PART A
Introductory Overview

1 Introduction

Welcome to mathematics and particularly to its language. You will find it to be a simple language, with only a little grammar and a limited vocabulary, but quite different from the other languages you know. Unlike natural languages such as English, its semantics are precisely defined and unambiguous. In particular, its complete lack of ambiguity enables exact reasoning, probably its greatest advantage. On the negative side, one cannot express such a wide variety of things in the Language of Mathematics as in English, and intentional vagueness, so important in English poetry and much prose, cannot be expressed directly. Nonetheless, it is often surprising what one can express in and with the help of the Language of Mathematics, especially when combined appropriately with English or some other natural language.

Vagueness cannot be expressed directly in the Language of Mathematics, but it can be modeled—precisely and unambiguously—with mathematics. Expressed differently, the Language of Mathematics enables one to make precise and unambiguous statements about vaguely determined things. Probability theory, statistics, and, more recently, fuzzy theory are the mathematical subdisciplines that enable one to talk and write about uncertainty and vagueness—but with precision and without ambiguity.

Although the Language of Mathematics is quite limited in the range of things that can be expressed in it directly, many things outside the Language of Mathematics can be related to mathematical objects as needed for specific applications. Thus, the Language of Mathematics is, in effect, a template language for such applications. Adapting it to the needs of a particular application extends its usefulness greatly and poses the main challenge in its application. This challenge is primarily linguistic, not mathematical, in nature. Helping the reader to meet this challenge is an important goal of this book and underlies essentially all of the material in it.

One of the limitations in the Language of Mathematics is the fact that the notion of time is absent from it completely. This fact is mentioned here, at the very beginning, because the lack of conscious awareness of it has led to many students becoming (and sometimes remaining) very confused without realizing this source of confusion. Time and dynamic processes can easily be and often are modeled mathematically, but this is part of the adaptation of the template Language of Mathematics to the particular application in question. How this can be done is covered in several places in the book, in particular in Section 7.5.

The Language of Mathematics: Utilizing Math in Practice, First Edition. Robert Laurence Baber.

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1.1 WHAT IS LANGUAGE?

A *language* is a medium for:

- Expressing or communicating:
  - Verbally or visually (e.g., in written form)
  - Facts, opinions, thoughts, ideas, feelings, desires, or commands
  - At one time or from one time to another
  - Between different people or from one person to her/himself at a different time
- Thinking
- Analyzing or reasoning

Every language employs abstract symbols—verbal, visual, and sometimes using other senses, such as touch—to represent things. In many natural languages, the visual form was developed to represent the verbal form, so that there is a close relationship between the spoken and written forms. Other languages, however, have developed spoken and written forms which are not directly related. Originally, their symbols were often pictorial in nature, albeit often rather abstractly. One can think of such a language as two distinct languages, a spoken language and a written language. In the case of Sumerian (the earliest known written language), written symbols (in cuneiform) represented what we think of today as words, so that there was no direct connection between the written and spoken forms of the language. Later, the cuneiform symbols were taken over by other languages (e.g., Akkadian) to represent syllables in the spoken language, establishing a direct connection between the spoken and written forms of the language. Still later, other languages introduced symbols for parts of a syllable, leading to the abstract symbols that we now call letters.

Mathematics exhibits the characteristics of a language described above. The range and distribution of topics communicated in natural languages such as English and those communicated in the Language of Mathematics differ in some significant ways, however. Feelings and emotions are rarely expressed in mathematical terms. Vague (i.e., imprecisely defined) terms are not permitted in the Language of Mathematics. Otherwise, all of the characteristics of a language mentioned above are found in the Language of Mathematics, albeit with different emphasis and importance.

Scientists and historians believe that language began by our distant ancestors communicating with one another via sounds made by using the vocal chords, the mouth, the lips, and the tongue (hence our term *language*, from *lingua*, Latin for "tongue"). This form of language was useful for communicating between individuals at one time and when they were physically close to one another. Sounds made in other ways (e.g., by drums) were used for communication over greater distances, but still between people at essentially one time. Visual signals of various kinds were also employed in much the same way.

Marks on bones apparently representing numbers are believed to be an early (perhaps the earliest) form of record keeping: communicating from one time to another. Gradually, this idea was extended to symbols for various things, ideas,

P1: TIX/XYZ P2: ABC
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concepts, and so on, leading in a long sequence of developmental steps to language as we know it today. It is noteworthy that even precursors to writing apparently included numbers, the basic objects of arithmetic, and hence of mathematics. The earliest forms of writing known to us today certainly included numbers. Thus, the development of languages included elements of mathematics from very early times.

Symbols and signs recorded physically in visually observable form constitute records that store information for later use. They are a major source of our knowledge of early civilizations. Their durability seriously limits our knowledge of those early civilizations. The most durable records known to date are clay tablets inscribed with cuneiform characters and inscriptions on stone monuments. Records of old societies that used less durable forms of writing have decomposed in the meantime and are no longer available. Those potentially interesting historical records are lost forever.

Recently, humans have begun to communicate with symbols they cannot observe visually but only with the help of technical equipment. The symbols are in the form of electrically and magnetically recorded analog signals and, still more recently, digital symbols. In some cases, these signals and symbols are direct representations of previous forms of human language. In other cases, they are not; rather, they represent new linguistic structures and forms.

Natural languages such as English, Chinese, and Arabic have evolved to enable people to communicate about all the kinds of things they encounter in everyday life. Therefore, the universes of discourse of natural languages overlap considerably. The Language of Mathematics has, however, evolved to fulfill quite different, very specific, and comparatively quite limited goals. It is a language dealing only with abstract things and concepts and having only a rather limited scope. Therefore, for the purposes of applications to real-world situations, the Language of Mathematics is not a finished language but, instead, a *template language*. When applying mathematics, the Language of Mathematics must be adapted for each application. This is done by specifying how the elements of the mathematical description are to be *interpreted* in the terminology of the application area. A new *interpretation* must normally be given for each application, or at least for each group of closely related applications.

1.2 WHAT IS MATHEMATICS?

The archaeological record suggests that mathematics probably originated with counting and measuring things and recording those quantities. Soon, however, people began to pose and answer questions about the quantities of the things counted or needed for some purpose; that is, they began to reason about quantities and to solve related problems. As early as about 4000 years ago, mathematics included the study of geometrical figures: in particular, of relationships between their parts and between numerical measures of their parts. Later, mathematicians turned their attention to ever more abstract things and concepts, including ones not necessarily of a numerical or geometrical nature.

The description of language at the beginning of Section 1.1. also applies to mathematics. The relative emphasis on communication on the one hand and on reasoning

P1: TIX/XYZ P2: ABC
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and analysis on the other hand is perhaps different, but there is much common ground. Some would say that mathematics itself, in the narrow sense, concentrates on concepts and techniques for reasoning and analyzing, and that mathematics is therefore not itself a language. However, mathematics does use extensively a particular language that has evolved to facilitate reasoning and analyzing. Such reasoning and analyzing is performed primarily by manipulating the symbols of mathematical language mechanistically, according to precise rules. The Language of Mathematics is also used extensively for expressing and communicating both over time and between people. It is also used for thinking.

What is mathematics today? Someone once answered that question with "What mathematicians do." That, of course, begs the question "What do mathematicians do?" Answered most succinctly, they reason logically about things—artificial, abstract things—not just about quantities, numbers, or numerical properties of various objects.

Many of those things, although artificial and abstract, are useful in modeling actual things in the real world; for example:

- Structures of buildings, dams, bridges, and other engineering artifacts
- Materials of all kinds and their properties
- Mechanical devices and equipment
- Machines, engines, and all kinds of energy conversion devices and systems
- Vehicles of all types: land, underwater, water surface, air, space
- Electrical circuits and systems composed of them
- Communication systems—wired and wireless—and their components
- Systems for cryptography
- Molecules, atoms, nuclei, and subatomic particles
- Chemical reactions and chemical reactors
- Systems for generating and distributing electrical power
- Nuclear decay and interaction processes and nuclear reactors
- Heating and cooling systems
- Computer software
- Prices in financial markets
- Sales and markets
- Order processing and billing systems
- Inventory control systems
- Various business assets and liabilities
- Data and information of all types, including names and addresses
- Social, economic, business and technical systems
- Relationships among objects, properties, and values of all (not just numerical) types
- Structural aspects of languages, natural and artificial

P1: TIX/XYZ P2: ABC
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Such models better enable us to describe, to understand, and to predict things in the real world—to our considerable benefit.

It is important that the reader always be consciously aware that mathematics today consists of much more than numbers and arithmetic. Important as these are, they constitute only a part of mathematics. The main goal of mathematics is not to work with numbers but to reason about objects, properties, values, and so on, of all types. Mathematicians work mostly with relationships between these things. Mathematicians actually spend very little of their time calculating with numbers. They spend most of their time reasoning about abstract things. Logic is an important part of that work.

Different subdisciplines of mathematics have been created in the course of time. Numbers, counting, geometric figures, and quantitative analyses constituted the first subdisciplines. Among the more recent is logic. Unfortunately, especially for the novice learning mathematics, logic used its own terminology and symbols, and this distinction is still evident in the ways in which mathematical logic is often taught today. This leads many beginners to believe that logic is somehow fundamentally different from the other subdisciplines and that a different notation and point of view must be learned. The linguistic approach presented in this book integrates these views and notational schemes, so that the beginner need learn only one mathematical language. Although this integrating view is already present in some mathematical work, it is not really widespread yet, especially not in teaching mathematics.

1.3 WHY USE MATHEMATICS?

Among the several reasons for using mathematics in practice, the two most important are:

- To find a solution to a problem. The statement of a problem or the requirements that a solution must fulfill can often be transformed into the solution itself.
- To understand something better and more thoroughly: for example, to identify all possibilities that must be considered when defining a problem and solving it.

Examples are given in Chapter 2, in Section 6.13.2, and in Chapter 8.

The author and many others have found in the course of their work that mathematics frequently enables them to think effectively about and solve problems they could not have come to grips with in any other way. As long as a problem is expressed in English, one can reason about the problem and deduce its solution only when one constantly keeps the precise meaning of the words, phrases, and sentences consciously in mind. If the text is at all long, this becomes unworkable and very subject to error. It is likely that some important detail will be overlooked. After formulating the problem in mathematical language, the expressions can be transformed in ways reflecting and representing the reasoning about the corresponding English sentences. However, the expressions can be transformed mechanistically according to generally applicable rules without regard to the meaning of the expressions. This effectively reduces

P1: TIX/XYZ P2: ABC
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reasoning to transformations independent of the interpretation of the expressions being transformed, simplifying the process considerably and enabling much more complicated problems to be considered and solved. People who are specialists in transforming mathematical expressions but who are not specialists in the application area can find solutions. In this way, the mental work of reasoning can be largely reduced to the mechanistic manipulation of symbols. In the words of Edsger W. Dijkstra, a well-known computer scientist whose areas of special interest included mathematics and logic, one can and should "let the symbols do the work."

For the reasons cited above, one should convert from English to the Language of Mathematics at as early a stage as possible when reasoning about anything. Transforming the mathematically formulated statement of a problem into its solution sounds easy. Although it is, in principle, straightforward, it can be computationally intensive and tedious to do manually. For a great many applications, algorithms for solving the problem and computer programs for calculating the numerical solutions exist. Where such solutions do not already exist, mathematicians can often develop them.

The usefulness of the Language of Mathematics for the purposes listed above derives from its precision, the absence of ambiguity, and rules for transforming mathematical expressions into various equivalent forms while preserving meaning. These characteristics are unique to the Language of Mathematics. Natural languages, lacking these characteristics, are much less satisfactory and useful for the purposes noted above.

1.4 MATHEMATICS AND ITS LANGUAGE

In order to reason logically about things, mathematicians have developed a particular language with particular characteristics. That language—the Language of Mathematics—and other languages developed by human societies—such as English—are similar in some respects and different in some ways.

Distinguishing characteristics of the Language of Mathematics are its precision of expression and total lack of ambiguity. These characteristics make it particularly useful for exact reasoning. They also make it useful for specifying technical things. The Language of Mathematics is a language of *uninterpreted* expressions, which are described in Section 3.4. This does not imply that mathematical expressions are uninterpretable. They can be and often are interpreted when applying mathematics in the real world: when associating mathematical values, variables, and expressions with entities in the application area (see Chapters 6 and 7, especially Section 6.13).

Within the Language of Mathematics, however, expressions are never interpreted. When transforming mathematical expressions in order to reason or analyze, one should be very careful not to interpret them, as doing so takes one out of the Language of Mathematics and into English. This can result in the loss of precision and the introduction of ambiguity—the loss of the very reasons for using mathematics—without one being aware that the loss is occurring. Reasoning must be conducted only and strictly within the abstract world of uninterpreted expressions, applying only mathematically valid transformations to the expressions without interpreting them. In

P1: TIX/XYZ P2: ABC
JWBS073-01 JWBS073-Baber August 4, 2011 22:52 Printer: Markono

this way, mathematical expressions represent the ultimate form of abstraction of the logically essential aspects of a practical problem, containing only the logical relationships between its various aspects and without any inherent reference to the real world represented. The final results of the transformations representing reasoning are mathematical expressions representing solutions. The latter expressions are, of course, interpreted in order to implement the solution in the application domain.

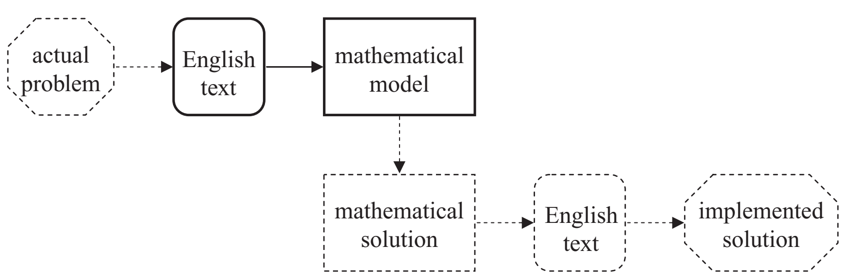
Mathematics and the Language of Mathematics are not the same thing. Facility with the language is a prerequisite for understanding and applying mathematics effectively. Unfortunately, mathematics is usually taught without explicitly introducing the language used. The student of mathematics is left to discover the language unassisted. Although this is possible for some people, it makes learning mathematics unnecessarily difficult and time consuming for many. For others, it constitutes the difference between learning mathematics and giving up before getting very far.

My experience learning, using, and teaching others mathematics and how to use it in practice has convinced me that looking at mathematics consciously as a language can facilitate the learning process, understanding, and the ability to apply mathematics in practice. I believe that it can even enable some people to learn how to use mathematics effectively who would otherwise be turned off mathematics completely by their early exposure to it—and unfortunately, there are many such people in today's world.

One must distinguish between the Language of Mathematics on the one hand and that part of English that is used to talk and write about mathematics on the other hand. The Language of Mathematics itself builds expressions upon values, variables, functions, and structures of these components. To communicate with other people about mathematics, one typically uses a combination of normal English and specialized mathematical terminology and jargon, just as is done in other specialized disciplines, such as the several scientific and engineering fields, medicine, and law. This distinction is discussed in greater depth in Section 6.10.

1.5 THE ROLE OF TRANSLATING ENGLISH TO MATHEMATICS IN APPLYING MATHEMATICS

The steps in the overall process of applying mathematics to a problem are illustrated in the following diagram, in which translating English to mathematics is highlighted.



Translating from an English description of a problem to the Language of Mathematics is the second step in the process of applying mathematics to a problem. The mathematical model is needed in order to reason logically about the problem, to analyze it systematically, accurately, and precisely, and to find a solution.

The mathematical model itself represents an interface between:

- The English language–oriented analysis and identification of the problem and the requirements for its solution, and
- The purely mathematical analysis and determination of one or more solutions

The mathematical model, being written in the Language of Mathematics, is an unambiguous statement of the problem and the requirements that any solution must satisfy. Its meaning in terms of the application is defined by the interpretation of the values, variables, and functions in the mathematical model, but its meaning in terms of the subsequent mathematical analysis is independent of that interpretation and the application. Thus, the mathematical model represents a boundary between the English language view of the application and the mathematical view of the application. The mathematical model connects, couples, the application and the mathematical worlds with each other, and at the same time it separates, insulates, isolates each from the other.

This, in turn, means that a solution can be determined in the mathematical world without regard to the application world, and correspondingly, any solution that satisfies the mathematical model will be applicable to the application world, without regard to how that solution was found. In the extreme, the specialists who find the mathematical solution do not really need to know anything about the application world to the left of the mathematical model in the diagram above. Correspondingly, the specialists in the application domain do not need to know or understand how the solution was found in the mathematical world below the mathematical model in the diagram at the beginning of this section.

Both specialist groups must, however, be able to read and understand the mathematical model (the interface specification) itself. Two factors are critical:

- That the application specialists agree that the mathematical model is an appropriate statement of the application problem and the requirements any solution must satisfy
- That the mathematical specialists agree that the mathematical model is a syntactically correct and semantically meaningful mathematical expression in the Language of Mathematics

That is the extent of the need for communication between the two groups of specialists. Lest the reader think that this is an unrealistic, utopian view, it must be pointed out that exactly this type of interface specification underlies all engineering work. Such interface specifications enable—and are prerequisites for—the division of labor required for the efficient and effective realization of any large-scale task,

P1: TIX/XYZ P2: ABC
JWBS073-01 JWBS073-Baber August 4, 2011 22:52 Printer: Markono

such as the design of a system for generating and distributing electricity regionally, nationally, or internationally; the design of a vehicle of any type (ship, automobile, truck, airplane, etc.); the design of a building; or the design of international telephone and communication systems.

Expressed differently but equivalently, communication and mutual understanding among the people involved is the key in such efforts. It is not necessary that every team member have the mathematical ability to solve all aspects of a problem or that every team member be an expert in all aspects of the application area. What is important is that they all understand what the problem is: what problem is being solved. The ability to read the expressions in a mathematical model and understand their meaning is sufficient; actually finding a solution can be left to specialists. That is one of the advantages of a mathematical formulation of the problem: Finding a solution depends only on the unambiguous mathematical expressions, not on what they are interpreted to mean in the application domain.

Although problems regarding accuracy, discrepancies, and errors can have their origins in any and all steps shown in the earlier diagram, particularly severe consequences arise from inaccuracies in translating the English text into the mathematical model. The especially important step of translating from English into the Language of Mathematics is the subject of this book.

1.6 THE LANGUAGE OF MATHEMATICS VS. MATHEMATICS VS. MATHEMATICAL MODELS

The Language of Mathematics, mathematical models, and mathematics are three different but closely related entities. The *Language of Mathematics* is the language of the notational forms used in mathematics. *Mathematical models* express relationships among the various variables, values, and functions that describe some part of the world to which mathematics is being applied. *Mathematics*, what one does in and with the Language of Mathematics, includes the notational forms, that is, the Language of Mathematics, and definitions of many different mathematical objects, techniques for transforming mathematical expressions and the proofs of their general validity, the theory underlying such techniques, and proofs of characteristics of the various mathematical objects.

A rough comparison will perhaps help to make these distinctions clearer. Corresponding to the Language of Mathematics, the English language can be thought of as the definitions of English words together with the grammar and accepted conventions for forming variations of the words (e.g., conjugating verbs, forming plurals and participles) and for combining words into sentences. Corresponding to mathematics, English in general can be considered to be the collection of the language itself together with what one does in and with the language, that is, the literature written (and spoken) in English and the associated culture. Corresponding to a mathematical model is an individual piece of English literature.

The distinction between the English language, English in general (i.e., together with its literature and culture), and particular pieces of English literature is commonly

P1: TIX/XYZ P2: ABC
JWBS073-01 JWBS073-Baber August 4, 2011 22:52 Printer: Markono

made in teaching and learning. The goals and contents of a course in English grammar are different from the goals and contents of a course in English literature and literary culture. The goals and contents of a course in an individual piece of literature, or in a collection of closely related literature (such as by one author), are different again. The approaches employed in such different types of courses are, correspondingly, different.

Unfortunately, the corresponding distinction between the Language of Mathematics, mathematical models, and mathematics is usually not made in teaching or learning any of these topics. The Language of Mathematics is, for the most part, treated implicitly and the student is, also implicitly, expected to absorb intuitively the linguistic aspects of the Language of Mathematics on his or her own. The nominal topics of courses are either mathematics or particular application areas. Courses on mathematics deal with specific subdisciplines of mathematics, such as differential calculus, integral calculus, linear algebra, real analysis, analytical geometry, and number theory. Definitions of mathematical objects relevant to the subdiscipline and techniques for manipulating expressions typically arising in the subdiscipline make up the content of those courses. Courses on particular application areas deal with phenomena in the application area in question and present the relevant mathematical models together with relevant aspects of mathematics. The mathematical models are presented as the relevant mathematics, not explicitly as models as such. Some examples of such application-oriented courses are physics, atomic physics, nuclear reactor physics, chemistry, mechanics (statics and dynamics), operations research, inventory control, electrical circuit theory, switching circuits, control theory, thermodynamics, heat transfer, and fluid mechanics. The mathematical flavor varies among such courses, but all emphasize the mathematical models relevant to the particular application area.

Notable in both the mathematics courses and the application-oriented courses are (1) that linguistic aspects of notation—the Language of Mathematics—are either absent or only implicit; and (2) that formulating new mathematical models (e.g., translating an English text into a mathematical model) is not dealt with.

The material in this book distinguishes consciously between these three topics: the Language of Mathematics, mathematical models, and mathematics. The reader should pay conscious attention to the distinction between them and to each individually. A major goal of the material in this book is to provide explicit guidelines for formulating new mathematical models based on descriptive English text.

1.7 GOALS AND INTENDED READERSHIP

By presenting the Language of Mathematics explicitly and systematically, this book is intended to help its readers to improve their ability to apply mathematics beneficially in their own work: in particular, by improving their ability to translate English descriptions into the Language of Mathematics. This book is not intended as a textbook on mathematics itself or on any subdiscipline of mathematics.

In summary, this book is written for the following people:

- Those who would like to improve their ability to apply mathematics effectively, systematically, and efficiently to practical problems
- Teachers of mathematics who would like to improve their ability to convey to their students a better understanding and appreciation of mathematics and how to apply it in practice
- Those who are curious about the linguistic nature and aspects of mathematical notation

More specifically, the intended readership includes:

- Engineers, consultants, managers, scientists, technicians, and others who could benefit vocationally and professionally by a greater ability to use and apply mathematics in their work
- Students in tertiary educational institutions
- Students in secondary schools especially interested in mathematics, science, or languages
- Educators designing mathematics curricula, course content, and teaching materials for students at all levels
- Teachers of mathematics, science, or languages in tertiary educational institutions (universities, polytechnics, and vocational and technical schools)
- Teachers of mathematics, science, or languages in secondary schools
- Teachers in primary schools who introduce pupils to mathematics and especially to word problems
- Persons with a general or an intellectual interest in mathematics, science, or language

The prerequisites for reading this book are a recognition and conscious awareness that mathematics might be useful in your work or other activities and a desire to realize its potential benefits. Basic knowledge of English grammar is also necessary; the essentials needed are summarized in Section 6.2. This book is self-contained in the sense that no particular mathematical background is assumed or needed.

Readers with an extensive mathematical background will find much of the mathematical notation presented in this book familiar. Their earlier mathematical courses will have given them the mathematical models needed for classical professional practice, but will not have taught them how to formulate mathematical models for new or significantly different types of problems themselves. Some, but not all, readers will have developed this ability intuitively and implicitly. This book will show all of them explicitly how to formulate new mathematical models based on English descriptions of problems to be analyzed and solved. Logical mathematical expressions will also be new to some readers with a mathematical background,

P1: TIX/XYZ P2: ABC
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especially to those whose mathematics concentrated on differential and integral calculus.

Readers with limited or no prior mathematical knowledge will find both the mathematical notation and the mathematics presented in this book largely new. Of particular importance to this group of readers is this book's goal of helping them to develop their ability to contribute actively to the translation of English descriptions of application requirements into the Language of Mathematics: that is, to formulate mathematical models. Part of this is helping them to become familiar with mathematical notation—with the Language of Mathematics itself.

This book is not about mathematics as a subject and is not intended to help you learn mathematics itself or any particular subdiscipline of mathematics. The book does not deal with the various topics typically covered in texts on mathematics. If you encounter mathematical topics in this book that you want to know more about, refer to an appropriate book on the relevant area of mathematics.

1.8 STRUCTURE OF THE BOOK

The **Preface** outlines the societal background and environment in which mathematics and its application are relevant and useful. It also describes the author's experience, observations, and thoughts leading to the decision to write the book and to the selection of its contents.

Part A, ***Introductory Overview*** (Chapters 1 and 2), deals with the subject of the book. Chapter 1 introduces the topics covered in the book: language, mathematics, reasons for applying mathematics to practical problems, the distinction between mathematics, its language (notational forms), and mathematical models. Chapter 1 also states the goals and outlines the intended readership. Guidelines for the reader are presented. Chapter 2 gives examples of the application of mathematics and mathematical models.

Part B, *Mathematics and Its Language* (Chapters 3, 4, and 5): An important purpose and goal of mathematical notation—the Language of Mathematics—is to enable ideas and concepts to be expressed unambiguously and to enable and encourage a corresponding way of thinking. In addition to mathematical notation, Part B presents a number of concepts that have been found in the course of time to be beneficial and important for many applications of mathematics and that have therefore become fundamental parts of mathematics. They are presented here because they are some of the reasons for the form and structure that the Language of Mathematics has acquired. Some acquaintance with these mathematical concepts is a prerequisite for understanding the nature of the Language of Mathematics and for acquiring even a passive knowledge of it.

In short, Part B is an introductory overview of those things that mathematicians and nonmathematicians who apply mathematics in their work often use, think, talk, and write about.

Part C, *English, the Language of Mathematics, and Translating Between Them* (Chapters 6, 7, and 8):

P1: TIX/XYZ P2: ABC
JWBS073-01 JWBS073-Baber August 4, 2011 22:52 Printer: Markono

- Compares important characteristics of English and the Language of Mathematics
- Identifies their similarities and differences and the implications of their differences for translating
- Describes how to translate between English and the Language of Mathematics, giving extensive guidelines and illustrating this process with extensive examples.

Part D, ***Conclusion*** (Chapter 9), summarizes the main points developed in the book.

The **appendices** present various aspects of mathematics that some readers will find interesting and useful as additional background. **Appendix B** comprises a list of mathematical symbols used in the book, gives their meanings, and refers the reader to sections of the book describing them in detail. **Appendix G** is a **glossary** of English terms and their usual translations into the Language of Mathematics. The other appendices give additional information on numbers, selected structures in mathematics, mathematical logic, the mathematical treatment of waves, and programming languages in contrast with the Language of Mathematics. Finally, recommendations are given to the reader for finding works on selected subtopics among the vast literature on mathematics.

1.9 GUIDELINES FOR THE READER

Everything in this book is simple, and some of it is trivial. While reading, look for generality and simplicity—the simple things, structures, concepts—not complexity. Don't look for complicated things because you will not find them. If you expect them, you will be confused by their absence. If something looks complicated, you are reading complexity into the material where there is none. Read it again, looking for the simplicity. What you encounter may seem strange, unfamiliar, and lengthy, sometimes tedious, but it is not complicated.

Although every step in this book's development of the description of the Language of Mathematics is simple, there are many such steps. Especially in the mathematically oriented Chapters 3, 4, and 5 of Part B, try to understand the material in each step before proceeding to the next. Only partial understanding at one stage will usually be followed by a weaker understanding at the next stage, and your degree of understanding will lessen progressively. The material will then seem to be complicated.

On the other hand, it is sometimes useful to skip over material you do not at first understand, read other parts of the book, and return to that material later. This strategy is particularly useful to newcomers to material in any book who are looking for challenging material to stretch themselves and to widen their horizons. It is also useful to newcomers who seek only selected subtopics on their first reading.

Some readers will find some, perhaps much, of the material in the book to be intuitive. That intuitive, unconscious knowledge will be transferred into conscious, explicit knowledge. Experience shows that people can apply knowledge more

P1: TIX/XYZ P2: ABC
JWBS073-01 JWBS073-Baber August 4, 2011 22:52 Printer: Markono

effectively, more extensively, and to more complicated problems when they are explicitly and consciously aware of that knowledge than when that knowledge is only intuitive.

While and after reading the book, you will find it helpful to refer to standard books on mathematics and particular subdisciplines for more specific information on particular areas of mathematics. Select books and articles on those mathematical topics of relevance to the applications in which you are interested.

The reader already familiar with mathematics and its application will find some material in the book to be old and familiar, especially the contents of Chapter 3. These readers should scan those parts of the book, however, as some of the material is presented in new, different, and unconventional ways.

Readers will find that the topic of the book is presented in ways quite different from traditional mathematics teaching, and some might therefore question its validity. The very point of the book is that mathematics can be viewed from different standpoints and in different ways and that some of these approaches are absent from traditional mathematics teaching and learning. Those missing approaches can be useful, even critical, for some people. Starting by viewing the linguistic aspects of mathematics is the most important of these approaches.

Nothing in the book is mathematically incorrect, at least not intentionally so. If you do find an error, please let me know so that I can correct it.

Mathematics and the Language of Mathematics are like literature and the language in which that literature is written. If you don't understand the language, you will not understand the literature written in that language. Similarly, if you are not familiar with the Language of Mathematics, you will not get very far with your study of mathematics. If you try to study the literature (e.g., mathematics) anyway, without first learning its language—as newcomers and students of mathematics are too often forced today to do—your progress will at best be unnecessarily slow and frustrating. You will have to learn the language implicitly as you try to study the subject. The result will be that it will take you unnecessarily long to learn either the language or the subject, your knowledge of both will be incomplete, and the level of knowledge of both that you will attain will be unnecessarily limited. So begin your study of mathematics by examining explicitly its language, and you will find that you will be able to proceed faster and go farther in mathematics than you would otherwise be able to do.

At this point it is worthwhile to consider how people learn their first language (i.e., their native language) and how they learn subsequent languages. "Native" ability in a person's first language is acquired initially from the parents, especially the mother, who does not try so much to teach the child the language but simply to use it to communicate with the child. Later, in play and in school, this native ability is developed further by communicating with other speakers of the language, especially peers. Later, formal instruction complements and refines the native ability already acquired. When learning a second language, this learning process is usually reversed: First, basic aspects of the new language are learned in formal instruction. Then the learners develop their capability and fluency further by communicating in the language: the more, the better. This suggests that the best way to develop a thorough

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knowledge of and ability in the Language of Mathematics is to learn the basics and then, as soon as possible, to use it to communicate, both with others and with yourself, as much as possible.

Given serious motivation and a reasonable amount of time to spend on learning, most people find that they can learn a new language faster than they learned their first language. (A question for the reader to ponder: How many years elapsed before you were really fluent in your native language?) However, developing a truly native ability in a subsequent language can take much longer than developing fluency. In fact, many people who develop complete fluency in a foreign language never develop truly native ability, that intuitive feeling for colloquial, formal, and common usage patterns and many other aspects of the language: for those aspects of the language never covered in school. Knowing a language and living with it, living in it, embracing it, are not the same things.

Whenever one learns a new language (e.g., French), one must read, write, and talk about something. That something is typically taken from the daily lives of the speakers of the language in question. So, when learning French, one will read about France, typical cultural aspects of France and its people, its politics, life in France, situations arising there, its history, its literature, and so on. When learning the Language of Mathematics, one similarly must read, write, and talk about things about which mathematicians and people who apply mathematics in practice typically read, write, and talk: the culture of the field of mathematics, objects defined and used in mathematics and in its typical application areas, and so on. Therefore, this book presents both the language and some aspects of the subject of mathematics, hand in hand. The language is the initial and guiding factor. Mathematics itself is presented here only in conjunction with linguistic aspects of the Language of Mathematics and insofar as is necessary or helpful to proceed with an examination of the language. After completing the book, the reader will be in a much better position to read traditional mathematical books and articles and thereby learn more mathematics.

The reader who wishes to learn this material in order to be able to apply it actively and effectively to practical problems should examine the passages in the book thoroughly, filling in the sketches of proofs presented here. On the other hand, the reader who wishes only to become passively aware of the ideas and concepts presented may believe them uncritically, reading them somewhat cursorily and noting only the main points developed. Practice in reading mathematical expressions and models is still required.

The book may be read in various ways. Depending on the interests and prior knowledge of the reader, different balances between scanning cursorily, reading, and studying the several parts, chapters, and sections are appropriate.

It is recommended that the reader begin by scanning or reading the Preface and Part A, Introductory Overview, consisting of Chapters 1 and 2. While perusing the examples in Chapter 2, the reader should keep in mind the questions posed at the beginning of the Chapter.

Chapters 3 through 7 provide the theoretical basis for the Language of Mathematics and for translating from English to the Language of Mathematics. Sections 3.1, 3.2, and 3.3 present the foundations of the Language of Mathematics and are therefore required reading. At least one subsection of Section 3.4 on expressions should be read and understood and the others at least scanned.

Chapter 8, with its many examples, can be viewed as "practice." When reading each of the longer examples, scan it first, noting only a few details of each kind, to gain an overview of the overall structure of the problem and of its solution. Then reread the example in more detail. If you do not first gain an overview of a lengthy example, there is a danger that you will become bogged down in the middle of a long series of details and lose sight of the path through which you are being led. This can lead to confusion and frustration. First acquiring and then maintaining a good overview of the structure of the example will enable you to maintain your orientation on your way from the beginning through to the end of the example.

Whether one reads the theory or the practice first, or skips between them while reading, is a matter of personal choice; pick the sequence that you feel enables you to make the best progress. Ultimately, both are necessary to acquire full understanding and satisfactory application skills, so your choice is not theory or practice, it is the sequence in which you cover both. It has long been recognized that in the practically oriented professions such as medicine, law, and engineering, a good balance between both theory and practice, with a thorough, well-developed knowledge of both, is necessary for success. (The Roman architect and engineer Vitruvius recognized the need for a balanced and thorough knowledge of both theory and practice over 2000 years ago.) If either theory or practice is lacking, the results will be unsatisfactory. Furthermore, a little knowledge and skill can get one into trouble, while more extensive knowledge and ability are required to get one out. Worse yet, with inadequate knowledge, one will often not even recognize that one is in trouble and how and why one got there.

Finally, Chapter 9, the summary, should be read—again if the reader read it earlier.

The appendices are included for reference and, if desired, additional insight into certain selected topics.