	"Do you like me?"

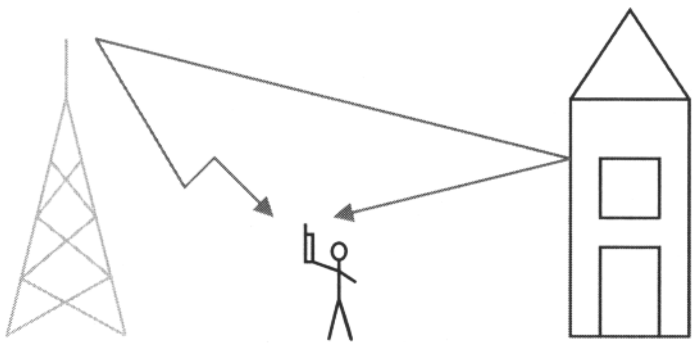

Figure 1.1 Dispersive scenario.

Which message was sent? How would *you* figure it out? Would it help if symbol s_0 were known or thought to be +1? Think about different approaches for determining the transmitted symbols. Try them out. Do they give the same answer? Do they give *valid* answers (the sequence $s_1 = -1$ $s_2 = +1$ is not in the table)?

1.1 THE IDEA

Channel equalization is about solving the problem of *intersymbol interference* (ISI). What is ISI? First, information can be represented as digital *symbols*. Letters and words on computers are represented using the symbols 0 and 1. Speech and music are represented using integers by sampling the signal, as shown in Fig. 1.2. These numbers can be converted into base 2. Thus, the number 6 becomes 110 $(0 \times 1 + 1 \times 2 + 1 \times 4)$. There are different ways of mapping the symbols 0 and 1 into values for transmission. One mapping is to represent 0 with +1 and 1 with −1. Thus, 110 is transmitted as using the series −1 −1 +1. The symbols 0 and 1 are often referred to as *Boolean* values. The transmitted values are called *modem* symbols or simply symbols.

ISI is the interference between symbols that can occur at the receiver. In the Alice and Bob example, we saw that one symbol was interfered by a previous symbol due to a second signal path. This is a problem in cell phone communications, and we will refer to it as the *dispersive channel* scenario. A cell tower transmitter sends

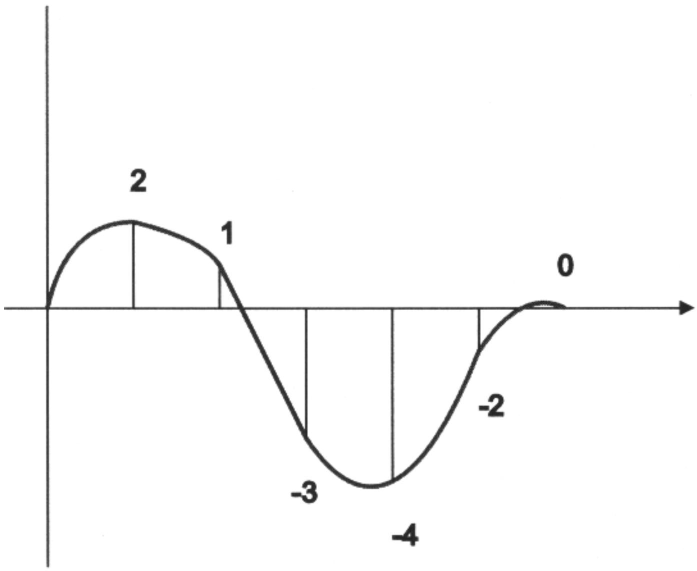

Figure 1.2 Sampling and digitizing speech.

a series or *packet* of digital symbols to a cell phone. The transmitted signal travels through the air, often bouncing off of walls and buildings, before arriving at the cell phone receiver. The receiver's job is to figure out what symbols were sent. This is an example of the channel equalization problem.

To solve this problem, we would like a mathematical model of what is happening. The model should be based on the laws of physics. Cell phone signals are transmitted using electromagnetic (radio) waves. The signal travels through the air, along a path to the receiver. From the laws of physics, the effect of this "channel" is multiplication by a channel coefficient. Thus, if s is the transmitted symbol, then cs is the received symbol, where c is a channel coefficient. To keep things simple, we will assume c is a real number (e.g., -10), though in practice it is a complex number with real and imaginary parts (amplitude and phase).

Sometimes the channel is *dispersive*, so that the signal travels along multiple paths with different path lengths, as illustrated in Fig. 1.1. The first path goes directly from the transmitter to the receiver and has channel coefficient $c = -10$. The second path bounces off a building, so it is longer, which delays the signal like an echo. It has channel coefficient $d = 9$. There is also noise present. The overall mathematical model of the received signal values is given in (1.1). The portion of the received signal containing the transmitted symbols is illustrated in Fig. 1.3.

Notice that the model includes terms n_1, n_2 to model random noise. The laws of physics tell us that electrons bounce around randomly, more so at higher temperatures. We call this *thermal noise*. Such noise adds to the received signal.

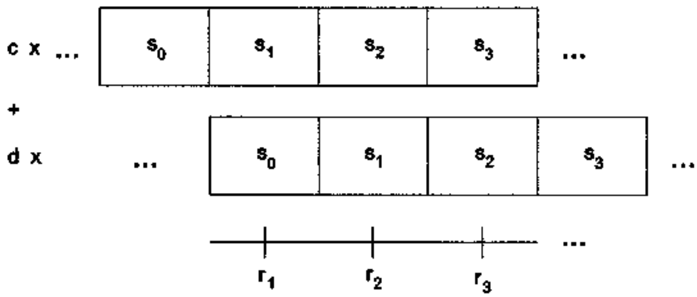

Figure 1.3 Received signal example.

While we don't know the noise values, we do know that they are usually small. In fact, physics tells us that the likelihood of noise taking on a particular value is given by the histogram in Fig. 1.4. Such noise is called *Gaussian*, named after the scientist Gauss. The average noise value is 0. The average of the square of a noise value is denoted σ^2 (the average of n_1^2 or n_2^2). We call the average of the square *energy* or *power* (energy per sample). We will assume we know this power. If needed, it would be estimated in practice. One more assumption regarding the noise terms. We will assume different noise values are unrelated (uncorrelated). Thus, knowing n_1 would tell us nothing about n_2.

1.2 MORE DETAILS

How well an equalizer performs depends on how large the noise power is, relative to the signal power. A useful measure of this is the signal-to-noise ratio (SNR). It is defined as the ratio of signal power (S) to noise power (N), i.e., S/N. If we are told that the noise power is $\sigma^2 = 100$, we just need to figure out the signal power S.

We can use the model for r_2 in (1.1) to determine S. The input signal power S is the average of the signal component $(-10s_2 + 9s_1)^2$, averaged over the possible values of s_1 and s_2. This turns out to be 181, which can be computed one of two ways. One way is to consider all possible combinations of s_1 and s_2. For example, the combination $s_1 = +1$ and $s_2 = +1$ gives a signal term of $-10(+1)+9(+1) = -1$ which has power $(-1)^2 = 1$. Assuming all combinations are possible[1], the average power becomes

$$S = (1/4)[(-1)^2 + (-19)^2 + (19)^2 + 1^2] = 181. \tag{1.3}$$

Another way to compute S is to use the fact that s_1 and s_2 are assumed to be unrelated. When two terms are unrelated, their powers add. The power in $-10s_1$

[1]This is not quite true, because one combination does not occur according to Table 1.1. However, for most practical systems, this aspect can be ignored.

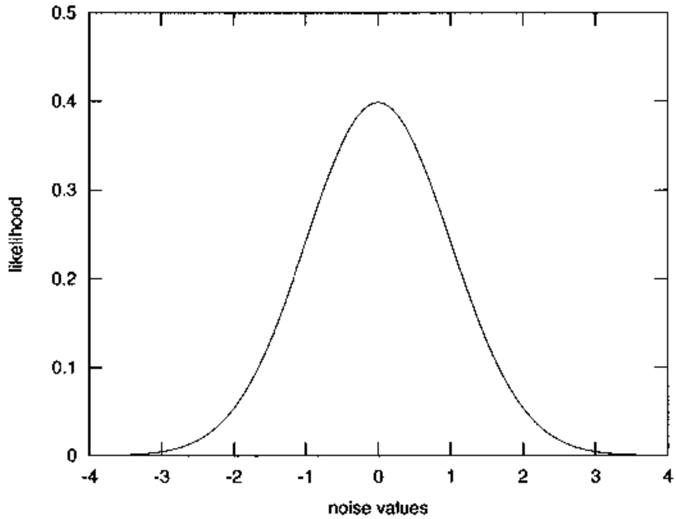

Figure 1.4 Noise histogram for noise power $\sigma^2 = 1$.

is the average of $[(-10)(+1)]^2$ and $[(-10)(-1)]^2$, which is 100. We could have used the property that the average of cs is c^2 times the average of s^2. The power in $9s_1$ is 81, so the total signal power is 181. Thus, the input SNR is

$$\text{SNR} = 181/100 = 1.81. \tag{1.4}$$

It is common to express SNR in units of *decibels*, abbreviated dB. These units are obtained by taking the base 10 logarithm and then multiplying by 10. Thus, the SNR of 1.81 becomes $10\log_{10}(1.81) = 2.6$ dB.

We will be interested in two extremes: low input SNR and high input SNR. When input SNR is low, performance is limited by noise. When input SNR is high, performance is limited by ISI.

1.2.1 General dispersive and MIMO scenarios

In general, we can write the received values in terms of channel coefficients c and d, keeping in mind that we know the values for c and d. Thus, for the dispersive scenario, we have

$$r_m = cs_m + ds_{m-1} + n_m; \quad m = 1, 2, \text{etc.}, \tag{1.5}$$

where the noise power is σ^2. The corresponding SNR is

$$\text{SNR} = (c^2 + d^2)/\sigma^2. \tag{1.6}$$

A block diagram of this scenario is given in Fig. 1.5.

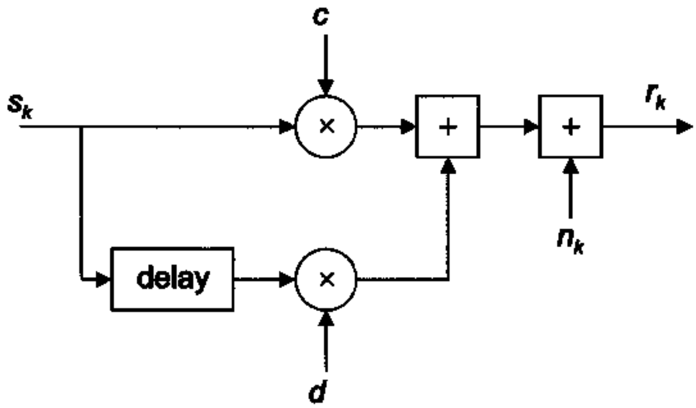

Figure 1.5 Dispersive scenario block diagram.

We will also consider a second ISI scenario, the multiple-input multiple-output (MIMO) scenario, illustrated in Fig. 1.6. Two symbols (s_1 and s_2) are transmitted, each from a different transmit antenna. Both are received at two receive antennas. There is only a single, direct path from each transmit antenna to each receive antenna. The two received values are modeled as

$$\begin{aligned} r_1 &= -10s_1 + 9s_2 + n_1 \\ r_2 &= 7s_1 - 6s_2 + n_2. \end{aligned} \tag{1.7}$$

Thus, we have ISI from another symbol transmitted at the same time on the same channel. In this case we have two input SNRs, one for each symbol. For each symbol, signal power is the sum of the squares of the channel coefficients associated with that symbol. Thus,

$$\text{SNR}(1) = ((-10)^2 + 7^2)/100 = 1.49 = 1.7 \text{ dB} \tag{1.8}$$

$$\text{SNR}(2) = (9^2 + (-6)^2)/100 = 1.17 = 0.7 \text{ dB}. \tag{1.9}$$

In general, the MIMO scenario can be modeled as

$$\begin{aligned} r_1 &= cs_1 + ds_2 + n_1 \\ r_2 &= es_1 + fs_2 + n_2. \end{aligned} \tag{1.10}$$

This is sometimes written in matrix form as

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \tag{1.11}$$

or simply

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}. \tag{1.12}$$

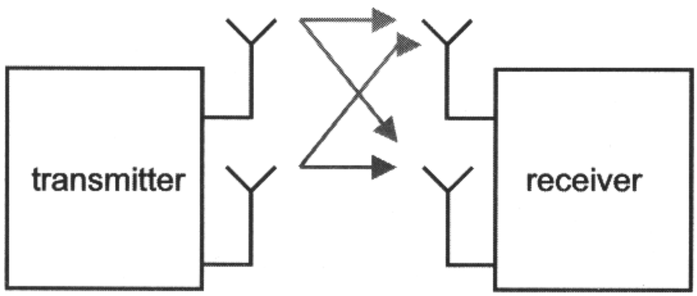

Figure 1.6 MIMO scenario.

The corresponding SNR values are

$$\text{SNR}(1) = (c^2 + e^2)/\sigma^2 \tag{1.13}$$

$$\text{SNR}(2) = (d^2 + f^2)/\sigma^2. \tag{1.14}$$

1.2.2 Use of complex numbers

Finally, in radio applications, the received values are actually complex numbers, with real and imaginary parts. We refer to the real part as the in-phase (I) component and the imaginary part as the quadrature (Q) component. At the transmitter, the I component is used to *modulate* a cosine waveform, and the Q component is used to modulate the negative of a sine waveform. These two waveforms are orthogonal (do not interfere with one another), so it is convenient to use complex numbers, as the real and imaginary parts are kept separate. Also, the arithmetic of complex numbers corresponds to the phase shift relationship between sine and cosine.

We can send one bit on the I component (the I bit) as $+1$ or -1 and one bit on the Q component (the Q bit) as $+j$ or $-j$, where j (i is often used in mathematics textbooks) indicates the Q component and behaves like $\sqrt{-1}$. This leads to a *constellation* of four possible symbol values: $1+j$, $1+j$, $-1-j$, and $+1-j$. This is shown in Fig. 1.7 and is called Quadrature Phase Shift Keying (QPSK).

1.3 THE MATH

In this section, a model is developed for the transmitter and channel, and sources of ISI at the receiver are discussed. To keep the math simple, we consider time-division multiplexing (TDM), in which symbols are transmitted sequentially in time. There is only one transmit antenna and one receive antenna, which is sometimes referred to as single-input single-output (SISO). A block diagram showing the system and notation is given in Fig. 1.8. A notation table is given at the end of the book.

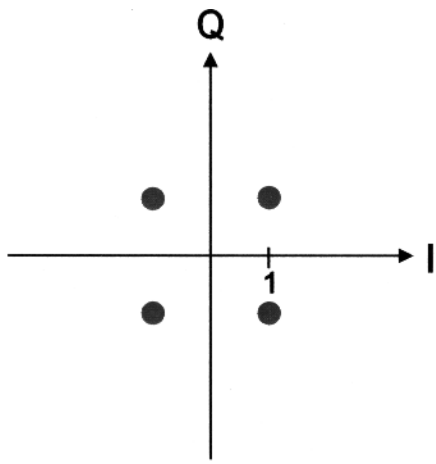

Figure 1.7 QPSK.

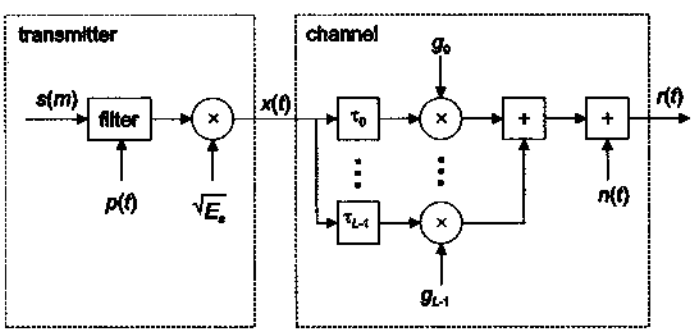

Figure 1.8 System block diagram showing notation.

We will use a complex, baseband equivalent of the system. A radio signal can be written as the sum of cosine component and a sine component, i.e.,

$$x(t) = u_r(t)\sqrt{2}\cos(2\pi f_c t) - u_i(t)\sqrt{2}\sin(2\pi f_c t), \tag{1.15}$$

where f_c is the carrier frequency in Hertz (cycles per second). The two components are orthogonal (occupy different signal dimensions) under normal assumptions. The $\sqrt{2}$ is included so that the power is the average of $u_r^2(t) + u_i^2(t)$. We can rewrite (1.15) as

$$\text{Re}\{u(t)\sqrt{2}\exp(j2\pi f_c t)\}, \tag{1.16}$$

where $u(t) = u_r(t) + ju_i(t)$ is the *complex envelope* of the radio signal. We can model the system at the complex envelope level, referred to as complex baseband, rather than having to include the carrier frequency term.

We will assume the receiver radio extracts the complex envelope from the received signal. For example, the real part of the complex envelope can be obtained by multiplying by $\sqrt{2}\cos(2\pi f_c t)$ and using a baseband filter that passes the signal. Mathematically,

$$y_r(t) = x(t)\sqrt{2}\cos(2\pi f_c t) = u_r(t)2\cos^2(2\pi f_c t) - u_i(t)2\sin(2\pi f_c t)\cos(2\pi f_c t). \tag{1.17}$$

Using the fact that $\cos^2(A) = 0.5(1 + \cos(2A))$, we obtain

$$y_r(t) = u_r(t) + u_r(t)\cos(2\pi 2 f_c t) - u_i(t)2\sin(2\pi f_c t)\cos(2\pi f_c t) \tag{1.18}$$

A filter can be used to eliminate the second and third terms on the right-hand side (r.h.s.). Similarly, the imaginary part of the complex envelope can be obtained by multiplying by $\sqrt{2}\sin(2\pi f_c t)$ and using a baseband filter that passes the signal.

Notice that we have switched to a *continuous time* waveform $u(t)$. Thus, when we send symbols one after another, we have to explain how we transition from one symbol to the next. We will see that each discrete symbol has a *pulse shape* associated with it, which explains how the symbol gets started and finishes up in time.

1.3.1 Transmitter

At the transmitter, modem symbols are transmitted sequentially as

$$x(t) = \sqrt{E_s}\sum_{m=-\infty}^{\infty} s(m)p(t - mT), \tag{1.19}$$

where

- E_s is the average received energy per symbol,
- $s(m)$ is the complex (modem) symbol transmitted during symbol period m, and
- $p(t)$ is the symbol waveform or pulse shape (usually purely real).

The symbols are normalized so that $\mathrm{E}\{|s(m)|^2\} = 1$, where $\mathrm{E}\{\cdot\}$ denotes expected value.[2] The pulse shape is also normalized so that $\int_{-\infty}^{\infty} |p(t)|^2\, dt = 1$.

In (1.19) we have assumed a continuous (infinite) stream of symbols. In practice, a block of N_s symbols is usually transmitted as a packet. Usually N_s is sufficiently large that the infinite model is reasonable for most symbols in the block. Theoretically, symbols on the edge of the block should be treated differently. However, in most cases, it is reasonable (and simpler) to treat all the symbols the same.

In general, a symbol can be one of M possible values, drawn from the set $S = \{S_j; j = 1 \ldots M\}$. These M possible complex symbol values can have different

[2] In this case, expectation is taken over all possible symbol values.

phases (phase modulation) and/or different amplitudes (amplitude modulation). For good receiver performance, we would like these symbol values to be as different from one another as possible for a given average symbol power. Note that with M possible symbol values, we can transmit $\log_2(M)$ bits (e.g., 3 bits have $M = 8$ possible combinations)

Modulation is typically Gray-mapped Quadrature Amplitude Modulation (QAM), such as Quadrature Phase Shift Keying (QPSK) (illustrated in Fig. 1.7) and 16-QAM (illustrated in Fig. 1.9). These can be viewed as Binary Phase Shift Keying (BPSK) and 4-ary Amplitude Shift Keying (4-ASK) on the in-phase (I) and quadrature (Q) axes. The 4-ASK constellation, illustrated in Fig. 1.10, conveys two modem bits: a most significant bit (MSB) and a least significant bit (LSB). The MSB has better distance properties, giving it a lower error rate than the LSB.

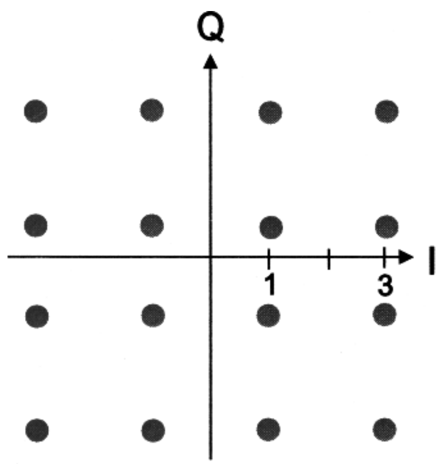

Figure 1.9 16-QAM.

As for pulse shaping, root-Nyquist pulse shapes are typically used, which have the property that their sampled autocorrelation function is given by

$$R_p(mT) \triangleq \int_{-\infty}^{\infty} p(t + mT)p^*(t)\, dt = \delta(m), \tag{1.20}$$

where superscript "*" denotes complex conjugation and $\delta(m)$ is the Kronecker delta function (1 for $m = 0$ and 0 for other integer values of m). (The pulse shape $p(t)$ is typically purely real.) Such pulse shaping prevents ISI at the receiver when the channel is not dispersive and the receiver initially filters the signal using a filter matched to the pulse shape (see Chapter 2). Sometimes *partial-response* pulse shaping is used, in which ISI is intentionally introduced at the transmitter to enable higher data rates.

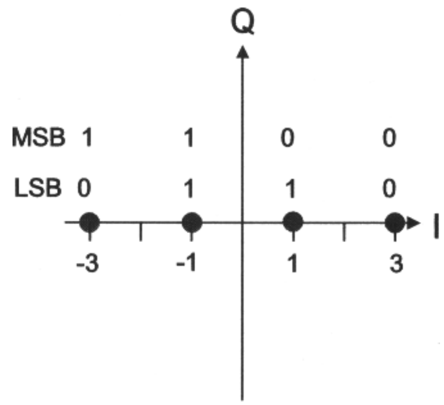

Figure 1.10 4-ASK with Gray mapping.

A commonly used root-Nyquist pulse shape is root-raised cosine. Its autocorrelation function is given by

$$R_p(t) = \left(\frac{\sin(\pi t/T)}{\pi t/T}\right)\left(\frac{\cos(\beta\pi t/T)}{1-(2\beta t/T)^2}\right). \tag{1.21}$$

where β is the *rolloff*. The RRC waveform and its autocorrelation function are shown in Fig. 1.11 for a rolloff of 0.22 (22% excess bandwidth).

1.3.2 Channel

The transmitted signal passes through a communications channel on the way to the receive antenna of a particular device. We can model this aspect of the channel as a linear filter and characterize this filter by its impulse response. The actual, physical channel may consist of hundreds of paths on a continuum of path delays. Fortunately, for an arbitrary channel, the channel response can be modeled as a finite-impulse-response (FIR) filter, using a tap-spacing that meets the Nyquist sampling criterion (sampling rate at least twice the bandwidth) for the transmitted signal (typically between 1 and 2 samples per symbol period). The accuracy of this model depends on how many tap delays are used.

Regulatory bodies typically limit the amount of bandwidth a wireless signal is allowed to occupy. Thus, the channel is bandlimited. Theoretically, for root-Nyquist pulse shaping, the radio bandwidth must be at least as large as the symbol rate (baud rate) (the baseband equivalent bandwidth is half the baud rate, giving a Nyquist sampling period of one symbol period). Conversely, for a given bandwidth, the symbol rate with root-Nyquist pulse shaping is limited to the radio bandwidth or twice the baseband bandwidth. This limit in symbol rate is sometimes referred to as the Nyquist rate.

However, in most systems, a slightly larger bandwidth is used, giving rise to the notion of excess bandwidth. When excess bandwidth is low, it is reasonable to

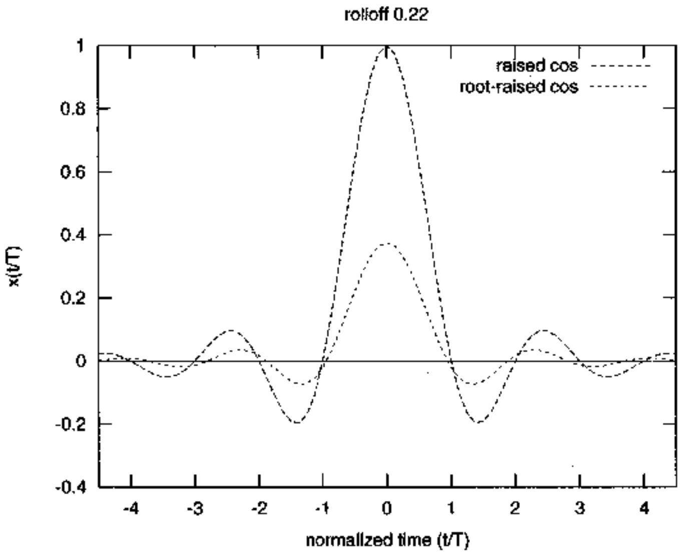

Figure 1.11 Raised cosine function.

approximate the channel with a symbol-spaced channel model, especially when the channel is highly dispersive (signal energy spread out in time due to the channel).

Consider an example in which the transmitter uses RRC pulse shaping with rolloff 0.22. The Nyquist sampling period is 1/1.22 or 0.82 symbol periods. Thus, for an arbitrary channel, we would need a tap spacing of $0.82T$ for smaller. As most simulation programs work with a sampling rate that is a power of 2 times the symbol rate, a convenient tap spacing would be $0.75T$. If the channel is well-modeled with a single tap at delay 0, the received signal (after filtering with a RRC filter) would give us the raised cosine function shown in Fig. 1.11. To recover the symbol at time 0, we would sample at time 0, where the raised cosine function is at its maximum. Notice that when recovering the next symbol, we would sample at time 1, and the effect of the symbol at time 0 would be 0 (no ISI). In fact, we can see that when recovering any other symbol, the effect of symbol 0 would be 0, as the zero crossings are symbol-spaced relative to the peak.

Suppose, instead, that the channel is well-modeled by two taps $0.75T$ apart. An example with path coefficients 0.5 and 0.5 is shown in Fig. 1.12 (the x axis is normalized so that the peak occurs at time 0). Relative to Fig. 1.11, we see that the symbol is spread out more in time, or *dispersed*. Hence, the channel is considered *dispersive*. Observe that when recovering the next symbol at time 1, there is ISI from symbol 0.

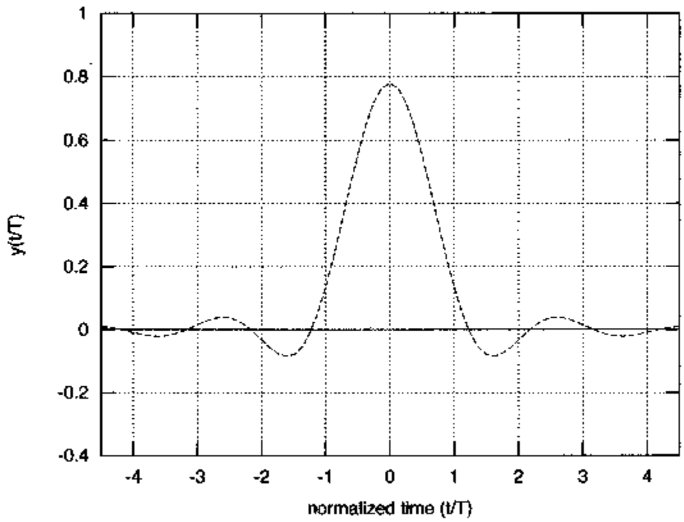

Figure 1.12 Effect of dispersion due to two, $0.75T$-spaced, equal amplitude paths on raised cosine with 0.22 rolloff.

Another aspect of the channel is noise, which can be modeled as an additive term to the received signal. Characterization of the noise is discussed in the next subsection.

Putting these two aspects together, the received signal can be modeled as

$$r(t) \models \sum_{\ell=0}^{L-1} g_\ell x(t - \tau_\ell) + n(t), \tag{1.22}$$

where L is the number of taps or (resolvable) paths, g_ℓ is the *medium* response or path coefficient for the ℓth path, and τ_ℓ is the path delay for the ℓth path. Note that we use $\models$ to emphasize that this is a model. This means we think of $n(t)$ as a stochastic process rather than a particular realization of the noise.

By substituting (1.19) into (1.22), we obtain the following model for the received signal:

$$r(t) \models \sqrt{E_s} \sum_{m=-\infty}^{\infty} h(t - mT)s(m) + n(t), \tag{1.23}$$

where

$$h(t) = \sum_{\ell=0}^{L-1} g_\ell p(t - \tau_\ell) \tag{1.24}$$

is the "channel" response, which includes the symbol waveform at the transmitter as well as the medium response.

1.3.2.1 Noise and interference models The term $n(t)$ models noise. Here we will assume this noise is additive, white Gaussian noise (AWGN). Such noise is implicitly assumed to have zero mean, i.e.,

$$m_n(t) \triangleq \mathrm{E}\{n(t)\} = 0. \tag{1.25}$$

The term "white" noise means two things. First, it means that different samples of the noise are uncorrelated. It also means that its moments are not a function of time. That is, the covariance function is given by

$$C_n(t_1, t_2) \triangleq \mathrm{E}\{[n(t_1) - m_n(t_1)][n^*(t_2) - m_n^*(t_2)]\} = N_0 \delta_D(t_1 - t_2), \tag{1.26}$$

where $\delta_D(\tau)$ denotes the Dirac delta function (a unity-area impulse at $\tau = 0$).

Another implicit assumption with AWGN is that it is *proper*, also referred to as *circular*. This has to do with the relation between the real and imaginary parts of an arbitrary noise sample $n(t_0) = n = n_r + jn_i$. With circular noise, the real and imaginary components of $n(t_0)$ are uncorrelated and have the same distribution. With AWGN, this distribution is assumed to be *Gaussian*, which is a good model for thermal noise. A circular, complex Gaussian random variable (r.v.) has probability density function (PDF)

$$f_n(x) = \frac{1}{\pi N_0} \exp\left\{\frac{-|x - m_n|^2}{N_0}\right\}, \tag{1.27}$$

where m_n is the mean, assumed to be zero, and N_0 is the one-sided power spectral density of the original radio signal (noise on the I and Q components has variance $\sigma^2 = N_0/2$). If we write $n = n_r + jn_i$, where n_r and n_i are real random variables, then n_r is Gaussian with PDF

$$f_{n_r}(x) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{\frac{-(x - m_r)^2}{N_0}\right\} \tag{1.28}$$

and has cumulative distribution function (CDF)

$$F_{n_r}(x) \triangleq \Pr\{n_r \le x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi N_0}} \exp\left\{\frac{a^2}{N_0}\right\} da \tag{1.29}$$

$$= 1 - (1/2)\mathrm{erfc}\left(\frac{x}{\sqrt{\pi N_0}}\right), \tag{1.30}$$

where

$$\mathrm{erfc}(y) \triangleq \frac{2}{\sqrt{\pi}} \int_{y}^{\infty} e^{-u^2} du \tag{1.31}$$

and $\mathrm{erfc}(-y) = 2 - \mathrm{erfc}(y)$. There are tables and software routines for evaluating the erfc function.

Bandwidth (BW) limitations and the presence of noise limit the rate information can be reliably transmitted. For Gaussian noise, Shannon showed that the information rate (in bits per second) is limited by the *capacity* (C) of the channel, which is given by

$$C = \mathrm{BW}\log_2(1 + \mathrm{SNR}). \tag{1.32}$$

The area of *information theory* includes the development of modulation and coding procedures that approach this limit. For our purposes, it is important to note that increasing the symbol rate beyond the Nyquist rate and using equalization to address the resulting ISI has its limits.

1.3.3 Receiver

At the receiver, the medium-filtered, noisy signal is processed to *detect* which message was sent. One way to do this is to first detect the modem symbols (*demodulation*). The term "equalization" is usually reserved for a form of demodulation that directly addresses ISI in some way.

Based on our system model, there are several sources of ISI at the receiver.

1. Interference from different symbol periods. Symbols sent before are after a particular symbol can interfere because of
   (a) the transmit pulse shape,
   (b) a dispersive medium, and/or
   (c) the receive filter response.
2. Interference from different transmitters. Symbols sent from other transmitters are either
   (a) also intended for the receiver (MIMO scenario) or
   (b) intended for another receiver or another user (cochannel interference).

   In a single-path channel, such interference can be synchronous (time-aligned) or asynchronous.

Noise and ISI cause the receiver to make errors. For example, it can detect the incorrect modem symbol, which can give rise to an incorrect bit value. This may lead to incorrect detection of which message was sent. In later chapters, we will compare receivers based on their bit error rate (BER), which will be defined as the probability that a detected bit value is in error. It will be measured by counting the fraction of bits that are in error (e.g., a $+1$ was transmitted and the received detected a -1). Other useful measures of performance are symbol error rate (SER) and frame erasure rate (FER). The latter refers to the probability that a message or frame is in error.

Throughout this book, we will focus on *coherent* forms of equalization, in which it is assumed that the medium response can be estimated to determine the amplitude and phase effects of the medium. This is typically done by transmitting some known reference (pilot) symbols. We will not consider *noncoherent* forms, which only work for certain modulation schemes. Also, we will not consider *blind equalization*, in which there are no pilot symbols being transmitted.

1.4 MORE MATH

In this section, more elaborate system models and scenarios are considered. Additional sources of ISI at the receiver are identified.

The system model is extended by considering several multiplicities. The transmitter multiplexes multiple symbols in parallel, such as code-division multiplexing (CDM) and orthogonal frequency-division multiplexing (OFDM) of symbols. TDM can be viewed as a special case in which the number of symbols sent in parallel is one.

Multiple transmit and receive antennas are also introduced, covering the cases of cochannel interference and MIMO. This also introduces the notion of code-division multiple access (CDMA) and time-division multiple access (TDMA), in which different transmitters access the channel using different spreading codes or different time slots.

1.4.1 Transmitter

We assume there are N_t transmit antennas. At transmit antenna i, modem symbols are transmitted in parallel using K parallel multiplexing channels (PMCs). For CDM, K is the number of spreading codes in use; for OFDM, K is the number of subcarriers. TDM can be viewed as a special case of CDM in which $K = 1$.

The transmitted signal is given by

$$x^{(i)}(t) = \sum_{k=0}^{K-1} \sqrt{E_s^{(i)}(k)} \sum_{m=-\infty}^{\infty} s_k^{(i)}(m) a_{k,m}^{(i)}(t - mT), \tag{1.33}$$

where

- $E_s^{(i)}(k)$ is the average received symbol energy on PMC k of transmit antenna i,
- $s_k^{(i)}(m)$ is the (modem) symbol transmitted on PMC k of transmit antenna i during symbol period m, and
- $a_{k,m}^{(i)}(t)$ is the symbol waveform for the symbol transmitted on PMC k of transmit antenna i during symbol period m.

Symbols are normalized so that $\mathrm{E}\{|s_k^{(i)}(m)|^2\} = 1$. The symbol waveforms are also normalized so that $\int_{-\infty}^{\infty} |a_{k,m}^{(i)}(t)|^2 dt = 1$. A block diagram is shown in Fig. 1.13 for the case of a single transmitter (transmitter superscript i has been omitted).

1.4.1.1 TDM For TDM, symbols are sent one at a time ($K = 1$), and the symbol waveform is simply

$$a_{0,m}^{(i)}(t) = p(t), \tag{1.34}$$

where $p(t)$ is the symbol pulse shape. Notice that the symbol waveform is the same for each symbol period m.

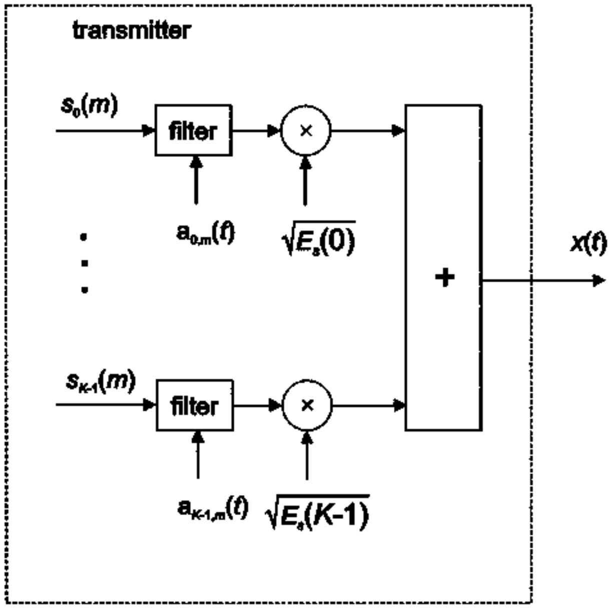

Figure 1.13 Transmitter block diagram showing parallel multiplexing channels.

1.4.1.2 CDM For CDM, symbols are sent in parallel on different spreading waveforms. The symbol waveform is formed from a spreading code or sequence of "chip" values, i.e.,

$$a_{k,m}^{(i)}(t) = (1/\sqrt{N_c}) \sum_{n=0}^{N_c-1} c_{k,m}^{(i)}(n)p(t - nT_c), \tag{1.35}$$

where

- N_c is the number of chips used (the spreading factor),
- $c_{k,m}^{(i)}(n)$ is the nth chip value for the spreading code for symbol transmitted on spreading code k of transmit antenna i during symbol period m, and
- $p(t)$ is the chip pulse shape.

Chip values are assumed to have unity average energy and are typically unity-amplitude QPSK symbols. For transmitter i, the spreading codes are typically orthogonal when time-aligned, i.e.,

$$\sum_{n=0}^{N_c-1} [c_{k_1,m}^{(i)}(n)]^* c_{k_2,m}^{(i)}(n) = N_c\delta(k_1 - k_2). \tag{1.36}$$

A commonly used set of orthogonal sequences is the Walsh/Hadamard or Walsh code set. There are K codes of length K, where $K = 2^k\alpha$ and *alpha* is the *order*.

For $K = 1$ (order 0), the single Walsh code is $+1$. Higher-order code sets can be generated as rows of a matrix $\mathbf{W}(\alpha)$ which is formed order-recursively using

$$\mathbf{W}(\alpha) = \begin{bmatrix} \mathbf{W}(\alpha-1) & \mathbf{W}(\alpha-1) \\ \mathbf{W}(\alpha-1) & -\mathbf{W}(\alpha-1) \end{bmatrix}. \tag{1.37}$$

The $K = 4$ Walsh codes for $N_c = 4$ are given in Table 1.2.

Table 1.2 Walsh codes of length 4

Index	Code			
0	+1	+1	+1	+1
1	+1	−1	+1	−1
2	+1	+1	−1	−1
3	+1	−1	−1	+1

In cellular communication systems, spreading sequences are formed by scrambling a set of Walsh codes with a pseudo-random QPSK scrambling sequence that is much longer than the symbol period, so that each symbol period uses a different set of orthogonal spreading sequences. This is referred to as *longcode* scrambling. Using the same orthogonal codes for each symbol period is referred to as *short codes*. For good performance in possibly dispersive channels, scrambled Walsh codes are used. We will assume longcode scrambling throughout, as use of short codes is a special case in which $a_{k,m}^{(i)}(t)$ is the same for each m.

Now we have two ways to view TDM. As suggested earlier, we can think of TDM as a special case of CDM in which one symbol is sent at a time, so that $K = 1$, $N = 1$, $T_c = T$, $c_{k,m}^{(i)}(n) = 1$, and (1.34) holds. This is the most common way to think of TDM.

However, sometimes it is useful to think of TDM as sending $K > 1$ symbols in parallel using special spreading codes. For example, we can think of TDM as sending $K = 4$ symbols in parallel using the codes in Table 1.3.

Table 1.3 TDM codes of length 4

Index	Code			
0	1	0	0	0
1	0	1	0	0
2	0	0	1	0
3	0	0	0	1

1.4.1.3 OFDM For OFDM, symbols are sent in parallel on different subcarriers. The symbol waveform is similar in structure to CDM, except the "spreading sequences" are related to complex sinusoidal functions. While there are different forms of OFDM, we will consider a form in which each symbol period can be divided into a cyclic prefix (CP) or guard interval followed by a main block (MB). An example is given in Fig. 1.14.

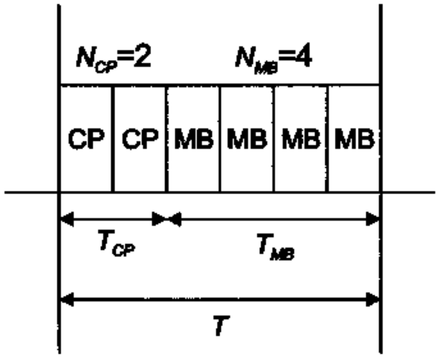

Figure 1.14 OFDM symbol block.

The symbol waveform can be expressed as

$$a_{k,m}^{(i)}(t) = (1/\sqrt{N_c}) \sum_{n=0}^{N_c-1} c_k(n)p(t - nT_c)\alpha(t - mT), \tag{1.38}$$

where

- $N_c = N_{CP} + N_{MB}$ is the number of nonzero chips in the symbol waveform,
- N_{CP} is the number of chips in the cyclic prefix,
- N_{MB} is the number of chips in the main block,
- $c_k(n)$ is the nth unity-amplitude chip value for the symbol transmitted on subcarrier k, independent of transmit antenna i,
- $p(t)$ is the chip pulse shape, and
- $\alpha(t)$ is a rectangular windowing function.

The first N_{CP} values are the cyclic prefix values and the remaining N_{MB} values are the main block values. The total symbol period is given by $T = N_cT_c$.

A reasonable approximation is to ignore the windowing effects at the edges of each symbol period. The symbol waveform simplifies to

$$a_{k,m}^{(i)}(t) \approx (1/\sqrt{N_c}) \sum_{n=0}^{N_c-1} c_k(n)p(t - nT_c), \tag{1.39}$$

which we recognize as the same form as CDM with short codes. Thus, the CDM model can be used to obtain results for both CDM and OFDM. The difference is the particular spreading sequences used.

With OFDM, the main block sequences are given by

$$f_k(n) = \exp(j2\pi kn/K), \quad n = 0 \ \ldots \ N_{MB} - 1. \tag{1.40}$$

The $K = 4$ main block sequences of length $N_{MB} = 4$ are given in Table 1.4. Similar to CDM, the main block sequences are orthogonal when time aligned, i.e.,

$$\sum_{n=0}^{N_{MB}-1} c_{k_1}^*(n) c_{k_2}(n) = N_{MB}\delta(k_1 - k_2). \tag{1.41}$$

They have an additional property in that a circular shift of the sequence is equivalent to applying a phase shift to the original sequence. Specifically,

$$\begin{aligned} f_k(n \ominus \ell) &= \exp\left(j2\pi k(n-\ell)/K\right) \\ &= \exp\left(-j2\pi k\ell/K\right)\exp\left(j2\pi kn/K\right) \\ &= \exp\left(-j2\pi k\ell/K\right) f_k(n). \end{aligned} \tag{1.42}$$

where $\ominus$ denotes subtraction modulus N_{MB}. This property implies that the sequences are also orthogonal with circular shifts of one another, i.e.,

$$\sum_{n=0}^{N_{MB}-1} c_{k_1}^*(n) c_{k_2}(n \oplus \ell) = N_{MB}\delta(k_1 - k_2), \tag{1.43}$$

where $\oplus$ denotes modular addition using modulus N_{MB}. We will see in the next chapter that the use of a cyclic prefix and discarding of certain receive samples makes delayed versions of the symbol appear as circular shifts. This allows orthogonality to be preserved in a dispersive channel. (From a CDM point of view, the CP makes interference a function of periodic crosscorrelations, which are "perfect" in this case.)

Table 1.4 Main block OFDM sequences of length 4

Index	Subcarrier chip sequence			
0	+1	+1	+1	+1
1	+1	$+j$	−1	$-j$
2	+1	−1	+1	−1
3	+1	$-j$	−1	$+j$

The CP is obtained by repeating the last N_{CP} chip values and pre-appending them. Thus, the overall chip sequence is given by

$$c_k(n) = \begin{cases} f_k(N_{MB} - N_{CP} + n - 1), & 0 \le n \le N_{CP} - 1 \\ f_k(n - N_{CP}), & N_{CP} \le n \le N_c - 1 \end{cases} \tag{1.44}$$

Though less common, it is possible to have a CP in a CDM system. In this case, a windowing function $\alpha(t)$ would not be used. A CP can also be used when transmitting a block of TDM symbols. The uplink of the Long Term Evolution (LTE) system [Dah08] can be interpreted as a form of TDM with a cyclic prefix.

1.4.2 Channel

The model used in the previous section is extended to allow for multiple transmit and receive antennas. The received vector (N_r receive antennas) can be modeled as

$$\mathbf{r}(t) \models \sum_{i=1}^{N_t} \sum_{\ell=0}^{L-1} \mathbf{g}_\ell^{(i)} x^{(i)}(t - \tau_\ell) + \mathbf{n}(t), \tag{1.45}$$

where $\mathbf{g}_\ell^{(i)}$ is a vector of *medium* response coefficients, one per receive antenna. Also, unless otherwise indicated, all vectors are column vectors.

In general, the medium responses from transmit antennas in different locations will have different path delays. We can handle this case by modeling all possible path delays and setting some of the coefficient vectors to zero.

By substituting (1.33) into (1.45), we obtain the following model for the received signal:

$$\mathbf{r}(t) \models \sum_{i=1}^{N_t} \sum_{k=0}^{K-1} \sqrt{E_s^{(i)}(k)} \sum_{m=-\infty}^{\infty} \mathbf{h}_{k,m}^{(i)}(t - mT) s_k^{(i)}(m) + \mathbf{n}(t), \tag{1.46}$$

where

$$\mathbf{h}_{k,m}^{(i)}(t) = \sum_{\ell=0}^{L-1} \mathbf{g}_\ell^{(i)} a_{k,m}^{(i)}(t - \tau_\ell) \tag{1.47}$$

is the channel response.

1.4.2.1 Noise and interference models Here the noise model is extended for multiple receive antennas, and more general noise models are considered. We will still assume the noise has zero mean, i.e.,

$$\mathbf{m}_n(t) \triangleq \mathrm{E}\{\mathbf{n}(t)\} = \mathbf{0}, \tag{1.48}$$

where boldface is used for column vectors. All vectors are $N_r \times 1$.

The noise may be *colored*, meaning that there may be correlation from one time instance to another as well as from one antenna to another, and the covariance function may be a function of time. For multiple receive antennas, the correlation is defined as

$$\mathbf{C}_n(t_1, t_2) \triangleq \mathrm{E}\{[\mathbf{n}(t_1) - \mathbf{m}_n(t_1)][\mathbf{n}(t_2) - \mathbf{m}_n(t_2)]^H\}, \tag{1.49}$$

where superscript "H" denotes conjugate transpose (Hermitian transpose). If $t_1 = t_2 + \tau$ and the correlation depends on both t_2 and τ, then it is considered *nonstationary*. If it only depends on τ, it is *stationary* and is then written as $\mathbf{C}_n(\tau)$.

We will still assume the noise is *proper*, also known as *circular*. With circular Gaussian noise, the I and Q components of $\mathbf{n}(t)$ are uncorrelated and have the same autocorrelation function, i.e.,

$$\begin{aligned} \mathrm{E}\{\mathbf{n}_r(t_1)\mathbf{n}_r^*(t_2)\} &= \mathrm{E}\{\mathbf{n}_i(t_1)\mathbf{n}_i^*(t_2)\} = (0.5)\mathbf{C}_n(t_1, t_2) && (1.50) \\ \mathrm{E}\{\mathbf{n}_r(t_1)\mathbf{n}_i^*(t_2)\} &= \mathrm{E}\{\mathbf{n}_i(t_1)\mathbf{n}_r^*(t_2)\} = 0. && (1.51) \end{aligned}$$

A sample of the noise $\mathbf{n} = \mathbf{n}(t_0)$ is a complex Gaussian random vector. Assuming stationary noise, the noise vector has probability density function (PDF)

$$f_n(\mathbf{x}) = \frac{1}{\pi^{N_r}|\mathbf{C}_n(0)|} \exp\left\{(\mathbf{x} - \mathbf{m}_n)^H \mathbf{C}_n^{-1}(0)(\mathbf{x} - \mathbf{m}_n)\right\}, \tag{1.52}$$

where $\mathbf{m}_n$ is the mean, assumed to be zero, $\mathbf{C}_n(0)$ is the noise correlation function at zero lag, sometimes called the spatial covariance, and $|\cdot|$ denotes determinant of a matrix.

When we assume AWGN, we will assume the noise is uncorrelated across receive antennas, so that

$$\mathbf{C}_n(\tau) = N_0 \mathbf{I} \delta_D(\tau). \tag{1.53}$$

where $\mathbf{I}$ is the identity matrix.

1.4.2.2 Scenarios In discussing approaches and the literature, it helps to consider two scenarios. In the first scenario, there is a set of symbols during a given symbol period, and each symbol in the set interferes with all other symbols in the set (but not symbols from other symbol periods). We will call this the MIMO/Cochannel scenario as it includes the following.

1. MIMO scenario. In TDM and CDM, this occurs if the transmit pulse is root-Nyquist, the medium is not dispersive, and the receiver uses a filter matched to the transmit pulse and samples at the appropriate time. In the CDM case, we will assume that codes transmitted from the same antenna are orthogonal. In OFDM, the medium can be dispersive as long as the delay spread is less than the length of the cyclic prefix. If there are N_t transmit antennas, then a set of N_t symbols interfere with one another.

2. Synchronous cochannel scenario. This is similar to the MIMO case, except that the different transmitted streams are intended for different users. Also, the transmitters may be at different locations. For TDMA and CDMA, in addition to the requirements for TDM and CDM in the MIMO case, the different transmitted signals are assumed to be synchronized to arrive at the receiver at the same time. For CDMA, an example of this is the synchronous uplink. For OFDM, the synchronization must be close enough so that subcarriers remain orthogonal, even if transmitted from different antennas. Again, there are N_t symbols that interfere with one another. In the CDMA case, nonorthogonal codes are typically assumed in the synchronous uplink, so that there are $N_t K$ symbols interfering with one another. However, in this case, it is usually assumed that $K = 1$, giving N_t interfering symbols.

In the nondispersive case, the channel coefficients are typically assumed to be independently fading (fading channel) or nonfading and unity (AWGN channel).

With these assumptions, the received sample vector corresponding to the set of symbols interfering with one another can be modeled as

$$\mathbf{r} \models \mathbf{HAs} + \mathbf{n}, \tag{1.54}$$

where $\mathbf{n}$ is a vector of Gaussian r.v.s with zero mean and covariance $\mathbf{C}_n$. While $\mathbf{C}_n = N_0\mathbf{I}$ in this specific case, we will allow other values for $\mathbf{C}_n$ to keep the model general.

As for the other terms, we have stacked the symbols from different transmitters into one symbol vector $\mathbf{s}$. Matrix $\mathbf{A}$ is a diagonal matrix given by

$$\mathbf{A} = \text{diag}\{E_s^{(1)}\ E_s^{(2)}\ \ldots\}, \tag{1.55}$$

where the index k has been dropped. The $N_t \times N_t$ matrix $\mathbf{H}$ is the channel matrix. For example, in the TDM and CDM cases, it can be shown that the ith column of $\mathbf{H}$ is given by

$$\mathbf{h}_i = \mathbf{g}_0^{(i)}. \tag{1.56}$$

The model in (1.54) is also appropriate in other scenarios. It can be used to model the entire block of received data when there is ISI between symbol periods. It can also be used to model a window or sub-block of data of a few symbol periods when there is ISI between symbol periods. If all symbols are included in $\mathbf{s}$, then $\mathbf{H}$ can have more columns than rows. Sometimes we move the symbols at the edges of the sub-block out of $\mathbf{s}$ and fold them into $\mathbf{n}$, changing $\mathbf{C}_n$. This gives $\mathbf{H}$ fewer columns.

In the second scenario, not all symbols interfere with one another. In addition, there is a structure to how symbols interfere with one another, because we will assume the interference is due to a dispersive medium, partial response pulse shaping, or asynchronous transmission of different transmitters. Thus, for TDM, each symbol experiences interference from a window of symbols in time. For TDM with MIMO, a sub-block of N_t symbols experiences interference from a window of sub-blocks in time. In CDM, a sub-block of K symbols experience interference from a window of symbol sub-blocks in time. We will call this the *dispersive/asynchronous scenario*. Note that asynchronous transmission can be modeled as a dispersive channel in which different paths have zero energy depending on the transmitter.

In this scenario, we usually assume the block size is large, so that using (1.54) to design a block equalizer would lead to large matrices. However, if we were to use (1.54), we would see that the channel matrix $\mathbf{H}$ has nonzero elements along the middle diagonals and zeros along the outer diagonals.

1.4.3 Receiver

At the receiver, there are several sources of ISI. For the CDM case, the sources are the same as the TDM case, with an additional source being ISI from other symbols sent in parallel. In the CDM case, this can be due to the symbol waveform (chip pulse shape not root-Nyquist or spreading codes not orthogonal) or the medium response (dispersive). Typically the spreading waveforms are orthogonal (after chip pulse matched filtering), so that ISI from symbols in parallel is due to a dispersive medium response.

For the OFDM case, the cyclic prefix is used to avoid ISI from symbols sent in parallel as well as symbols sent sequentially. We will see in the next chapter that this is achieved by discarding part of the received signal before performing matched filtering.

For both CDM and OFDM, ISI between symbols in parallel can result from time variation of the medium response (not included in our model). If the variation is significant within a symbol period, orthogonality is lost.

The receiver may have multiple receive antennas. We will assume that the fading medium coefficients are different on the different receive antennas. Common assumptions are uncorrelated fading at the mobile terminal and some correlation (e.g., 0.7) at the base station. Such an array of antennas is sometimes called a *diversity array*. By contrast, if the fading is completely correlated (magnitude of the complex correlation is one), it is sometimes called a *phased array*. In this case, the medium coefficients on one antenna are phased-rotated versions of the medium coefficients on another antenna. The phase depends on the direction of arrival.

1.5 AN EXAMPLE

The examples in the remainder of the book will be wireless communications examples, specifically radio communications such as cellular communications. In such systems, there are several standard models used for the medium response. In this section we will discuss some of the standard models and provide a set of reference models for performance results in other chapters.

One is the *static* channel, in which the channel coefficients do not change with time. A special case is the AWGN channel, which implies not only that AWGN is present, but that there is a single path ($L = 1$, $\tau_0 = 0$ and $g_0 = 1$). This model makes sense when there are no scattering objects nearby and nothing is in motion. Thus, there is a Line of Sight (LOS) between the transmitter and receiver.

Another one-tap channel is the flat fading channel for which $L = 1$, $\tau_0 = 0$ and g_0 is a complex Gaussian random variable with unity power, i.e.,

$$E\{g_0 g_0^*\} = 1. \tag{1.57}$$

The channel coefficient is random because it is the result of the signal bouncing off of objects (scatterers) and adding at the receiver either constructively or destructively. If there is are many signal paths, the central limit theorem tells us that the coefficient should be Gaussian.

The fading is referred to as Rayleigh fading because the magnitude of the medium coefficient is Rayleigh distributed. The phase is uniformly distributed. This model makes sense when the *delay spread* of the actual channel (maximum path delay minus minimum path delay) is much smaller than the symbol (TDM) or chip (CDM, OFDM) period. The random channel coefficient changes with motion of the transmitter, environment, and/or receiver.

A *block fading model* will be assumed, for which the random fading value remains constant for a block of data then changes to an independent value for each subsequent block of data. Such a model is realistic when short bursts of data are transmitted.

We will also consider static and fading dispersive channels for which $L > 1$. All models will have fixed values for the path delays. The dispersive static channel will be specified in terms of fixed values for the medium coefficients. For the dispersive fading channel, each medium coefficient is a complex, Gaussian random variable. We can collect medium coefficients from different path delays into a vector $\mathbf{g} = [g_0 \quad \ldots \quad g_{L-1}]^T$, where superscript T denotes transpose. We will assume these coefficients are uncorrelated, so that

$$\mathrm{E}\{\mathbf{g}\mathbf{g}^H\} = \mathrm{Diag}\{\alpha_0, \ldots, \alpha_{L-1}\}, \tag{1.58}$$

where α_ℓ is the average path strength or power for the ℓth path. The path strengths are assumed normalized so that they sum to one. For example, a channel with two paths of relative strengths 0 and -3 dB would have path strengths of 0.666 and 0.334.

So what are realistic values for the path delays and average path strengths? Propagation theory tells use that path strengths tend to exponentially decay with delay, so that their relative strengths follow a decaying line in log units. Sometimes there is a large reflecting object in the distance, giving rise to a second set of path delays starting at an offset delay relative to the first set. The Typical Urban (TU) channel model is based on this.

What about path delays? In wireless channels, the reality is often that there is a continuum of path delays. From a Nyquist point of view, we can show that such a channel can be accurately modeled using Nyquist-spaced path delays. The Nyquist spacing depends on the bandwidth of the signal relative to the symbol rate. If the pulse shape has zero excess bandwidth, then a symbol-spaced channel model is highly accurate. In practical systems, there is usually some excess bandwidth, so the use of a symbol-spaced channel model is an approximation. Sometimes the approximation is reasonable. Otherwise, a fractionally spaced channel model is used, in which the path delay spacing exceeds the Nyquist spacing. Typically, $T/2$ (TDM) or $T_c/2$ (CDM,OFDM) spacing is used.

Though not considered here, other fading channel models exist. Sometimes one of the medium coefficients vectors is modeled as having a Rice distribution, which is complex Gaussian with a nonzero mean. This models a strong LOS path. Also, in addition to block fading, time-correlated fading models exist which capture how the fading changes gradually with time.

The medium response models can be extended to multiple receive antennas. For the flat static channel, $L = 1$, $\tau_0 = 0$ and $\mathbf{g}_0 = \mathbf{a}$, where $\mathbf{a}$ is a vector of unity-magnitude complex numbers. The angles of these numbers depend on the direction of arrival and the configuration of the receive antennas. For the flat fading channel, $L = 1$, $\tau_0 = 0$ and $\mathbf{g}_0$ is a set of uncorrelated complex Gaussian random variables with unity power. Note that this implies E_s is the average receive symbol energy *per antenna*.

For the dispersive static channel, we will specify fixed values for the medium coefficients. For the dispersive fading channel, we will specify relative average powers for the medium coefficients.

In CDM and OFDM, the Nyquist criterion is applied to the chip rate and the chip pulse shape excess bandwidth. In CDM systems, the amount of excess bandwidth depends on the particular system, though it is usually fairly small. Experience suggests that fractionally spaced models are needed with light dispersion, whereas chip-spaced models are sufficiently accurate when there is heavy dispersion. For OFDM, chip-spaced models are usually sufficient.

1.5.1 Reference system and channel models

In later chapters, we will use simulation to compare different equalization approaches for a TDM system. Notes on how these simulations were performed are given in Appendix A. Most results will be for QPSK. The pulse shape is root-raised cosine with rolloff (β) 0.22 (22% excess bandwidth).

The following channel models will be used.

TwoTS Dispersive medium with two, nonfading symbol-spaced paths with relative powers 0 and -1 dB (sum of path energies normalized to unity) and angles 0 and 90 degrees.

TwoFS Dispersive medium with two, nonfading half-symbol-spaced paths with relative powers 0 and -1 dB (sum of path energies normalized to unity) and angles 0 and 90 degrees.

TwoTSfade Similar to the TwoTS channel, except that each path experiences independent, Rayleigh fading, i.e., each path is a complex Gaussian random variable. The variances of the random variables are set so that $E\{\mathbf{g}^H\mathbf{g}\} = 1\}$, and the relative average powers are 0 and -1 dB.

1.6 THE LITERATURE

The general system model and its notation are based on [Wan06b, Ful09]. Real and complex Gaussian random variables are addressed in a number of places, including [Wha71].

Digital communications background material, including modulation, channel modeling, and performance analysis, can be found in [Pro89, Pro01]. The notion of Nyquist rate for distortionless transmission is developed in [Nyq28]. Nyquist rate is the result of the fact that if one is given bandwidth B and time duration T, there are 2TW independent dimensions or degrees of freedom [Nyq28, Sha49]. Sending more symbols than independent dimensions leads to ISI. The notion of channel capacity is developed in [Sha48, Sha49].

Cellular communications is described in [Lee95, Rap96]. Background material on OFDM and CDMA can be found in [Sch05, Dah08]. For OFDM, use of the FFT can found in [Wei71], and application to mobile radio communications is discussed in [Cim85]. Using a cyclic prefix in CDM systems is considered in [Bau02]. Information on the discrete Fourier transform can be found in standard signal processing textbooks, such as [Rob87].

While modeling the channel as linear is fairly general, the assumption of Rayleigh or Rice fading is particular to wireless communications. Accurate modeling is important, because equalization design is usually targeted to particular scenarios for which reliable communications is desirable. In the literature, channel modeling information can be found for

- wireless (radio) communications [Tur72, Suz77],
- wireline communications (twisted pair) [Fis95],

- optical communications over fiber [Aza02],
- underwater acoustic communications [Sin09],
- underwater optical communications [Jar08], and
- magnetic recording [Kum94, Pro98].

OFDM equalization when the delay spread exceeds the length of the cyclic prefix is considered in [Van98]. We will not consider it further, though results for the CDM case are applicable by redefining the spreading sequences. Equalization when the channel varies within a symbol period is considered in [Jeo99, Wan06a]. We will not consider it further.

PROBLEMS

The idea

1.1 Suppose a transmitted symbol, either +1 or −1, passes through a channel, which multiplies the symbol by −10 and introduces a very small amount of noise. Suppose the received value is 8.

a) What most likely is the transmitted symbol?

b) What most likely is the noise value?

c) What is the other possible noise value?

1.2 Suppose a transmitted symbol, either +1 or −1, passes through a channel, which multiplies the current symbol by 1, adds the previous symbol multiplied by 2, and introduces a very small amount of noise. Suppose you know that the previous symbol is −1 and the current received value is −1. What most likely is the current symbol?

1.3 Suppose a transmitted symbol s, either +1 or −1, passes through a channel which scales the symbol by 5 and adds −10 and introduces a very small amount of noise. Suppose the current received value is −3. What most likely is the current symbol?

1.4 Suppose a transmitted symbol s, either +1 or −1, passes through a nonlinear channel, which produces $20s^2 + 10s$, and introduces a very small amount of noise. Suppose the current received value is +9. What most likely is the current symbol?

More details

1.5 Suppose we have the MIMO scenario in which $c = 1$, $d = 0$, $e = 0$, and $f = 1$. Also, suppose the two received values are $r_1 = -1.2$ and $r_2 = -0.8$.

a) What most likely was the symbol s_1?

b) What most likely was the symbol s_2?

1.6 Suppose we have the MIMO scenario in which $c = 1$, $d = 0$, $e = 2$, and $f = 1$. Also, suppose the two received values are $r_1 = -1.2$ and $r_2 = -0.8$.

a) What most likely was the symbol s_1?

b) Assuming you detected s_1 correctly, what most likely was the symbol s_2?

1.7 Suppose we have the dispersive scenario in which $c = 4$, $d = 2$, $r_2 = 2.1$, and r_1 is not available.

a) Each symbol can take one of two values. For each of the four combinations of s_1 and s_2, determine the corresponding noise value for n_2.

b) Which combination corresponds to the smallest magnitude noise value?

1.8 Suppose we have the dispersive scenario in which $c = -2$ and $d = 0$. Suppose QPSK is sent and $r_1 = -1.8 + j2.3$.

a) What most likely is the I component of s_1?

b) What most likely is the Q component of s_1?

1.9 Sometimes we want to send two bits in one symbol period. One way to do this is to send one of four possible symbol values: -3, -1, $+1$ or $+3$. Consider the mapping $00 = -3$, $01 = -1$, $10 = +1$ and $11 = +3$. At the receiver, when mistakes are made due to noise, they typically involve mistaking a symbol for one if its nearest neighbors. For example, -3 is detected as -1, its nearest neighbor.

a) When -3 is mistaken as -1, how many bit errors are made?

b) When -1 is mistaken as $+1$, how many bit errors are made?

c) What is the average signal power, assuming each symbol is equi-likely to occur?

d) Suppose the symbol passes through a channel, which multiplies the symbol by -10 and introduces a very small amount of noise. Suppose the received value is -11. What most likely was the transmitted symbol? What were the transmitted bits?

1.10 Suppose we change the mapping to $00 = -3$, $01 = -1$, $11 = +1$ and $10 = +3$, referred to as Gray-mapping.

a) When -1 is mistaken as $+1$, how many bit errors are made?

b) Is there a case where a nearest neighbor mistake causes two bit errors?

The math

1.11 Consider a TDM transmitter using a root-Nyquist pulse shape. The signal passes through a single-path medium with delay $\tau_0 = 0$. Suppose the receiver initially filters the received signal using $v(qT) = \int_{-\infty}^{\infty} r(\tau)p^*(\tau - qT - t_0)\, d\tau$.

a) For $t_0 = 0$, how many symbols does $v(qT)$ depend on?

b) For $t_0 = T/2$, how many symbols does $v(qT)$ depend on?

c) For $t_0 = T$, how many symbols does $v(qT)$ depend on?

d) Suppose the medium consists of *two* paths, with path delays 0 and T seconds. Now how many symbols does $v(qT)$ depend on for $t_0 = 0$?

1.12 Suppose the pulse shape is a rectangular pulse shape, so that $p(t)$ is $1/T$ on the interval $[0,T)$ and zero otherwise.

a) What is $R_p(\tau)$?

b) Is this pulse shape root-Nyquist?

c) Suppose the receiver initially filters the received signal using $v(qT) = \int_{-\infty}^{\infty} r(\tau)p^*(\tau - qT)\, d\tau$. How many symbols does $v(qT)$ depend on?

d) Suppose the receiver initially filters the received signal using $v(qT) = \int_{-\infty}^{\infty} r(\tau)p^*(\tau - qT - T/2)\, d\tau$. How many symbols does $v(qT)$ depend on?

1.13 Consider BPSK, in which a detect static can be modeled as $z \models \sqrt{E_b}s + n$, where s is $+1$ or -1 and n is a Gaussian random variable with zero mean and variance $N_0/2$. Derive (A.8).