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HEAT CONDUCTION FUNDAMENTALS

> No subject has more extensive relations with the progress of industry and the natural sciences; for the action of heat is always present, it penetrates all bodies and spaces, it influences the processes of the arts, and occurs in all the phenomena of the universe.
>
> —Joseph Fourier, *Théorie Analytique de la Chaleur,* 1822 [1]

All matter when considered at the macroscopic level has a definite and precise energy. Such a state of energy may be quantified in terms of a thermodynamic energy function, which partitions energy at the atomic level among, for example, electronic, vibrational, and rotational states. Under local equilibrium, the energy function may be characterized by a measurable scalar quantity called *temperature*. The energy exchanged by the constituent particles (e.g., atoms, molecules, or free electrons) from a region with a greater local temperature (i.e., greater thermodynamic energy function) to a region with a lower local temperature is called *heat*. The transfer of heat is classically considered to take place by conduction, convection, and radiation, and although it cannot be measured directly, the concept has physical meaning because of the direct relationship to temperature. *Conduction* is a specific mode of heat transfer in which this energy exchange takes place in solids or quiescent fluids (i.e., no convective motion resulting from the macroscopic displacement of the medium) from the region of high temperature to the region of low temperature due to the presence of a *temperature gradient* within the system. Once the temperature distribution $T(\hat{r}, t)$ is known within the medium as a function of space (defined by the position vector $\hat{r}$) and time (defined by scalar t), the flow of heat is then prescribed from the governing heat transfer laws. The study of heat conduction provides an enriching

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combination of fundamental science and mathematics. As the prominent thermodynamicist H. Callen wrote: "The history of the concept of heat as a form of energy transfer is unsurpassed as a case study in the tortuous development of scientific theory, as an illustration of the almost insuperable inertia presented by accepted physical doctrine, and as a superb tale of human ingenuity applied to a subtle and abstract problem" [2]. The science of heat conduction is principally concerned with the determination of the temperature distribution and flow of energy within solids. In this chapter, we present the basic laws relating the heat flux to the temperature gradient in the medium, the governing differential equation of heat conduction, the boundary conditions appropriate for the analysis of heat conduction problems, the rules of coordinate transformation needed for working in different orthogonal coordinate systems, and a general discussion of the various solution methods applicable to the heat conduction equation.

1-1   THE HEAT FLUX

Laws of nature provide accepted descriptions of natural phenomena based on observed behavior. Such laws are generally formulated based on a large body of empirical evidence accepted within the scientific community, although they usually can be neither proven nor disproven. To quote Joseph Fourier from the opening sentence of his *Analytical Theory of Heat*: "Primary causes are unknown to us; but are subject to simple and constant laws, which may be discovered by observation" [1]. These laws are considered *general laws*, as their application is independent of the medium. Well-known examples include Newton's laws of motion and the laws of thermodynamics. Problems that can be solved using only general laws of nature are referred to as deterministic and include, for example, simple projectile motion.

Other problems may require supplemental relationships in addition to the general laws. Such problems may be referred to as nondeterministic, and their solution requires laws that apply to the specific medium in question. These additional laws are referred to as *particular laws* or *constitutive relations*. Well-known examples include the ideal gas law, the relationship between shear stress and the velocity gradient for a Newtonian fluid, and the relationship between stress and strain for a linear-elastic material (Hooke's law).

The particular law that governs the relationship between the flow of heat and the temperature field is named after Joseph Fourier. For a homogeneous, isotropic solid (i.e., material in which thermal conductivity is independent of direction), *Fourier's law* may be given in the form

$$q''(\hat{r}, t) = -k\nabla T(\hat{r}, t) \quad \text{W/m}^2 \tag{1-1}$$

where the temperature gradient $\nabla T(\hat{r}, t)$ is a vector normal to the isothermal surface, the heat flux vector $q''(\hat{r}, t)$ represents the heat flow per unit time, per unit area of the isothermal surface in the direction of decreasing temperature gradient,

and k is the *thermal conductivity* of the material. The thermal conductivity is a positive, scalar quantity for a homogeneous, isotropic material. The minus sign is introduced in equation (1-1) to make the heat flow a positive quantity in the positive coordinate direction (i.e., opposite of the temperature gradient), as described below. This text will consider the heat flux in the SI units W/m^2 and the temperature gradient in K/m (equivalent to the unit °C/m), giving the thermal conductivity the units of W/(m · K). In the Cartesian coordinate system (i.e., rectangular system), equation (1-1) is written as

$$q''(x, y, z, t) = -\hat{i}k\frac{\partial T}{\partial x} - \hat{j}k\frac{\partial T}{\partial y} - \hat{k}k\frac{\partial T}{\partial z} \tag{1-2}$$

where $\hat{i}, \hat{j}$, and $\hat{k}$ are the unit direction vectors along the x, y, and z directions, respectively. One may consider the three components of the heat flux vector in the x, y, and z directions, respectively, as given by

$$q''_x = -k\frac{\partial T}{\partial x} \quad q''_y = -k\frac{\partial T}{\partial y} \quad \text{and} \quad q''_z = -k\frac{\partial T}{\partial z} \tag{1-3a,b,c}$$

Clearly, the flow of heat for a given temperature gradient is directly proportional to the thermal conductivity of the material. Equation (1-3a) is generally used for one-dimensional (1-D) heat transfer in a rectangular coordinate system. Figure 1-1 illustrates the sign convention of Fourier's law for the 1-D Cartesian coordinate system. Both plots depict the heat flux (W/m^2) through the plane at $x = x_0$ based on the local temperature gradient. In Figure 1-1(*a*), the gradient dT/dx is negative with regard to the Cartesian coordinate system; hence the resulting flux is mathematically positive, and by convention is in the *positive x direction*, as shown in the figure. In contrast, Figure 1-1(*b*) depicts a positive gradient dT/dx. This yields a mathematically negative heat flux, which by convention
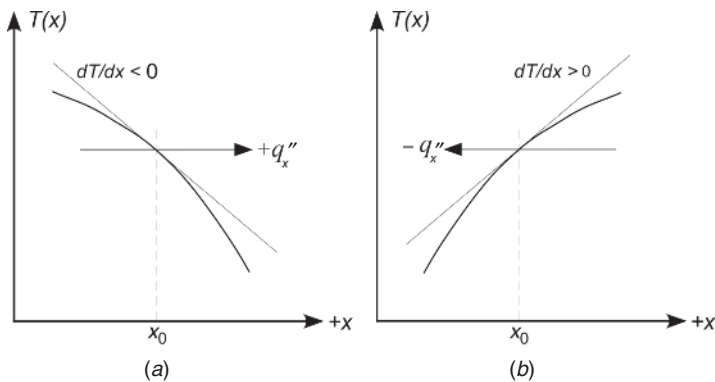
Figure 1-1   Fourier's law illustrated for a (a) positive heat flux and (b) a negative heat flux.

is in the negative x direction, as indicated in the figure. As defined, Fourier's law is directly tied to the coordinate system, with positive heat flux always flowing in the positive coordinate direction. While determining the actual direction of heat flow is often trivial for 1-D problems, multidimensional problems, and notably transient problems, can present considerable difficulty in determining the direction of the local heat flux terms. Adherence to the sign convention of Fourier's law will avoid any such difficulties of flux determination, which is useful in the context of overall energy conservation for a given heat transfer problem.

In addition to the heat flux, which is the flow of heat *per unit area* normal to the direction of flow (e.g., a plane perpendicular to the page in Fig. 1-1), one may define the total heat flow, often called the *heat rate*, in the unit of watts (W). The heat rate is calculated by multiplying the heat flux by the *total cross-sectional area* through which the heat flows for a 1-D problem or by integrating over the area of flow for a multidimensional problem. The heat rate in the x direction for one-, two-, and three-dimensional (1-D, 2-D, and 3-D) Cartesian problems is given by

$$q_x = -kA_x\frac{dT}{dx} \quad \text{W} \tag{1-4}$$

$$q_x = -kH\int_{y=0}^{L}\frac{\partial T(x,y)}{\partial x}dy \quad \text{W} \tag{1-5}$$

$$q_x = -k\int_{y=0}^{L}\int_{z=0}^{H}\frac{\partial T(x,y,z)}{\partial x}dz\,dy \quad \text{W} \tag{1-6}$$

where A_x is the total cross-sectional area for the 1-D problem in equation (1-4). The total cross-sectional area for the 2-D problem in equation (1-5) is defined by the surface from $y = 0$ to L in the second spatial dimension and by the length H in the z direction, for which there is no temperature dependence [i.e., $T \neq f(z)$]. The total cross-sectional area for the 3-D problem in equation (1-6) is defined by the surface from $y = 0$ to L and $z = 0$ to H in the second and third spatial dimensions, noting that $T = f(x,y,z)$.

1-2   THERMAL CONDUCTIVITY

Given the direct dependency of the heat flux on the thermal conductivity via Fourier's law, the thermal conductivity is an important parameter in the analysis of heat conduction. There is a wide range in the thermal conductivities of various engineering materials. Generally, the highest values are observed for pure metals and the lowest value by gases and vapors, with the amorphous insulating materials and inorganic liquids having thermal conductivities that lie in between. There are important exceptions. For example, natural type IIa diamond (nitrogen free) has the highest thermal conductivity of any bulk material ($\sim$2300 W/m·K at ambient

temperature), due to the ability of the well-ordered crystal lattice to transmit thermal energy via vibrational quanta called *phonons*. In Chapter 16, we will explore in depth the physics of energy carriers to gain further insight into this important material property.

To give some idea of the order of magnitude of thermal conductivity for various materials, Figure 1-2 illustrates the typical range for various material classes. Thermal conductivity also varies with temperature and may change with orientation for nonisotropic materials. For most pure metals the thermal conductivity decreases with increasing temperature, whereas for gases it increases with increasing temperature. For most insulating materials it increases with increasing temperature. Figure 1-3 provides the effect of temperature on the thermal conductivity for a range of materials. At very low temperatures, thermal conductivity increases rapidly and then exhibits a sharp decrease as temperatures approach absolute zero, as shown in Figure 1-4, due to the dominance of energy carrier scattering from defects at extreme low temperatures. A comprehensive compilation of thermal conductivities of materials may be found in references 3–6. We present in Appendix I the thermal conductivity of typical engineering materials together with the specific heat c_p, density ρ, and the thermal diffusivity α. These latter properties are discussed in more detail in the following section.
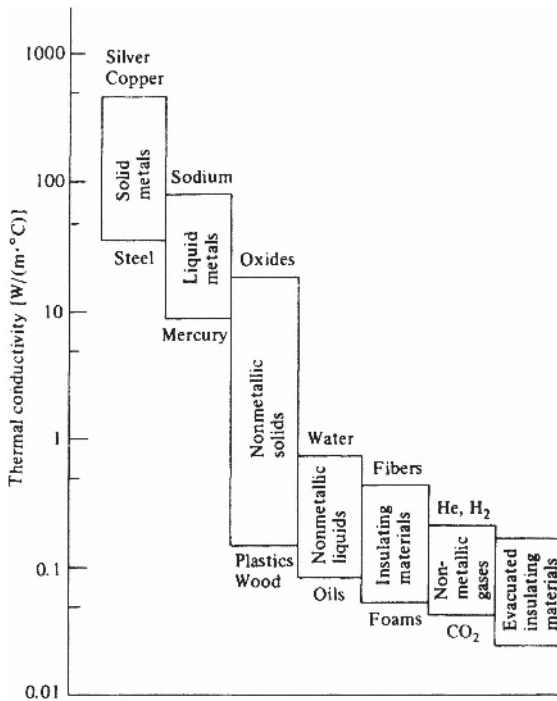

Figure 1-2 Typical range of thermal conductivity of various material classes.
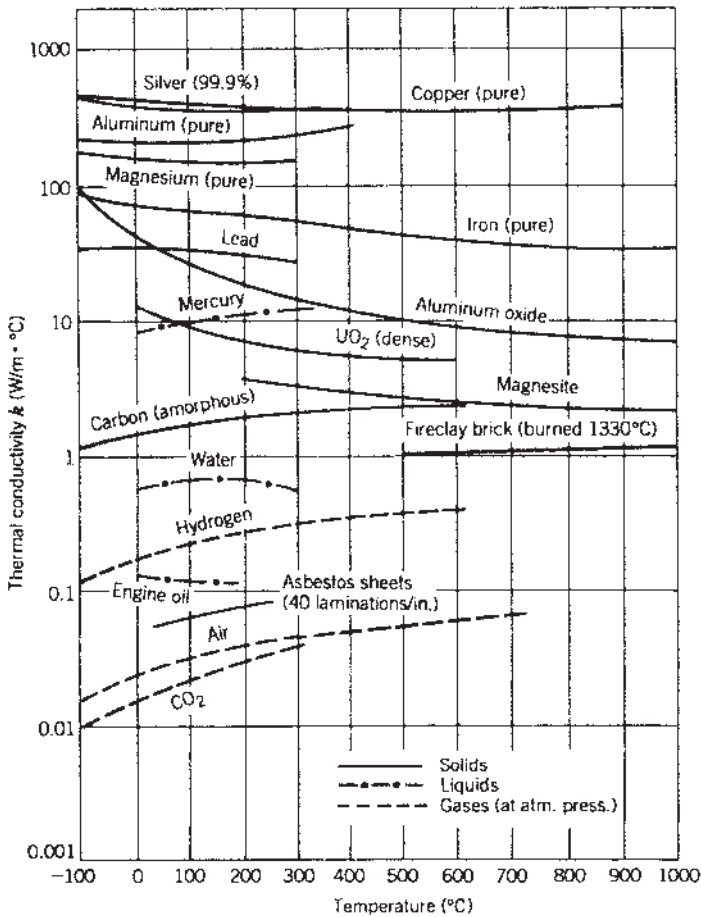
Figure 1-3   Effect of temperature on thermal conductivity.

1-3 DIFFERENTIAL EQUATION OF HEAT CONDUCTION

We now derive the differential equation of heat conduction, often called the *heat equation*, for a stationary, homogeneous, isotropic solid with heat generation within the body. *Internal heat generation* may be due to nuclear or chemical reactivity, electrical current (i.e., Joule heating), absorption of laser light, or other sources that may in general be a function of time and/or position. The heat equation may be derived using either a differential control volume approach or an integral approach. The former is perhaps more intuitive and will be presented first, while the latter approach is more general and readily extends the derivation to moving solids.
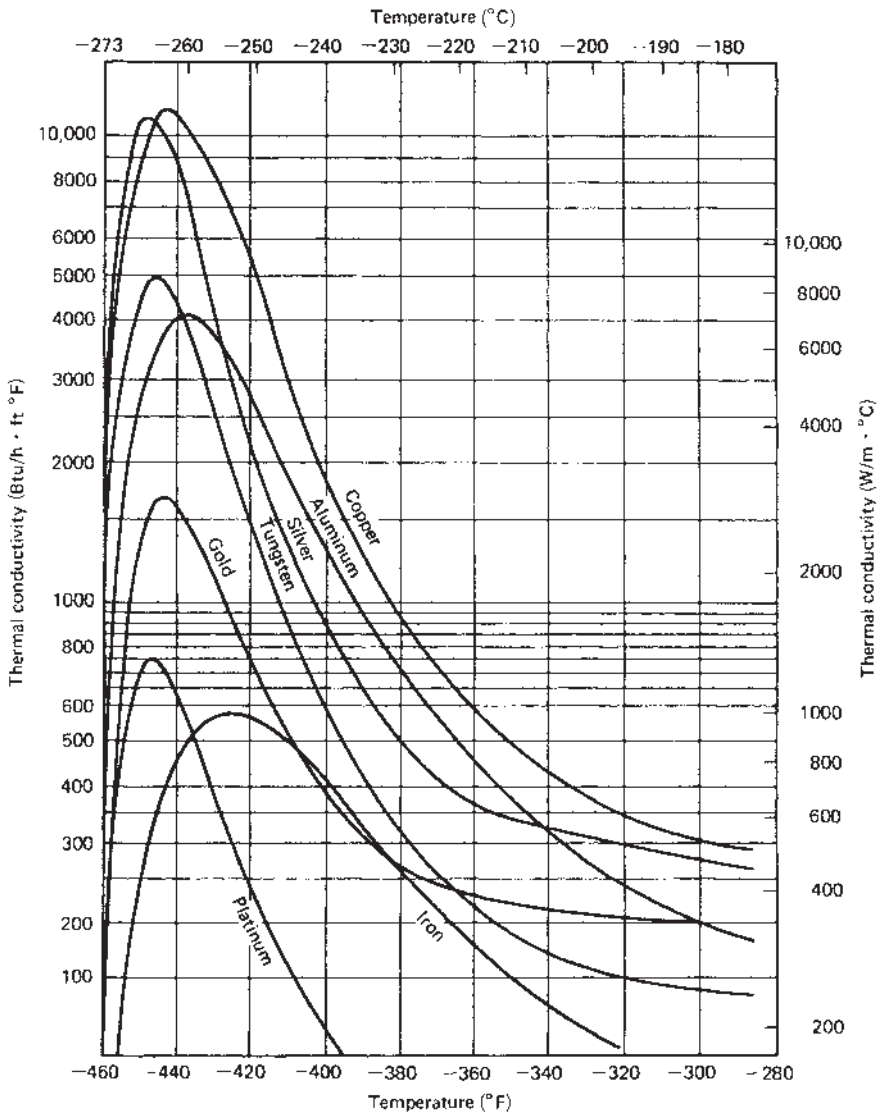
Figure 1-4   Thermal conductivity of metals at low temperatures.

The *differential control volume* is defined in Figure 1-5 for the Cartesian coordinate system. The corresponding volume and mass of the differential control volume are defined, respectively, as

$$dv = dx\ dy\ dz \quad \text{and} \quad dm = \rho dx\ dy\ dz \tag{1-7}$$

where ρ is the mass density (kg/m^3) of the control volume. The differential approach will assume a *continuum* such that all properties do not vary
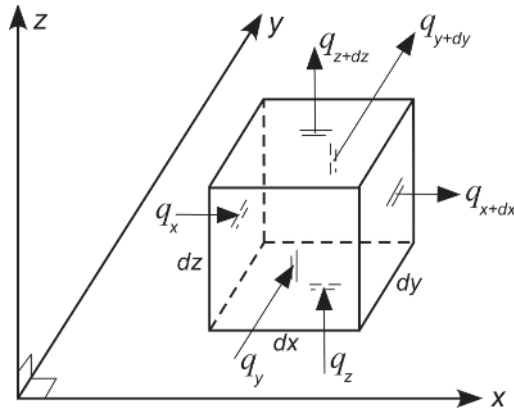
Figure 1-5 Differential control volume for derivation of the heat equation in Cartesian coordinates.

microscopically. The continuum assumption may be considered in terms of ε, a volume much larger than individual atoms. If we let L_c be the smallest length scale of interest for the heat transfer problem, then the continuum assumption is considered justified for the condition

$$\varepsilon \ll L_c^3 \tag{1-8}$$

Limitations on the continuum approach for the heat equation and Fourier's law are discussed in Chapter 16.

We begin with a general statement of *conservation of energy* based on the first law of thermodynamics, namely,

$$\left(h + \frac{1}{2}\overline{V}^2 + gz\right)_{\text{in}} \delta \dot{m}_{\text{in}} - \left(h + \frac{1}{2}\overline{V}^2 + gz\right)_{\text{out}} \delta \dot{m}_{\text{out}}$$
$$+ \delta \dot{Q} + \delta \dot{E}_{\text{gen}} - \delta \dot{W} = \frac{dE_{\text{cv}}}{dt} \tag{1-9}$$

where $\delta \dot{m}_{\text{in}}$ and $\delta \dot{m}_{\text{out}}$ represent the mass flow rates in and out of the differential control volume, respectively. We will derive the heat equation for a *quiescent medium*, hence the mass flow rates are zero, and assume that the rate of work done by the control volume is zero ($\delta \dot{W} = 0$). The rate of change of energy within the control volume may be expanded as

$$\frac{dE_{\text{cv}}}{dt} = \frac{d}{dt}\left[\left(u + \frac{1}{2}\overline{V}^2 + gz\right)_{\text{cv}} dm\right] \tag{1-10}$$

where u is the internal energy (J/kg), an intensive, scalar property associated with the thermodynamic state of the system. Neglecting any *changes* in the kinetic

and potential energy of the control volume, and applying the above assumptions, conservation of energy becomes

$$\delta \dot{Q} + \delta \dot{E}_{\text{gen}} = \frac{d(u\ dm)}{dt} \tag{1-11}$$

where $\delta \dot{E}_{\text{gen}}$ (W) represents the rate at which energy is generated within the control volume due to internal energy generation as described above, and $\delta \dot{Q}$ (W) represents the net rate of heat transfer into the control volume, with positive $\delta \dot{Q}$ representing heat transfer into the system. We may now consider equation (1-11) term by term.

The net rate of heat transfer is given in terms of the heat rate in and out of the control volume, namely,

$$\delta \dot{Q} = \left(q_x - q_{x+dx}\right) + \left(q_y - q_{y+dy}\right) + \left(q_z - q_{z+dz}\right) \tag{1-12}$$

where the individual, entering heat rate terms may be defined using Fourier's law and the respective cross-sectional areas, as given by equation (1-4) for the x direction:

$$q_x = -kA_x \frac{\partial T}{\partial x} \quad \text{where} \quad A_x = dy\ dz \tag{1-13}$$

$$q_y = -kA_y \frac{\partial T}{\partial y} \quad \text{where} \quad A_y = dx\ dz \tag{1-14}$$

$$q_z = -kA_z \frac{\partial T}{\partial z} \quad \text{where} \quad A_z = dx\ dy \tag{1-15}$$

The individual, exiting heat rate terms may be defined using a Taylor series expansion of the entering terms. Neglecting higher-order terms, for the x direction this term becomes

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx = -kA_x \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-kA_x \frac{\partial T}{\partial x}\right) dx \tag{1-16}$$

Using equations (1-13) and (1-16), the net heat rate entering the differential control volume from the x direction becomes

$$q_x - q_{x+dx} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) dx\ dy\ dz \tag{1-17}$$

Similarly, the net heat rate in the y and z directions becomes

$$q_y - q_{y+dy} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) dx\ dy\ dz \tag{1-18}$$

$$q_z - q_{z+dz} = \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) dx\ dy\ dz \tag{1-19}$$

Equations (1-17)–(1-19) may now be substituted in equation (1-12).

The rate of internal energy generation (W) may be directly calculated from the *volumetric rate* of internal energy generation g (W/m^3), noting that in general $g = g(\hat{r}, t)$, and the control volume, namely,

$$\delta \dot{E}_{gen} = g \; dx \; dy \; dz \qquad (1\text{-}20)$$

Finally, the rate of change of energy within the control volume may be defined by introducing the *constant volume specific heat* c_v (J/ kg · K), namely,

$$c_v \equiv \left. \frac{\partial u}{\partial T} \right|_v \quad \rightarrow \quad u = c_v T + u_{ref} \qquad (1\text{-}21)$$

noting that for an incompressible solid or fluid, $c_v = c_p = c$, with the middle quantity defined as the constant pressure specific heat. Equation (1-21) may be substituted into the right-hand side of equation (1-11), which along with the assumption of constant properties ρ and c yields the net rate of change of energy within the control volume as

$$\frac{d(u \; dm)}{dt} = \rho c \frac{\partial T}{\partial t} dx \; dy \; dx \qquad (1\text{-}22)$$

The above expressions may now be introduced into equation (1-11) to provide the *general heat equation* for the Cartesian coordinate system, which after cancelation of the *dx dy dz* terms yields

$$\boxed{\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + g = \rho c \frac{\partial T}{\partial t}} \qquad (1\text{-}23)$$

where each term has the units W/m^3. In simple terms, the heat equation expresses that the net rate of heat conducted per differential volume plus the rate of energy generated internally per volume is equal to the net rate of energy stored per differential volume. The heat equation may be expressed in several additional forms, including using vector notation:

$$\nabla \cdot (k \nabla T) + g = \rho c \frac{\partial T}{\partial t} \qquad (1\text{-}24)$$

where ∇ is the vector differential operator [$\nabla = \hat{i}(\partial/\partial x) + \hat{j}(\partial/\partial y) + \hat{k}(\partial/\partial z)$]. When the thermal conductivity is a constant, equation (1-23) may be written in the form

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}} \qquad (1\text{-}25)$$

where each term now has the units K/m^2. The *thermal diffusivity* (m^2/s), which appears on the right-hand side, is defined as

$$\alpha = \frac{k}{\rho c} \qquad (1\text{-}26)$$

and represents a thermal-physical property of the medium. The physical significance of thermal diffusivity is associated with the speed of propagation of heat into the solid during changes of temperature. In other words, the thermal diffusivity represents the flow of heat (i.e., conduction of heat) compared to the storage of energy (i.e., heat capacity). The higher the thermal diffusivity, the faster is the response of a medium to thermal perturbations, and the faster such changes propagate throughout the medium. This statement is better understood by referring to the following heat conduction problem: Consider a semi-infinite medium, $x \geq 0$, initially at a uniform temperature T_0. For times $t > 0$, the boundary surface at $x = 0$ is suddenly reduced to and kept at zero temperature. Clearly, the temperature within the medium will now vary with position and time. Suppose we are interested in the time required for the temperature to decrease from its initial value T_0 to half of this value, $\frac{1}{2}T_0$, at a position, say, 30 cm from the boundary surface. Table 1-1 gives the time required for this change for several different materials. It is apparent from these results that the greater the thermal diffusivity, the shorter is the time required for the boundary perturbation to penetrate into the depth of the solid. It is important to note, therefore, that the thermal response of a material depends not only on the thermal conductivity but also on the density and specific heat.

For a medium with constant thermal conductivity, no internal heat generation, and under steady-state conditions (i.e., $\partial T/\partial t = 0$), the heat equation takes the form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{or} \quad \nabla^2 T = 0 \qquad (1\text{-}27)$$

which is known as *Laplace's equation*, after the French mathematician Pierre-Simon Laplace.

We now present the *integral formulation* of the heat equation by considering the energy balance for a small control volume V, as illustrated in Figure 1-6. Conversation of energy may be stated as

$$\begin{bmatrix} \text{Rate of heat entering through} \\ \text{the bounding surface of } V \end{bmatrix} + \begin{bmatrix} \text{Rate of energy} \\ \text{generation in } V \end{bmatrix} = \begin{bmatrix} \text{Rate of energy} \\ \text{storage in } V \end{bmatrix}$$

$$(1\text{-}28)$$

Each term in equation (1-28) may now be evaluated individually, beginning with the rate of heat entering through the boundary. This can be calculated by

TABLE 1-1 Effect of Thermal Diffusivity on Rate of Heat Propagation

Material	Silver	Copper	Steel	Glass	Cork
$\alpha \times 10^6$ m^2/s	170	103	12.9	0.59	0.16
Time	9.5 min	16.5 min	2.2 h	2.0 days	7.7 days
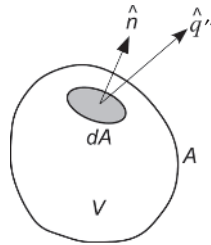
Figure 1-6 Control volume for integral formulation of the heat equation.

integrating the surface heat flux normal to the surface over the entire surface area of the control volume, as given by

$$\left[\begin{array}{l}\text{Rate of heat entering through} \\ \text{the bounding surface of } V\end{array}\right] = -\int_A q'' \cdot \hat{n} dA = -\int_V \nabla \cdot q'' dv \quad (1\text{-}29)$$

where A is the surface area of the control volume V, $\hat{n}$ is the outward-drawn normal vector to the surface element dA, and $q''(\hat{r}, t)$ is the heat flux vector at the surface element dA. The minus sign is introduced to ensure that the heat flow is positive into the control volume in consideration of the negative sign in Fourier's law per equation (1-1). The *divergence theorem*, also known as Gauss' theorem, is then used to transform the surface integral into a volume integral, yielding the final form above.

We next consider the rate of energy generation within the control volume, which is readily evaluated by integrating the volumetric energy generation, as defined above, over the control volume

$$\left[\begin{array}{l}\text{Rate of energy} \\ \text{generation in } V\end{array}\right] = \int_V g(\hat{r}, t) dv \quad (1\text{-}30)$$

For the rate of storage within the control volume, it is first useful to define the *material* or *total derivative* in terms of Eulerian derivatives for a generic property $\lambda(\hat{r}, t)$, namely,

$$\frac{D\lambda}{Dt} \equiv \frac{\partial \lambda}{\partial t} + u\frac{\partial \lambda}{\partial x} + v\frac{\partial \lambda}{\partial y} + w\frac{\partial \lambda}{\partial z} \quad (1\text{-}31)$$

where the velocity vector $\hat{u}$ has the components

$$\hat{u} = u\hat{i} + v\hat{j} + w\hat{k} \quad (1\text{-}32)$$

We now introduce *Reynolds transport theorem*, which allows one to readily calculate the material derivative of a volume integral. Again using our generic property

λ, Reynolds transport theorem may be expressed as

$$\frac{D}{Dt} \int_V \lambda(\hat{r}, t) dv = \int_V \left[\frac{\partial \lambda}{\partial t} + \nabla \cdot (\lambda \hat{u}) \right] dv \qquad (1\text{-}33)$$

For the rate of change of energy within the control volume, we want our generic property λ to specifically equal the energy per unit volume (J/m^3). Using our definition of constant volume specific heat, this is accomplished by letting $\lambda(\hat{r}, t) = \rho c T(\hat{r}, t)$. Substituting into equation (1-33) yields

$$\left[\begin{matrix} \text{Rate of energy} \\ \text{storage in } V \end{matrix} \right] = \frac{D}{Dt} \int_V \rho c T(\hat{r}, t) dv = \int_V \rho c \left[\frac{\partial T}{\partial t} + \nabla \cdot (T \hat{u}) \right] dv \quad (1\text{-}34)$$

Now all three rate terms of equation (1-28) are expressed as volume integrals, namely equations (1-29), (1-30), and (1-34), which may be brought into a common integral, yielding

$$\int_V \left\{ -\nabla \cdot q'' + g - \rho c \left[\frac{\partial T}{\partial t} + \nabla \cdot (T \hat{u}) \right] \right\} dv = 0 \qquad (1\text{-}35)$$

Because equation (1-35) is derived for an arbitrary control volume V, the only way it is satisfied for all choices of V is if the integrand itself is zero. With this condition, equation (1-35) becomes

$$-\nabla \cdot q'' + g = \rho c \left[\frac{\partial T}{\partial t} + \nabla \cdot (T \hat{u}) \right] \qquad (1\text{-}36)$$

This equation can now be simplified further by expanding the rightmost term,

$$\nabla \cdot (T \hat{u}) = T(\nabla \cdot \hat{u}) + \hat{u} \cdot \nabla T \qquad (1\text{-}37)$$

and noting that $\nabla \cdot \hat{u} = 0$ per continuity for an incompressible medium. We then insert equation (1-1) for the heat flux vector in the left-hand side (LHS). Making these substitutions yields the desired final form of the conduction heat equation,

$$\nabla \cdot (k \nabla T) + g = \rho c \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right] \qquad (1\text{-}38)$$

Equation (1-38) is valid for an *incompressible, moving solid*, assuming constant ρc. Overall, the bulk motion of the solid is regarded to give rise to convective or enthalpy fluxes, namely, in the x, y, and z directions, in addition to the conduction fluxes in these same directions. With these considerations, the components of the heat flux vector $q''(\hat{r}, t)$ are taken as

$$q''_x = -k \frac{\partial T}{\partial x} + \rho c T u \qquad (1\text{-}39a)$$

$$q_y'' = -k\frac{\partial T}{\partial y} + \rho c T\, v \qquad (1\text{-}39b)$$

$$q_z'' = -k\frac{\partial T}{\partial z} + \rho c T\, w \qquad (1\text{-}39c)$$

Clearly, on the right-hand sides (RHS) of the above three equations, the first term is the conduction heat flux and the second term is the convective heat flux due to the bulk motion of the solid. For the case of no motion (i.e., quiescent medium), the terms $u = v = w = 0$ and equation (1-38) reduces exactly to equation (1-24).

1-4 FOURIER'S LAW AND THE HEAT EQUATION IN CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

Here we present Fourier's law and the heat conduction equation for other *orthogonal curvilinear coordinate systems*, namely, cylindrical and spherical coordinates. The heat equations may be directly derived using a differential control volume in the respective coordinate systems, following the approach of Section 1-3, or they may be obtained using the appropriate coordinate transformation into cylindrical or spherical coordinates. The results are presented here without derivation, although the respective differential control volumes are defined.

The expression for the heat flux vector (i.e., Fourier's law) in each new coordinate system may be given by the three principal components

$$q_i'' = -k\frac{1}{a_i}\frac{\partial T}{\partial u_i} \quad \text{for } i = 1, 2, 3, \ldots \qquad (1\text{-}40)$$

where $u_1, u_2,$ and u_3 are the curvilinear coordinates, and the coefficients $a_1, a_2,$ and a_3 are the *coordinate scale factors*, which may be constants or functions of the coordinates. The expressions for the scale factors are derived for a general orthogonal curvilinear system by Özisik [7].

We will first consider the *cylindrical coordinate system*, as shown in Figure 1-7 along with a representative differential control volume. Using the appropriate scale factors, the three components of heat flux in the r, ϕ and z directions, respectively, become

$$q_r'' = -k\frac{\partial T}{\partial r} \quad q_\phi'' = -\frac{k}{r}\frac{\partial T}{\partial \phi} \quad \text{and} \quad q_z'' = -k\frac{\partial T}{\partial z} \qquad (1\text{-}41a,b,c)$$

By inspection, it is seen that the scale factors a_r and a_z are unity, while the scale factor $a_\phi = r$. This scale factor also provides the correct units for flux, as the gradient term, $\partial T/\partial \phi$, in the ϕ direction (K/rad) is missing the spatial dimension. In the cylindrical coordinate system, the heat equations (1-23) and (1-25) become, respectively,
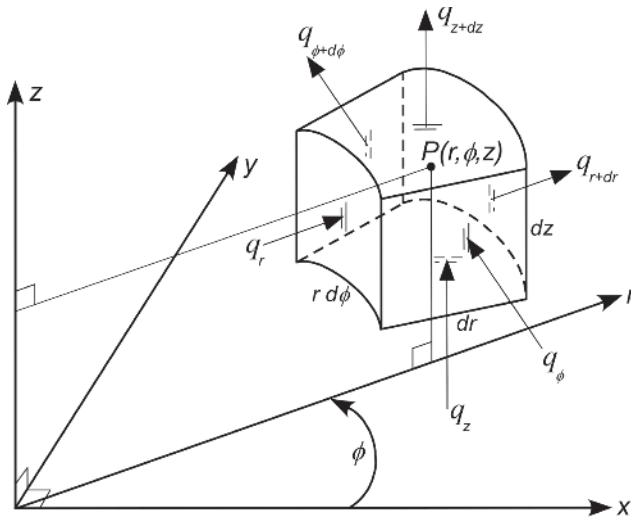
Figure 1-7   Cylindrical coordinate system and representative differential control volume.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+g=\rho c\frac{\partial T}{\partial t} \qquad (1\text{-}42)$$
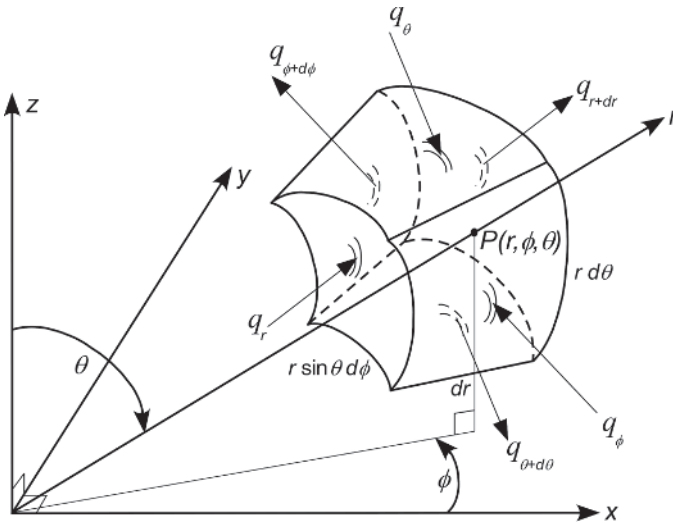
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (1\text{-}43)$$

We now consider the *spherical coordinate system*, as shown in Figure 1-8 along with a representative differential control volume. Using the appropriate scale factors, the three components of heat flux in the r, ϕ and θ directions become, respectively,

$$q_r'' = -k\frac{\partial T}{\partial r} \quad q_\phi'' = -\frac{k}{r\,\sin\theta}\frac{\partial T}{\partial \phi} \quad \text{and} \quad q_\theta'' = -\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad (1\text{-}44a,b,c)$$

By inspection, it is seen now that only the scale factor a_r is unity, while the scale factors $a_\phi = r\sin\theta$ and $a_\theta = r$. As before, these scale factors also provide the correct units for flux, as the gradient terms in both the ϕ and θ directions (K/rad) are missing the spatial dimension. In the spherical coordinate system, the heat equations (1-23) and (1-25) become, respectively,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+g=\rho c\frac{\partial T}{\partial t}$$

$$(1\text{-}45)$$

Figure 1-8　Spherical coordinate system and representative differential control volume.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$(1\text{-}46)$$

1-5　GENERAL BOUNDARY CONDITIONS AND INITIAL CONDITION FOR THE HEAT EQUATION

The differential equation of heat conduction, see, for example, equation (1-25), will require two boundary conditions for each spatial dimension, as well as one initial condition for the non–steady-state problem. The initial condition specifies the temperature distribution in the medium at the origin of the time coordinate, that is, $T(\hat{r}, t = 0)$, and the boundary conditions specify the temperature or the heat flux at the boundaries of the region. For example, at a given boundary surface, the temperature distribution may be prescribed, or the heat flux distribution may be prescribed, or there may be heat exchange by convection and/or radiation with an environment at a prescribed temperature. The boundary condition can be derived by writing an energy balance equation at the surface of the solid. Prior to considering formal boundary conditions, it is useful to define two additional particular laws for heat transfer, namely, for radiation and convection heat transfer.

The *Stefan–Boltzmann law* [8, 9] describes the heat flux emitted from a surface by radiation heat transfer

$$q''_{\text{rad}} = \varepsilon\sigma T^4 \quad \text{W/m}^2 \tag{1-47}$$

where ε is the *total, hemispherical emissivity* of the surface, and σ is the Stefan–Boltzmann constant, given as $\sigma = 5.670 \times 10^{-8}$ W/(m$^2 \cdot$ K^4). As presented, equation (1-47) represents energy radiated into all directions and over all wavelengths. The total, hemispherical emissivity represents an integration of the spectral, directional emissivity over all directions and wavelengths. Because the weighting of the spectral emissivity is by the Planck distribution, ε is in general a function of the surface temperature for a *nongray surface* (i.e., a surface for which the properties vary with wavelength).

A commonly used approximation for the *net* radiation heat flux between a surface and a surrounding medium is given as

$$q''_{\text{rad}} = \varepsilon \sigma (T^4 - T_\infty^4) \quad \text{W/m}^2 \tag{1-48}$$

where T_∞ is the temperature of the ambient, surrounding medium. Equation (1-48) assumes that the ambient surroundings form an *ideal enclosure*, which is satisfied if the surroundings are isothermal and of a much larger surface area, and assumes that the emitting surface is a *gray body* (i.e., neglect the wavelength dependency of the surface's emissivity and absorptivity).

Newton's law of cooling describes the heat flux to or from a surface by convection heat transfer

$$q''_{\text{conv}} = h(T - T_\infty) \quad \text{W/m}^2 \tag{1-49}$$

where T_∞ is the reference temperature of the surrounding ambient fluid (e.g., liquid or gas) and h is the *convection heat transfer coefficient* of units W/(m$^2 \cdot$ K). Equation (1-49) is not tied to the overall coordinate system; hence positive heat flux is considered in the direction of the surface normal (i.e., away from the surface). Unlike Fourier's law and the Stefan–Boltzmann law, Newton's law is not so much a particular law as it is the *definition* of the heat transfer coefficient, namely,

$$h = \frac{q''_{\text{conv}}}{T - T_\infty} \quad \text{W/(m}^2 \cdot \text{K)} \tag{1-50}$$

which reflects the dependency of h on the actual heat flux, and the difference between the surface temperature and a suitable *reference temperature*. Here the heat transfer coefficient h varies with the type of flow (laminar, turbulent, etc.), the geometry of the body and flow passage area, the physical properties of the fluid, the average surface and fluid temperatures, and many other parameters. As a result, there is a wide difference in the range of values of the heat transfer coefficient for various applications. Table 1-2 lists the typical values of h, in our units W/(m$^2 \cdot$ K), encountered in some applications.

To now develop the general boundary conditions, we consider conservation of energy at the surface, assumed to be stationary, noting that no energy can be accumulated (i.e., stored) at an infinitely thin surface. Figure 1-9 depicts a

TABLE 1-2   Typical Values of Convective Heat Transfer Coefficient

Type of Flow[a]	h, W/(m^2 · K)
Free Convection, $\Delta T = 25$ K	
0.25-m vertical plate in:	
Atmospheric air	5
Engine oil	40
Water	440
0.02-m (OD) horizontal cylinder in:	
Atmospheric air	10
Engine oil	60
Water	740
Forced Convection	
Atmospheric air at 25 K with $U_\infty = 10$ m/s over $L = 0.1$-m flat plate	40
Flow at 5 m/s across 1-cm (OD) cylinder of:	
Atmospheric air	90
Engine oil	1,800
Water flow at 1 kg/s inside 2.5-cm (ID) tube	10,500
Boiling of Water at 1 atm	
Pool boiling in a container	3,000
Pool boiling at peak heat flux	35,000
Film boiling	300
Condensation of Steam at 1 atm	
Film condensation on horizontal tubes	9,000–25,000
Film condensation on vertical surfaces	4,000–11,000
Dropwise condensation	60,000–120,000

[a]OD = outer diameter and ID = inner diameter.
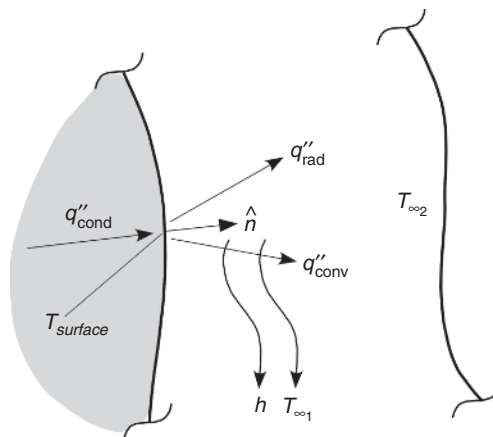
Figure 1-9   Energy balance at surface of a solid.

surface having an outward-drawn unit normal vector, $\hat{n}$, that is in the positive coordinate direction, subjected to convection heat transfer with some fluid, and to radiation heat transfer with an ideal surrounding. Conservation of energy at the surface boundary takes the form

$$q''_{in} = q''_{out} \tag{1-51}$$

or

$$-k \left.\frac{\partial T}{\partial n}\right|_{surface} = h(T|_{surface} - T_{\infty_1}) + \varepsilon\sigma(T^4|_{surface} - T^4_{\infty_2}) \tag{1-52}$$

In equation (1-52), Fourier's law follows our sign convention of positive flux in the positive coordinate direction, while Newton's law follows our convention of positive flux in the direction of the surface normal. The Stefan–Boltzmann law yields a positive flux away from the surface when the surface temperature is greater than the surrounding medium. Note also that the dependent variable T on the right-hand side is considered the value of T *at the surface*.

It is also useful to classify a given boundary or initial condition as either *homogeneous* or *nonhomogeneous*. A homogeneous condition is one in which all nonzero terms in the expression contain the dependent variable, $T(\hat{r}, t)$ in our case, or its derivative. The concept of homogeneous and nonhomogeneous boundary and initial conditions lies at the very core of the *method of separation of variables* that will be considered in the following chapters. In our treatment, for the analytic solution of linear heat conduction problems, we shall consider the following three types of linear boundary conditions.

1. ***Boundary Condition of the First Type (Prescribed Temperature)***. This is the situation when the temperature is prescribed at the boundary surface, that is,

$$T|_{surface} = T_0 \tag{1-53a}$$

or

$$T|_{surface} = f(\hat{r}, t) \tag{1-53b}$$

where T_0 is a prescribed constant temperature, and where $f(\hat{r}, t)$ is the prescribed surface temperature distribution that is, in general, a function of position and time. The special case of zero temperature on the boundary

$$T|_{surface} = 0 \tag{1-54}$$

is called the *homogeneous boundary condition of the first type*. In mathematics, boundary conditions of the first type are called *Dirichlet boundary conditions*.

2. ***Boundary Condition of the Second Type (Prescribed Heat Flux)***. This is the situation in which the heat flux is prescribed at the boundary surface, that is,

$$-k\left.\frac{\partial T}{\partial n}\right|_{surface} = q''_0 \tag{1-55a}$$

or

$$-k\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = f(\hat{r}, t) \qquad (1\text{-}55b)$$

where $\partial T/\partial n$ is the derivative along the outward-drawn normal to the surface, q_0'' is a prescribed constant heat flux (W/m^2), and $f(\hat{r}, t)$ is the prescribed surface heat flux distribution that is, in general, a function of position and time. The special case of zero heat flux at the boundary

$$\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = 0 \quad \text{(perfectly insulated or adiabatic)} \qquad (1\text{-}56)$$

is called the *homogeneous boundary condition of the second type*. In mathematics, boundary conditions of the second type (i.e., prescribed derivative values) are called *Neumann boundary conditions*.

3. ***Boundary Condition of the Third Type (Convection)***. This is the pure convection boundary condition, which is readily obtained from equation (1-52) by setting the radiation term to zero, that is,

$$-k\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = h\left[T|_{\text{surface}} - T_\infty\right] \qquad (1\text{-}57a)$$

For generality, the ambient fluid temperature T_∞ may assumed to be a function of position and time, yielding

$$-k\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = h\left[T|_{\text{surface}} - T_\infty(\hat{r}, t)\right] \qquad (1\text{-}57b)$$

The special case of zero fluid temperature ($T_\infty = 0$), as given by

$$-k\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = h\,T|_{\text{surface}} \qquad (1\text{-}58)$$

is called the *homogeneous boundary condition of the third type*, since the dependent variable or its derivative now appears in all nonzero terms. This represents convection into a fluid medium at zero temperature, noting that a common practice is to redefine or shift the temperature scale such that the fluid temperature is now zero, as discussed in more detail in Section 1-8. A convection boundary condition is physically different than type 1 (prescribed temperature) or type 2 (prescribed flux) boundary conditions in that the temperature gradient within the solid at the surface is now *coupled* to the convective flux at the solid–fluid interface. Neither the flux nor the temperature are prescribed, but rather, a balance between conduction and convection is forced, see equation (1-51), with the exact surface temperature and surface heat flux determined by the

combination of convection coefficient, thermal conductivity, and ambient fluid temperature. Clearly, the boundary conditions of the first and second type can be obtained from the type 3 boundary condition as special cases if k and h are treated as coefficients. For example, by setting $hT_\infty(\hat{r}, t) \equiv f(\hat{r}, t)$ and then letting $h = 0$ in the first term of the right-hand side, equation (1-57b) reduces to equation (1-55b).
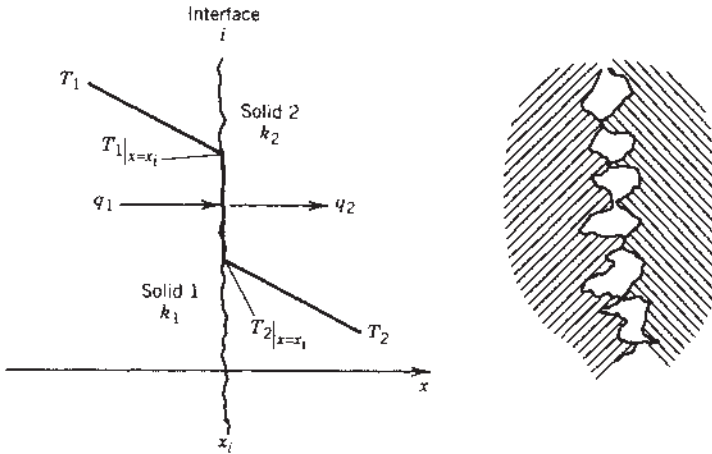
A few final words are offered with regard to these three important boundary conditions. Mathematically speaking, convection boundary conditions provide the greatest complexity; however, from a physical point of view they are the simplest to realize in that many actual systems are governed by a natural energy balance between conduction and convection; hence no active control is necessary. In contrast, prescribed temperature boundary conditions, while mathematically simple, are actually rather difficult to realize in practice in that they are nearly always associated with surface heat flux. Therefore, for a transient problem, a constant temperature boundary condition necessitates a controlled, time-dependent surface heat flux to maintain the prescribed temperature. This is often difficult to achieve in practice. A prescribed temperature boundary condition is perhaps best realized when a physical phase change (e.g., evaporation/boiling) occurs on the surface. Such is the case of spray cooling with phase change heat transfer in which the surface will remain constant at the boiling point (i.e., saturation temperature) of the coolant fluid provided that sufficient coolant is applied to maintain a wetted surface and sufficient heat flux is present. Alternatively, the constant temperature boundary condition may be thought of as the limiting case of a convective boundary condition as $h \to \infty$, yielding $T_{\text{surface}} = T_\infty =$ constant. Boundary conditions of the second type may physically correspond to heaters (e.g., thin electric strip heaters) attached to the surface, which with low contact resistance and proper control can provide a prescribed heat flux condition.

In addition to the three linear boundary conditions discussed above, other boundary conditions are now considered here.

4. *Interface Boundary Conditions*. When two materials having different thermal conductivities k_1 and k_2 are in *imperfect thermal contact* and have a common boundary as illustrated in Figure 1-10, the temperature profile through the solids experiences a sudden drop across the interface between the two materials. The physical significance of this temperature drop is envisioned better if we consider an enlarged view of the interface as shown in this figure, and note that actual solid-to-solid contact takes place at a limited number of spots, and that the void between them is filled with air (or other interfacial fluid), which is the surrounding fluid. As the thermal conductivity of air is much smaller than that of many solids (e.g., metals), a steep temperature drop occurs across the gap.

To develop the boundary condition for such an interface, we write the energy balance as

$$\begin{pmatrix} \text{Heat conduction} \\ \text{in solid 1} \end{pmatrix} = \begin{pmatrix} \text{Heat transfer} \\ \text{across the gap} \end{pmatrix} = \begin{pmatrix} \text{Heat conduction} \\ \text{in solid 2} \end{pmatrix} \qquad (1\text{-}59\text{a})$$

Figure 1-10   Boundary conditions at interface of two contacting solid surfaces.

$$q_i'' = -k_1 \left.\frac{\partial T_1}{\partial x}\right|_i = h_c(T_1 - T_2)_i = -k_2 \left.\frac{\partial T_2}{\partial x}\right|_i \qquad (1\text{-}59b)$$

where subscript i denotes the interface, and h_c, in units W/(m$^2 \cdot$ K), is called the *contact conductance* for the interface. Equation (1-59b) provides two expressions for the boundary condition at the interface of two contacting solids, which together are generally called the *interface boundary conditions*. It is also common to consider the reciprocal of the contact conductance as the *thermal contact resistance*, R_c'', in units of (m$^2 \cdot$ K)/W.

For the special case of *perfect thermal contact* between the surfaces, we have $h_c \to \infty$, and equation (1-59b) is replaced with the following:

$$T_1\big|_i = T_2\big|_i \quad \text{at the surface interface} \qquad (1\text{-}60a)$$

$$-k_1 \left.\frac{\partial T_1}{\partial x}\right|_i = -k_2 \left.\frac{\partial T_2}{\partial x}\right|_i \quad \text{at the surface interface} \qquad (1\text{-}60b)$$

where equation (1-60a) is the continuity of temperature, and equation (1-60b) is the continuity of heat flux at the interface (i.e., conservation of energy).

Overall, the surface roughness, the interface contact pressure and temperature, thermal conductivities of the contacting solids, and the type of fluid in the gap are the principal factors that affect contact conductance. The experimentally determined values of contact conductance for typical materials in contact can be found in references 10–12.

To illustrate the effects of various parameters such as the surface roughness, the interface temperature, the interface pressure, and the type of material, we present in Table 1-3 the *interface thermal contact conductance h_c* for various material combinations. The results show that interface conductance increases with

TABLE 1-3   Interface Contact Conductance for Representative Solid–Solid Interfaces

Interface	Contact Pressure, atm	Interfacial Fluid	h_c, W/(m^2 · K)
Stainless steel to stainless steel	10	Air	9,000–11,500
[10] (0.76 μm roughness)	20	Air	10,000–12,000
Stainless steel to stainless steel	10	Air	2,800–4,000
[10] (2.5 μm roughness)	20	Air	3,100–4,200
Aluminum to aluminum [10]	10	Air	6,000–15,000
(3 μm roughness)	20	Air	10,500–28,000
Stainless steel to stainless steel	1	Vacuum	400–1,600
[13]	100	Vacuum	2,500–14,000
Copper to copper [13]	1	Vacuum	1,000–10,000
	100	Vacuum	20,000–100,000
Aluminum to aluminum [13]	1	Vacuum	2,000–6,600
	100	Vacuum	25,000–50,000
Aluminum to aluminum [13]	1	Air	3,600
(10 μm roughness)	1	Helium	10,000
	1	Silicone oil	19,000
Aluminum to aluminum [13, 14]	1	Dow Corning 340 grease	140,000
Stainless steel to stainless steel [13, 14]	35	Dow Corning 340 grease	250,000

increasing interface pressure, increasing interface temperature, and decreasing surface roughness. As might be expected, the interface conductance is higher with a softer material (e.g., aluminum) than with a harder material (e.g., stainless steel). The smoothness of the surface is another factor that affects contact conductance; a joint with a superior surface finish may exhibit lower contact conductance owing to waviness. The adverse effect of waviness can be overcome by introducing between the surfaces an interface shim from a soft material such as lead. Contact conductance also is reduced with a decrease in the ambient air pressure because the effective thermal conductance of the gas entrapped in the interface is lowered.

5. *Other Boundary Conditions and Relations*. Two additional boundary conditions are frequently used during the solution of the heat conduction equation. When symmetry is present in a given coordinate direction, it is often desirable to limit the domain to one-half of the problem and use the *line of symmetry* as an alternative boundary condition. Since the net heat flux is zero across a line of symmetry, the boundary condition becomes

$$\left. \frac{\partial T}{\partial n} \right|_{\text{boundary}} = 0 \quad \text{(symmetry condition)} \tag{1-61}$$

which acts like an adiabatic (i.e., perfectly insulated) boundary in keeping with thermodynamic equilibrium. Care should be taken, however, when imposing symmetry as a boundary condition. For example, the initial condition $T(\hat{r}, t = 0)$ or nonuniform internal energy generation $g(\hat{r})$ may break the symmetry, even if the outer boundary conditions appear symmetric.

A second condition, more of a pseudoboundary condition, to consider here concerns the necessity for *finite temperature* throughout the domain of the problem. With curvilinear coordinate systems, as will be seen in later chapters, the solution of the heat equation often contains functions that tend to infinity as their argument approaches zero. Because such behavior violates the condition of finite temperature, these functions are eliminated from the general solution if zero is within the spatial variable domain of the problem. Under this scenario, the equivalent boundary condition at the coordinate origin ($r = 0$) may be stated as

$$\lim_{r \to 0} T(r) \neq \pm\infty \quad \text{(finite temperature condition)} \tag{1-62}$$

which implies that a finite temperature limit exists at the origin.

Example 1-1   Problem Formulation for 1-D Cylinder

Consider a hollow cylinder (i.e., thick-walled pipe) subjected to convection boundary conditions at the inner $r = a$ and outer $r = b$ surfaces into ambient fluids at constant temperatures $T_{\infty 1}$ and $T_{\infty 2}$, with heat transfer coefficients $h_{\infty 1}$ and $h_{\infty 2}$, respectively, as illustrated in Fig. 1-11. Write the boundary conditions.
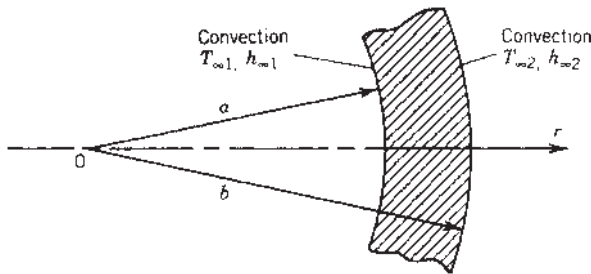

Figure 1-11   Boundary conditions for cylinder in Example 1-1.

The domain of the problem is $a \leq r \leq b$; hence boundary conditions are required at $r = a$ and $r = b$. The appropriate convection boundary condition is given by equation (1-57a), which is written here in the general cylindrical form

$$-k\frac{\partial T}{\partial r} = \pm h(T - T_{\infty}) \tag{1-63}$$

The positive conductive heat flux is always in the positive r direction per Fourier's law, while the outward-drawn surface normal at the boundary surfaces

$r = a$ and $r = b$ are in the negative r and positive r directions, respectively. Hence *positive* convection at the inner surface ($r = a$) is in the opposite direction of positive conduction, while positive conduction and convection are both in the same direction at the outer surface ($r = b$). Accordingly, we have to introduce a sign change to Newton's law for the inner surface, what we consider to be convection on the *back side* with respect to the coordinate direction. With these considerations in mind, the two boundary conditions (BC) become

BC1:
$$-k \left. \frac{\partial T}{\partial r} \right|_{r=a} = -h_{\infty 1} \left(T|_{r=a} - T_{\infty 1} \right) \tag{1-64a}$$

BC2:
$$-k \left. \frac{\partial T}{\partial r} \right|_{r=b} = h_{\infty 2} \left(T|_{r=b} - T_{\infty 2} \right) \tag{1-64b}$$

1-6 NONDIMENSIONAL ANALYSIS OF THE HEAT CONDUCTION EQUATION

In general, the solution and analysis of engineering problems benefit by first non-dimensionalizing the governing equations. This process often yields important nondimensional groups, such as the Fourier number and Biot number, and reduces the dependency of the solution from a potentially large number of dimensional parameters. We consider the general 1-D Cartesian coordinate system, which from equation (1-25) is written here as

$$\frac{\partial^2 T}{\partial x^2} + \frac{g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{1-65}$$

over the domain $0 \leq x \leq L$, with the following initial (IC) and boundary conditions:

IC:
$$T(x, t = 0) = T_0 \tag{1-66a}$$

BC1:
$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \tag{1-66b}$$

BC2:
$$-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h(T|_{x=L} - T_{\infty}) \tag{1-66c}$$

It is now possible to define the *nondimensional independent variables*, denoted with an asterisk, using the available dimensional parameters of the problem. The independent variables become

$$x^* = \frac{x}{L} \tag{1-67a}$$

$$t^* = \frac{\alpha t}{L^2} \tag{1-67b}$$

where the nondimensional time is known as the *Fourier number*. The most common approach to define the *nondimensional temperature* is to use the reference temperature (e.g., the fluid temperature) in combination with the temperature difference between the initial and fluid temperatures, namely,

$$T^* = \frac{T - T_\infty}{T_0 - T_\infty} \tag{1-68}$$

A formal change of variables is now done via the chain rule, namely,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial T^*} \frac{\partial T^*}{\partial x} = \left(T_0 - T_\infty\right) \frac{\partial T^*}{\partial x} \tag{1-69a}$$

and

$$\frac{\partial T^*}{\partial x} = \frac{\partial T^*}{\partial x^*} \frac{\partial x^*}{\partial x} = \frac{\partial T^*}{\partial x^*} \frac{1}{L} \tag{1-69b}$$

Combining equations (1-69a) and (1-69b) yields

$$\frac{\partial T}{\partial x} = \frac{T_0 - T_\infty}{L} \frac{\partial T^*}{\partial x^*} \tag{1-70}$$

Further differentiating equation (1-70) yields

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{T_0 - T_\infty}{L} \frac{\partial T^*}{\partial x^*}\right) = \frac{T_0 - T_\infty}{L} \frac{\partial^2 T^*}{\partial x^{*2}} \frac{\partial x^*}{\partial x} = \frac{T_0 - T_\infty}{L^2} \frac{\partial^2 T^*}{\partial x^{*2}} \tag{1-71}$$

It is seen that the necessary dimension of the second derivative on the left-hand side of equation (1-71), namely K/m^2, is now supplied by the scaling factor of the right-hand side, since the right-hand side second derivative is dimensionless. In a similar manner,

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial T^*} \frac{\partial T^*}{\partial t} = T_0 - T_\infty \frac{\partial T^*}{\partial t} \tag{1-72a}$$

$$\frac{\partial T^*}{\partial t} = \frac{\partial T^*}{\partial t^*} \frac{\partial t^*}{\partial t} = \frac{\partial T^*}{\partial t^*} \frac{\alpha}{L^2} \tag{1-72b}$$

which together yield

$$\frac{\partial T}{\partial t} = \frac{\alpha \left(T_0 - T_\infty\right)}{L^2} \frac{\partial T^*}{\partial t^*} \tag{1-73}$$

Inserting equations (1-71) and (1-73) into the heat equation yields

$$\boxed{\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{g L^2}{k \left(T_0 - T_\infty\right)} = \frac{\partial T^*}{\partial t^*}} \tag{1-74}$$

All terms in equation (1-74) are now without dimension; hence it represents the *nondimensional form of the* 1-*D heat equation*. In a similar manner, the initial and boundary conditions are readily transformed:

IC: $$T^*(x^*, t^* = 0) = \frac{T_0 - T_\infty}{T_0 - T_\infty} = 1 \qquad (1\text{-}75a)$$

BC1: $$\left.\frac{\partial T^*}{\partial x^*}\right|_{x^*=0} = 0 \qquad (1\text{-}75b)$$

BC2: $$-\left.\frac{\partial T^*}{\partial x^*}\right|_{x^*=1} = \text{Bi } T^*\big|_{x^*=1} \qquad (1\text{-}75c)$$

where $\text{Bi} = hL/k$ is defined as the nondimensional *Biot number*, named after the physicist and mathematician Jean-Baptiste Biot. The Biot number is an important heat transfer parameter relating the conduction of heat within a solid to the convection of heat across the boundary, and is discussed in greater detail in Section 1-8. Examination of equations (1-74) and (1-75) reveals that the nondimensional temperature T^* depends on only two nondimensional parameters. If the internal energy generation is set to zero ($g = 0$), then

$$T^*(x^*, t^*) = f(\text{Bi}) \qquad (1\text{-}76)$$

such that the nondimensional temperature profile depends only on the Biot number. Clearly nondimensionalization is a powerful tool for engineering analysis.

1-7 HEAT CONDUCTION EQUATION FOR ANISOTROPIC MEDIUM

So far we considered the heat flux law for isotropic media, namely, the thermal conductivity k is independent of direction, and developed the heat conduction equation accordingly. However, there are natural as well as synthetic materials in which thermal conductivity varies with direction. For example, in a tree trunk the thermal conductivity may vary with direction; specifically, the thermal conductivities along the grain and across the grain are different. In lamellar materials, such as graphite and molybdenum disulfide, the thermal conductivity along and across the laminations may differ significantly. For example, in graphite, the thermal conductivity varies by about two orders of magnitude between the two principal orientations. Other examples include sedimentary rocks, fibrous reinforced structures, cables, heat shielding for space vehicles, and many others.

Orthotropic Medium

First we consider a situation in the rectangular coordinates in which the thermal conductivities k_x, k_y, and k_z in the *x, y*, and *z* dimensions, respectively,

are different. Then the heat flux vector $q''(\hat{r}, t)$ given by equation (1-2) is modified as

$$q''(x, y, z, t) = -\left(\hat{i} k_x \frac{\partial T}{\partial x} + \hat{j} k_y \frac{\partial T}{\partial y} + \hat{k} k_z \frac{\partial T}{\partial z} \right) \tag{1-77}$$

and the three components of the heat flux vector in the x, y, and z directions, respectively, become

$$q_x'' = -k_x \frac{\partial T}{\partial x} \qquad q_y'' = -k_y \frac{\partial T}{\partial y} \qquad \text{and} \qquad q_z'' = -k_z \frac{\partial T}{\partial z} \tag{1-78a,b,c}$$

Similar relations can be written for the heat flux components in the cylindrical and spherical coordinates. The materials in which thermal conductivity varies in the (x, y, z), (r, θ, z), or (r, θ, ϕ) directions are called *orthotropic materials*. The heat conduction equation for an orthotropic medium in the rectangular coordinate system is obtained by introducing the heat flux vector given by equation (1-77) into equation (1-23), which for a quiescent medium yields

$$\frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right) + g = \rho c \frac{\partial T}{\partial t} \tag{1-79}$$

Thus the thermal conductivity has three distinct components.

Anisotropic Medium

In a more general situation encountered in heat flow through *crystals*, at any point in the medium, each component q_x'', q_y'', and q_z'' of the heat flux vector is considered a linear combination of the temperature gradients $\partial T/dx$, $\partial T/dy$, and $\partial T/dz$, that is,

$$q_x'' = -\left(k_{11} \frac{\partial T}{\partial x} + k_{12} \frac{\partial T}{\partial y} + k_{13} \frac{\partial T}{\partial z} \right) \tag{1-80a}$$

$$q_y'' = -\left(k_{21} \frac{\partial T}{\partial x} + k_{22} \frac{\partial T}{\partial y} + k_{23} \frac{\partial T}{\partial z} \right) \tag{1-80b}$$

$$q_z'' = -\left(k_{31} \frac{\partial T}{\partial x} + k_{32} \frac{\partial T}{\partial y} + k_{33} \frac{\partial T}{\partial z} \right) \tag{1-80c}$$

Such a medium is called an *anisotropic medium*, and the thermal conductivity for such a medium has nine components, k_{ij}, called the *conductivity coefficients*, that are considered to be the components of a second-order tensor $\overline{\overline{k}}$:

$$\overline{\overline{k}} \equiv \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} \tag{1-81}$$

Crystals are typical examples of anisotropic materials involving nine conductivity coefficients [15]. The heat conduction equation for anisotropic solids in the rectangular coordinate system is obtained by introducing the expressions for the three components of heat flux given by equations (1-80) into the energy equation (1-23). Again for a quiescent medium, we find

$$k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + \left(k_{12} + k_{21}\right)\frac{\partial^2 T}{\partial x \, \partial y} + \left(k_{13} + k_{31}\right)\frac{\partial^2 T}{\partial x \, \partial z}$$

$$+ \left(k_{23} + k_{32}\right)\frac{\partial^2 T}{\partial y \, \partial z} + g\left(x, y, z, t\right) = \rho c \frac{\partial T\left(x, y, z, t\right)}{\partial t} \qquad (1\text{-}82)$$

where $k_{12} = k_{21}$, $k_{13} = k_{31}$, and $k_{23} = k_{32}$ by the reciprocity relation. This matter will be discussed further in Chapter 15.

1-8   LUMPED AND PARTIALLY LUMPED FORMULATION

The transient heat conduction formulations considered up to this point assume a general temperature distribution varying both with time and position. There are many engineering applications in which the spatial variation of temperature within the medium can be neglected, and temperature is considered to be a function of time only. Such formulations, called *lumped system formulation* or *lumped capacitance method*, provide a great simplification in the analysis of transient heat conduction; but their range of applicability is very restricted. Here we illustrate the concept of the lumped formulation approach and examine its range of validity in terms of the dimensionless Biot number.

Consider a small, high-conductivity material, such as a metal, initially at a uniform temperature T_0, and then suddenly immersed into a well-stirred hot bath maintained at a uniform temperature T_∞. Let V be the volume, A the surface area, ρ the density, c the specific heat of the solid, and h the convection heat transfer coefficient between the solid's surface and the fluid. We assume that the temperature distribution within the solid remains sufficiently uniform for all times due to its small size and high thermal conductivity. Under such an assumption, the uniform temperature $T(t)$ of the solid can be considered to be a function of time only. The energy balance equation, taking the entire solid as the control volume, is stated as

$$\begin{pmatrix} \text{Rate of heat flow from the} \\ \text{solid through its boundaries} \end{pmatrix} = \begin{pmatrix} \text{Rate of change of the} \\ \text{internal energy of the solid} \end{pmatrix} \qquad (1\text{-}83)$$

Considering convection as the only means for heat to enter or leave the control volume, the energy equation (1-83) takes the form

$$-hA[T(t) - T_\infty] = \rho Vc\frac{dT(t)}{dt} \qquad (1\text{-}84)$$

which is rearranged to yield

$$\frac{dT(t)}{dt} + \frac{hA}{\rho Vc}[T(t) - T_\infty] = 0 \qquad \text{for} \qquad t > 0 \tag{1-85a}$$

IC: $\qquad\qquad\qquad T(t = 0) = T_0 \tag{1-85b}$

Equation (1-85a) is a nonhomogeneous ordinary differential equation, which is readily solved using the sum of the homogeneous and particular solutions. However, it is useful to remove the nonhomogeneity by defining the excess temperature $\theta(t)$ as

$$\theta(t) = T(t) - T_\infty \tag{1-86}$$

With this substitution, the lumped formulation becomes

$$\frac{d\theta(t)}{dt} + m\theta(t) = 0 \qquad \text{for} \qquad t > 0 \tag{1-87a}$$

IC: $\qquad\qquad\qquad \theta(t = 0) = T_0 - T_\infty = \theta_0 \tag{1-87b}$

where

$$m = \frac{hA}{\rho Vc} \tag{1-87c}$$

The solution of equations (1-87) becomes

$$\theta(t) = \theta_0 e^{-mt} \tag{1-88}$$

This is a very simple expression for the temperature of the solid as a function of time, noting that the parameter m has the unit of s^{-1} and may be thought of as the inverse of the *thermal time constant*. The physical significance of the parameter m is better envisioned if its definition is rearranged in the form

$$\frac{1}{m} = (\rho cV)\left(\frac{1}{hA}\right) \tag{1-89}$$

which is the product of the *thermal heat capacitance* and the resistance to convection heat transfer. It follows that the smaller the thermal capacitance and/or the convective resistance, the larger is the value of m, and hence the faster is the rate of change of temperature $\theta(t)$ of the solid according to equation (1-88).

In order to establish some criterion for the range of validity of such a straightforward method for the analysis of transient heat conduction, we consider the definition of the Biot number, and rearrange it in the form

$$\text{Bi} \equiv \frac{hL_c}{k_s} = \frac{L_c/k_s A}{1/hA} = \frac{\text{internal conductive resistance}}{\text{external convective resistance}} \tag{1-90}$$

where k_s is the thermal conductivity of the solid. L_c is the *characteristic length* of the solid and is generally defined as $L_c = V/A$.

We recall that the lumped system analysis is applicable if the temperature distribution within the solid remains sufficiently uniform during the transients.

This may be interpreted as the condition in which the internal resistance to conduction within the solid is negligible as compared to the external resistance to convection heat transfer at the solid–fluid boundary. Now we refer to the above definition of the Biot number and note that the internal conductive resistance of the solid is very small in comparison to the external convective resistance if the Biot number is *much less than unity*, say one order of magnitude smaller. We therefore conclude that the lumped system analysis is valid only for small values of the Biot number, namely

$$\text{Bi} = \frac{hL_c}{k_s} < 0.1 \quad \text{(lumped analysis criterion)} \tag{1-91}$$

We note that equation (1-91) is a *guideline* for validity of lumped analysis, but must emphasize that spatial gradients *gradually diminish* with decreasing Bi number for transient problems. Hence a Bi = 0.1 does not represent an abrupt transition from the presence of spatial gradients to the absence of spatial gradients, but rather is considered a reasonable interpretation of Bi ≪ 1. For example, exact analytic solutions of transient heat conduction for solids in the form of a slab, cylinder or sphere, subjected to convective cooling show that for a Bi < 0.1, the variation of temperature within the solid during transients is less than about 5%. Hence it may be concluded that the lumped system analysis may be applicable for most engineering applications if the Biot number is less than about 0.1.

It is useful here to examine graphically the behavior of the transient temperature profile for three different values of the Biot number, namely Bi ≪ 1, Bi ≈ 1, and Bi ≫ 1. This is done in Figure 1-12 for a symmetric plane wall with a
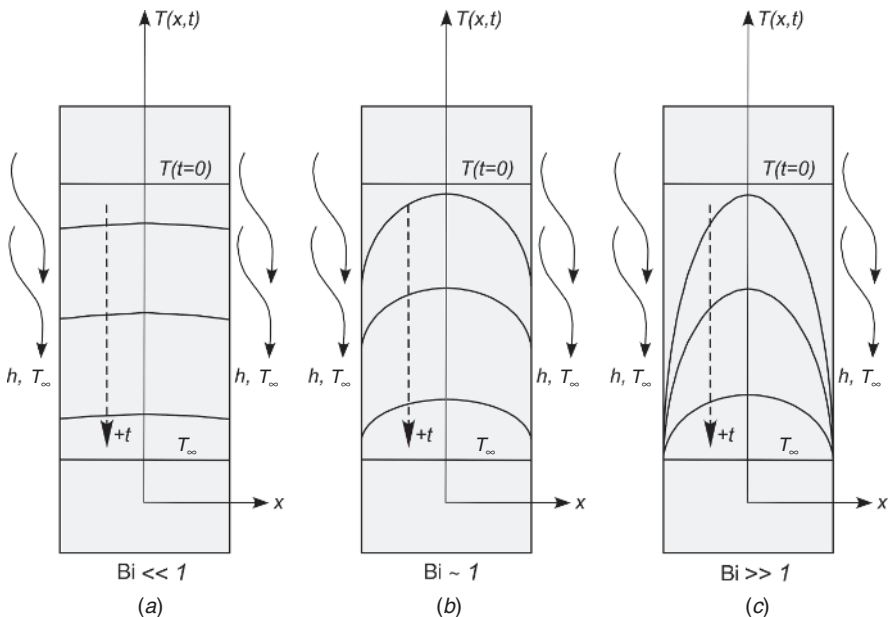
Figure 1-12 Temperature distribution $T(x,t)$ for a symmetric plane wall cooled by convection heat transfer for various Biot numbers.

uniform initial temperature of T_0, which is then subjected to cooling via convection heat transfer at both surfaces from a fluid at T_∞.

For the case of Bi $\ll$ 1, Figure 1-12(a), the wall is observed to cool *uniformly* from the initial temperature T_0 to the steady-state temperature, which would be T_∞. This is consistent with spatial gradients being dominated by convective resistance, hence $T \simeq T(t)$. Figure 1-12(b) shows the case of Bi $\approx$ 1; hence comparable conductive and convective resistances lead to temperature gradients within the solid and temperature differences between the solid and fluid. Finally, Figure 1-12(c) depicts the Bi $\gg$ 1, which in the limiting case corresponds to $h \rightarrow \infty$, or essentially the case of prescribed surface temperature. Under this condition, the spatial temperature gradients dominate the problem. In particular, the gradient near the surfaces at $t \simeq 0$ is very steep.

Example 1-2   Lumped Analysis for a Solid Sphere

The temperature of a gas stream is to be measured with a thermocouple. The junction may be approximated as a sphere of diameter and properties: $D = \frac{3}{4}$ mm, $k = 30$ W/(m $\cdot$ K), $\rho = 8400$ kg/m^3, and $c = 400$ J/(kg $\cdot$ K). If the heat transfer coefficient between the junction and the gas stream is $h = 600$ W/(m^2 $\cdot$ K), how long does it take for the thermocouple to record 99% of the temperature difference between the gas temperature and the initial temperature of the thermocouple? Here we neglect any radiation losses.

The characteristic length L_c is

$$L_c = \frac{V}{A} = \frac{(\frac{4}{3})\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{D}{6} = \frac{\frac{3}{4}}{6} = \frac{1}{8} \text{ mm} = 1.25 \times 10^{-4} \text{ m}$$

The Biot number becomes

$$\text{Bi} = \frac{hL_c}{k} = \frac{600}{30}1.25 \times 10^{-4} = 2.5 \times 10^{-3}$$

The lumped system analysis is applicable since Bi $<$ 0.1. From equation (1-88) we have

$$\frac{T(t) - T_\infty}{T_0 - T_\infty} = \frac{1}{100} = e^{-mt}$$

or

$$e^{mt} = 100 \quad \rightarrow \quad mt = 4.605$$

The value of m is determined from its definition:

$$m = \frac{hA}{\rho Vc} = \frac{h}{\rho cL_c} = \frac{600}{8400 \times 400 \times 0.000125} = 1.429 \text{ s}^{-1}$$

Then

$$t = \frac{4.605}{m} = \frac{4.605}{1.429} \cong 3.2 \text{ s}$$

That is, about 3.2 s is needed for the thermocouple to record 99% of the applied temperature difference.

Partial Lumping

In the lumped system analysis described above, we considered a total lumping in all the space variables; as a result, the temperature for the fully lumped system became a function of the time variable only.

It is also possible to perform a *partial lumped analysis*, such that the temperature variation is retained in one of the space variables but lumped in the others. For example, if the temperature gradient in a solid is steep in the x direction and very small in the y and z directions, then it is possible to lump the system with regard to the y and z variables and let $T = T(x)$. To illustrate this matter we consider a solid with thermal conductivity k as shown in Figure 1-13 in which temperature gradients are assumed to be large along the x direction, but small over the y–z plane perpendicular to the x axis. This would be valid by considering the length scale δ as the width of the solid in the y and z directions, such that $\text{Bi}_y = \text{Bi}_z = {}^{h\delta}/k < 0.1$. Let the solid dissipate heat by convection from its lateral surfaces into an ambient fluid at a constant temperature T_∞ with a uniform heat transfer coefficient h over the entire exposed surface.

To develop the steady-state heat conduction equation with lumping over the plane perpendicular to the x axis, we consider an energy balance for a differential disk of thickness dx located about the axial location x given by

$$\begin{pmatrix} \text{Net rate of heat} \\ \text{gain by conduction} \\ \text{in the } x \text{ direction} \end{pmatrix} - \begin{pmatrix} \text{Rate of heat loss} \\ \text{by convection from} \\ \text{the lateral surfaces} \end{pmatrix} = 0 \qquad (1\text{-}92)$$
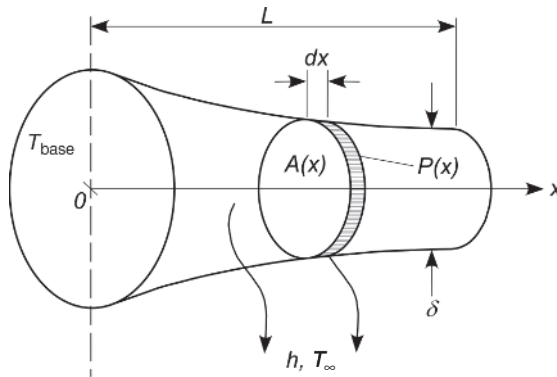
Figure 1-13 Nomenclature for derivation of partially lumped heat conduction equation.

When the appropriate rate expressions are introduced for each of these terms, we obtain

$$\left(q_x - q_{x+dx}\right) - hP(x)dx\left(T(x) - T_\infty\right) = 0 \tag{1-93}$$

where the heat rates q_x and q_{x+dx} are given by

$$q_x = -kA(x)\frac{\partial T}{\partial x} \tag{1-94a}$$

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x}dx = -kA(x)\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left[-kA(x)\frac{\partial T}{\partial x}\right]dx \tag{1-94b}$$

where we have again used a Taylor series expansion for q_{x+dx}, and where the other quantities are defined as

$$A(x) = \text{cross-sectional area of the disk}$$

$$P(x) = \text{perimeter of the disk}$$

$$h = \text{convection heat transfer coefficient}$$

$$k = \text{thermal conductivity of the solid}$$

$$T_\infty = \text{ambient fluid temperature}$$

We now introduce equations (1-94) into (1-93) to yield

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}\left(T - T_\infty\right) = 0 \tag{1-95}$$

We have left equation (1-95) to reflect a variable cross-sectional area; hence $A(x)$ and $P(x)$ both are prescribed functions of x. If we further assume that the cross-sectional area is constant, namely, $A(x) = A_0 = \text{constant}$ and $P(x) = P_0 = \text{constant}$, then equation (1-95) reduces to

$$\frac{d^2T}{dx^2} - \frac{hP_0}{kA_0}\left(T - T_\infty\right) = 0 \tag{1-96}$$

which is the *fin equation* for fins of uniform cross section. Rather than solving this second-order ordinary differential equation (ODE), it is useful to once again define the excess temperature $\theta(x)$:

$$\theta(x) = T(x) - T_\infty \tag{1-97}$$

which yields the final form of the 1-D constant-area fin equation, namely

$$\frac{d^2\theta}{dx^2} - \frac{hP_0}{kA_0}\theta = 0 \tag{1-98a}$$

or

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (1\text{-}99)$$

where the parameter m is defined as

$$m^2 = \frac{hP_0}{kA_0} \qquad (1\text{-}99)$$

The solution to the fin equation (1-98b) can be expressed in the form

$$\theta(x) = C_1 e^{-mx} + C_2 e^{mx} \qquad (1\text{-}100a)$$

or

$$\theta(x) = C_1 \cosh(mx) + C_2 \sinh(mx) \qquad (1\text{-}100b)$$

We note here that the general solution form of the second-order ode in equation (1-98) is determined by the roots of the *auxiliary or characteristic equation*

$$\lambda^2 - m^2 = 0 \qquad (1\text{-}101)$$

The present case of two real roots (i.e., $\lambda_{1,2} = \pm m$) yields the exponential solution given by equation (1-100a). However, for the case of a *conjugate pair* of real roots, the solution may also be formed using the *hyperbolic functions* as given by equation (1-100b). For this case, it is readily seen that a linear combination of the two exponential solutions yields the hyperbolics. The two unknown coefficients C_1 and C_2 are determined by the application of boundary conditions at $x = 0$ and $x = L$, which correspond to the base and tip of the fin, respectively. Typically, the temperature at the base of the fin is prescribed

$$\theta(x = 0) = T_b - T_\infty = \theta_b \qquad (1\text{-}102)$$

However, there are several choices of boundary conditions at the tip of the fin, including the following four cases:

1. $-k \left.\dfrac{\partial T}{\partial x}\right|_{x=L} = h(T|_{x=L} - T_\infty)$ (convective tip) $\qquad (1\text{-}103a)$

 which yields $\qquad -k \left.\dfrac{\partial \theta}{\partial x}\right|_{x=L} = h\,\theta|_{x=L} \qquad (1\text{-}103b)$

2. $\left.\dfrac{\partial T}{\partial x}\right|_{x=L} = 0$ (insulated/symmetric tip) $\qquad (1\text{-}104a)$

 which yields $\qquad \left.\dfrac{\partial \theta}{\partial x}\right|_{x=L} = 0 \qquad (1\text{-}104b)$

3. $T(x = L) = T_{\text{tip}}$ (prescribed tip)   (1-105a)

which yields $\theta(x = L) = T_{\text{tip}} - T_\infty = \theta_{\text{tip}}$ (1-105b)

4. $T(x \to \infty) = T_\infty$ (long/infinite fin)   (1-106a)

which yields $\theta(x \to \infty) = 0$ (1-106b)

Generally, it is preferable to use the hyperbolic solution, equation (1-100b), for the finite domain problems, that is, boundary equations (1-103) – (1-105), and to use the exponential solution, equation (1-100a), for the infinite domain problem, namely boundary equation (1-106).

The solution of equation (1-95) for fins of a variable cross section is more involved, as the resulting ode has nonconstant coefficients. Analytic solutions of fins of various cross sections can be found in references 16 and 17.

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PROBLEMS

1-1 Derive the heat conduction equation (1-43) in cylindrical coordinates using the differential control approach beginning with the general statement of conservation of energy. Show all steps and list all assumptions. Consider Figure 1-7.

1-2 Derive the heat conduction equation (1-46) in spherical coordinates using the differential control approach beginning with the general statement of conservation of energy. Show all steps and list all assumptions. Consider Figure 1-8.

1-3 Show that the following two forms of the differential operator in the cylindrical coordinate system are equivalent:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}$$

1-4 Show that the following three different forms of the differential operator in the spherical coordinate system are equivalent:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = \frac{1}{r}\frac{d^2}{dr^2}(rT) = \frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr}$$

1-5 Set up the mathematical formulation of the following heat conduction problems. Formulation includes the simplified differential heat equation along with boundary and initial conditions. Do not solve the problems.

   a. A slab in $0 \leq x \leq L$ is initially at a temperature $F(x)$. For times $t > 0$, the boundary at $x = 0$ is kept insulated, and the boundary at $x = L$ dissipates heat by convection into a medium at zero temperature.

   b. A semi-infinite region $0 \leq x < \infty$ is initially at a temperature $F(x)$. For times $t > 0$, heat is generated in the medium at a constant, uniform rate of g_0 (W/m^3), while the boundary at $x = 0$ is kept at zero temperature.

c. A hollow cylinder $a \leq r \leq b$ is initially at a temperature $F(r)$. For times $t > 0$, heat is generated within the medium at a rate of $g(r)$, (W/m^3), while both the inner boundary at $r = a$ and outer boundary $r = b$ dissipate heat by convection into mediums at fluid temperature T_∞.

d. A solid sphere $0 \leq r \leq b$ is initially at temperature $F(r)$. For times $t > 0$, heat is generated in the medium at a rate of $g(r)$ (W/m^3), while the boundary at $r = b$ is kept at a uniform temperature T_0.

1-6 A solid cube of dimension L is originally at a uniform temperature T_0. The cube is then dropped into a large bath where the cube rapidly settles flat on the bottom. The fluid in the bath provides convection heat transfer with coefficient h (W/m^2 K) from the fluid at constant temperature T_∞. Formulate the heat conduction problem. Formulation includes the simplified differential heat equation along with appropriate boundary and initial conditions. Include a sketch with your coordinate axis position. Do not solve the problem.

1-7 For an anisotropic solid, the three components of the heat conduction vector q_x, q_y, and q_z are given by equations (1-80). Write the similar expressions in the cylindrical coordinates for q_r, q_ϕ, q_z and in the spherical coordinates for q_r, q_ϕ, q_θ.

1-8 An infinitely long, solid cylinder (D = diameter) has the ability for uniform internal energy generation given by the rate g_0 (W/m^3) by passing a current through the cylinder. Initially ($t = 0$), the cylinder is at a uniform temperature T_0. The internal energy generation is then turned on (i.e., current passed) and maintained at a constant rate g_0, and at the same moment the cylinder is exposed to convection heat transfer with coefficient h (W/m^2 K) from a fluid at constant temperature T_∞, noting that $T_\infty > T_0$. The cylinder has uniform and constant thermal conductivity k (W/m K). The Biot number $hD/k \ll 1$. Solve for time t at which point the surface heat flux is exactly zero. Present your answer in variable form.

1-9 A long cylindrical iron bar of diameter $D = 5$ cm, initially at temperature $T_0 = 650°$C, is exposed to an air stream at $T_\infty = 50°$C. The heat transfer coefficient between the air stream and the surface of the bar is $h = 80$ W/(m$^2 \cdot$ K). Thermophysical properties are constant: $\rho = 7800$ kg/m^3, $c = 460$ J/(kg $\cdot$ K), and $k = 60$ W/(m $\cdot$ K). Determine the time required for the temperature of the bar to reach 250°C by using the lumped system analysis.

1-10 A thermocouple is to be used to measure the temperature in a gas stream. The junction may be approximated as a sphere having thermal conductivity $k = 25$ W/(m $\cdot$ K), $\rho = 8400$ kg/m^3, and $c = 0.4$ kJ/(kg $\cdot$ K). The heat transfer coefficient between the junction and the gas stream is

$h = 560$ W/(m$^2 \cdot$ K). Calculate the diameter of the junction if the thermocouple should measure 95% of the applied temperature difference in 3 s.

1-11   Determine the constants C_1 and C_2 for the constant area fin solution of equations (1-100) for the case of the prescribed base temperature of equation (1-102), and the following tip conditions:

a. Convective tip per equation (1-103)

b. Insulated or symmetric tip per equation (1-104)

c. Prescribed temperature tip per equation (1-105)

d. Infinitely long fin per equation (1-106)