# 1

# Forces and Moments

## 1.1   Introduction

Mechanics of solids is concerned with the analysis and design of solid bodies under the action of applied forces in order to ensure "acceptable" behavior. These solid bodies are the components and the assemblies of components that make up the structures of aircraft, automobiles, washing machines, golf clubs, roller blades, buildings, bridges, and so on, that is, of many manufactured and constructed products. If the solid body is suitably restrained to exclude "rigid body" motion it will deform when acted upon by applied forces, or loads, and internal forces will be generated in the body. For "acceptable" behavior:

1. Internal forces must not exceed values that the materials can withstand.
2. Deformations must not exceed certain limits.

   In later chapters of this text we shall identify, define, and examine the various quantities, such as internal forces, stresses, deformations, and material stress-strain relations, which determine acceptable behavior. We shall study methods for analyzing solid bodies and structures when loaded and briefly study ways to design solid bodies to achieve a desired behavior.

   All solid bodies are three dimensional objects and there is a general theory of mechanics of solids in three dimensions. Because understanding the behavior of three dimensional objects can be difficult and sometimes confusing we shall work primarily with objects that have simplified geometry, simplified applied forces, and simplified restraints. This enables us to concentrate on the process instead of the details. After we have a clear understanding of the process we shall consider ever increasing complexity in geometry, loading, and restraint.

   In this introductory chapter we examine three categories of force. First are *applied forces* which act on the surface or the mass of the body. Next are *restraint forces*, that is, forces on the surfaces where displacement is constricted (or restrained). Thirdly, *internal forces* generated by the resistance of the material to deformation as a result of applied and restraint forces.

   Forces can generate moments acting about some point. For the most part we carefully distinguish between forces and moments; however, it is common practice to include both forces and moments when referring in general terms to the *forces acting on the body* or the *forces at the restraints*.

## 1.2   Units

The basic quantities in the study of solid mechanics are length ($L$), mass ($M$), force ($F$), and time ($t$). To these we must assign appropriate units. Because of their prominent use in every day life in the

United States, the so-called English system of units is still the most familiar to many of us. Some engineering is still done in English units; however, global markets insist upon a world standard and so a version of the International Standard or SI system (from the French Système International d'Unités) prevails. The standard in SI is the meter, *m*, for length, the Newton, *N*, for force, the kilogram, *kg*, for mass, and the second, *s*, for time. The Newton is defined in terms of mass and acceleration as

$$1N = 1\ kg \cdot 1\frac{m}{s^2} \tag{1.2.1}$$

For future reference the acceleration due to gravity on the earth's surface, *g*, in metric units is

$$g = 9.81\frac{m}{s^2} \tag{1.2.2}$$

The standard English units are the foot, *ft*, for length, the pound, *lb*, for force, the slug, *slug*, for mass, and the second, *s*, for time. The pound is defined in terms of mass and acceleration as

$$1\ lb = 1\ slug \cdot 1\frac{ft}{s^2} \tag{1.2.3}$$

The acceleration due to gravity on the earth's surface, *g*, in English units is

$$g = 32.2\frac{ft}{s^2} \tag{1.2.4}$$

We shall use SI units as much as possible.

Most of you are still thinking in English units and so for quick estimates you can note that a meter is approximately 39.37 inches; there are approximately 4.45 Newtons in a pound; and there are approximately 14.59 kilograms in a slug. But since you are not used to thinking in slugs it may help to note that a kilogram of mass weighs about 2.2 pounds on the earth's surface. For those who must convert between units there are precise tables for conversion. In time you will begin to think in SI units.

Often we obtain quantities that are either very large or very small and so units such as millimeter are defined. One millimeter is one thousandth of a meter, or 1 *mm* = 0.001 *m,* and, of course, one kilogram is one thousand grams, or 1 *kg* = 1000 *g*. The following table lists the prefixes for different multiples:

| Multiple | Prefix | Symbol |
|----------|--------|--------|
| $10^9$ | giga | *G* |
| $10^6$ | mega | *M* |
| $10^3$ | kilo | *k* |
| $10^{-3}$ | milli | *m* |
| $10^{-6}$ | micro | $\mu$ |
| $10^{-9}$ | nano | *n* |

One modification of SI is that it is common practice in much of engineering to use the millimeter, *mm*, as the unit of length. Thus force per unit length is often, perhaps usually, given as Newtons per millimeter or *N/mm*. Force per unit area is given as Newtons per millimeter squared or *N/mm²*. One *N/m²* is called a Pascal or *Pa*, so the unit of 1 *N/mm²* is called 1 mega Pascal or 1 *MPa*. Mass density has the units of kilograms per cubic millimeter or *kg/mm³*. Throughout we shall use millimeter, Newton, and kilogram in all examples, discussions, and problems.

As noted in the above table: Only multiples of powers of three are normally used; thus, we do not use, for example, centimeters, decimeters, or other multiples that are the power of one or two. These are conventions, of course, so in the workplace you will find a variety of practices.

## 1.3   Forces in Mechanics of Materials

There are several types of forces that act on solid bodies. These consist of forces applied to the mass of the body and to the surface of the body, forces at restraints, and internal forces.

In Figure 1.3.1 we show a general three dimensional body with forces depicted acting on its surface and on its mass.
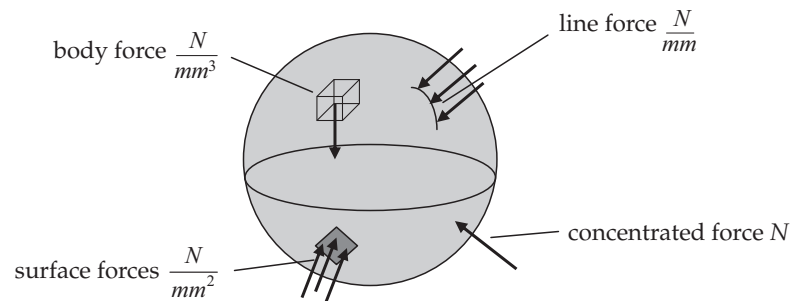
body force $\dfrac{N}{mm^3}$     line force $\dfrac{N}{mm}$

concentrated force $N$

surface forces $\dfrac{N}{mm^2}$

**Figure 1.3.1**

Forces that are volume or mass related are called *body forces*. In the system of units we are using they have the units of Newtons per cubic millimeter ($N/mm^3$). Gravity forces are a good example. Inertia forces generated by accelerations are another.

S*urface forces* can be specified in terms of force per unit area distributed over a surface and have the units of Newtons per square millimeter ($N/mm^2$). As noted one Newton per square millimeter is also called one mega Pascal (*MPa*).

If a force is distributed along a narrow band it is specified as a *line force*, that is, a force per unit length or Newtons per millimeter ($N/mm$).

If the force acts at a point it is a *concentrated force* and has the units of Newtons ($N$). Concentrated forces and line forces are usually idealizations or resultants of distributed surface forces. We can imagine an ice pick pushing on a surface creating a concentrated force. More likely the actual force acts on a small surface area where small means the size of the area is very small compared to other characteristic dimensions of the surface. Likewise a line force may be the resultant of a narrow band of surface forces.

When a concentrated, line, surface, or body force acts on the solid body or is applied to the body by means of an external agent it is called an *applied force*. When the concentrated, line, or surface force is generated at a point or region where an external displacement is imposed it is called a *restraint force*. In addition, for any body that is loaded and restrained, a force per unit area can be found on any internal surface. This particular distributed force is referred to as *internal* or simply as *stress*.

Generally, in the initial formulation of a problem for analysis, the geometry, applied forces, and physical restraints (displacements on specified surfaces) are known while the restraint forces and internal stresses are unknown. When the problem is formulated for design, the acceptable stress limits may be specified in advance and the final geometry, applied forces, and restraints may initially be unknown. For the most part the problems will be formulated for analysis but the subject of design will be introduced from time to time.

The analysis of the interaction of these various forces is a major part of the following chapters. For the most part we shall use rectangular Cartesian coordinates and resolve forces into components with respect to these axes. An exception is made for the study of torsion in Chapter 6. There we use cylindrical coordinates.

In the sign convention adopted here, applied force components and restraint force components are positive if acting in the positive direction of the coordinate axes. Positive stresses and internal forces will be defined in different ways as needed.

We start first with a discussion of concentrated forces.

## 1.4   Concentrated Forces

As noted, concentrated forces are usually idealizations of distributed forces. Because of the wide utility of this idealization we shall first examine the behavior of concentrated forces. In all examples we shall use the Newton ($N$) as our unit of force.

Force is a vector quantity, that is, it has both magnitude and direction. There are several ways of representing a concentrated force in text and in equations; however, the pervasive use of the digital computer in solving problems has standardized how forces are usually represented in formulating and solving problems in the behavior of solid bodies under load.

First, we shall consider a force that can be oriented in a two dimensional right handed rectangular Cartesian coordinate system and we shall define positive unit vectors **i** and **j** in the $x$, and $y$ directions, respectively, as shown in Figure 1.4.1. Using boldface has been a common practice in representing vectors in publications.
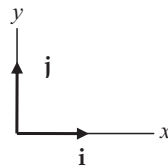


**Figure 1.4.1**

A force is often shown in diagrams as a line that starts at the point of application and has an arrowhead to show its direction as shown in Figure 1.4.2.
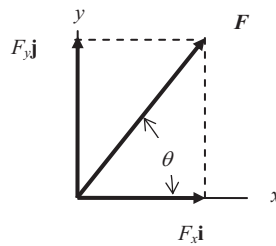


**Figure 1.4.2**

The concentrated force, **F,** can be represented by its components in the $x$ and $y$ directions.

$$\boldsymbol{F} = F_x\mathbf{i} + F_y\mathbf{j} \tag{1.4.1}$$

In keeping with the notation most commonly used for later computation we represent this force vector by a column matrix $\{F\}$ as shown in Equation 1.4.2.

$$\{F\} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{1.4.2}$$

In matrix notation the unit vector directions are implied by the component subscripts. From the properties of a right triangle the magnitude of the vector is given by

$$F = \sqrt{F_x^2 + F_y^2} \qquad (1.4.3)$$

The orientation of the force can be represented by the angle between the force and either axis. For example, with respect to the $x$ axis

$$\tan \theta = \frac{F_y}{F_x} \quad \rightarrow \quad \theta = \tan^{-1} \frac{F_y}{F_x} \qquad (1.4.4)$$

Quite often we must sum two or more forces such as those shown in Figure 1.4.3 as solid lines.
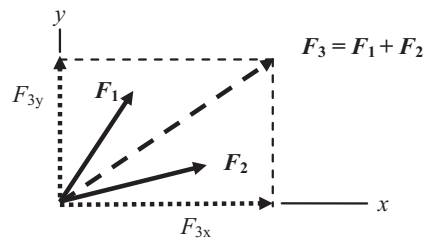


**Figure 1.4.3**

To add or subtract vectors is simply to add or subtract components. For example,

$$\{F_3\} = \{F_1\} + \{F_2\} = \begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix} = \begin{bmatrix} F_{1x} + F_{2x} \\ F_{1y} + F_{2y} \end{bmatrix} = \begin{bmatrix} F_{3x} \\ F_{3y} \end{bmatrix} \qquad (1.4.5)$$

The sum is shown by the dashed line and its components in the two coordinate directions by the dotted lines.

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**Example 1.4.1**

*Problem:* Two forces are acting at a point at the origin of the coordinate system as shown in Figure (a). Sum the two to find the resultant force and its direction.
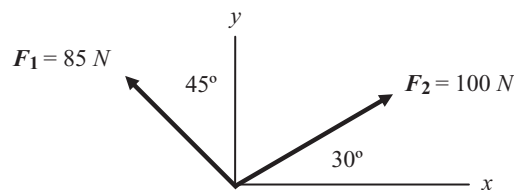


**Figure (a)**

*Solution:* Resolve the forces into components and sum. Solve for the resultant force and its orientation. The components are

$$\{F_1\} = \begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} = \begin{bmatrix} -85\sin 45° \\ 85\cos 45° \end{bmatrix} = \begin{bmatrix} -60.1 \\ 60.1 \end{bmatrix} \quad \{F_2\} = \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix} = \begin{bmatrix} 100\cos 30° \\ 100\sin 30° \end{bmatrix} = \begin{bmatrix} 86.6 \\ 50 \end{bmatrix} N \quad \text{(a)}$$

The sum is

$$\{F_3\} = \{F_1\} + \{F_2\} = \begin{bmatrix} -60.1 \\ 60.1 \end{bmatrix} + \begin{bmatrix} 86.6 \\ 50 \end{bmatrix} = \begin{bmatrix} 26.5 \\ 110.1 \end{bmatrix} N \quad \text{(b)}$$

The total magnitude of the force is

$$F_3 = \sqrt{F_x^2 + F_y^2} = \sqrt{(26.5)^2 + (110.1)^2} = 113.2 \ N \quad \text{(c)}$$

The resultant force vector makes an angle with respect to the *x* axis,

$$\theta = \tan^{-1} \frac{F_y}{F_x} = \tan^{-1} \frac{110.1}{26.5} = \tan^{-1} 4.15 = 76.5° \quad \text{(d)}$$

The resultant force is shown as a dashed line and its components as dotted lines in Figure (b).
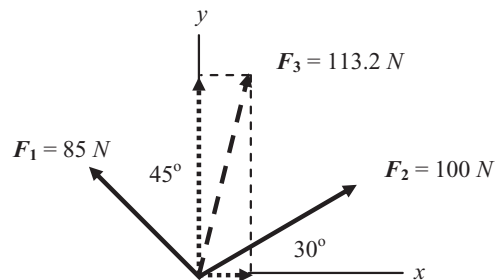


**Figure (b)**

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This can be extended to three dimensions. We shall define positive unit vectors **i, j, k** in the *x, y, z* directions, respectively, as shown in Figure 1.4.4.



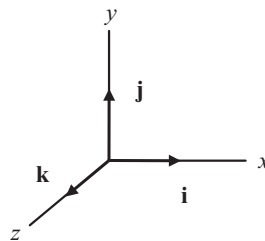**Figure 1.4.4**

The concentrated force, $F$, can be represented by its components in the $x$, $y$, and z directions as in Equation 1.4.6.

$$F = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \tag{1.4.6}$$

This is shown graphically in Figure 1.4.5.



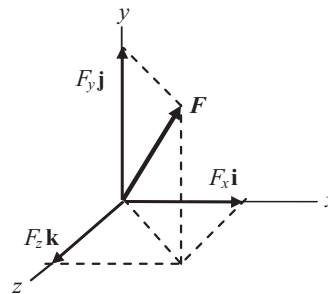**Figure 1.4.5**

The components in matrix form are

$$\{F\} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{1.4.7}$$

The magnitude of the vector $F$ is given by

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{1.4.8}$$

The angular orientation of the force $F$ with respect to each axis is given by

$$\cos\alpha = \frac{F_x}{F} \qquad \cos\beta = \frac{F_y}{F} \qquad \cos\gamma = \frac{F_z}{F} \tag{1.4.9}$$

The angle between the force, $F$, and the $x$ axis is $\alpha$, between the force $F$ and the $y$ axis is $\beta$, and between the force $F$ and the $z$ axis is $\gamma$. The quantities in Equation 1.4.9 are called the *direction cosines*.

As noted in the two dimensional case, to add or subtract vectors is simply to add or subtract components. For example, given three forces acting at a point the force representing the sum is

$$\{F_4\} = \{F_1\} + \{F_2\} - \{F_3\} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \\ F_{2z} \end{bmatrix} - \begin{bmatrix} F_{3x} \\ F_{3y} \\ F_{3z} \end{bmatrix} = \begin{bmatrix} F_{1x} + F_{2x} - F_{3x} \\ F_{1y} + F_{2y} - F_{3y} \\ F_{1z} + F_{2z} - F_{3z} \end{bmatrix} = \begin{bmatrix} F_{4x} \\ F_{4y} \\ F_{4z} \end{bmatrix} \tag{1.4.10}$$

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**Example 1.4.2**

*Problem:* Two forces act in perpendicular planes as shown in Figure (a). Sum the two to find the resultant force and its direction.
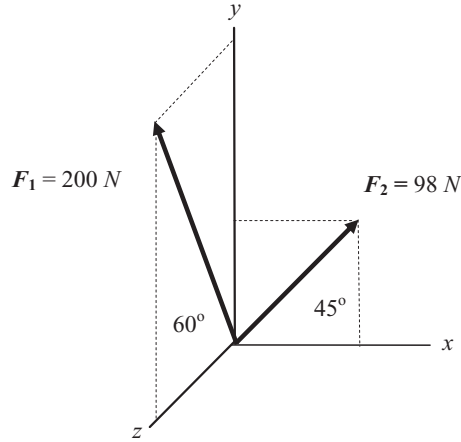
**Figure (a)**

*Solution:* Resolve the forces into components and sum. Solve for the value of the resultant force and its orientation.

The components of the forces are

$$\{F_1\} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \end{bmatrix} = \begin{bmatrix} 0 \\ 200\sin 60° \\ 200\cos 60° \end{bmatrix} = \begin{bmatrix} 0 \\ 173.2 \\ 100 \end{bmatrix} N \qquad \{F_2\} = \begin{bmatrix} F_{2x} \\ F_{2y} \\ F_{2z} \end{bmatrix} = \begin{bmatrix} 98\cos 45° \\ 98\sin 45° \\ 0 \end{bmatrix} = \begin{bmatrix} 69.3 \\ 69.3 \\ 0 \end{bmatrix} N$$

(a)

The sum of the forces is

$$\{F\} = \{F_1\} + \{F_2\} = \begin{bmatrix} 0 \\ 173.2 \\ 100 \end{bmatrix} + \begin{bmatrix} 69.3 \\ 69.3 \\ 0 \end{bmatrix} = \begin{bmatrix} 69.3 \\ 242.5 \\ 100 \end{bmatrix} N$$

(b)

The magnitude of the total force is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(69.3)^2 + (242.5)^2 + (100)^2} = 271.3 \ N$$

(c)

The direction cosines are

$$\cos\alpha = \frac{F_x}{F} = \frac{69.3}{271.3} = 0.255 \quad \rightarrow \quad \alpha = 75.2°$$

$$\cos\beta = \frac{F_y}{F} = \frac{242.5}{271.3} = 0.662 \quad \rightarrow \quad \beta = 26.6°$$

$$\cos\gamma = \frac{F_z}{F} = \frac{100}{271.3} = 0.369 \quad \rightarrow \quad \gamma = 68.4°$$
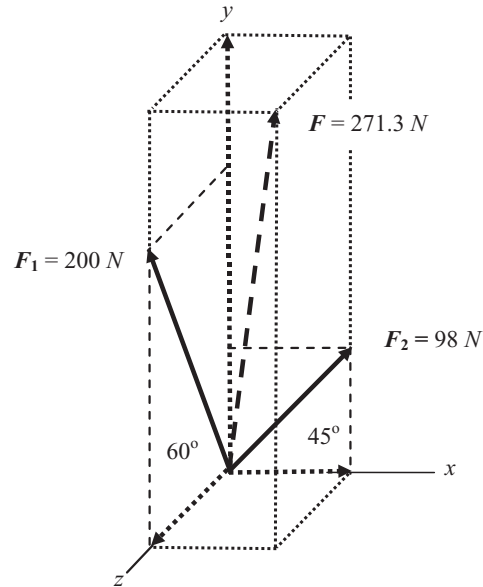
(d)

The final result is shown in Figure (b).

**Figure (b)**

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Another property of a matrix that we shall use shortly is multiplication of a matrix by a scalar. It is simply

$$a\{F\} = a\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} aF_x \\ aF_y \\ aF_z \end{bmatrix} \tag{1.4.11}$$

Additional matrix operations will be introduced as needed. They are summarized in Appendix A.

## 1.5   Moment of a Concentrated Force

A concentrated force can produce a moment about any given axis. In all examples we shall use Newton millimeter $(N \cdot mm)$ as our unit for moments. Consider the force applied to the rigid bar at point B as shown in Figure 1.5.1.



**Figure 1.5.1**

If we take moments about points A and B we get

$$M_A = FL \qquad M_B = M - FL = FL - FL = 0 \qquad (1.5.1)$$

Now consider the force has been moved to point A and a concentrated moment equal to $FL$ is added at point A as shown in Figure 1.5.2.
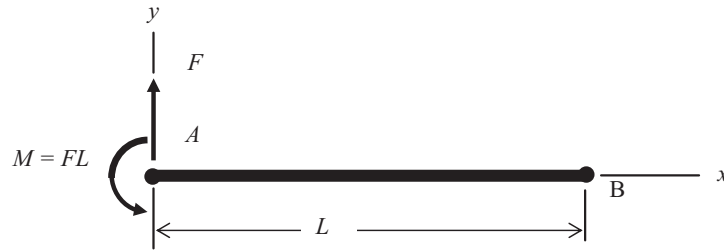


**Figure 1.5.2**

The moments about points A and B in this new configuration are the same as for the first configuration. Summing moments about each point we get

$$M_A = F \cdot 0 + M = M = FL \qquad M_B = M - FL = FL - FL = 0 \qquad (1.5.2)$$

We can, in fact, take moments about any point in the $xy$ plane and get the same result for both configurations. For example, take moments about the point C as shown in Figure 1.5.3 located at

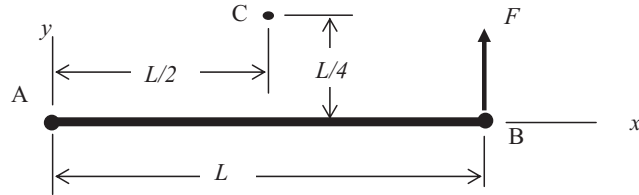$$x_C = \frac{L}{2} \qquad y_C = \frac{L}{4} \qquad (1.5.3)$$



**Figure 1.5.3**

From the configuration in Figure 1.5.3 we get

$$M_C = F\frac{L}{2} \qquad (1.5.4)$$

From the configuration in Figure 1.5.2 we get

$$M_C = M - F\frac{L}{2} = FL - F\frac{L}{2} = F\frac{L}{2} \qquad (1.5.5)$$

The problem can be posed in another way: If you move a force, what moment must be added to achieve an equivalent balance of moments?

Let us consider the rigid bar in Figure 1.5.4 with the force initially at point B. We shall call this configuration 1. It is then moved to point C (shown by a dashed line). This we call configuration 2. What moment must be added (and at what location) to provide equivalence?
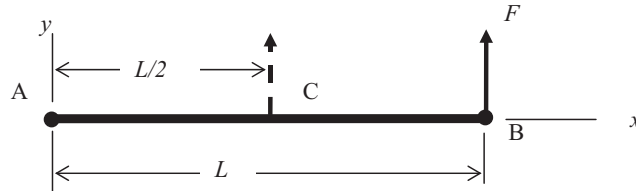


**Figure 1.5.4**

Let us take moments about point A for configuration 1 and 2.

$$M_{A1} = F \cdot L \qquad M_{A2} = F \cdot \frac{L}{2} \tag{1.5.6}$$

For equivalence we must add a moment that is equal to the difference in the two values or

$$M = M_{A1} - M_{A2} = F \cdot L - F \cdot \frac{L}{2} = F \cdot \frac{L}{2} \tag{1.5.7}$$

Now where should it be added? The answer is anywhere. In this case anywhere along the bar, for example, at point A, or point C, or point B, or any point in between.
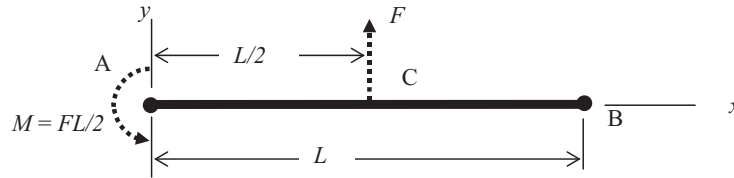


**Figure 1.5.5**

Just to be sure let us place the new applied moment at point A and sum moments about points A, B, and C in Figure 1.5.5.

$$M_A = \frac{FL}{2} + \frac{FL}{2} = FL \qquad M_B = -\frac{FL}{2} + \frac{FL}{2} = 0 \qquad M_C = \frac{FL}{2} \tag{1.5.8}$$

If you compare this with the original configuration in Figure 1.5.1 you will see that the moments about points A, B, and C agree.

The use of the half circle symbol in Figures 1.5.2 and 1.5.5 is one way of representing a concentrated moment in diagrams. It is used when the moment is about an axis perpendicular to the plane of the page. A common practice is to use a vector with a double arrowhead shown here in an isometric view to represent a moment. The vector is parallel to the axis about which the moment acts. The right hand rule of the thumb pointed in the vector direction and the curve fingers of the right hand showing the direction of the moment is implied here. The moment of Figure 1.5.2 is repeated in Figure 1.5.6 using a double arrowhead notation.
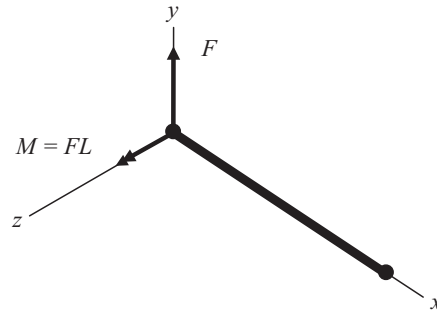
**Figure 1.5.6**

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## Example 1.5.1

*Problem:* A force is applied to a rigid body at point A as shown in Figure (a). If the force is moved to point B what moment must be applied at point C (origin of coordinates) to produce the same net moment about all points in space?
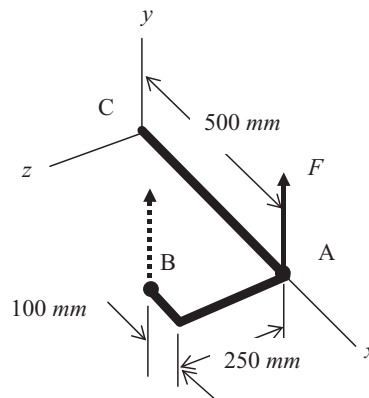


**Figure (a)**

*Solution:* Find the moment components at point C due to the force at point B and add the necessary moments so the total is equivalent to the moment components generated by the force at point A.

The force at point A produces the following moment components about the origin (point C)

$$M_{CAx} = 0 \qquad M_{CAy} = 0 \qquad M_{CAz} = 500 \cdot F \tag{a}$$

This can be written in matrix form as a column vector.

$$\{M_{CA}\} = F \begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix} \tag{b}$$

The same force at point B would produce the following moment components at point C.

$$M_{CBx} = -250 \cdot F \qquad M_{CBy} = 0 \qquad M_{CBz} = 400 \cdot F \tag{c}$$

or

$$\{M_{CB}\} = F \begin{bmatrix} -250 \\ 0 \\ 400 \end{bmatrix} \tag{d}$$

The needed moment components would be

$$\{M_C\} = \begin{bmatrix} M_{Cx} \\ M_{Cy} \\ M_{Cz} \end{bmatrix} = \{M_{CA}\} - \{M_{CB}\} = F \begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix} - F \begin{bmatrix} -250 \\ 0 \\ 400 \end{bmatrix} = F \begin{bmatrix} 250 \\ 0 \\ 100 \end{bmatrix} \tag{e}$$
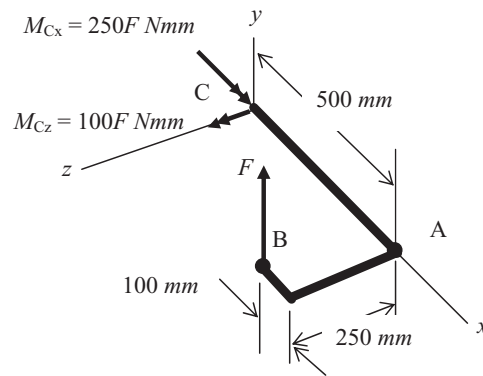
This is illustrated in Figure (b).



**Figure (b)**

Let us check our answer by summing moments about the origin. Using Figure (b)

$$\sum M_x = 250F - 250F = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 400F + 100F = 500F \tag{f}$$

In matrix form we get the same answer as Equation (a)

$$\{M\} = F \begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix} \tag{g}$$

Summing moments about any point in space will prove that the answer is always the same.

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So far we have considered a single force parallel to one of the axes. Consider now a force with components in all three dimensions acting at a point in space. We select a point about which we wish to find the moment components of this force with respect to a set of rectangular Cartesian coordinate axes.
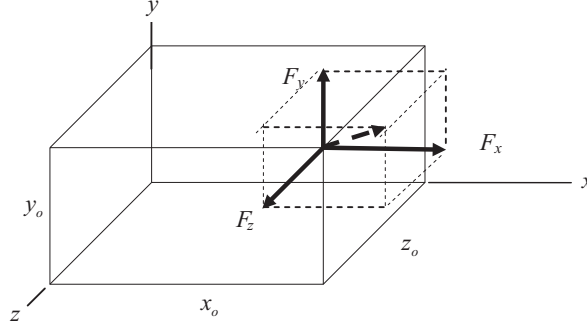
**Figure 1.5.7**

In Figure 1.5.7 we show the components of a force $\{F\}$ and the axes about which we wish to find the moment. If we take the moment at the origin of the coordinate system about each axis in turn we get

$$\sum M_x = F_z y_o - F_y z_o \qquad \sum M_y = F_x z_o - F_z x_o \qquad \sum M_z = F_y x_o - F_x y_o \qquad (1.5.9)$$

The moment components in matrix form about all three axes are

$$\{M\} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} F_z y_o - F_y z_o \\ F_x z_o - F_z x_o \\ F_y x_o - F_x y_o \end{bmatrix} \qquad (1.5.10)$$

This same information is often presented in the form of vector notation. The cross product of two vectors is stated as

$$C = A \times B \qquad (1.5.11)$$

The magnitude of the cross product is

$$C = AB \cos \theta \qquad (1.5.12)$$

where $\theta$ is the angle between the two vectors. The vector $C$ has a direction that is perpendicular to the plane containing $A$ and $B$ and its direction is defined by the right hand rule. Since the angle between unit vectors is either $90^\circ$, $-90^\circ$, or $0^\circ$ the cross product of unit vectors is found to be

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{i} \times \mathbf{i} = 0 \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = 0 \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = 0 \end{array} \qquad (1.5.13)$$

With this definition in mind the moment of the force in Figure 1.5.7 is represented as the cross product of a position vector and the force. Thus

$$M = d \times F \qquad (1.5.14)$$

If the moment is taken with respect to the origin of the coordinates in Figure 1.5.7 the position vector is

$$d = x_o \mathbf{i} + y_o \mathbf{j} + z_o \mathbf{k} \qquad (1.5.15)$$

The moment is then

$$
\begin{aligned}
\boldsymbol{M} = \boldsymbol{d} \times \boldsymbol{F} &= (x_o\mathbf{i} + y_o\mathbf{j} + z_o\mathbf{k}) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= x_o F_x(\mathbf{i} \times \mathbf{i}) + x_o F_y(\mathbf{i} \times \mathbf{j}) + x_o F_z(\mathbf{i} \times \mathbf{k}) \\
&\quad + y_o F_x(\mathbf{j} \times \mathbf{i}) + y_o F_y(\mathbf{j} \times \mathbf{j}) + z_o F_z(\mathbf{j} \times \mathbf{k}) \\
&\quad + z_o F_x(\mathbf{k} \times \mathbf{i}) + z_o F_x(\mathbf{k} \times \mathbf{j}) + z_o F_x(\mathbf{k} \times \mathbf{k})
\end{aligned}
\tag{1.5.16}
$$

By combining terms this reduces to

$$
\boldsymbol{M} = \boldsymbol{d} \times \boldsymbol{F} = (y_o F_z - z_o F_y)\mathbf{i} + (x_o F_z - z_o F_x)\mathbf{j} + (x_o F_y - y_o F_x)\mathbf{k}
\tag{1.5.17}
$$

This is often presented in the form of a determinant.

$$
\boldsymbol{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_o & y_o & z_o \\ F_x & F_y & F_z \end{vmatrix}
\tag{1.5.18}
$$

Equations 1.5.17 and 1.5.18 convey exactly the same information as that contained in Equation 1.5.10.

The matrix formulation of vectors very often has replaced the boldfaced vector representation more common in past treatises. We shall discontinue the use of boldface in representing vectors since we shall be using the matrix representation in all future work. The representation of any vector quantity will be clear from the context of its use.

As noted before we denote a moment using a vector with a double arrowhead as shown in Figure 1.5.8.
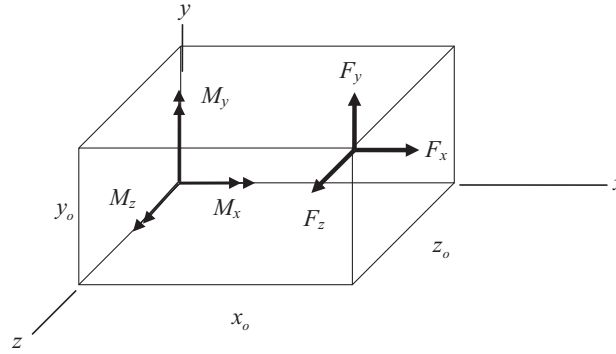


**Figure 1.5.8**

The vector components can be combined to obtain the total value of the moment.

$$
M = \sqrt{M_x^2 + M_y^2 + M_z^2}
\tag{1.5.19}
$$

and the orientation can be given by the direction cosines.

$$
\cos\alpha = \frac{M_x}{M} \qquad \cos\beta = \frac{M_y}{M} \qquad \cos\gamma = \frac{M_z}{M}
\tag{1.5.20}
$$

In all our deliberations applied forces are positive if they are in the positive directions of the axes and applied moments resulting from applied forces are positive by the right hand rule. Right handed rectangular Cartesian coordinate systems are used for the most part. Cylindrical coordinates will be used when we study torsion in Chapter 6.

##########

**Example 1.5.2**

***Problem:*** Find the moment components of the force shown in Figure (a) about the origin of the coordinate axes and the total value of the moment.
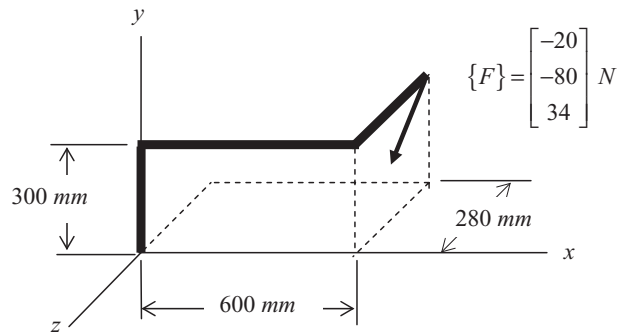


**Figure (a)**

***Solution:*** Use Equations 1.5.18–20.

The components are

$$\{F\} = \begin{bmatrix} -20 \\ -80 \\ 34 \end{bmatrix} N \qquad \{d\} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} 600 \\ 300 \\ -280 \end{bmatrix} mm \tag{a}$$

The moment can be obtained from

$$M = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 600 & 300 & -280 \\ -20 & -80 & 34 \end{vmatrix} \rightarrow \{M\} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 34 \cdot 300 - 80 \cdot 280 \\ -34 \cdot 600 + 20 \cdot 280 \\ -80 \cdot 600 + 20 \cdot 300 \end{bmatrix} = \begin{bmatrix} -12200 \\ -14800 \\ -42000 \end{bmatrix} N \cdot mm \tag{b}$$

The total value of the moment is

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{(12200)^2 + (14800)^2 + (42000)^2} = 46172.3 \ Nmm \tag{c}$$

The orientation of the resultant moment is given by the direction cosines.

$$\cos \alpha = \frac{M_x}{M} = \frac{-12200}{46172.3} = -0.2642 \quad \rightarrow \quad \alpha = 105.32°$$

$$\cos \beta = \frac{M_y}{M} = \frac{-14800}{46172.3} = -0.3205 \quad \rightarrow \quad \beta = 108.7° \tag{d}$$

$$\cos \gamma = \frac{M_z}{M} = \frac{-42000}{46172.3} = -0.9096 \quad \rightarrow \quad \gamma = 155.46°$$

##########

To find the moment about some point other than the origin of the coordinate system requires only defining a new position vector. For example suppose we wish to find the moment components about point A as shown in Figure 1.5.9.
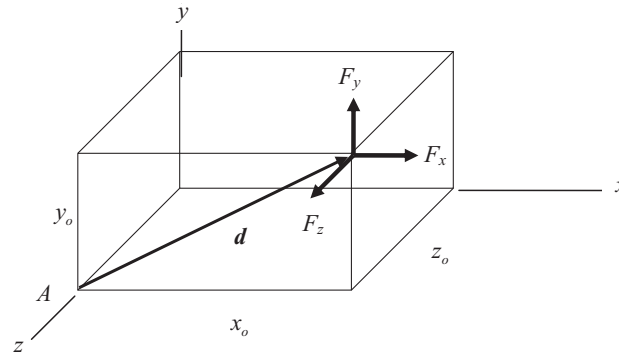


**Figure 1.5.9**

The components of the new position vector would be

$$\{d\} = \begin{bmatrix} x_o \\ y_o \\ 0 \end{bmatrix} \tag{1.5.21}$$

A special case of a moment caused by forces occurs when there are two parallel forces of equal and opposite direction separated by a distance $a$. This might occur, for example, with the loads applied to a member as shown in Figure 1.5.10.
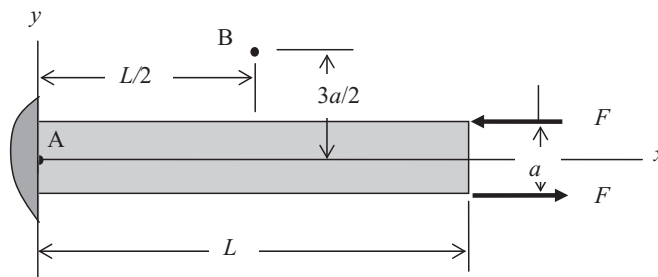


**Figure 1.5.10**

The resultant force of the two forces is zero. The resultant moment about any point in the plane is

$$M = Fa \tag{1.5.22}$$

To illustrate that the location of the point in the plane is of no effect take moments about point A which is at the origin of the coordinates and about point B which is at $x = L/2$ and $y = 3d/2$.

$$M_A = F\frac{a}{2} + F\frac{a}{2} = Fa \tag{1.5.23}$$

$$M_B = -\left(\frac{3a}{2} - \frac{a}{2}\right)F + \left(\frac{3a}{2} + \frac{a}{2}\right)F = Fa$$

Such a force combination as shown in Figure 1.5.10 is called a *couple*. When $d$ is small it can often be represented as a concentrated moment as in Figure 1.5.11.
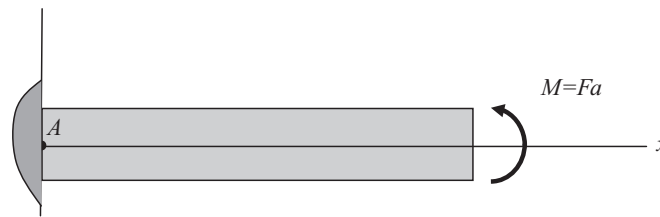


**Figure 1.5.11**

###########

**Example 1.5.3**

*Problem:* Find the moment components about the axes of the set of couples shown in Figure (a).
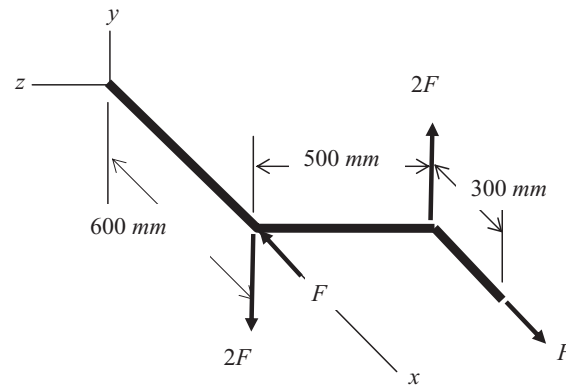


**Figure (a)**

*Solution:* Use the definition of a couple.

The moment components about the axes are simply

$$M_x = 500 \cdot 2F \qquad M_y = -500 \cdot F \qquad M_z = 0 \tag{a}$$

**Example 1.5.4**

*Problem:* A force system consists of a couple and another force as shown in Figure (a). Find the moment about the $z$ axis at point A.
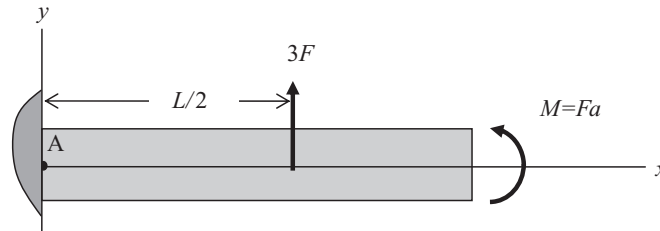
*Solution:* Sum moments about the $z$ axis.



**Figure (a)**

The sum of moments about point A is

$$M_A = 3F \cdot \frac{L}{2} + M = 3F \cdot \frac{L}{2} + Fa = \left( \frac{3L}{2} + a \right) F \tag{a}$$

###########

## 1.6    Distributed Forces—Force and Moment Resultants

Forces may be distributed along a line, on a surface, or throughout a volume. It is often necessary to find the total force resultant of the distributed force and also find through what point it is acting. Consider the area shown in Figure 1.6.1 and the force per unit area acting upon it. We have chosen a planar rectangular area and a particular distribution for ease of explanation. Real surfaces with loads will be found in many shapes and sizes and can be external or internal surfaces.
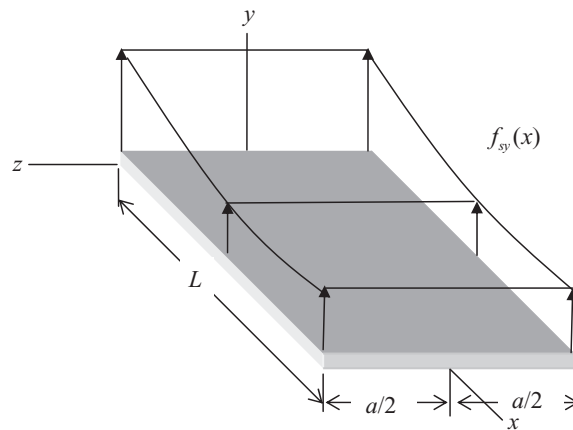


**Figure 1.6.1**

This particular surface loading acts in the $y$ direction, varies in the $x$ direction, and is uniform in the $z$ direction. We label this force $f_{sy}(x, z)$ where the subscript $s$ denotes a surface force and the $y$ its

component direction. We note that its units are force per unit area ($N/mm^2$) and since it is uniform in the $z$ direction we can multiply the surface force by the width, $a$, of the planar surface and obtain an equivalent line force with units of force per unit length ($N/mm$) as shown in Figure 1.6.2. This line force acts in the plane of symmetry, that is, in the $xy$ plane at $z = 0$. Our coordinate system was selected conveniently with this in mind. Bear in mind that this is the resultant of the distributed surface force and not the actual force acting on the surface. It may be used for establishing equilibrium of the body.
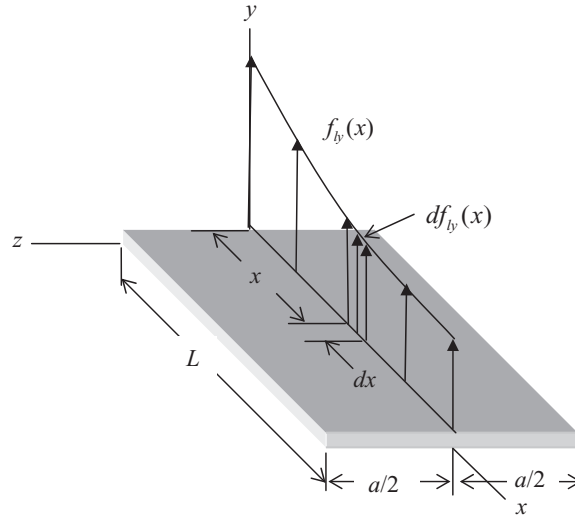


**Figure 1.6.2**

We label this force $f_{ly}(x)$ where the subscript $l$ denotes a line force and note that

$$f_{ly}(x) = a f_{sy}(x, z) \qquad (1.6.1)$$

Now consider an infinitesimal length, $dx$, along the line force at location $x$, as shown in Figure 1.6.2. On the length, $dx$, the force in the $y$ direction is

$$dF_y = df_{ly}(x) \qquad (1.6.2)$$

If we sum the forces on all such $dx$ lengths, ranging for $x = 0$ to $x = L$, we obtain the total resultant of the distributed force

$$F_y = \int_0^L f_{ly}(x)dx = a \int_0^L f_{sy}(x)dx \qquad (1.6.3)$$

We can find the location, or *line of action*, of the force resultant by equating the moments of the distributed force to the moment of the force resultant as follows. Again, the sum of all the moments of all the forces on all the infinitesimal elements $dx$ is

$$\bar{x} F_y = \int_0^L x f_{ly}(x)dx \qquad (1.6.4)$$

And the location of the resultant is

$$\bar{x} = \frac{1}{F_y} \int_0^L x f_{ly}(x)dx = \frac{\int_0^L x f_{ly}(x)dx}{\int_0^L f_{ly}(x)\,dx} \qquad (1.6.5)$$

The use of symmetry to locate the line force and the resultant force at $z = 0$ can be confirmed by equating the moments of the distributed force to the moment of the resultant force about the $z$ axis.

$$\bar{z}F_y = \int_0^L \int_{-\frac{a}{2}}^{\frac{a}{2}} z f_{sy}(x)dzdx \quad \rightarrow \quad \bar{z} = \frac{1}{F_y} \int_{-\frac{a}{2}}^{\frac{a}{2}} zdz \int_0^L f_{sy}(x)\,dx \qquad (1.6.6)$$

Clearly the integral

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} zdz = 0 \qquad (1.6.7)$$

and therefore $\bar{z} = 0$.

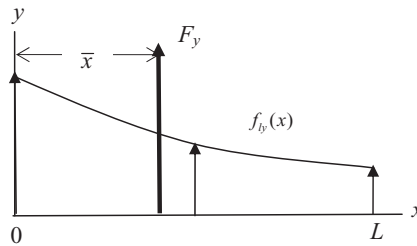The line force and the resultant force and its location are depicted in Figure 1.6.3.



**Figure 1.6.3**

###########

**Example 1.6.1**

*Problem:* A distributed force per unit area is applied to the surface as shown in Figure (a). The force is uniform in the $z$ direction.

$$f_{sy}(x, z) = f_0 \frac{x}{L} \qquad (a)$$

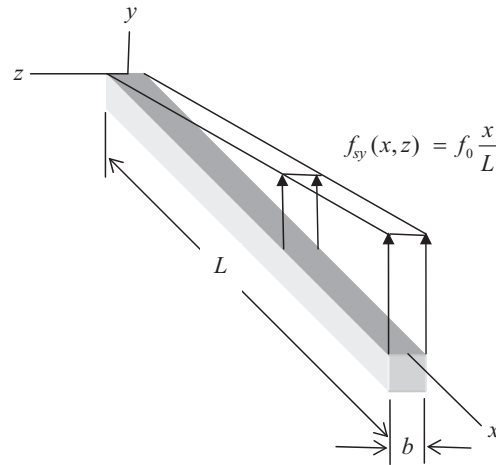Find the total value and its line of action.

**Figure (a)**

*Solution:* Convert the surface force ($N/mm^2$) to a line force ($N/mm$) and integrate to find the total resultant force ($N$). Then equate moments to find its line of action.

The distributed line force and resultant force are

$$f_{ly}(x) = bf_0 \frac{x}{L} \qquad F_y = bf_0 \int_0^L \frac{x}{L} dx = bf_0 \left. \frac{x^2}{2L} \right|_0^L = bf_0 \frac{L}{2} \tag{b}$$

Find the line of action by equating moments.

$$\bar{x} F_y = \int_0^L x bf_0 \frac{x}{L} dx \quad \rightarrow \quad \bar{x} = \frac{1}{F_y} \int_0^L x bf_0 \frac{x}{L} dx = \frac{1}{F_y} \left. \frac{bf_0 x^3}{3L} \right|_0^L = \frac{2}{3} L \tag{c}$$

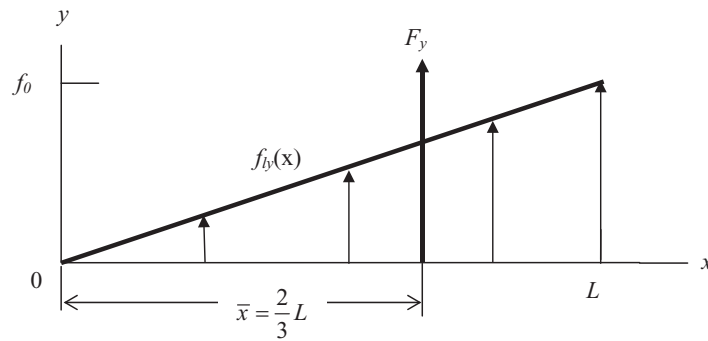The line force and the location of the force resultant are shown in Figure (b).



**Figure (b)**

###########

Symmetry may be used to identify quickly the location of a resultant force. The two line forces shown in Figure 1.6.4 are symmetrical about their midpoints and so the location of the resultant is known instantly as shown.
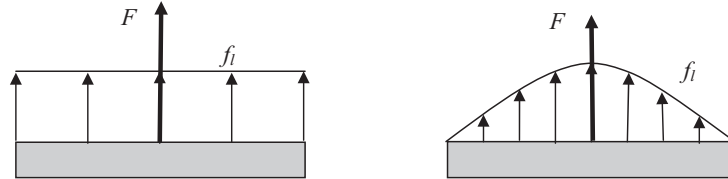


**Figure 1.6.4**

Distributed surface forces may be functions of two variables; for example, as shown in Figure 1.6.5.
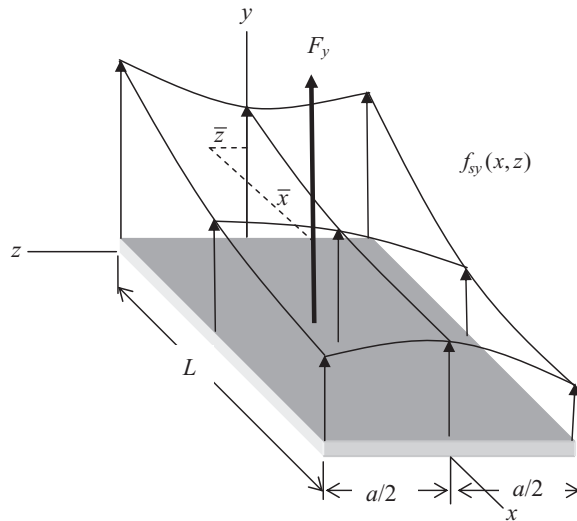


**Figure 1.6.5**

Then the resultant force for the area shown, using the same procedure as before, is

$$F_y = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L f_{sy}(x,z) dx dz \qquad (1.6.6)$$

and its line of action is found by

$$\bar{x}F_y = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L x f_{sy}(x,z) dx dz \rightarrow \bar{x} = \frac{1}{F_y} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L x f_{sy}(x,z) dx dz = \frac{\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L x f_{sy}(x,z) dx dz}{\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L f_{sy}(x,z) dx dz}$$

$$\bar{z}F_y = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L z f_{sy}(x,z) dx dz \rightarrow \bar{z} = \frac{1}{F_y} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L z f_{sy}(x,z) dx dz = \frac{\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L z f_{sy}(x,z) dx dz}{\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_0^L f_{sy}(x,z) dx dz}$$

$$(1.6.7)$$

The location of the resultant force is also depicted in Figure 1.6.5.

Of course, surfaces are not always planar and rectangular and surface forces may have components in all coordinate directions. We shall be satisfied with simplified geometry and forces until and unless the need arises for more complicated cases.

<div align="center">###########</div>

**Example 1.6.2**

***Problem:*** A distributed force per unit area is applied to the surface shown in Figure (a). The surface force is represented by the function in Equation (a). Find the resultant force and its line of action.

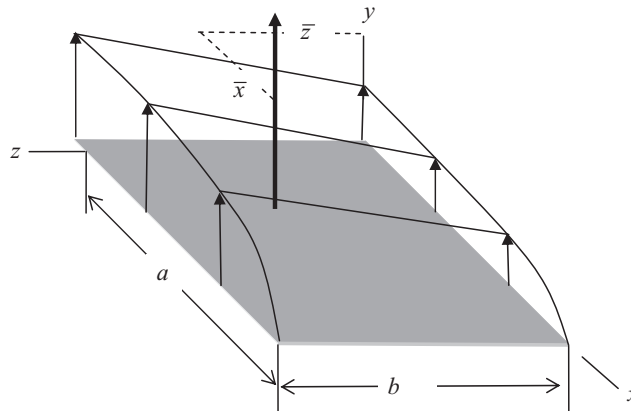$$f_{sy}(x, z) = f_0 \left(1 - \frac{x^2}{a^2}\right)\left(\frac{1}{2} + \frac{z}{2b}\right) \tag{a}$$



<div align="center">**Figure (a)**</div>

***Solution:*** Integrate to find the resultant force. Equate moments to find its line of action.

The resultant force is given by Equation (b)

$$\begin{aligned}
F_y &= \int_0^a \int_0^b f_{sy}(x, z)\,dz\,dx = f_0 \int_0^a \int_0^b \left(1 - \frac{x^2}{a^2}\right)\left(\frac{1}{2} + \frac{z}{2b}\right)dz\,dx \\
&= f_0 \left(x - \frac{x^3}{3a^2}\right)\Big|_0^a \left(\frac{z}{2} + \frac{z^2}{4b}\right)\Big|_0^b = f_0 \frac{ab}{2}
\end{aligned} \tag{b}$$

Its location is given by Equations (c) and (d).

$$\begin{aligned}
\bar{x}F_y = \int_0^a \int_0^b x f_{sy}(x, z)\,dx\,dz \quad &\rightarrow \quad \bar{x} = \frac{1}{F_y} f_0 \int_0^a \int_0^b x\left(1 - \frac{x^2}{a^2}\right)\left(\frac{1}{2} + \frac{z}{2b}\right)dz\,dx \\
&= \frac{2}{ab}\left(\frac{x^2}{2} - \frac{x^4}{4a^2}\right)\Big|_0^a \left(\frac{z}{2} + \frac{z^2}{4b}\right)\Big|_0^b = \frac{3a}{8}
\end{aligned} \tag{c}$$

$$\bar{z}F_y = \int_0^a \int_0^b z f_{sy}(x, z) dx \,\mathrm{d}z \quad \rightarrow \quad \bar{z} = \frac{1}{F_y} f_0 \int_0^a \int_0^b z \left(1 - \frac{x^2}{a^2}\right)\left(\frac{1}{2} + \frac{z}{2b}\right) dz dx$$

$$= \frac{2}{ab}\left(x - \frac{x^3}{3a^2}\right)\Bigg|_0^a \left(\frac{z^2}{4} + \frac{z^3}{6b}\right)\Bigg|_0^b = \frac{5b}{9} \tag{d}$$

###########

We may also need to find the resultants of body forces and their location. Body forces have the units of force per unit volume ($N/mm^3$) and may be distributed over the volume. Consider the general solid shown in Figure 1.6.6.
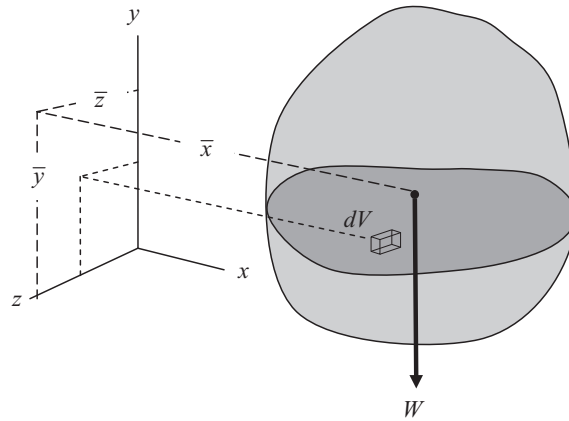


**Figure 1.6.6**

The most common body force is that due to gravity, that is, weight. The weight ($dW$) of a infinitesimal volume $dV$ is given by

$$dW = \rho(x, y, z) g dV \tag{1.6.8}$$

where $\rho(x, y, z)$ is the mass density ($kg/mm^3$) and $g$ is the acceleration due to gravity ($mm/s^2$). (See Section 1.2, Equation 1.2.2.)

If the $y$ axis is oriented parallel to the gravity vector then the total weight is given by

$$W = g \int_V \rho(x, y, z) dV \tag{1.6.9}$$

and its line of action, or, in this case, the point through which it acts is given by

$$\bar{x} = \frac{\int_V x \rho(x, y, z) dV}{\int_V \rho(x, y, z) dV} \qquad \bar{y} = \frac{\int_V y \rho(x, y, z) dV}{\int_V \rho(x, y, z) dV} \qquad \bar{z} = \frac{\int_V z \rho(x, y, z) dV}{\int_V \rho(x, y, z) dV} \tag{1.6.10}$$

The location of this resultant force is called the *center of gravity* and is commonly abbreviated as C.G. Notice that the acceleration due to gravity is a constant that cancels out of the integrals for finding the C.G. and the resulting equations depended only upon the mass density and the volume. This is also called the *center of mass*.

In many situations the solid body is homogenous, that is, the mass density is a constant. In such cases the mass density terms also cancel. Equations 1.6.10 then become

$$\bar{x} = \frac{\int_V x\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V y\, dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V z\, dV}{\int_V dV} \qquad (1.6.11)$$

This locates the *centroid* of the volume. In such cases the centroid, the center of mass, and the center of gravity are the same point.

When a homogenous body can be divided into sub volumes with simple geometry so that the centroids of the sub volumes are known we can find the centroid of the total using the following formulas.

$$\bar{x} = \frac{\sum x_s V_s}{\sum V_s} \qquad \bar{y} = \frac{\sum y_s V_s}{\sum V_s} \qquad \bar{z} = \frac{\sum z_s V_s}{\sum V_s} \qquad (1.6.12)$$

The quantities $x_s$, $y_s$, and $z_s$ represent the distances from the base axes to the centroids of the sub volumes $V_s$. For a body with uniform mass density you can replace the volume in Equation 1.6.12 with the mass or the weight to find the center of mass or the center of gravity. All are at the same location.

<div align="center">###########</div>

**Example 1.6.3**

**Problem:** A cylindrical bar has a portion hollowed out as shown in Figure (a). It is made of aluminum which has a mass density of 2.72 *E-06 kg/mm³*. Find its total weight and center of gravity. The *y* axis is aligned with the gravity vector.
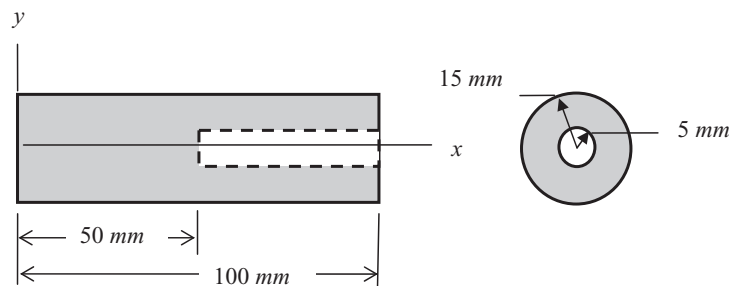


**Figure (a)**

**Solution:** The total weight is the volume times the mass density times the acceleration due to gravity. From axial symmetry we know the center of gravity will lie on the centerline of the cylinder. We find the *x* location by summing moments about the *z* axis.

To find the total weight we find the weight of the outer cylinder and subtract the weight of the inner cylinder.

$$W_{total} = W_{outer} - W_{inner} = \rho g \left( \pi r_{outer}^2 L - \frac{\pi r_{inner}^2 L}{2} \right) \qquad (a)$$
$$= 2.72 \cdot 10^{-6} \cdot 9.81 \left( \pi (15)^2\, 100 - \pi (5)^2\, 50 \right) = (1.886 - 0.105)\, N = 1.781 N$$

Using symmetry the center of gravity location is located on the centerline of the cylinder and at an $x$ position given by

$$\bar{x}W_{total} = 50W_{outer} - 75W_{inner} = 50 \cdot 1.886 - 75 \cdot 0.105 = 86.425$$

$$\rightarrow \bar{x} = \frac{86.425}{1.781} = 48.526 \; mm \qquad \text{(b)}$$

$$\bar{y} = \bar{z} = 0$$

###########

## 1.7   Internal Forces and Stresses—Stress Resultants

As we have said, when forces are applied to a solid body that is suitably restrained to eliminate rigid body motions it will deform and internal forces will be generated. It has been found convenient on any internal surface to define *stress* as the force per unit area ($N/mm^2$). The stress on any internal surface is usually divided into components normal to that surface and tangent to it. The stress and stress components are depicted in Figure 1.7.1.

Consider a force, $\Delta p$, acting on an element of area, $\Delta A$, in the interior of the solid. If we resolve the force $\Delta p$ into components normal and tangential to the surface, $\Delta p_n$ and $\Delta p_t$, respectively, we can define a *normal* component of stress, $\sigma$, and a *tangential* or *shearing* (or *shear*) stress component, $\tau$, as shown in Figure 1.7.1.
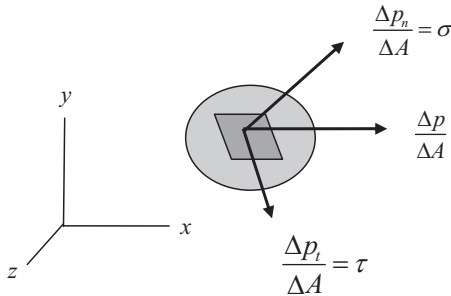


**Figure 1.7.1**

If we allow $\Delta A$ to shrink to an infinitesimal size, then,

$$\lim_{\Delta A \to 0} \frac{\Delta p_n}{\Delta A} = \sigma \qquad \lim_{\Delta A \to 0} \frac{\Delta p_t}{\Delta A} = \tau \qquad (1.7.1)$$

To illustrate internal stresses let us first consider a uniform slender bar in equilibrium with equal but opposite distributed loads on each end as shown in Figure 1.7.2. These distributed forces have units of force per unit area and act on the end surfaces. There are no force components in the $y$ and $z$ directions.

The applied force resultants in the $x$ direction are

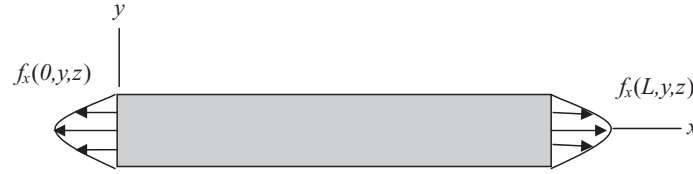$$F_0 = \int_A f_x(0, y, z)dA = -F \qquad F_L = \int_A f_x(L, y, z)dA = F \qquad (1.7.2)$$

**Figure 1.7.2**

Let us choose distributed forces for which the applied moment resultants are zero, that is

$$M_{y0} = \int_A f_x(0, yz)z\, dA = 0 \qquad M_{yL} = \int_A f_x(L, yz)z\, dA = 0$$

$$M_{z0} = \int_A f_x(0, yz)y\, dA = 0 \qquad M_{zL} = \int_A f_x(L, yz)y\, dA = 0$$

(1.7.3)

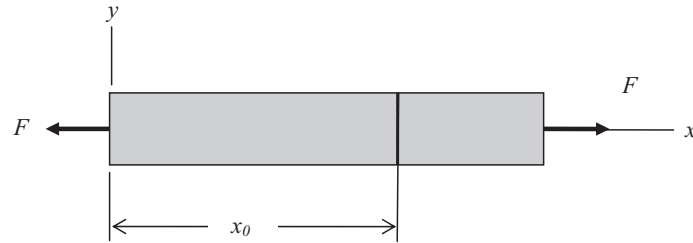This is illustrated in Figure 1.7.3.



**Figure 1.7.3**

Now let us ask what the distributed internal force or stress is on an interior surface. Let us choose a flat surface normal to the $x$ axis at, say, $x = x_0$ and examine the stress on that surface by considering the equilibrium of the segment of the bar between $x = x_0$ and $x = L$, as shown in Figure 1.7.4. As noted above, in the usual notation for stress, we designate the normal stress component with the symbol $\sigma$ and the tangential or shearing stress component with the symbol $\tau$. To further specify the specific components of stress we use a double subscript notation. The first subscript refers to the direction of the normal to the surface and the second to the direction of the stress.

Thus we show the two possible stress components on this particular surface and label them $\sigma_{xx}$ and $\tau_{xy}$. Since normal stress components always have the same subscript repeated we usually use only one and so we use $\sigma_x$ for $\sigma_{xx}$ in subsequent applications.



**Figure 1.7.4**

The sign convention for stresses is different from that for external and restraint forces; the normal and shear stress components are positive as shown in Figure 1.7.4. Sign conventions for stress will be discussed in greater detail shortly.

At this time we do not yet know the distribution of these stresses on that surface; however, we can define stress resultants on the surface as

$$P = \int_A \sigma_x \, dA \qquad V = \int_A \tau_{xy} \, dA \tag{1.7.4}$$

This is shown in Figure 1.7.5.



**Figure 1.7.5**

If we sum the forces acting on the segment in the $x$ and $y$ directions we get

$$\sum F_x = F - P = 0 \quad \rightarrow \quad P = F \qquad \sum F_y = V = 0 \quad \rightarrow \quad V = 0 \tag{1.7.5}$$

While we do not yet know the actual distribution of stress on this surface we can define an average stress as

$$(\sigma_x)_{ave} = \frac{F}{A} \qquad (\tau_{xy})_{ave} = \frac{V}{A} = 0 \tag{1.7.6}$$

Now consider that the chosen internal surface is not normal to the axis but is defined by a local coordinate system, $tn$, at an angle $\alpha$ to the $xy$ coordinate system as shown in Figure 1.7.6.
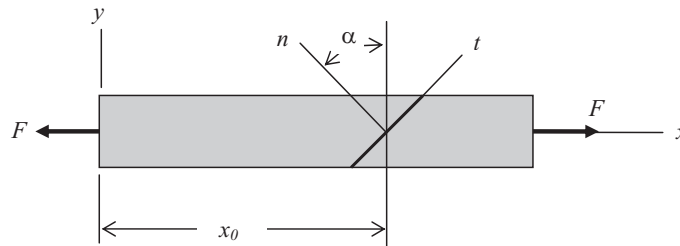


**Figure 1.7.6**

The stresses on this surface, that we have labeled $A_\alpha$, are shown in Figure 1.7.7.
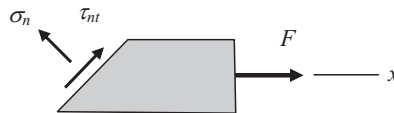


**Figure 1.7.7**

Once again we can define stress resultants on this surface as

$$P_n = \int_{A_\alpha} \sigma_n \, dA \qquad V_t = \int_{A_\alpha} \tau_{nt} \, dA \tag{1.7.7}$$
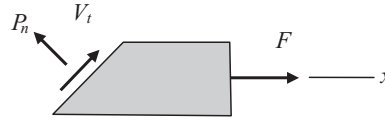
This is shown in Figure 1.7.8.



**Figure 1.7.8**

When we sum the forces acting on the segment in the $x$ and $y$ directions we get

$$\sum F_x = F + V_t \cos\alpha - P_n \sin\alpha = 0 \quad \rightarrow \quad V_t \cos\alpha - P_n \sin\alpha = -F$$
$$\sum F_y = V_t \sin\alpha + P_n \cos\alpha = 0 \quad \rightarrow \quad V_t \sin\alpha + P_n \cos\alpha = 0$$

(1.7.8)

Solving we get

$$P_n = F \sin\alpha \quad \rightarrow \quad V_t = -F \cos\alpha$$

(1.7.9)

And the average stresses are

$$(\sigma_n)_{ave} = \frac{P_n}{A_\alpha} = \frac{F}{A_\alpha} \sin\alpha \qquad (\tau_{nt})_{ave} = \frac{V_t}{A_\alpha} = -\frac{F}{A_\alpha} \cos\alpha$$

(1.7.10)

Stresses are always defined by their magnitude, direction, and the surface upon which they act.

The actual distribution of the stresses on the interior surfaces in this example is discussed in detail in Chapter 4.

To continue our discussion of stress we consider a thin flat plate acted upon by forces in the plane of the plate. We orient this plate in the $xy$ plane of the coordinate system. With forces only in the plane of the plate on the thin plate edge surfaces and no forces on the plate surfaces in the $z$ direction we can assume that the only stress components are those acting in the plane of the plate.

To consider the stress at a point in the plate we take a small rectangular element, $dxdy$, of the plate as shown in Figure 1.7.9 and note the stress components as the size of the element approaches zero.
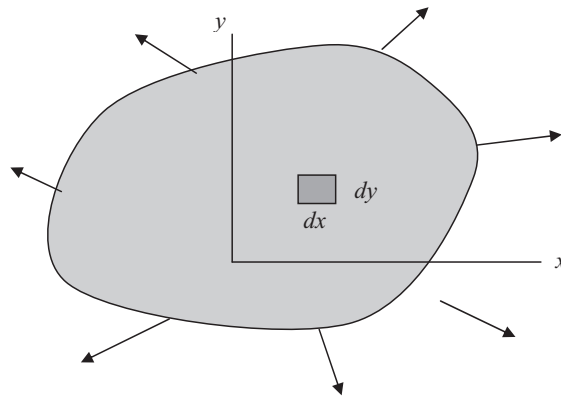


**Figure 1.7.9**

The stresses on this rectangular element are shown in Figure 1.7.10. This is called a two dimensional state of stress. Remember this is a uniform stress through the thickness at a point where the edges $dx$ and $dy$ approach zero.
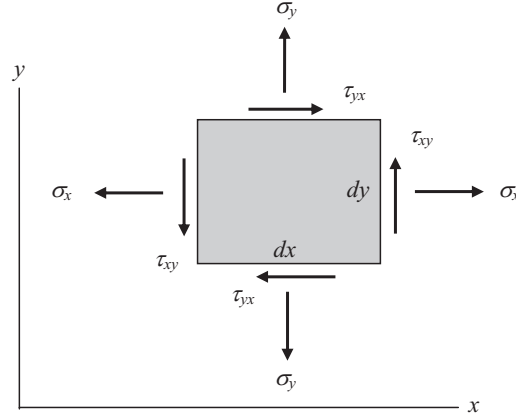
**Figure 1.7.10**

Stresses are positive as shown. If the normal to the surface upon which the stress acts points in the positive $x$ direction the normal stress is positive in the positive $x$ direction. If the normal points in the negative $x$ direction the normal stress is positive in the negative $x$ direction. If the normal points in the positive $x$ direction the shear stress is positive in the positive $y$ direction. If the normal points in the negative $x$ direction the shear stress is positive in the negative $y$ direction. We see that the forces called stress resultants have a different sign convention than the applied forces.

At this time one would conclude that there are four components of stress in a two dimensional solid - $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx}$. We note in Figure 1.7.10 that the $\tau_{xy}$ and $\tau_{yx}$ components each form a couple and as the sides of the rectangle shrink to a point by applying moment equilibrium let us conclude that

$$\tau_{xy} = \tau_{yx} \tag{1.7.11}$$

This 2D state of stress is often depicted in matrix form as

$$\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{1.7.12}$$

To depict a complete state of stress at a point in a three dimensional solid with respect to rectangular Cartesian coordinates consider a rectangular element $dxdydz$ oriented as shown in Figure 1.7.11. Stress components are shown on the three faces with positive normal directions and on the one face with a negative normal ($x$) direction. Similar components are on the negative $y$ and $z$ directions but are not shown to avoid confusion.

On a face whose normal is in the positive $x$ direction we have three stress components as follows:

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A} \qquad \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A} \tag{1.7.13}$$

On a face whose normal is in the negative direction of the axis the positive directions of both normal and shearing stresses are in the negative direction of their respective axes.

At this time one would conclude that there are nine components of stress in a three dimensional solid - $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}$. By the same argument used in the two dimensional case each of the three pairs of shear stresses form a couple and as the element shrinks to a point, we can conclude from
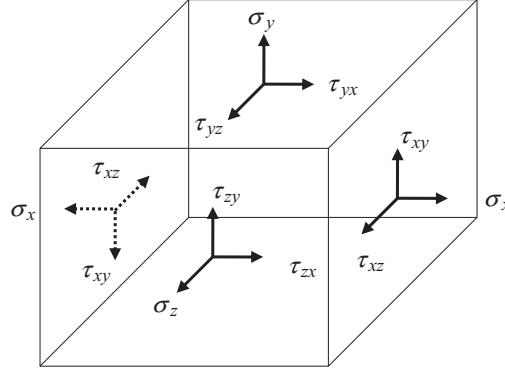
**Figure 1.7.11**

moment equilibrium that

$$\tau_{xy} = \tau_{yx} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz} \tag{1.7.14}$$

Thus, there are six independent components of stress at a material point within a solid. These are variously arranged in matrix form as a column or square matrix according to their use in equations.

$$\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \quad \text{or} \quad [\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{1.7.15}$$

At the beginning of our deliberation the stresses, generally, are unknown. Finding stresses in particular situations is a major part of the task before us. The state of stress on surfaces at other orientations than normal to the *xyz* axis is discussed in great detail in Chapter 9.

Acceptable magnitudes of internal forces are limited by the properties of the material of the solid body. We all know that if they are too great the body will fail either from undesirable permanent deformations or from fracture. Material properties are introduced in Chapter 3 and discussed in greater detail in Chapter 9.

## 1.8   Restraint Forces and Restraint Force Resultants

Solid bodies can be restrained at a point, they can be restrained along a line, or over a surface. When the body is subjected to applied forces there must be forces at the restraints. In a common statement of the problem these forces are unknown. Part of the purpose of our study in the following chapters is to find the value of these forces. In some circumstances it is possible to predict the value of the resultants of these forces without finding the actual distribution of the forces. This, in fact, is an important lesson of Chapter 2. Just how the restraint forces are distributed will be considered in Chapters 4 and beyond.

## 1.9    Summary and Conclusions

We have introduced the definition and notation for forces and moments of forces. An important part of the study of the mechanics of material is the interaction among applied forces, restraint forces, and internal forces and the moments generated by them. These forces come in various forms such as concentrated forces with units of Newtons ($N$), forces per unit length or line forces with units of Newtons per millimeter ($N/mm$), surface forces with units of Newtons per millimeter squared ($N/mm^2$), and body forces with units of Newtons per millimeter cubed. ($N/mm^3$).

We have shown how force resultants of distributed forces are found and how their location is determined. We have defined internal forces and stresses, and restraint forces.

Now we shall put all this to good use in Chapter 2 in establishing static equilibrium, that is, in satisfying Newton's laws as the various forces interact.