CHAPTER 1

QUALITY FACTORS OF ESA

1.1 INTRODUCTION

In a 1947 paper, Harold Wheeler defined an electrically small antenna (ESA) as an antenna that could be enclosed within a radian sphere (Wheeler, 1947). The radian sphere was a sphere of radius equal to $\lambda/2\pi$, where λ is the wavelength. The antennas used by Marconi and Fessenden in the early years of wireless telegraphy were electrically small antennas even though they were very large physical structures, often involving wires strung as an inverted fan or cone from masts several hundred feet tall. These antennas were electrically small antennas since in order to achieve long-distance transmission the wavelengths used, typically greater than 3000 m, were much longer than the antenna heights. These electrically small antennas were characterized by a very low radiation resistance and a large capacitive input reactance. The purpose of the large inverted fans and cones was to increase the antenna capacitance and thus reduce the capacitive reactance. Wheeler introduced the radiation power factor (RPF) as a figure of merit for these electrically small antennas. He considered two basic antenna types: the magnetic dipole or loop antenna consisting of a solenoid coil with N turns, length b, and radius a; and a short electric dipole antenna consisting of a thin wire of length b and with capacitive loading at each end by means of circular conducting disks of radius a. The radiation power factor was defined as the ratio of the radiation resistance to the reactance of the antenna. For the solenoid loop antenna, the radiation resistance is given by $R_{\mathrm{m}} = 320N^2\pi^6(a/\lambda)^4$. An approximate expression for the inductance of the solenoid

Small Antenna Handbook, Robert C. Hansen and Robert E. Collin.

coil is $L = \mu_0 N^2(\pi a^2/b)$. For the small electric dipole, the radiation resistance is given by $R_e = 80\pi^2(b/\lambda)^2$ and the capacitance between the two circular plates is $C = \varepsilon_0 \pi a^2/b$ when the fringing effects are neglected. From these expressions, we can calculate the radiation power factors, p_m for the magnetic dipole and p_e for the electric dipole, as follows:

$$p_m = \frac{R_m}{\omega L} = \frac{4}{3}\frac{\pi^3 a^2 b}{\lambda^3} \tag{1.1a}$$

$$p_e = R_e \omega C = \frac{4}{3}\frac{\pi^3 a^2 b}{\lambda^3} \tag{1.1b}$$

Wheeler modified Equation 1.1 by magnetic and electric shape factors derived from statics. For ESA, the radiation power factors are very small. Also see Wheeler (1975).

Wheeler's radiation power factors are related to the Q parameter introduced by Kenneth S. Johnson, an authority on wire transmission at Bell Telephone Laboratories. Initially, Johnson used the symbol K to represent the ratio of the inductive reactance to the resistance of a coil, $K = \omega L/R$. In 1920, while working on wave filters, invented by G. A. Campbell, he replaced the symbol K by the symbol Q and introduced the lowercase symbol q for the analogous quantity $\omega C/G$ for a capacitor C, where G is the parallel conductance of a capacitor (called condenser in those days). Later on in 1927 he used Q for both in his U.S. Patent No. 1,628,983. His introduction of this symbol was adopted by most people working with tuned circuits in radio receivers in the early days of radio broadcasting.

It is easily shown that the response of a tuned circuit, consisting of a parallel or series connection of an inductor and a capacitor, is reduced by the factor $1/\sqrt{2}$ when the circuit is detuned by a fractional amount $\Delta\omega/\omega = 1/2Q$, provided the Q is equal to 10 or more. Thus, $1/Q$ is the 3 dB bandwidth (BW) of the tuned circuit. If a resistive load is connected across the tuned circuit such that a maximum amount of power can be obtained from the circuit, the Q of the loaded circuit is reduced by a factor of 2 and the 3 dB bandwidth is increased by a factor of 2. Tuned circuits with high values of Q were needed in order to achieve high selectivity in the tuned radio frequency radios. This led to extensive efforts to design radio frequency coils with low loss resistance. During the World War II years, the Q became widely used to describe the sharpness of the resonance curve of both electric and mechanical resonators such as microwave cavities, quartz crystal resonators, and so on. If we have a tuned circuit with a capacitor C in parallel with an inductor L, it is known that the resonant frequency of the circuit is given by $\omega_0 = 1/\sqrt{LC}$ and that at resonance the time-averaged energy stored in the capacitor is equal to that in the inductor. If the inductor has a series resistance R and the current in the inductor is I, then the average energy stored in the magnetic field around the inductor is given by $W_m = I^2L/4$ and the average power dissipated in the resistor is $P_L = I^2R/2$. When we introduce these into the definition of Q, as used by Johnson, we can express Q in the form

$$Q = \frac{\omega L}{R} = \frac{2\omega(I^2 L/4)}{I^2 R/2} = \frac{2\omega W_{\mathrm{m}}}{P_{\mathrm{L}}} = \frac{\omega(W_{\mathrm{m}} + W_{\mathrm{e}})}{P_{\mathrm{L}}}$$

$$Q = \frac{\omega(\text{average energy stored in resonant circuit})}{(\text{average energy dissipated per second})} \tag{1.2a}$$

This latter definition of the quality factor or Q of a resonant circuit is the most commonly used one.

For ESA, it is approximately

$$Q \simeq \frac{\omega\, dX/d\omega}{2R} \tag{1.2b}$$

Harrington (1965) and Rhodes (1966, 1974) extended the bandwidth relationship for circuits, $\mathrm{BW} = 1/Q$, to dipole-type antennas. Dipole bandwidth and $1/Q$ were compared by Hansen (2007); the match was excellent for $ka \leq 0.3$ and good for $ka \leq 0.5$.

The Q of electrically small antennas represents a fundamental limit on the performance of these antennas, in particular their bandwidth. If the designer of electrically small antennas is cognizant of this fundamental limit, he (she) will not expend excessive time designing an antenna to achieve what cannot be achieved. Although the operational bandwidth of electrically small antennas can be increased by the use of multiple tuned circuits in the matching network or by inclusion of magnetic materials, this invariably introduces significant additional loss and a reduced efficiency of the antenna. Fano's theory of broadband matching, which will be briefly discussed in a later section, shows that a maximum increase in the 3 dB bandwidth is by a factor of 3.2. A realistic goal is to increase the bandwidth by a factor of 2. The double-tuned coupled circuits used in intermediate-frequency (IF) transformers in superheterodyne radio receivers, to increase the audio fidelity of these amplifiers, increased the bandwidth by a factor of around 2 (see Terman, 1943). Active circuits, called non-Foster networks, can in principle provide broadband matching. These are discussed in Chapter 2 but have their own limitations.

Canonical types of ESA are loaded dipoles, patch antennas with uncommon substrates, loop antennas with air or magnetic cores, dielectric resonator antennas, as well as bent conductors of unusual shapes, and antennas incorporating metamaterials. Many of the latter cannot be realistically fabricated and do not work according to the theories proposed for them. These antennas are discussed in later sections. Some clever ideas have been found to be impractical. A long list of antenna ideas that resemble science fiction is given in Chapter 5.

L. J. Chu (1948) applied the concept of Q to small antennas and used the definition given by Equation 1.2a to find a lower bound for Q of small antennas whose radiated fields could be expressed in terms of spherical waves. The results obtained by Chu provide a more accurate measure of the limitations of ESA than Wheeler's power factors do. Since the pioneering work of Chu, many other authors have contributed to the evaluation of antenna Q.

1.2 CHU ANTENNA *Q*

Chu considered a hypothetical antenna that was contained entirely within a sphere of radius a. The electromagnetic field outside this sphere can be described in terms of infinite series of spherical transverse electric (TE) and transverse magnetic (TM) modes. The vector wave functions in a spherical coordinate system were derived by W. W. Hansen (1935). These modes consist of two sets of transverse (divergence-free) modes and a set of longitudinal modes (modes with zero curl). The former are designated by the symbols $\mathbf{M}_{nm}(\mathbf{r})$ and $\mathbf{N}_{nm}(\mathbf{r})$ while the latter are represented by the symbol $\mathbf{L}_{nm}(\mathbf{r})$. In regions external to the source region, only the transverse modes are required in the expansion of an arbitrary electromagnetic field. Thus, in the region external to a sphere of radius a that completely encloses the small antenna, the electric and magnetic fields can be represented in the following form (Stratton, 1941; Collin, 1990):

$$\mathbf{E}(\mathbf{r}) = \sum_{n,m} C^e_{nm}\mathbf{M}_{nm}(\mathbf{r}) + \sum_{n,m} D^e_{nm}\mathbf{N}_{nm}(\mathbf{r}) \tag{1.3a}$$

$$\mathbf{H}(\mathbf{r}) = \sum_{n,m} C^h_{nm}\mathbf{M}_{nm}(\mathbf{r}) + \sum_{n,m} D^h_{nm}\mathbf{N}_{nm}(\mathbf{r}) \tag{1.3b}$$

where

$$\mathbf{M}_{nm}(\mathbf{r}) = \nabla \times \mathbf{a}_r \left[P^m_n(\cos\theta)k_0 r h^2_n(k_0 r)^{\cos}_{\sin}\,\phi\right] \tag{1.3c}$$

$$\mathbf{N}_{nm}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{a}_r \left[P^m_n(\cos\theta)k_0 r h^2_n(k_0 r)^{\cos}_{\sin}\,\phi\right] \tag{1.3d}$$

In these expressions, $P^m_n(\cos\theta)$ are the associated Legendre polynomials, $h^2_n(k_0 r)$ is the spherical Hankel function of order n, and $k_0 = 2\pi/\lambda_0$ is the free space wave number. $C^e_{nm}, C^h_{nm}, D^e_{nm}$, and D^h_{nm} are amplitude constants. For each individual mode, the stored electric and magnetic energy divided by the radiated power is independent of the azimuthal index m. Also, the transverse electric or TE_{nm} and the transverse magnetic or TM_{nm} modes are duals of each other with the electric and magnetic fields interchanged, so their Q's are the same. Thus, it is sufficient to consider only the TM_{n0} mode, which is what Chu did.

For the TM_{n0} modes, the electric and magnetic fields are given by

$$E_\theta = C^h_n \frac{-k_0 \sin\theta}{j\omega\varepsilon_0}\frac{dP_n(\cos\theta)}{d(\cos\theta)}\frac{d[k_0 r h^2_n(k_0 r)]}{d(k_0 r)} \tag{1.4a}$$

$$E_r = C_n^h \frac{n(n+1)}{j\omega\varepsilon_0 r^2} P_n(\cos\theta)\left[k_0 r h_n^2(k_0 r)\right] \tag{1.4b}$$

$$H_\phi = C_n^h \frac{\sin\theta}{r} \frac{dP_n(\cos\theta)}{d(\cos\theta)} \left[k_0 r h_n^2(k_0 r)\right] \tag{1.4c}$$

where C_n^h is an amplitude constant. The mode wave impedance at $r = a$ is given by

$$Z_{\mathrm{w},n} = \frac{E_\theta}{H_\phi} = jZ_0 \frac{d\left[k_0 a h_n^2(k_0 a)\right]/d(k_0 a)}{k_0 a h_n^2(k_0 a)} \tag{1.5}$$

Instead of evaluating the stored reactive electric and magnetic energy and the radiated power from the electric and magnetic fields, Chu expanded the mode wave impedance into a continued fraction that could be interpreted as a ladder network with $2n$ elements consisting of alternating capacitors and inductors, and terminated in a normalized resistance of 1 Ω. Conventional circuit analysis could then be used to determine the stored electric energy in the capacitors and the stored magnetic energy in the inductors, as well as the radiated power, which equals the power dissipated in the terminating resistor.

The mode normalized wave impedance can be expressed in the form

$$\frac{Z_{\mathrm{w},n}}{Z_0} = \left[\frac{j}{\rho} + \frac{j}{h_n^2}\frac{dh_n^2}{d\rho}\right] \tag{1.6}$$

where $\rho = k_0 a$. We now use the following recurrence relation for spherical Bessel functions $f_n(\rho)$:

$$f_n = \frac{2n-1}{\rho} f_{n-1} - f_{n-2}$$

and the relation

$$\frac{df_n}{d\rho} = f_{n-1} - \frac{n+1}{\rho} f_n$$

where $\rho = k_0 a$, to get

$$\frac{Z_{\mathrm{w},n}}{Z_0} = \frac{j}{\rho} + \frac{j}{h_n^2}\left[h_{n-1}^2 - \frac{n+1}{\rho}h_n^2\right] = \frac{n}{j\rho} + \frac{1}{\dfrac{h_n^2}{jh_{n-1}^2}} = \frac{n}{j\rho} + \frac{1}{\dfrac{1}{jh_{n-1}^2}\left[\dfrac{2n-1}{\rho}h_{n-1}^2 - h_{n-2}^2\right]}$$

$$= \frac{n}{j\rho} + \frac{1}{\dfrac{2n-1}{j\rho} + \dfrac{1}{\dfrac{h_{n-1}^2}{jh_{n-2}^2}}} = \frac{n}{j\rho} + \cfrac{1}{\cfrac{2n-1}{j\rho} + \cfrac{1}{\cfrac{2n-3}{j\rho} + \ddots \cfrac{1}{\cfrac{3}{j\rho} + \cfrac{1}{\cfrac{1}{j\rho} + 1}}}} \tag{1.7}$$

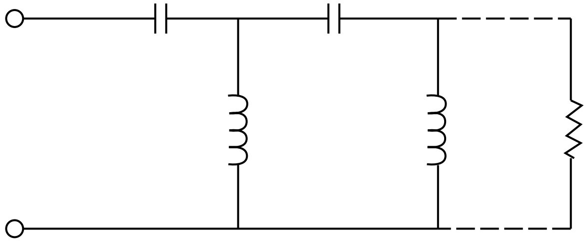

FIGURE 1.1 Chu ladder network.

This expansion can be interpreted as the impedance of the ladder network illustrated in Figure 1.1 for TM modes. L and C values decrease with each step.

The TM_{n0} modes store more electric energy than magnetic energy and hence the mode must be tuned to resonance by adding some additional magnetic energy, that is, by an inductive reactance. If we assume that this is done, then the total stored reactive energy will be twice the electric energy W_e, which equals the energy stored in the capacitors in the equivalent electric circuit. The stored electric energy and the power dissipated in the $1\,\Omega$ terminating resistance can be determined by conventional circuit analysis, but it becomes very tedious to carry out for $n > 3$. For the $n = 1$ mode, the Q was found to be given by

$$Q_1 = \frac{2\omega W_e}{P_r} = \frac{1}{(k_0 a)^3} + \frac{1}{k_0 a} \tag{1.8}$$

Note that the Q in Hansen (1981) has a typo error.

Chu evaluated the Q of the higher order modes by using an approximate equivalent circuit that was obtained as follows. For a series RLC circuit, the input reactance is $X = \omega L - 1/\omega C$ and $dX/d\omega - X = L + 1/\omega^2 C - L = 1/\omega^2 C$. When the input current to the circuit is I, the power dissipated in R is $|I|^2 R/2$ and the electric energy stored in C is $|I|^2/4\omega^2 C$. Hence, the Q is given by

$$Q = \frac{2\omega W_e}{P_r} = \frac{\omega}{\omega^2 CR} = \frac{1}{2R}\left[\frac{dX}{d\omega} - \frac{X}{\omega}\right] \tag{1.9}$$

Chu equated X to the imaginary part of $Z_{w,n}/Z_0$ given by Equation 1.6. Chu also evaluated the Q of the higher order modes using this method. In addition, Chu also evaluated the Q of a combination of a TE_{n1} and a TM_{n1} mode. When the amplitudes of the two modes are equal, a circularly polarized field can be produced. By using his approximate value of the Q, it was found that the Q of the combined modes was one-half that of a single mode.

The ratio of the directive gain to the Q for an antenna radiating a total of N modes was also determined by Chu but he did not find the optimum gain for a given Q, a problem that later authors solved. Any antenna that is contained within a sphere of radius a will have additional energy storage within the enclosing sphere and will consequently have a higher Q. Thus, the Q that Chu found is a lower bound on the Q

of any lossless antenna. Many ESA have a Q that is considerably larger than Chu's lower bound. When the antenna Q is large, one can infer that the bandwidth of the antenna will be small but one cannot always assume that it will be equal to $1/Q$ since the tuning circuit and losses may provide a larger bandwidth. Harrington (1958, 1960, 1961) expanded on the work of Chu, but followed Chu's approximate method to obtain the Q's of TE_{n1} and TM_{n1} modes. Harrington showed that the maximum gain of an antenna, obtained by using only a finite number of TE_{n1} and TM_{n1} modes, was given by

$$G_{\max} = \sum_{n=1}^{N} (2n+1) = N^2 + 2N \tag{1.10}$$

If there was no constraint on the mode amplitudes, an arbitrarily large gain would theoretically be possible. However, high-order modes are very difficult to excite because their wave impedances (wave admittances for the TE_{n0} modes) are very large, so in practice unusually large gains cannot be achieved. The high-order modes also store large amounts of reactive energy, so high gain implies a large antenna Q and a narrow bandwidth. Other investigators have considered optimizing the ratio of gain divided by the antenna Q.

A different approach to Q was taken by Thiele et al. (2003), based on the far-field pattern of a small source. A "pattern" Q is based on the integral of pattern over visible space and that integral over all space, including invisible. Their Q values are higher than those of Chu. An electrically small dipole with sinusoidal current distribution was used to provide the fields for the integrations. These pattern Q values are eight times larger than the Chu results. Dipole bandwidth calculations were also made by Hujanen and Sten (2005). Kalafus (1969) calculates Q for higher modes as well, using series expansions for the integrals of energy. Then coefficients of polynomials representing Q are given. Another calculation of Q due to higher modes is done by Harrington (1960).

An egregious example of claiming antennas that violate the fundamental limitations on small antennas is provided by Underhill and Harper (2002, 2003). For a short folded dipole of length L, they have reactance proportional to kL. However, it is well known that the folded dipole reactance is four times that of the constituent dipole, which is proportional to $1/kL$. For a small loop of diameter D, they have radiation resistance proportional to kD^2, when it is widely accepted that it is proportional to k^4D^4. These errors appear to be due to applying static formulations to electromagnetic problems.

Another paper claims that orthogonal TE and TM modes produce a gain of 3 and a Q half that of either mode (Kwon, 2005). Both are in error; the input power and the peak power density are both doubled, leaving the gain at 1.5 and the Q that of one mode.

An erroneous calculation of bandwidth limitations occurred because of confusion between total energy, stored energy, and radiated energy. This resulted in a bandwidth of 16π times the fundamental limit for VSWR ≤ 2 (Chaloupka, 1992).

Geyi (2003a, 2003b) reexamines the task of maximizing the ratio D/Q, directivity divided by Q. He corrects some inconsistencies in Fante (1969), with the result that the maximum D/Q for a directive antenna (with both TE and TM modes) is $3/Q$, whereas that for an omnidirectional antenna is, as expected, $3/2Q$.

1.3 COLLIN AND ROTHSCHILD *Q* ANALYSIS

The next contribution to evaluating antenna Q was the paper by Collin and Rothschild (1964), where the stored energy was evaluated in terms of the electromagnetic fields. This work provided convenient closed-form formulas for the Q of any mode and was expanded upon by Fante (1969) and also by McLean (1996).

The configuration that will be analyzed consists of a spherical core of radius a with a current sheet located on the surface. The current sheet is chosen so as to excite only a single TE_{n0} mode for which the electric and magnetic fields in the region $r > a$ can be chosen to be

$$E_\phi = C_n^e \frac{\sin\theta}{r} \frac{dP_n(\cos\theta)}{d(\cos\theta)} \left[k_0 r h_n^2(k_0 r)\right] \tag{1.11a}$$

$$H_\theta = C_n^e \frac{k_0 \sin\theta}{j\omega\mu_0 r} \frac{dP_n(\cos\theta)}{d(\cos\theta)} \frac{d\left[k_0 r h_n^2(k_0 r)\right]}{d(k_0 r)} \tag{1.11b}$$

$$H_r = -C_n^e \frac{n(n+1)}{j\omega\mu_0 r^2} P_n(\cos\theta) \left[k_0 r h_n^2(k_0 r)\right] \tag{1.11c}$$

An electric current sheet proportional to $\mathbf{a}_\phi H_\theta(a)$ will support this mode. For simplicity, we will assume that the amplitude constants C_n^e are equal to unity. The TE_{nm} modes store more magnetic energy than electric energy. The Q can be expressed in terms of the total average stored reactive magnetic energy both inside and outside the spherical surface $r = a$ since we will assume that the antenna is tuned to resonance with an additional capacitive reactance. Thus, the Q is given by

$$Q_n = \frac{2\omega W_\mathrm{m}}{P_\mathrm{r}} \tag{1.12}$$

where P_r is the total radiated power. The total energy in the electromagnetic field is infinite since it includes the energy associated with the far-zone radiation field, which is energy that is being transmitted to infinity as radiated power. The radiated power is obtained by integrating the real part of the complex Poynting vector over the surface of a sphere with very large radius and is readily found to be given by

$$P_\mathrm{r} = \frac{k_0\pi}{\omega\mu_0} \frac{2n(n+1)}{2n+1} \tag{1.13}$$

The power flow at infinity is equal to the energy density in the electromagnetic field multiplied by the velocity $c = 1/\sqrt{\mu_0\varepsilon_0}$ integrated over θ and ϕ. The energy density in the radiation field is split equally between that in the electric field and that in the magnetic field. By using the asymptotic value of the Hankel function, the energy density for the electric and magnetic fields for very large values of r is found to be

$$w_e = w_m = \frac{\varepsilon_0 \sin^2\theta}{4r^2}\left[\frac{dP_n(\cos\theta)}{d(\cos\theta)}\right]^2 \tag{1.14}$$

After multiplying by $r^2\sin\theta$ and integrating over θ and ϕ, we will denote these energy densities by $W_{e,\mathrm{Rad}}$ and $W_{m,\mathrm{Rad}}$. It is found that

$$W_{e,\mathrm{Rad}} = W_{m,\mathrm{Rad}} = \frac{\pi\varepsilon_0 n(n+1)}{2n+1} \tag{1.15}$$

It is easy to verify that $c(W_{e,\mathrm{Rad}} + W_{m,\mathrm{Rad}}) = 2cW_{m,\mathrm{Rad}} = P_r$. In the original paper by Collin and Rothschild, they made the hypothesis that the energy density $w_e + w_m$ should be subtracted from the total energy density $(\varepsilon_0/4)|\mathbf{E}|^2 + (\mu_0/4)|\mathbf{H}|^2$ before integrating over the total volume in order to obtain the average stored reactive energy. After evaluating the total reactive energy, they used the integral of the complex Poynting vector over the surface $r = a$ to obtain an expression for $W_m - W_e$, which does not contain the energy associated with the radiation field. By this means, they obtained separate expressions for the average stored electric and magnetic reactive energy. McLean (1996) simplified this procedure by simply subtracting the energy densities w_e and w_m, respectively, from the total electric and magnetic field energy densities. We will follow McLean's procedure in the derivation given below.

By using the expressions for the magnetic field given in Equations 1.11b and 1.11c, the stored magnetic reactive energy in the volume outside the surface $r = a$ is found to be given by

$$W_m = \int_a^\infty \int_0^\pi \left\{\frac{\mu_0}{4}\left[|H_\theta|^2 + |H_r|^2\right] - \frac{\varepsilon_0 \sin^2\theta}{4r^2}\left[\frac{dP_n(\cos\theta)}{d(\cos\theta)}\right]^2\right\} 2\pi r^2 \sin\theta \, d\theta \, dr \tag{1.16}$$

When r becomes very large, $r^2|H_r|^2$ is asymptotic to $1/r^2$, so the integral of this term vanishes at the upper limit $r = \infty$. The asymptotic limit of the term $\mu_0|H_\theta|^2/4$ is $(\varepsilon_0 \sin^2\theta/4r^2)[dP_n(\cos\theta)/d(\cos\theta)]^2$ and is cancelled by the last term in the above integral. Hence, the integral over r converges as r tends to infinity. If the magnetic energy density in the radiation field had not been subtracted out, the integral would have diverged as r tended toward infinity.

The evaluation of W_{m} requires the following two integrals:

$$\int_0^{\pi} [P_n(\cos\theta)]^2 \sin\theta\, d\theta = \frac{2}{2n+1} \tag{1.17a}$$

$$\int_0^{\pi} \left[\frac{dP_n(\cos\theta)}{d\theta}\right]^2 \sin\theta\, d\theta = \frac{2n(n+1)}{2n+1} \tag{1.17b}$$

After completing the integrations over θ, the expression for W_{m} reduces to

$$W_{\mathrm{m}} = \frac{\pi\varepsilon_0 n(n+1)}{k_0(2n+1)} \int_{k_0 a}^{\infty} \left\{ \left[\frac{d\rho\, j_n(\rho)}{d\rho}\right]^2 + \left[\frac{d\rho\, y_n(\rho)}{d\rho}\right]^2 + n(n+1)\left[j_n^2(\rho) + y_n^2(\rho)\right] - 1 \right\} d\rho \tag{1.18}$$

where j_n and y_n are the spherical Bessel functions of the first and second kinds. In order to carry out the integrations in Equation 1.18, the following integral is used (Morse and Feshbach, 1953):

$$\int \rho^2 [f_n(\rho)]^2 d\rho = \frac{\rho^2}{2}(f_n^2 - f_{n-1} f_{n+1}) \tag{1.19a}$$

where f_n can be j_n or y_n. In order to transform all of the integrals into this form, the following relations are needed:

$$j_n = \frac{\rho}{2n+1}(j_{n-1} + j_{n+1}) \tag{1.19b}$$

$$\frac{d(\rho j_n)}{d\rho} = \frac{\rho}{2n+1}[(n+1)j_{n-1} - n j_{n+1}] \tag{1.19c}$$

From Equations 1.19b and 1.19c, we can derive the relations

$$j_{n+1} = \frac{n+1}{\rho} j_n - \frac{1}{\rho}\frac{dj_n(\rho)}{d\rho} \tag{1.19d}$$

$$j_{n-1} = \frac{1}{\rho}\left[n j_n + \frac{1}{\rho}\frac{dj_n(\rho)}{d\rho}\right] \tag{1.19e}$$

By using these relations, we can show that

$$\left[\frac{d(\rho j_n)}{d\rho}\right]^2 = \frac{\rho^2}{2n+1}\left[(n+1)j_{n-1}^2 + n j_{n+1}^2\right] - n(n+1) j_n^2 \tag{1.19f}$$

The integral in Equation 1.18 now becomes

$$W_{\mathrm{m}} = \frac{\pi\varepsilon_0 n(n+1)}{k_0(2n+1)} \int_{k_0 a}^{\infty} \left\{ \frac{(n+1)\rho^2}{2n+1} \left[j_{n-1}^2(\rho) + y_{n-1}^2(\rho) \right] + \frac{n\rho^2}{2n+1} \left[j_{n+1}^2(\rho) + y_{n+1}^2(\rho) \right] - 1 \right\} d\rho$$

$$= \frac{\pi\varepsilon_0 n(n+1)}{k_0(2n+1)} \left\{ k_0 a - \frac{(k_0 a)^3 (n+1)}{2(2n+1)} \left[j_{n-1}^2(k_0 a) - j_{n-2}(k_0 a) j_n(k_0 a) \right.\right.$$

$$\left.\left. + \frac{n}{n+1} j_{n+1}^2(k_0 a) - \frac{n}{n+1} j_n(k_0 a) j_{n+2}(k_0 a) \right] \right\} + \text{corresponding terms in}$$

$$y_n, y_{n-1}, y_{n-2}, y_{n+1}, \text{and } y_{n+2} \tag{1.20}$$

The quality factor Q for the nth mode, which we will designate by the symbol Q_n^{TE}, is obtained by multiplying by $2\omega/P_{\mathrm{r}}$ and is given by

$$Q_n^{\mathrm{TE}} = k_0 a - \frac{(k_0 a)^3 (n+1)}{2(2n+1)} \left[j_{n-1}^2(k_0 a) - j_{n-2}(k_0 a) j_n(k_0 a) + \frac{n}{n+1} j_{n+1}^2(k_0 a) \right.$$

$$\left. - \frac{n}{n+1} j_n(k_0 a) j_{n+2}(k_0 a) \right] + \text{ corresponding terms in}$$

$$y_n,\ y_{n-1},\ y_{n-2},\ y_{n+1},\ \text{and } y_{n+2}$$

This expression can be simplified by using the recurrence relation (Equation 1.19b) to eliminate the Bessel functions of order $n-1$, $n-2$, and $n+2$. When this is done, we obtain

$$Q_n^{\mathrm{TE}} = k_0 a - \left[\frac{(k_0 a)^2}{2} + (n+1) k_0 a \right] \left[j_n^2(k_0 a) + y_n^2(k_0 a) \right] - \frac{(k_0 a)^3}{2} \left[j_{n+1}^2(k_0 a) + y_{n+1}^2(k_0 a) \right]$$

$$+ \frac{2n+3}{2} (k_0 a)^2 \left[j_n(k_0 a) j_{n+1}(k_0 a) + y_n(k_0 a) y_{n+1}(k_0 a) \right] \tag{1.21}$$

which is the same expression as that given in the Collin and Rothschild's (1964) paper. The Q_n^{TE} can be expressed as a power series in inverse powers of $k_0 a$ by using

$$(k_0 a)^2 \left[j_n^2(k_0 a) + y_n^2(k_0 a) \right] = C_n^2 + D_n^2 \tag{1.22a}$$

$$(k_0 a)^2 \left[j_n(k_0 a) j_{n+1}(k_0 a) + y_n(k_0 a) y_{n+1}(k_0 a) \right] = C_n D_{n+1} - D_n C_{n+1} \tag{1.22b}$$

where

$$C_n = \sum_{m=0}^{2m \le n} \frac{(-1)^m (n+2m)!}{(2m)!(n-2m)! 2^{2m} (k_0 a)^{2m}}$$

and

$$D_n = \sum_{m=0}^{2m \le n-1} \frac{(-1)^m (n+2m+1)!}{(2m+1)!(n-2m-1)! 2^{2m+1} (k_0 a)^{2m+1}}$$

For the first three modes, the results are

$$Q_1 = \frac{1}{k_0 a} + \frac{1}{(k_0 a)^3} \tag{1.23a}$$

$$Q_2 = \frac{3}{k_0 a} + \frac{6}{(k_0 a)^3} + \frac{18}{(k_0 a)^5} \tag{1.23b}$$

$$Q_3 = \frac{6}{k_0 a} + \frac{21}{(k_0 a)^3} + \frac{135}{(k_0 a)^5} + \frac{675}{(k_0 a)^7} \tag{1.23c}$$

The Q of the first three modes is shown in Figure 1.2. Note that the Q rapidly becomes very large as soon as the parameter $k_0 a$ becomes less than unity, and that Equations 1.23a–1.23c are exact.

Collin and Rothschild (1964) applied the same method to calculate the Q of cylindrical modes excited outside the surface of a cylinder with radius a. The Q of cylindrical modes was found to have a similar dependence on the radius of the cylinder as the spherical modes have on the radius of the circumscribing sphere.

The TM_{n1} modes are the dual of the TE_{n1} modes and have the same value for the Q_n. The expressions for the electric and magnetic reactive energies are interchanged, so $W_e > W_m$. If the TE_{10} and TM_{10} modes are excited with equal amplitude from a one-port input network, then the electromagnetic field outside the sphere with radius a will contain equal amounts of reactive electric and magnetic energies. Since the TE_{n1} and TM_{n1} modes are orthogonal for energy storage and radiated power, these quantities can be summed when both sets of modes are excited. In general, there will also be some energy stored within the circumscribing sphere. We will let these be represented by W_e' and W_m'. For the purpose of discussion, we will assume that $W_m > W_e$. The following situations can occur:

$$W_m + W_m' > W_e + W_e'$$

$$W_m + W_m' < W_e + W_e'$$

For the first case, we would need to add some stored electric energy ΔW_e in order to tune the system to resonance, that is,

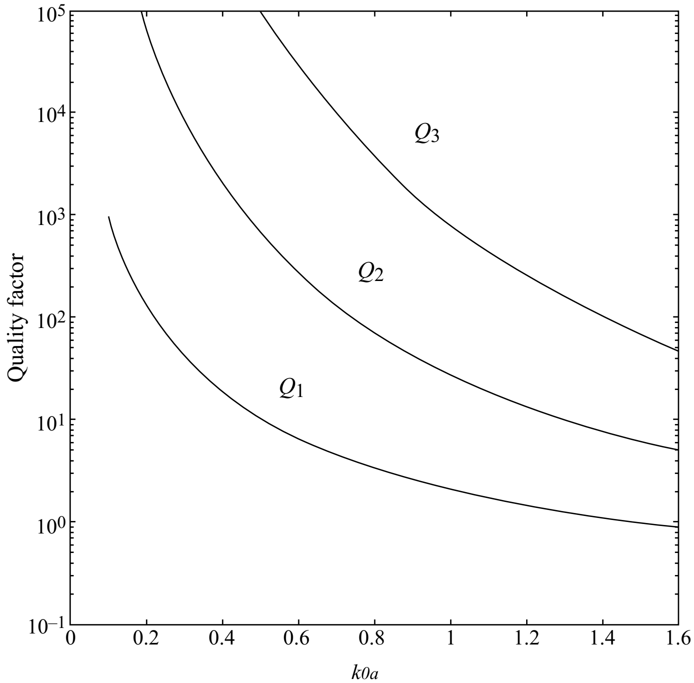

FIGURE 1.2 The quality factors (or the first three TE_{n0} and TM_{n0} modes. Only the stored reactive energy outside the circumscribing sphere of radius a is included.

$$W_m + W'_m = W_e + W'_e + \Delta W_e$$

Clearly, if the Q is calculated using W_m for the time-averaged stored magnetic energy, this will give a lower bound on the antenna Q since the total stored magnetic energy is larger because it includes the internal stored magnetic energy W'_m. For the second case, we would need to add additional magnetic energy ΔW_m such that

$$W_m + W'_m + \Delta W_m = W_e + W'_e$$

Again it is clear that using only the external stored magnetic energy will give a lower bound on the antenna Q.

If the TE_{10} mode is excited with a phase angle $\pi/2$ relative to that for the TM_{10} mode but with an equal amplitude, then the radiated field everywhere will be circularly polarized. The power across any spherical surface will be independent of time. If the internal stored electric and magnetic energies are also balanced, then the system will be resonant at all frequencies. The radiated power will be twice that of a single mode. The stored energy will be the sum of W_m from the TE_{10} mode and that from the TM_{10} mode, which is proportional to $1/k_0a$ and thus leads to the following lower bound on the Q (McLean, 1996; Collin, 1998):

$$Q_{\mathrm{TE+TM}} = \frac{1}{2}\left\{\left[\frac{1}{(k_0a)^3} + \frac{1}{k_0a}\right] + \frac{1}{k_0a}\right\} = \frac{1}{2(k_0a)^3} + \frac{1}{k_0a} \tag{1.24}$$

1.4 THAL ANTENNA Q

Any antenna that is contained within a sphere of radius a will have additional energy storage within the enclosing sphere and will consequently have a higher Q. Thus, the Q that Chu found is a lower bound on the Q of any lossless antenna. Many ESA have a Q that is considerably larger than Chu's lower bound. When the antenna Q is large, one can infer that the bandwidth of the antenna will be small but one cannot always assume that it will be equal to $1/Q$ since the tuning circuit and losses may provide for a larger bandwidth.

In order to complete the derivation of the new lower bound on antenna Q, we need to consider the effects of energy stored within the sphere of radius a. In two recent papers, Thal (1978, 2006) reevaluated the Q of TE_{n1} and TM_{n1} modes by assuming that the antenna consisted of a suitable current sheet on the surface of the sphere of radius a. This allowed the modes excited in the interior of the sphere to be included in the energy storage and hence led to larger values for the minimum achievable Q. This work was based on the use of continued fraction expansions for the mode impedances in both the internal and external regions. This current sheet can be chosen so as to excite a single TE_{n0} or TM_{n0} mode. The only boundary condition that needs to be applied is the continuity of the tangential electric field across the current sheet. Thal extended the circuit analysis of Chu by developing a ladder network that included the energy inside the enclosing sphere.

Hansen and Collin (2009) extended the exact formulation in terms of spherical modes to include the energy stored inside the sphere. The result is a quotient of spherical Bessel and Hankel functions. Numerical values are shown in Table 1.1, and as expected these agree with those published by Thal. Figure 1.3 shows the Chu-Q for TE_1 and TM_1, and the Thal-Q for TE_1 and Thal-Q for TM_1. Exact formulas are those in Section 1.5 for $\mu = 1$ and $\varepsilon = 1$.

TABLE 1.1 New Q Values

ka	Chu-Q (TM or TE)	Thal-Q (TM)	Thal-Q (TE)
0.1	1010.0	1506.0	3030.0
0.15	302.96	448.51	908.90
0.2	130.00	190.58	390.00
0.25	68.000	98.506	204.00
0.30	40.370	57.684	121.11
0.35	26.181	36.850	78.540
0.40	18.125	25.111	54.380
0.45	13.196	17.991	39.590
0.50	10.000	13.421	30.004

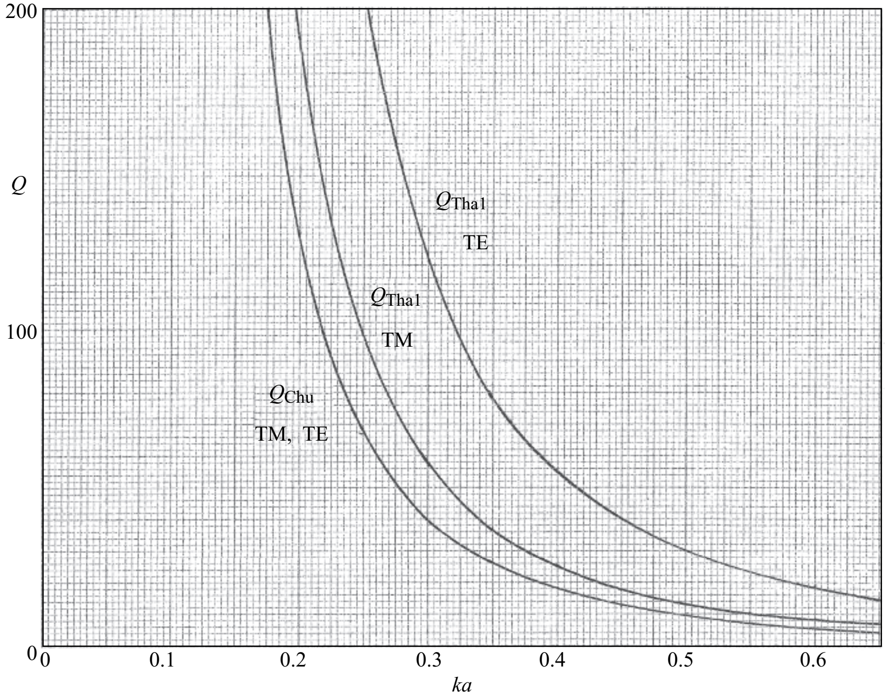

FIGURE 1.3 Q for lowest order modes.

The increase in Q due to the energy stored inside the sphere is (Hansen and Collin, 2009)

$$\Delta Q = \mathrm{SF}\left\langle \frac{(ka)^3}{2}\left[j_n{}^2 - j_{n-1}j_{n+1}\right] + \frac{(ka)^2}{2n+1}\left[(n+1)j_n j_{n-1} - nj_n j_{n+1}\right]\right\rangle \quad (1.25)$$

The scale factors are

$$\begin{aligned} \mathrm{SF_{TM}} &= 1 + \frac{[(n+1)y_{n-1} - ny_{n+1}]^2}{[(n+1)j_{n-1} - nj_{n+1}]^2} \\ \mathrm{SF_{TE}} &= \frac{[j_n(ka)]^2 + [y_n(ka)]^2}{[j_n(ka)]^2} \end{aligned} \quad (1.26)$$

Unlike the Chu-Q case, the new formulas do not have Q expressed as a two- or three-term formula. This was remedied by Hansen and Collin (2009) who performed a least pth fit to the exact values for both the TM_1 and TE_1 modes for two terms. Least-squares fitting has a limitation in that the errors at the interval ends are different from those in the middle of the interval. This was corrected by Bandler and Charalambous (1972) with the least pth fit. It takes the pth root of the sum of the

TABLE 1.2 TM Q Formula Errors

ka	Q_{new}	Q_{approx}	% Error
0.10	1506.0	1507.1	0.07
0.15	448.51	449.20	0.15
0.20	190.58	191.00	0.22
0.25	98.506	98.830	0.33
0.30	57.684	57.910	0.39
0.35	36.850	37.010	0.43
0.40	25.111	25.210	0.39
0.45	17.991	18.030	0.22
0.50	13.421	13.410	– 0.08

function values each raised to the pth power. For $p \geq 10$, the errors are evenly distributed. The calculated values were fitted with $p = 20$, for both TM_1 and TE_1 modes. The TM coefficients were close to 0.707 and 1.5, so these were used; TE coefficients were close to 3.

$$\begin{aligned} TM_1: \quad & Q = \frac{1}{\sqrt{2}ka} + \frac{1.5}{(ka)^3} \\ TE_1: \quad & Q = \frac{3}{ka} + \frac{3}{(ka)^3} \end{aligned} \tag{1.27}$$

Table 1.2 shows the errors versus ka for the TM_1 case; the maximum error is 0.43% over the important range of $ka \leq 0.5$; for the TE_1 formula, the errors are even smaller. These useful formulas will be easy to remember!

1.5 RADIAN SPHERE WITH MU AND/OR EPSILON: TE MODES

One could assume that the spherical core was a material with permittivity and permeability greater (or less) than those of free space. Indeed, Wheeler (1958) did evaluate a spherical antenna consisting of a coil wound on the surface of a sphere, with a permeability greater than that of free space. Two recent papers by Kim et al. (2010) and Kim and Breinbjerg (2011) also address this problem. See also McLean, Foltz, and Sutton, 2011. The analysis starts with the Collin and Rothschild's (1964) and Hansen and Collin's (2009) papers, modifying the analysis to allow a dielectric–magnetic core. The papers by Kim's group show that for TE modes and a magnetic core, a Q approaching the Chu lower bound can be realized when the radius of the core tends toward zero.

Now consider the case of the excitation of TE_{n0} modes and assume that the interior of the sphere is filled with lossless material having a relative permittivity ε_r and a relative permeability μ_r. The only case that can be analyzed without having to specify the details of the system of currents used to excite the electromagnetic field is the use of a current sheet located on the surface $r = a$. This current sheet can be chosen so as to excite a single TE_{n0} or TM_{n0} mode. The only boundary condition that needs to be applied is the continuity of the tangential electric field across the current sheet. For the case of TE_{n0} modes, the fields are given by Equations 1.11a–1.11c but with the Hankel functions replaced by the spherical Bessel functions $j_n(ka)$ and k_0 replaced by $k = \sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}}k_0$, thus

$$E_\phi = C_n^e \frac{\sin\theta}{r}\frac{dP_n(\cos\theta)}{d(\cos\theta)}[krj_n(kr)] \tag{1.28a}$$

$$H_\theta = C_n^e \frac{k\sin\theta}{j\omega\mu_\mathrm{r}\mu_0 r}\frac{dP_n(\cos\theta)}{d(\cos\theta)}\frac{d[krj_n(kr)]}{d(kr)} \tag{1.28b}$$

$$H_r = -C_n^e \frac{n(n+1)}{j\omega\mu_\mathrm{r}\mu_0 r^2}P_n(\cos\theta)[krj_n(kr)] \tag{1.28c}$$

The stored magnetic energy within the sphere is given by an integral similar to Equation 1.18 but with the terms involving y_n dropped, changing k_0 to k, and ε_0 to $\varepsilon_\mathrm{r}\varepsilon_0$, which gives

$$W_\mathrm{m} = |C_n^e|^2 \frac{\pi\varepsilon_\mathrm{r}\varepsilon_0 n(n+1)}{k(2n+1)}\int_0^{ka}\left\{\left[\frac{d\rho j_n(\rho)}{d\rho}\right]^2 + n(n+1)j_n^2(\rho)\right\}d\rho \tag{1.29a}$$

Note that $\rho = \sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}}k_0 a$ and there is no propagating energy density subtracted at infinity. In addition, the limits on the integral are now from 0 to ka. The evaluation of this integral is similar to that for Equation 1.18 and can be inferred to be

$$W_\mathrm{m} = |C_n^e|^2 \frac{\pi\varepsilon_\mathrm{r}\varepsilon_0 n(n+1)}{k(2n+1)}\left\{\frac{(ka)^3(n+1)}{2(2n+1)}\left[j_{n-1}^2(ka) - j_{n-2}(ka)j_n(ka) + \frac{n}{n+1}j_{n+1}^2(ka) - \frac{n}{n+1}j_n(ka)j_{n+2}(ka)\right]\right\} \tag{1.29b}$$

We can simplify this expression by using the Bessel function recurrence relations to eliminate the Bessel functions of order $n-2$ and $n+2$. This gives the result

$$W_\mathrm{m} = |C_n^e|^2 \frac{\pi\varepsilon_\mathrm{r}\varepsilon_0 n(n+1)}{k(2n+1)}\left\{\frac{\rho^3}{2}\left[j_n^2(\rho) - j_{n-1}(\rho)j_{n+1}(\rho)\right] + \frac{\rho^2}{2n+1}\left[(n+1)j_n(\rho)j_{n-1}(\rho) - nj_n(\rho)j_{n+1}(\rho)\right]\right\} \tag{1.29c}$$

When we multiply by 2ω and divide by the radiated power given by Equation 1.13, we obtain the change in the antenna Q due to the energy stored internal to the sphere. We will denote this change by ΔQ_n^TE, which is given by

$$\Delta Q_n^{\mathrm{TE}} = |C_n^e|^2 \sqrt{\frac{\varepsilon_{\mathrm{r}}}{\mu_r}} \left\{ \frac{\rho^3}{2} \left[j_n^2(\rho) - j_{n-1}(\rho) j_{n+1}(\rho) \right] \right.$$
$$\left. + \frac{\rho^2}{2n+1} \left[(n+1) j_n(\rho) j_{n-1}(\rho) - n j_n(\rho) j_{n+1}(\rho) \right] \right\} \tag{1.30}$$

The last step is to find the value of $|C_n^e|^2$, which is the scale factor that the interior field must be multiplied by, from the condition that the tangential electric field must be continuous across the current sheet. This condition gives

$$|C_n^e|^2 = \mathrm{SF}_{\mathrm{TE}} = \frac{k_0^2 \left[j_n^2(k_0 a) + y_n^2(k_0 a) \right]}{k^2 j_n^2(ka)} \tag{1.31}$$

The final result for the change in the Q for the TE_{n0} mode due to energy stored within the circumscribing sphere is

$$\Delta Q_n^{\mathrm{TE}} = \frac{k_0}{\mu_r k} \frac{j_n^2(k_0 a) + y_n^2(k_0 a)}{j_n^2(ka)} \left\{ \frac{\rho^3}{2} \left[j_n^2(\rho) - j_{n-1}(\rho) j_{n+1}(\rho) \right] \right.$$
$$\left. + \frac{\rho^2}{2n+1} \left[(n+1) j_n(\rho) j_{n-1}(\rho) - n j_n(\rho) j_{n+1}(\rho) \right] \right\} \tag{1.32a}$$

When $\varepsilon_{\mathrm{r}} = \mu_{\mathrm{r}} = 1$, the above result agrees with that given by Hansen and Collin (2009).

For the TE_{10} mode, a simplified expression is easily derived by using the sine and cosine expressions for the spherical Bessel functions, thus

$$\Delta Q_1^{\mathrm{TE}} = \sqrt{\frac{\varepsilon_{\mathrm{r}}}{\mu_{\mathrm{r}}}} \frac{(ka)^2}{(k_0 a)^2} \frac{1 + (k_0 a)^2}{\left[(ka)^2 \cos^2 ka - (ka) \sin 2ka + \sin^2 ka \right]}$$
$$\times \left[\frac{ka}{2} - \frac{\sin 2ka}{4} - \frac{\cos^2 ka}{ka} + \frac{\sin 2ka}{(ka)^2} - \frac{\sin^2 ka}{(ka)^3} \right] \tag{1.32b}$$

We can obtain an alternative expression by using

$$1 + (k_0 a)^2 = (k_0 a)^3 \left[\frac{1}{(k_0 a)^3} + \frac{1}{k_0 a} \right] = (k_0 a)^3 Q_{1,\mathrm{Chu}}$$

where $Q_{1,\mathrm{Chu}}$ is given by Equation 1.23a and is the contribution to the Q due to the external stored energy. We now find that the new lower bound on the total Q for the TE_{10} mode can be expressed in the form

$$Q_1^{\mathrm{TE}} = Q_{1,\mathrm{Chu}} + \Delta Q_1^{\mathrm{TE}}$$

$$= \left\{ 1 + \frac{2}{\mu_\mathrm{r}} \frac{\left[(ka)^4/4 - \left((ka)^3 \sin 2ka\right)/8 - \left((ka)^2 \cos^2(ka)\right)/2 + (ka \sin 2ka)/2 - \left(\sin^2(ka)\right)/2\right]}{\left[(ka)^2 \cos^2(ka) - ka \sin 2ka + \sin^2(ka)\right]} \right\} Q_{1,\mathrm{Chu}} \tag{1.33}$$

This result is the same as that given by Kim et al. (2010). However, the above authors do not give any formulas for the Q for the higher order TE_{n0} modes. The above results also support the result $Q_1 = (1 + 2/\mu_\mathrm{r})(k_0 a)^{-3}$ given many years ago by Wheeler (1958).

From Equation 1.33, it can be seen that the new lower bound for the total Q depends only on the permeability parameter μ_r and the size of the core through the parameter $ka = \sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}}(k_0 a)$. For very small values of ka, the contribution ΔQ_n^{TE} to the total Q of the antenna is very small provided that the permeability is very large. Hence, for a core with a small value of ka, the lower bound is very close to the value of $Q_{1,\mathrm{Chu}}$. As ka increases in value, the ΔQ_n^{TE} increases but the total Q decreases because $Q_{1,\mathrm{Chu}}$ is decreasing at a rapid rate. After Q reaches a minimum value, it begins to increase without limit as the resonant frequency of the internal spherical core is approached. The resonant frequency occurs when the denominator term $j_n(ka)$ in Equation 1.31 equals zero. At the resonant frequency, the tangential electric field equals zero on the interior side of the surface of the circumscribing sphere. Since the tangential electric field is continuous across the current sheet, it must also be zero on the exterior side of the current sheet and consequently there is no external radiation at the resonant frequencies. This is a manifestation of the nonuniqueness of the external scattering problem for a sphere at the internal resonances.

When ε_r and μ_r are equal to unity, ΔQ_1^{TE} for the TE_{10} mode is very close to $2Q_{1,\mathrm{Chu}}$ for $k_0 a < 0.7$, which makes the new lower bound on the antenna Q for this mode approximately three times as large as that obtained by considering only the energy stored outside of the circumscribing sphere. In Figure 1.4a, we show some typical results for the new lower bound on the Q as a function of $k_0 a$ for $\mu_\mathrm{r} = 4$, 16, and 64 and $\varepsilon_\mathrm{r} = 4$. The curve for $\mu_\mathrm{r} = 64$ shows a large increase in Q when the frequency approaches the resonance value. These resonances are also encountered for the lower values of μ_r, and also for the case of an air core, but at larger values of $k_0 a$. For μ_r equal to 16 and larger, the curves of Q versus $k_0 a$ are almost identical except in the near vicinity of the resonances, which are different for each case because these occur when $ka = \tan ka$ for the $k_0 a$ mode and thus depend on both core permittivity and permeability. From Equation 1.33, we can see that the new lower bound Q_1^{TE} for the TM_{10} mode depends only on ka and the relative permeability μ_r. For this reason, the same results shown in Figure 1.4a are shown in Figure 1.4b plotted as a function of ka. These curves show additional resonance points and illustrate the minimum values of Q_1^{TE} that can be achieved. Note that when Q_1^{TE} is plotted as a factor of $ka\, Q_1^{\mathrm{TE}}$ decreases with an increase in μ_r, but when plotted as a function of ka the quality factor increases with an increase in μ_r.

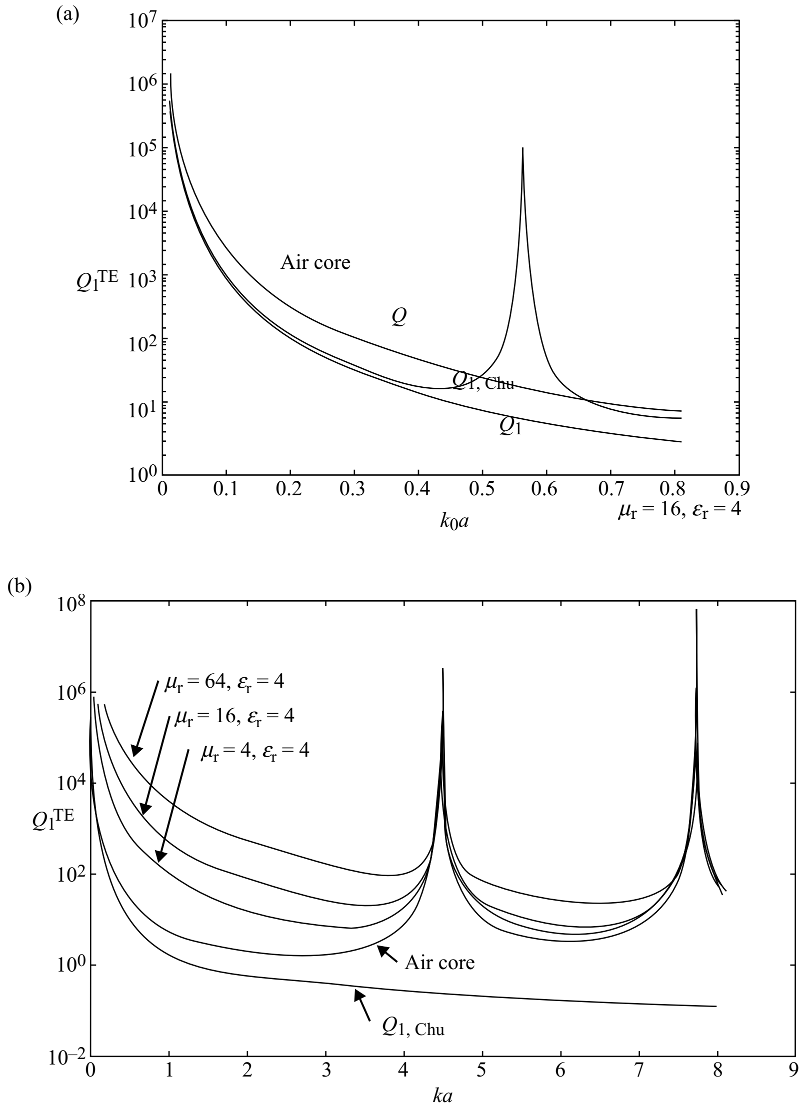

FIGURE 1.4 (a) A plot of lower bound Q_1^{TE} for TF_{10} mode as a function of ka for various electrical parameters for the spherical core. (b) Comparison of the new lower bound for Q_1^{TE} for the TE_{10} mode with the new lower bound taking into account internal energy stored in a core with relative permittivity of 16. Also shown is the Q for an air core and the original $Q_{1,\text{Chu}}$ that is based only on the external stored energy.

In the paper by Kim and Breinbjerg (2011), the ratio of the total stored magnetic field energy to the total stored electric field energy is plotted as a function of k_0a for μ_r equal to 1, 2, 8, and 100, with $\varepsilon_r = 1$. It was found that the stored electric field energy became equal to the stored magnetic field energy at the cavity resonant frequencies, but never exceeded the stored magnetic field energy. We have verified these calculations for ka up to 100 and also included ε_r values of 2, 4, 16, 64, and 100. The same property that the total stored electric field energy did not exceed the total stored magnetic field energy continued to hold. Thus, the formula given by Equation 1.33 for the Q of the TE_{10} will hold for all values of k_0a, which is contrary to a conclusion given by Kim and Breinbjerg (2011).

The next issue we wish to explore is whether or not the frequency dependence of the tuned antenna admittance will result in a 3 dB bandwidth that is equal to $2/Q_1^{TE}$, where Q_1^{TE} is the Q of the TE_{10} mode with the dielectric–magnetic core and no external conductive loading; that is, Q_1^{TE} is the unloaded antenna Q. The antenna configuration analyzed above consists of a dielectric–magnetic core of radius a and wound with a current sheet in the ϕ direction. The admittance presented to the current sheet source is the parallel combination of the wave admittance Y_e looking in the outward direction with the wave admittance Y_i looking inward from the surface at $r = a$. These wave admittances can be obtained from the expressions for the fields of the TE_{10} mode given in Equations 1.11a, 1.11b, 1.28a, and 1.28b and are, after normalization with respect to the characteristic admittance $Y_0 = (\varepsilon_0/\mu_0)^{1/2}$ of free space,

$$Y_e = -\frac{H_\theta}{E_\phi} = j\frac{d\left[(k_0a)h_1^2(k_0a)\right]/(dk_0a)}{(k_0a)h_1^2(k_0a)} \tag{1.34a}$$

$$Y_i = -j\sqrt{\frac{\varepsilon_r}{\mu_r}}\frac{d[(ka)j_1(ka)]/d(ka)}{(ka)j_1(ka)} \tag{1.34b}$$

We now use

$$xj_1(x) = \frac{\sin(x)}{x} - \cos(x)$$

and

$$xh_1^2(x) = \frac{\sin(x)}{x} - \cos(x) + j\left[\frac{\cos(x)}{x} + \sin(x)\right]$$

to obtain

$$Y_e = \frac{j\left\{k_0a\cos(k_0a) - \sin(k_0a) + (k_0a)^2\sin(k_0a) + j\left[(k_0a)^2\cos(k_0a) - \cos(k_0a) - k_0a\sin(k_0a)\right]\right\}}{k_0a\sin(k_0a) - (k_0a)^2\cos(k_0a) + j\left[k_0a\cos(k_0a) + (k_0a)^2\sin(k_0a)\right]}$$

$$= \frac{(k_0a)^3 - j}{k_0a\left[1 + (k_0a)^2\right]} \tag{1.35a}$$

$$Y_{\rm i} = j\sqrt{\frac{\varepsilon_{\rm r}}{\mu_{\rm r}}}\left[\frac{\sin(ka) - ka\cos(ka) - (ka)^2\sin(ka)}{ka\sin(ka) - (ka)^2\cos(ka)}\right] \tag{1.35b}$$

The external admittance $Y_{\rm e}$ is that of a lumped element circuit consisting of an inductive reactance jk_0a in parallel with the reactance of a series capacitor and resistor $1 + 1/jk_0a$ (Harrington, 1961). The internal admittance cannot be represented by a lumped element circuit, except when ka is small so that a first-order power series expansion of $Y_{\rm i}$ can be used. In the low-frequency range, $Y_{\rm i}$ can be approximated as an inductive reactance $jka = j\omega\mu_{\rm r}\mu_0 a\sqrt{\varepsilon/\mu}$, which is connected in parallel with $jk_0a = j\omega\mu_0\sqrt{\varepsilon_0/\mu_0}$. It can be seen that the parallel combination of these two inductive reactances will be dominated by the jk_0a one when $\mu_{\rm r}$ and $\varepsilon_{\rm r}$ are large. The admittance seen by the current source is $Y_{\rm e} + Y_{\rm i}$. This admittance is inductive. The antenna can be tuned to resonance by connecting a capacitive admittance $jB_{\rm c} = j\,{\rm Im}(Y_{\rm e} + Y_{\rm i})$ in parallel. We will assume that $B(\omega) = B(\omega_{\rm c})\omega/\omega_{\rm c}$ so that at the center frequency $\omega_{\rm c}$ the input admittance is a pure conductance equal to the radiation conductance of the antenna. Away from the center frequency, the antenna input admittance will no longer be a pure conductance. The issue we wish to explore is whether or not the frequency dependence of the tuned antenna admittance will result in a 3 dB bandwidth that is equal to $2/Q_1^{\rm TE}$, where $Q_1^{\rm TE}$ is the Q of the TE_{10} mode with the dielectric–magnetic core and no external conductive loading; that is, $Q_1^{\rm TE}$ is the unloaded antenna Q. A sample evaluation of the VSWR as a function of k_0a for the loaded antenna has been calculated for the case of an antenna with $k_0a = 0.25$ at the center frequency, with a core having $\varepsilon_{\rm r} = 4$ and $\mu_{\rm r} = 16$, and having a load conductance equal to that of the conductance of $Y_{\rm e}$ at the center frequency. For this antenna, the Q is very nearly equal to $(1 + 2/\mu_{\rm r})Q_{1,\rm Chu}$, which equals 76.5. The VSWR is shown in Figure 1.5. From the figure, it was estimated that the 3 dB bandwidth was 0.0065 in units of k_0a, which gives a fractional bandwidth of 0.0065/0.25 = 0.026. From this bandwidth, we can determine the circuit loaded Q, which is given by 1/0.026 = 38.45. The unloaded antenna Q is equal to twice this value and is 76.9, which is slightly greater than the theoretical lower bound given by Equation 1.33. This exmple suggests that an antenna using a core with a large value of permeability and with the excitation in the form of a surface winding in the azimuthal direction is an optimum design.

1.6 RADIAN SPHERE WITH MU AND/OR EPSILON: TM MODES

We will now analyze an antenna consisting of a dielectric–magnetic core that is excited by a current sheet in the θ direction so as to radiate only TM_{n1} modes. The external fields are given by Equations 1.4a–1.4c. The internal fields have the same form but with the Hankel functions replaced by Bessel functions, thus

$$E_\theta = D_n^h \frac{-k\sin\theta}{j\omega\varepsilon_{\rm r}\varepsilon_0 r}\frac{dP_n(\cos\theta)}{d(\cos\theta)}\frac{d[krj_n(kr)]}{d(kr)} \tag{1.36a}$$

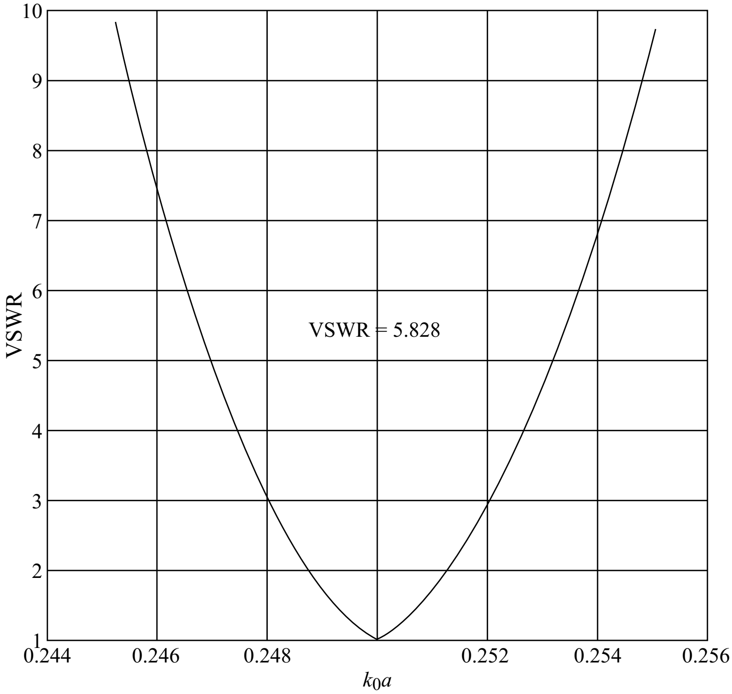

FIGURE 1.5 VSWR as a function of k_0a for an antenna tuned to resonance at the frequency corresponding to $k_0a = 0.25$. The core has a relative permittivity of 4 and a relative permeability of 16. The 3 dB loaded antenna Q is equal to 38.45. The unloaded antenna Q equals 76.9.

$$E_r = D_n^h \frac{n(n+1)}{j\omega\varepsilon_r\varepsilon_0 r^2} P_n(\cos\theta)[krj_n(kr)] \tag{1.36b}$$

$$H_\phi = D_n^h \frac{\sin\theta}{r} \frac{dP_n(\cos\theta)}{d(\cos\theta)} [krj_n(kr)] \tag{1.36c}$$

For these modes, the stored electric energy is greater than the stored magnetic energy. For the external fields, the stored electric energy is given by an expression similar to that in Equation 1.20 since the TM_{n1} modes are the dual of the TE_{n0} modes. Thus, the external stored energy leads to the same contribution to the antenna Q that is given by Equation 1.21. Consequently, we need to evaluate only the additional contribution to the antenna Q arising from the internal stored energy. The integral for the stored electric energy will be similar to that for the stored magnetic energy as given by Equation 1.29a. By using the expressions from Equations 1.36a and 1.36c, we obtain

$$
\begin{aligned}
W_{\mathrm{e}} &= \int_0^a\int_0^\pi \frac{\varepsilon_{\mathrm{r}}\varepsilon_0}{4}\left[|E_\theta|^2+|E_r|^2\right]2\pi r^2\sin\theta\, d\theta\, dr \\
&= \int_0^a\int_0^\pi |D_n^h|^2\frac{\mu_r\mu_0}{4}\left\{\left\{\frac{dP_n(\cos\theta)}{d(\cos\theta)}\frac{d[krj_n(kr)]}{d(kr)}\right\}^2\right. \\
&\qquad \left.+\left[n(n+1)P_n(\cos\theta)j_n(kr)\right]^2\right\}2\pi\sin^3\theta\, d\theta\, dr \\
&= \frac{\mu_{\mathrm{r}}\mu_0\pi}{k}\frac{n(n+1)}{2n+1}\int_0^{ka}|D_n^h|^2\left\{\left[\frac{d\rho j_n(\rho)}{d\rho}\right]^2+n(n+1)j_n^2(\rho)\right\}d\rho \qquad (1.37a)
\end{aligned}
$$

where $\rho = ka$. This integral is the same as that in Equation 1.26 and the result will be the same as that given by Equation 1.27. Thus, we have

$$
\begin{aligned}
W_{\mathrm{e}} = \left|D_n^h\right|^2\frac{\pi\mu_{\mathrm{r}}\mu_0 n(n+1)}{k(2n+1)}&\left\{\frac{(ka)^3(n+1)}{2(2n+1)}\left[j_{n-1}^2(ka)-j_{n-2}(ka)j_n(ka)\right.\right. \\
&\left.\left.+\frac{n}{n+1}j_{n+1}^2(ka)-\frac{n}{n+1}j_n(ka)j_{n+2}(ka)\right]\right\} \qquad (1.37b)
\end{aligned}
$$

The total radiated power is given by Equation 1.13 multiplied by μ_0/ε_0 and is $[2n(n+1)/(2n+1)]\pi\sqrt{\mu_0/\varepsilon_0}$, so the contribution ΔQ_n^{TM} to the antenna Q from the internal stored energy will be similar to that in Equation 1.29:

$$
\begin{aligned}
\Delta Q_n^{\mathrm{TM}} = \frac{2\omega W_{\mathrm{e}}}{P_{\mathrm{r}}} = \left|D_n^h\right|^2\sqrt{\frac{\mu_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}}}&\left\{\frac{\rho^3}{2}\left[j_n^2(\rho)-j_{n-1}(\rho)j_{n+1}(\rho)\right]\right. \\
&\left.+\frac{\rho^2}{2n+1}\left[(n+1)j_n(\rho)j_{n-1}(\rho)-nj_n(\rho)j_{n+1}(\rho)\right]\right\} \qquad (1.38)
\end{aligned}
$$

We must also choose the scale factor $\mathrm{SF_{TM}} = \left|D_n^h\right|^2$ so that the tangential electric field is continuous across the current sheet. This requires the constant D_n^h to be given by

$$
D_n^h = \sqrt{\frac{\varepsilon_{\mathrm{r}}}{\mu_{\mathrm{r}}}}\frac{d\left[\rho_0 h_n^2(\rho_0)/d\rho_0\right]}{d[\rho j_n(\rho)/d\rho]}
$$

which gives

$$
\mathrm{SF_{TM}} = \left|D_n^h\right|^2 = \frac{\varepsilon_{\mathrm{r}}}{\mu_{\mathrm{r}}}\frac{[d\rho_0 j_n(\rho_0)/d\rho_0]^2+[d\rho_0 y_n(\rho_0)/d\rho_0]^2}{[d\rho j_n(\rho)/d\rho]^2} \qquad (1.39)
$$

The final expression obtained for ΔQ_n^{TM} is

$$\Delta Q_n^{\mathrm{TM}} = \mathrm{SF}_{\mathrm{TM}}\sqrt{\frac{\mu_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}}}\left\{\frac{\rho^3}{2}\left[j_n^2(\rho) - j_{n-1}(\rho)j_{n+1}(\rho)\right] + \frac{\rho^2}{2n+1}\left[(n+1)j_n(\rho)j_{n-1}(\rho) - nj_n(\rho)j_{n+1}(\rho)\right]\right\} \quad (1.40)$$

Hence, for the TM_{n0} modes excited by a current sheet on a dielectric–magnetic spherical core, the new lower bound for the Q is

$$Q_n^{\mathrm{TM}} = \Delta Q_n^{\mathrm{TM}} + \left\{k_0a - \left[\frac{(k_0a)^2}{2} + (n+1)k_0a\right]\left[j_n^2(k_0a) + y_n^2(k_0a)\right] - \frac{(k_0a)^3}{2}\left[j_{n+1}^2(k_0a) + y_{n+1}^2(k_0a)\right] + \frac{2n+3}{2}(k_0a)^2\left[j_n(k_0a)j_{n+1}(k_0a) + y_n(k_0a)y_{n+1}(k_0a)\right]\right\} \quad (1.41)$$

For $n = 1$, this expression can be simplified to the form

$$Q_1^{\mathrm{TM}} = \Delta Q_1^{\mathrm{TM}} + \left[\frac{1}{(k_0a)^3} + \frac{1}{k_0a}\right] \quad (1.42)$$

where

$$\Delta Q_1^{\mathrm{TM}} = \sqrt{\frac{\varepsilon_{\mathrm{r}}}{\mu_{\mathrm{r}}}}\left[\frac{\rho^4(1+\rho_0^4-\rho_0^2)}{\rho_0^4\left[\rho^2\cos^2(\rho) + (1-\rho^2)^2\sin^2(\rho) - \rho(1-\rho^2)\sin(2\rho)\right]}\right] \times \left[\frac{\rho}{2} - \frac{\sin^2(\rho)}{\rho^3} + \frac{(4\rho-\rho^3)\sin(2\rho)}{4\rho^3} - \frac{\cos^2(\rho)}{\rho}\right] \quad (1.43)$$

Typical values of Q_1^{TM} are shown in Figure 1.6a for representative values of ε_{r} and μ_{r}. For an air core, the above formula shows that for small values of $k_0a = \rho_0$ the new lower bound on Q_1^{TM} is very nearly equal to 1.5 $Q_{1,\mathrm{Chu}}$ and decreasing to 1.34 $Q_{1,\mathrm{Chu}}$ for $k_0a = 0.5$, where $Q_{1,\mathrm{Chu}}$ is given by the term appearing after the term ΔQ_1^{TM} in the above equation. For the case of a relative permeability greater than 1, or a relative permittivity greater than 1, or when both are greater than 1, the Q is increased.

Overall, the relative permeability has very little effect on the Q as long as it does not result in a value for ka that is close to the resonant frequency for the core. Thus, an air core is the best choice for TM_{n1} modes. Resonances will occur when ε_{r} and μ_{r} are large enough to make $d[\rho j_n(\rho)]/d\rho = 0$ and these will occur for smaller values of k_0a. At the

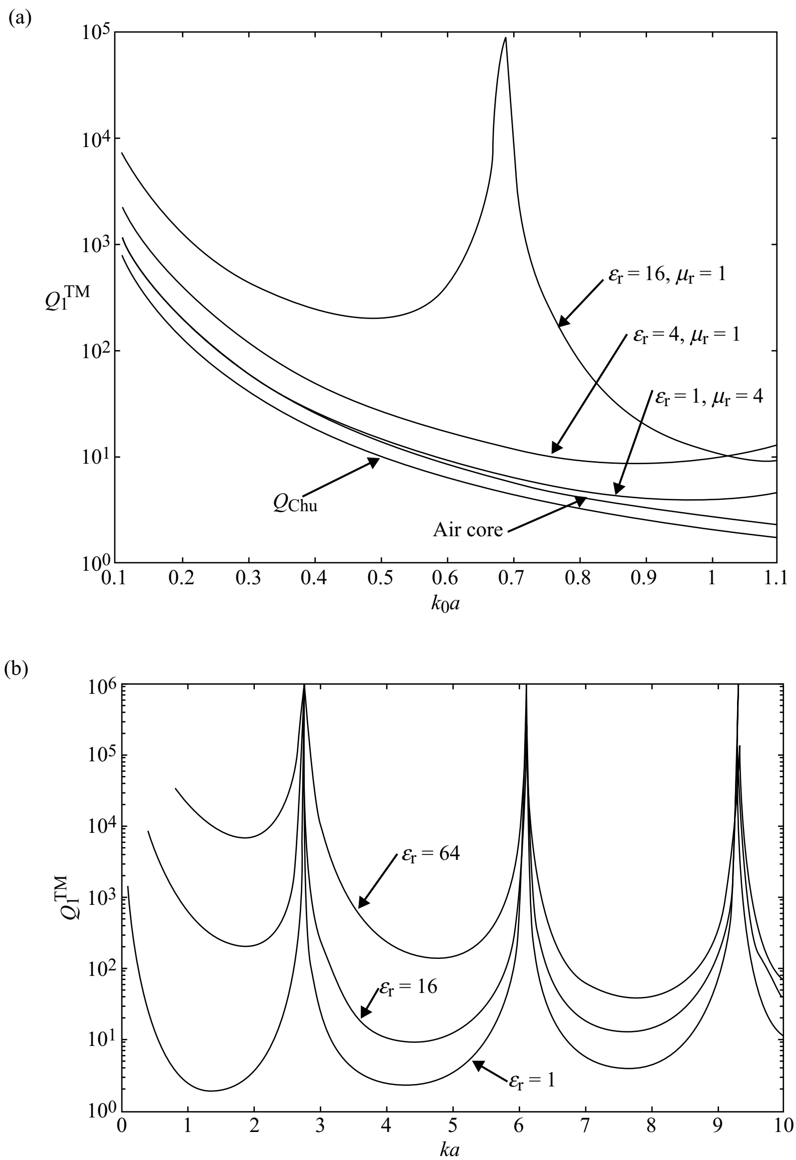

FIGURE 1.6 (a) Q_1^{TM} for TM_{10} mode as a function of k_0a for various electrical parameters of the spherical core. For an air core, the Q_1^{TM} is approximately 1.5 times the Q_{Chu} value. (b) Q_1^{TM} plotted as a function of ka for ε_r equal to 1, 16, and 64 with μ_r equal to 1.

resonant frequencies, the external radiation vanishes and since the core has been assumed to be lossless Q_1^{TM} becomes infinite. In Figure 1.6b, the value of Q_1^{TM} as a function of ka for ε_r equal to 16 and 64, with $\mu_r = 1$, is shown in order to illustrate the multiple resonances that occur.

The final topic we will examine is the evaluation of the Q of the antenna system, consisting of the dielectric–magnetic core wound with a current sheet in the θ direction, in terms of the frequency behavior of its equivalent circuit. It will be shown that the loaded 6 dB fractional bandwidth calculated from the equivalent circuit, when tuned to resonance by a shunt inductive reactance, is equal to twice the Q given by Equation 1.42, to a high degree of accuracy. This result is similar to that found for the TE_{10} mode driven by a current sheet. It is a verification of the generally held assumption that for simple antennas the fractional bandwidth is inversely proportional to the antenna Q.

For the TM_{10} mode, the wave admittance seen looking outward from the current sheet at $r = a$ may be found using Equations 1.4a and 1.4c and is given by

$$\begin{aligned} Y_e &= \frac{H_\phi}{E_\theta} = \frac{-j(k_0a)h_1^2(k_0a)}{d\left[k_0ah_1^2(k_0a)\right]/d(k_0a)} \\ &= \frac{\left[\rho_0\cos(\rho_0)+\rho_0^2\sin(\rho_0)-j\rho_0\sin(\rho_0)+j\rho_0^2\cos(\rho_0)\right]}{\left[\rho_0\cos(\rho_0)+\rho_0^2\sin(\rho_0)-\sin(\rho_0)\right]+j\left[\rho_0^2\cos(\rho_0)-\cos(\rho_0)-\rho_0\sin(\rho_0)\right]} \\ &= \frac{\rho_0^4+j\rho_0}{1+\rho_0^4-\rho_0^2} \end{aligned} \tag{1.44}$$

after normalization with respect to the characteristic admittance of free space. Note that $\rho_0 = k_0a$. The equivalent circuit for the external admittance consists of a capacitive admittance jk_0a connected in series to an inductive reactance jk_0a in parallel with a $1\,\Omega$ resistance (Harrington, 1961).

The normalized admittance seen looking inward from the current sheet can be found by using Equations 1.36a and 1.36c and is given by

$$Y_i = j\sqrt{\frac{\varepsilon_r}{\mu_r}}\frac{\rho j_1(\rho)}{d[\rho j_1(\rho)]/d\rho} = j\sqrt{\frac{\varepsilon_r}{\mu_r}}\frac{\rho\sin(\rho)-\rho^2\cos(\rho)}{\rho^2\sin(\rho)-\sin(\rho)+\rho\cos(\rho)} \tag{1.45}$$

where $\rho = ka$. Both Y_e and Y_i are capacitive for TM modes. The antenna can be tuned to resonance by connecting a parallel inductive admittance in parallel with $Y_e + Y_i$ such that $-jB_L + j\mathrm{Im}(Y_e + Y_i) = 0$ at the center frequency, where we have assumed that at the center frequency (resonant frequency) the net susceptance of the tuned circuit is zero. Away from the resonant frequency, we will assume that B_L is given by $B_L(\omega) = B_L(\omega_c)\omega_c/\omega$, where ω_c is the resonant frequency of the circuit. The last assumption will be that the circuit is loaded by a conductance equal to the radiation conductance $\mathrm{Re}(Y_e + Y_i)$ at the resonant frequency. We can now evaluate the input reflection coefficient seen by the current source and the resultant VSWR. From the fractional 6 dB bandwidth, the Q_1^{TM} for the antenna system described can be found from the relationship $Q_1^{TM} = 2/\mathrm{BW}$, where BW is the 6 dB fractional bandwidth of

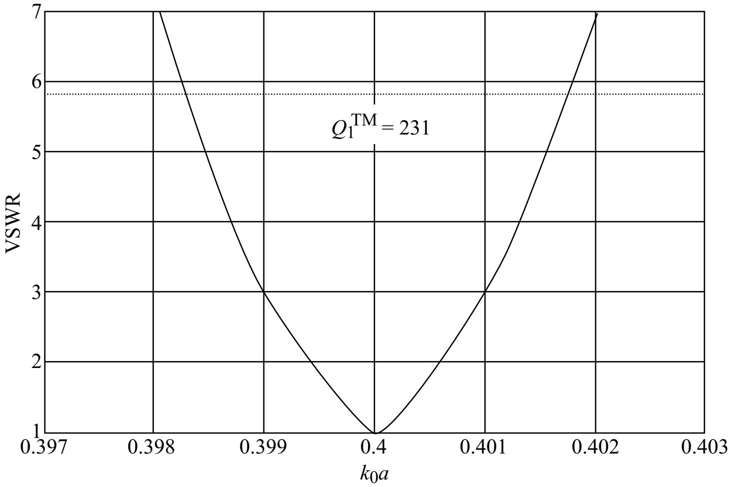

FIGURE 1.7 VSWR as a function of k_0a for the TM_{10} mode with a dielectric core with a relative permittivity of 16. The quality factor Q_1^{TM} is equal to 231 for the unloaded antenna.

the loaded tuned circuit. In Figure 1.7, we show the VSWR curve for the case where $\varepsilon_r = 16$ and the center frequency corresponds to k_0a equal to 0.4. From the VSWR curve, the estimate of the quality factor Q_1^{TM} is 231. The calculated value from Equation 1.42 is 230.2. Several other cases were also checked and gave similar very close agreements between the circuit-based Q and the value based on stored energy.

The preceding work assumed a spherical core of permeability material. Stuart and Yaghjian (2010) have shown that a solid core is not necessary. They used a monopole with a circular plate top hat; a high-μ cylindrical shell of diameter equal to that of the top hat, and height equal to that of the monopole, was added to the antenna. Results using $\mu > 100$ were close to the Chu limit. Figure 1.8 shows the decrease of Q as μ increases; also shown is the importance of an ε close to unity. See also Kim and Breinbjerg (2011).

1.7 EFFECTS OF CORE LOSSES

The results derived above showed that for TE_{n0} and TM_{n0} modes excited by means of current sheets located on the surface of a sphere of solid dielectric or magnetic material, the resultant Q would be dependent on the constituent parameters of the core material. The use of dielectric material with a relative permittivity greater than unity increased the quality factor of TM_{n0} modes and thus would not be a useful way to increase the bandwidth of the antenna. On the other hand, the use of magnetic material with a relative permittivity greater than unity had the effect of significantly reducing the Q of excited TE_{n0} modes. Since physical materials will have some loss,

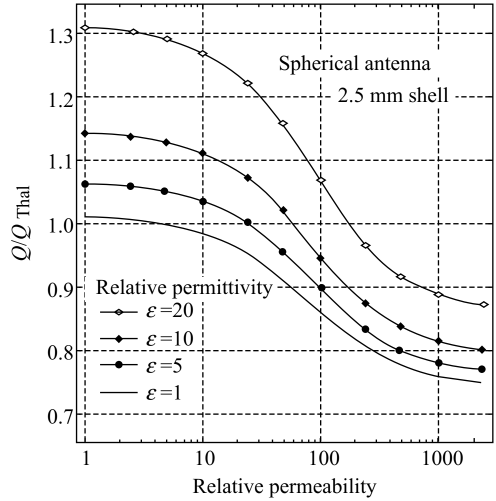

FIGURE 1.8 Top hat monopole with cylindrical permeable shell. Courtesy of Stuart and Yaghjian (2010).

an inevitable penalty to be paid by the use of a solid core is the increase in the loss and a decrease in the efficiency of the antenna. The reduction in the Q results in an increase in the bandwidth, which is normally a desirable effect, but it is offset by the decrease in efficiency.

For isotropic lossy material, the power loss is related to the average stored electric and magnetic energy within the core through the imaginary parts of the constitutive parameters. Consider the integral of the inward directed complex Poynting vector over the surface of the core at $r = a$,

$$-\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_r\sin\theta\,d\theta\,d\phi=\frac{j\omega}{2}\int_0^{a}\int_0^{2\pi}\int_0^{\pi}\Big[\mu_0(\mu_r'-j\mu_r'')\mathbf{H}\cdot\mathbf{H}^* - \varepsilon_0(\varepsilon_r'+j\varepsilon_r'')\Big]\sin\theta\,d\theta\,d\phi\,dr$$

The real part of the right-hand side represents the average power loss within the core. The imaginary part represents $2j\omega(W_{\mathrm{m}} - W_{\mathrm{e}})$, where W_{m} and W_{e} are the time-averaged stored magnetic and electric energies in the core material, respectively. When the core losses are small, the stored energy may be evaluated by assuming that the losses are zero. In terms of the stored energy, we can express the power loss in the following form:

$$P_{\mathrm{L}} = \frac{2\omega\mu_{\mathrm{r}}''}{\mu_{\mathrm{r}}'}W_{\mathrm{m}} + \frac{2\omega\varepsilon_{\mathrm{r}}''}{\varepsilon_{\mathrm{r}}'}W_{\mathrm{e}} \tag{1.46}$$

The new reduced Q of the system is given by Q', where

$$Q' = \frac{P_\mathrm{r}}{P_\mathrm{r} + P_\mathrm{L}} Q \tag{1.47}$$

and Q is the quality factor in the absence of core losses. P_r is the average radiated power from the excited TE_{n0} or TM_{n0} mode. The reduced efficiency of the antenna is given by the factor $P_\mathrm{r}/(P_\mathrm{r} + P_\mathrm{L})$. For the TE_{n0} and TM_{n0} modes, we have already obtained expressions for W_m and W_e, respectively. The other stored energy function can be obtained by direct evaluation of the volume integral of the corresponding energy density.

For the TE_{n0} modes, the stored energy in the magnetic field in the core is given by Equation 1.29c, which is repeted below:

$$\begin{aligned} W_\mathrm{m} = \left|C_n^e\right|^2 \frac{\pi \varepsilon_\mathrm{r}' \varepsilon_0 n(n+1)}{k(2n+1)} \Bigg\{ & \frac{\rho^3}{2} \left[j_n^2(\rho) - j_{n-1}(\rho) j_{n+1}(\rho)\right] \\ & + \frac{\rho^2}{2n+1} \left[(n+1) j_n(\rho) j_{n-1}(\rho) - n j_n(\rho) j_{n+1}(\rho)\right] \Bigg\} \end{aligned} \tag{1.48}$$

The stored electric field energy is given by (see Equations 1.17b and 1.19a)

$$\begin{aligned} W_\mathrm{e} &= \frac{\varepsilon_\mathrm{r}' \varepsilon_0}{4} \int_0^\pi \int_0^a \left|E_\phi\right|^2 2\pi r^2 \sin\theta \, d\theta \, dr \\ &= \frac{\varepsilon_\mathrm{r}' \varepsilon_0}{4k} \int_0^\pi \int_0^a |C_{ne}|^2 \left[\frac{dP_n(\cos\theta)}{d\theta}\right]^2 k^2 r^2 j_n^2(kr) 2\pi \sin\theta \, d\theta \, d(kr) \\ &= \frac{\varepsilon_\mathrm{r}' \varepsilon_0}{2k} \pi |C_{ne}|^2 \frac{n(n+1)}{2n+1} (kr)^3 \left[j_n^2(kr) - j_{n-1}(kr) j_{n+1}(kr)\right] \end{aligned} \tag{1.49}$$

The loss arising from the magnetic material is obtained by multiplying W_m given by Equation 1.48 by the factor $2\omega\mu_\mathrm{r}''/\mu_\mathrm{r}'$. Similarly, the power loss in the core due to the lossy dielectric material is given by multiplying Equation 1.49 by the factor $2\omega\varepsilon_\mathrm{r}''/\varepsilon_\mathrm{r}'$. For the TE_{10} mode, the expressions for the power loss in the core are given by

$$P_\mathrm{Lm} = \frac{4\pi}{3} \sqrt{\frac{\varepsilon_\mathrm{r}' \varepsilon_0}{\mu_\mathrm{r}' \mu_0}} \frac{\mu_\mathrm{r}''}{\mu_\mathrm{r}'} |C_{ne}|^2 \left[\frac{\rho^3}{2} \left(j_1^2 - j_0 j_2\right) + \frac{\rho^2}{3} \left(2 j_1 j_0 - j_1 j_2\right)\right] \tag{1.50}$$

$$P_\mathrm{Le} = \frac{4\pi}{3} \sqrt{\frac{\varepsilon_\mathrm{r}' \varepsilon_0}{\mu_\mathrm{r}' \mu_0}} \frac{\varepsilon_\mathrm{r}''}{\varepsilon_\mathrm{r}'} |C_{ne}|^2 \left[\frac{\rho^3}{2} \left(j_1^2 - j_0 j_2\right)\right] \tag{1.51}$$

where $\rho = ka$ is the argument for the Bessel functions.

A useful parameter is the ratio of power loss to power radiated

$$\eta_{\mathrm{TE}} = \frac{P_{\mathrm{Lm}} + P_{\mathrm{Le}}}{P_{\mathrm{r}}} \tag{1.52}$$

in terms of which the new lower Q of the antenna with a lossy core is given by

$$Q' = \frac{Q'_{\mathrm{TE}}}{1 + \eta_{\mathrm{TE}}} \tag{1.53}$$

For the special case of the TE_{10} mode when $n = 1$, we obtain

$$\eta_{\mathrm{TE}} = \frac{P_{\mathrm{L}}}{P_{\mathrm{r}}} = \left|C_1^e\right|^2 \sqrt{\frac{\varepsilon'_{\mathrm{r}}}{\mu'_{\mathrm{r}}}} \left\{ \frac{\mu''_{\mathrm{r}}}{\mu'_{\mathrm{r}}} \left[\frac{\rho}{2} - \frac{\sin(2\rho)}{4} - \frac{\sin^2(\rho)}{\rho^3} + \frac{\sin(2\rho)}{\rho^2} - \frac{\cos^2(\rho)}{\rho} \right] \right.$$
$$\left. + \frac{\varepsilon''_{\mathrm{r}}}{\varepsilon'_{\mathrm{r}}} \left[\frac{\rho}{2} + \frac{\sin(2\rho)}{4} - \frac{\sin^2(\rho)}{\rho} \right] \right\} \tag{1.54}$$

where

$$\left|C_1^e\right|^2 = \frac{\rho^2}{\rho_0^2} \frac{(1 + \rho_0^2)}{\left[\sin^2\rho + \rho^2 \cos^2 \rho - \rho \sin(2\rho)\right]}$$

A similar derivation may be carried out to obtain an expression for the total core loss for the TE_{10} mode. The factor η_{TM} for the TE_{10} mode is given by

$$\eta_{\mathrm{TM}} = \frac{P_{\mathrm{L}}}{P_{\mathrm{r}}} = \left|D_1^h\right|^2 \sqrt{\frac{\mu'_{\mathrm{r}}}{\varepsilon'_{\mathrm{r}}}} \left\{ \frac{\varepsilon''_{\mathrm{r}}}{\varepsilon'_{\mathrm{r}}} \left[\frac{\rho}{2} - \frac{\sin(2\rho)}{4} - \frac{\sin^2(\rho)}{\rho^3} + \frac{\sin(2\rho)}{\rho^2} - \frac{\cos^2(\rho)}{\rho} \right] \right.$$
$$\left. + \frac{\mu''_{\mathrm{r}}}{\mu'_{\mathrm{r}}} \left[\frac{\rho}{2} + \frac{\sin(2\rho)}{4} - \frac{\sin^2(\rho)}{\rho} \right] \right\} \tag{1.55}$$

where

$$\left|D_1^h\right|^2 = \sqrt{\frac{\varepsilon'_{\mathrm{r}}}{\mu'_{\mathrm{r}}}} \frac{\rho^4}{\rho_0^4} \frac{\left(\rho_0^4 - \rho_0^2 + 1\right)}{\left[\rho^2 \cos^2 \rho + (1 - \rho^2)^2 \sin^2\rho + (\rho^3 - \rho)\sin(2\rho)\right]}$$

Representative values of Q_1^{TM} and Q_1^{TE} are plotted in Figure 1.9a and b as a function of $k_0 a$ for a core with relative permittivity of 4 and a relative permeability of 16, and with loss tangents $\varepsilon'_{\mathrm{r}}/\varepsilon''_{\mathrm{r}} = \mu'_{\mathrm{r}}/\mu''_{\mathrm{r}}$ equal to 0.01, 0.001, and 0.0001. For the TE_{10} mode, the Q is about 5–10 times larger than that for the TE_{10} mode with the same core parameters.

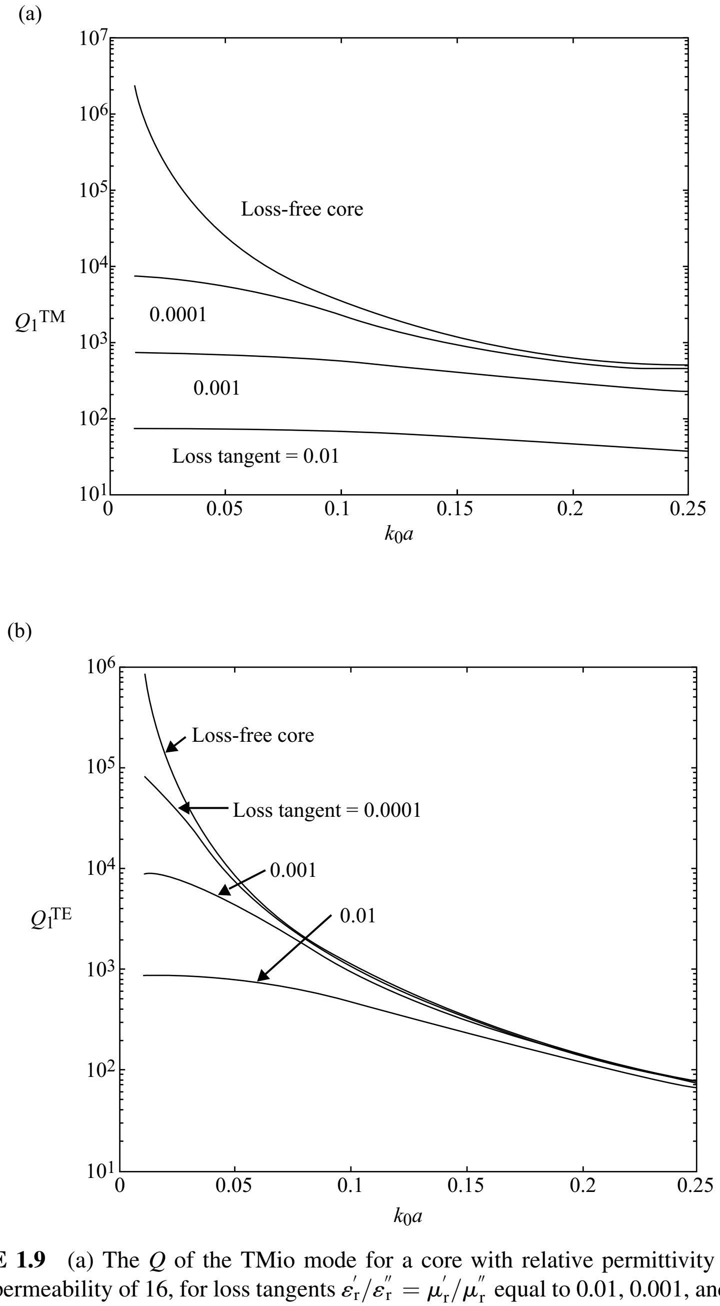

FIGURE 1.9 (a) The Q of the TMio mode for a core with relative permittivity of 4 and relative permeability of 16, for loss tangents $\varepsilon_r'/\varepsilon_r'' = \mu_r'/\mu_r''$ equal to 0.01, 0.001, and 0.0001. (b) The Q of the TE_{10} mode for a core with relative permittivity of 4 and relative permeability of 16, for loss tangents $\varepsilon_r'/\varepsilon_r'' = \mu_r'/\mu_r''$ equal to 0.01, 0.001, and 0.0001.

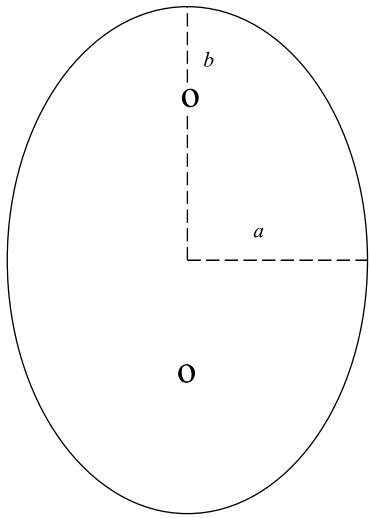

FIGURE 1.10 Prolate spheroid.

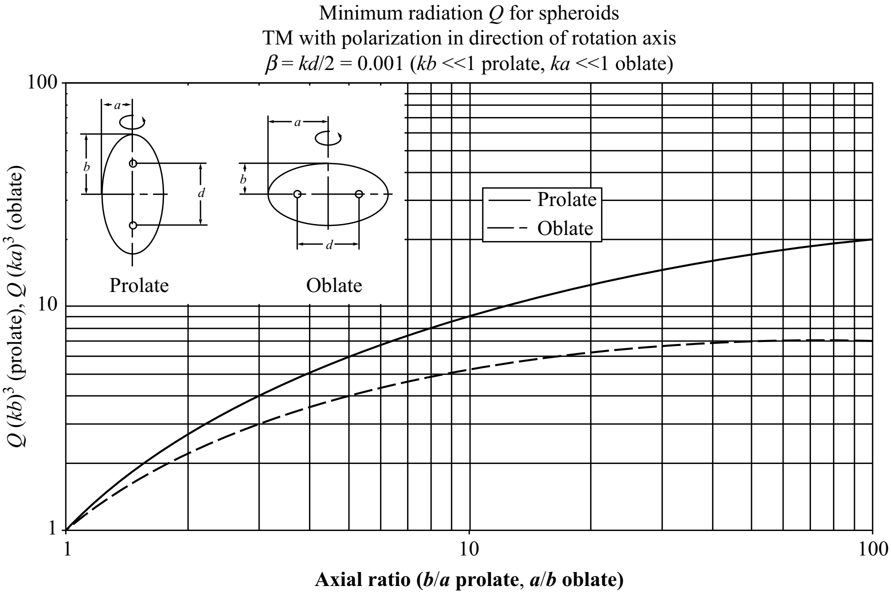

FIGURE 1.11 Q of spheroid in sphere. Courtesy of Peder Hansen, SPAWAR, San Diego, CA.

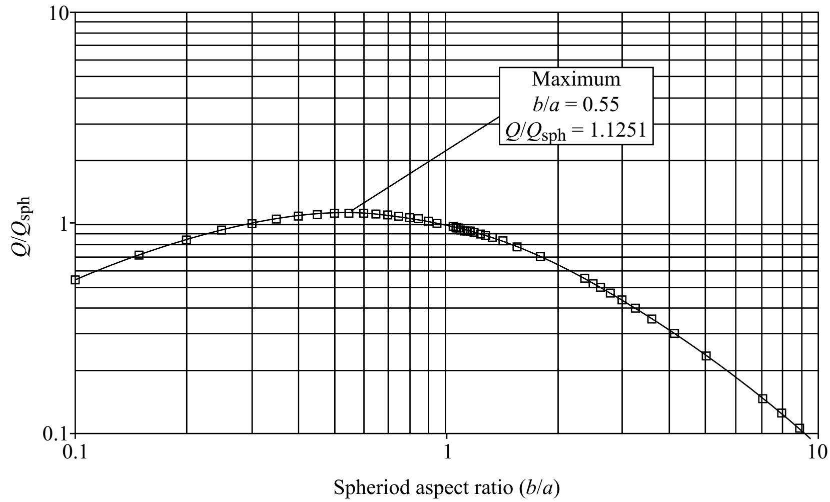

FIGURE 1.12 Q ratio: equal-volume sphere and spheroid. Courtesy of Hansen and Adams (2010).

1.8 *Q* FOR SPHEROIDAL ENCLOSURES

A rigorous Chu-type fundamental limitation-type formulation for nonspherical volumes would require that a low-order mode exists in the enclosure, and that stored and radiated energy can be calculated. The prolate spheroid, which approximates a cylinder, and the oblate spheroid, which approximates a flat disk, are candidates. The first effort was made by Foltz and McLean (1999) who attempted a numerical partial fraction expansion (ala Chu) of the prolate spheroid modal impedance. A Herculean spheroidal mode approach has been underway by Adams and Hansen (2004, 2008) of SPAWAR. Spheroidal functions were described by Hobson (1931), Stratton (1941), and Morse and Feshbach (1953). EM applications have been discussed by Li et al. (2002). Tables of functions are provided by Stratton et al. (1956), Flammer (1957), and Chang and Yeh (1966).

The spherical mode theory developed by Collin utilized closed-form integrals and recursion relationships (Equations 1.17 and 1.19) to develop the energy formulas. The spheroidal mathematical toolbox is much more sparse, so that numerical methods were used by Adams and Hansen to supplement the few available functional relationships. Figure 1.10 shows the prolate spheroid geometry, where the spheroid height is $2b$ and the diameter is $2a$. Because the Q of a spheroid depends upon both the dimensions in wavelengths and the prolateness, data in only two dimensions can be provided by comparing the spheroid with a sphere. Figure 1.11 shows the ratio of spheroid Q to Q of the sphere that just encloses the spheroid. It can be observed that the relative Q increases as the prolate spheroid becomes thinner, or as the oblate spheroid becomes fatter. Figure 1.12 shows the Q ratio for a sphere and a spheroid of equal volume. As shown, the Q peaks slowly at $b/a = 0.55$, with a Q ratio of 1.1251.

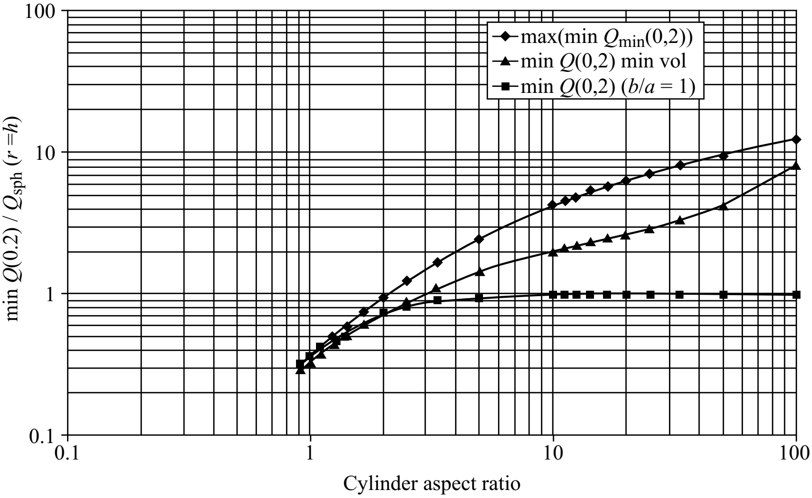

FIGURE 1.13 Q ratio: cylinder in prolate spheroid. Courtesy of Adams and Hansen (2008).

The Q/Q_{sphere} ratio versus cylinder aspect ratio for a cylinder enclosed in a prolate spheroid is shown in Figure 1.13. Data are shown for three cases: maximum Q, Q for maximum volume, and Q for the cylinder enclosed in a hemisphere.

All these data are summed up in Figure 1.14, which shows normalized Q versus aspect ratio, when sphere and spheroid have equal Q, for several dimensional equivalences.

Wheeler (1975), as mentioned earlier, used electrostatics and magnetostatics to develop Q boundaries for cylindrical boundaries. This work was continued by Gustafsson and colleagues (Gustafsson and Nordebo, 2006; Gustafsson et al.,

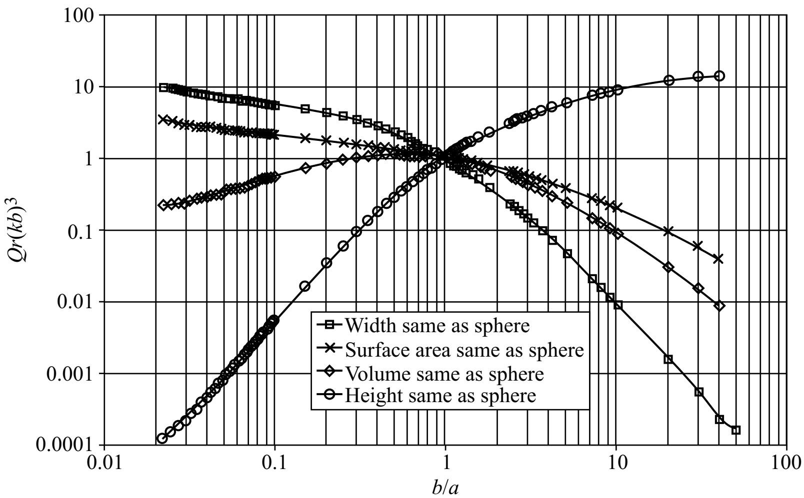

FIGURE 1.14 Sphere and spheroid of equal Q. Courtesy of Hansen and Adams (2010).

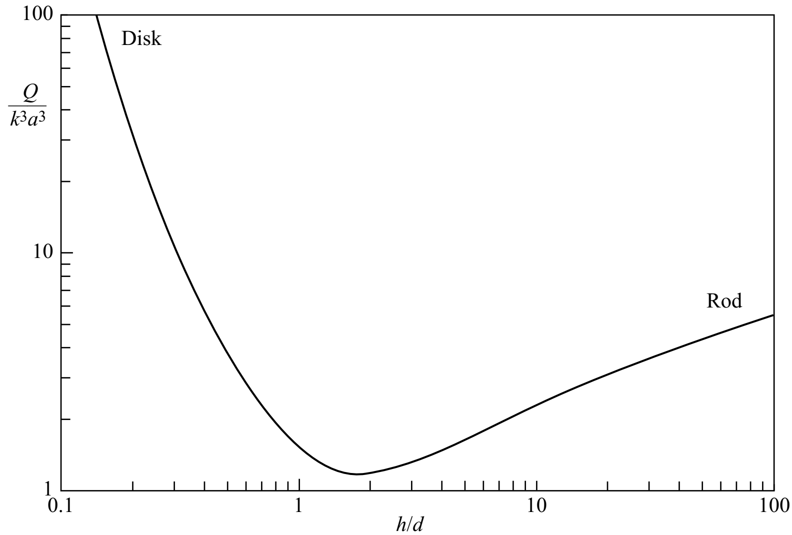

FIGURE 1.15 Static Q ratio for cylinder versus height/diameter. Courtesy of Gustafsson et al. (2009).

2007, 2009). Figure 1.15 shows Q/Q_{Chu} as a function of cylinder height/diameter. See also Yaghjian and Stuart (2010). These data, based on static fields, are accurate only for $ka \ll 1$.

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