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Governing Equations of Fluid and Structural Mechanics

In this chapter, we introduce the partial differential equations that govern the fluid and structural mechanics parts of the fluid–structure interaction (FSI) problem. The fluid and structural mechanics equations are complemented by the applicable boundary conditions and constitutive models. For the structural mechanics part of the problem, we adopt mostly a 3D solid description, but include a discussion of thin structures such as shells and membranes. Both strong and weak (or variational) forms of the fluid and structural mechanics equations are presented. We conclude the section with an Arbitrary Lagrangian–Eulerian (ALE) description of the fluid mechanics equations suitable for moving-domain computations.

1.1 Governing Equations of Fluid Mechanics

The fluid mechanics part of the FSI problem is governed by the Navier–Stokes equations of incompressible flows. In what follows, we present the strong and weak forms of these equations and discuss the applicable boundary conditions.

1.1.1 Strong Form of the Navier–Stokes Equations of Incompressible Flows

Let $\Omega_t \in \mathbb{R}^{n_{\text{sd}}}$, $n_{\text{sd}} = 2, 3$, be the spatial fluid mechanics domain with boundary Γ_t at time $t \in (0, T)$ (see Figure 1.1 for an illustration). The subscript t indicates that the fluid mechanics spatial domain is time-dependent. The Navier–Stokes equations of incompressible flows[1] may be written on Ω_t and $\forall\, t \in (0, T)$ as

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma}) - \rho \mathbf{f} = \mathbf{0}, \tag{1.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{1.2}$$

[1] Although the term "incompressible Navier–Stokes equations" is often employed instead of "Navier–Stokes equations of incompressible flows," we prefer the latter because it is the flows and not the equations that are incompressible.

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Yuri Bazilevs, Kenji Takizawa and Tayfun E. Tezduyar.

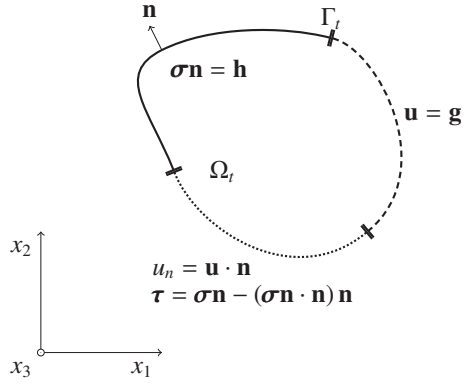

Figure 1.1 Fluid mechanics spatial domain and its boundary

where ρ, $\mathbf{u}$, and $\mathbf{f}$ are the density, velocity, and the external force (per unit mass), respectively, and the stress tensor $\boldsymbol{\sigma}$ is defined as

$$\boldsymbol{\sigma}(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}). \tag{1.3}$$

Here p is the pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain-rate tensor given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T\right). \tag{1.4}$$

Equations (1.1) and (1.2) represent the local balance of linear momentum and mass, respectively, and the momentum balance equation is written in the so-called *conservative form*. The local mass balance for incompressible flows states that the velocity field must be divergence-free at every point in space and time, which is also known as the incompressibility constraint (see Equation (1.2)).

REMARK 1.1 *In Equation (1.1), and everywhere in this book, we denote by* $\boldsymbol{\nabla}$ *the gradient with respect to the spatial coordinates* $\mathbf{x}$*. We also denote by* $\frac{\partial}{\partial t}$ *the time derivative that is taken holding* $\mathbf{x}$ *fixed. If coordinates other than* $\mathbf{x}$ *are used, the gradient operator will be assigned the appropriate subscript.*

For incompressible flows, we can write the momentum equation also as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{u} \otimes \mathbf{u}) - \mathbf{f}\right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{1.5}$$

For constant density, Equation (1.5) represents the conservative form of the momentum equation. Starting from Equation (1.1) and using the conservation of mass, or starting from Equation (1.5) and using $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, we can obtain

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{f}\right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{1.6}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{1.7}$$

In this case, the momentum balance equation is written in the so-called *convective form*. We use the convective form of the Navier–Stokes equations in the rest of the section. The convective form of the Navier–Stokes equations can further be simplified by using the incompressibility constraint in the viscous part of the stress tensor. In this case, assuming constant viscosity:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{u}, p) = -\boldsymbol{\nabla} p + \mu \Delta \mathbf{u}, \tag{1.8}$$

which leads to

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} - \mathbf{f} \right) + \boldsymbol{\nabla} p - \mu \Delta \mathbf{u} = \mathbf{0}. \tag{1.9}$$

Although this form of the linear-momentum balance equation is often used in the computations reported in the literature, we do not favor this choice because it leads to a non-objective definition of the Cauchy stress. For the importance of objectivity in fluid mechanics simulations, see, e.g., Limache *et al.* (2008) and references therein.

Assuming a fixed Cartesian basis on $\mathbb{R}^{n_{\text{sd}}}$, we let indices i and j take on the values $1, \ldots, n_{\text{sd}}$. We focus on the case of the spatial dimension $n_{\text{sd}} = 3$. We let u_i denote the i^{th} Cartesian component of $\mathbf{u}$, and let x_i denote the i^{th} component of $\mathbf{x}$. We denote differentiation by a comma (e.g., $u_{i,j} = u_{i,x_j} = \partial u_i / \partial x_j$). We will also use the summation convention, in which repeated indices imply summation; e.g., in $\mathbb{R}^3$,

$$u_{i,jj} = u_{i,11} + u_{i,22} + u_{i,33} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}. \tag{1.10}$$

Using index notation, the Navier–Stokes equations of incompressible flows, as given by Equations (1.6) and (1.7), can be rewritten as:

$$\rho \left(u_{i,t} + u_j u_{i,j} - f_i \right) - \sigma_{ij,j} = 0, \tag{1.11}$$

$$u_{i,i} = 0, \tag{1.12}$$

where

$$\sigma_{ij} = -p \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{1.13}$$

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right), \tag{1.14}$$

and δ_{ij} is the Kronecker delta (i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$).

REMARK 1.2 *Note the use of indices on the Cauchy stress σ_{ij} in Equation (1.11). Here, and in what follows, we adopt the convention where the first index, in this case i, indicates the direction in which the stress is acting, while the second index, j, indicates that the stress is acting on a plane normal to the x_j-axis. This is the opposite of the convention used in the papers written by the second and third authors of the book. However, we also note that as a consequence of the local moment equilibrium, the Cauchy stress is symmetric, which implies $\sigma_{ij} = \sigma_{ji}$. This, in turn, implies that the roles of indices in the Cauchy stress may be interchanged without any effect on the governing equations.*

REMARK 1.3 *We also declare the notations we have adopted for the representations related to the gradient and divergence operations on vectors and tensors. The gradient of a vector field is represented as*

$$\nabla \mathbf{a} \equiv \partial \mathbf{a}/\partial \mathbf{x} = \frac{\partial a_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j, \tag{1.15}$$

where $\frac{\partial a_i}{\partial x_j}$ *are the Cartesian components of* $\nabla \mathbf{a}$*. We note that the first index corresponds to the vector component, while the second index corresponds to the spatial derivative. The divergence of a tensor is represented as*

$$\nabla \cdot \mathbf{A} = \frac{\partial A_{ij}}{\partial x_j} \mathbf{e}_i, \tag{1.16}$$

where the contraction occurs on the second index. In the special case $\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$*, that representation becomes*

$$\nabla \cdot (\mathbf{a} \otimes \mathbf{b}) = \frac{\partial (a_i b_j)}{\partial x_j} \mathbf{e}_i. \tag{1.17}$$

The advective term, however, is represented as

$$\mathbf{b} \cdot \nabla \mathbf{a} = b_j \frac{\partial a_i}{\partial x_j} \mathbf{e}_i. \tag{1.18}$$

To complete the statement of the fluid mechanics problem, we need to specify the boundary conditions. In general, on a given part of the spatial boundary, either kinematic or traction boundary conditions are prescribed. Kinematic boundary conditions are also referred to as essential or Dirichlet, while traction boundary conditions are also called natural or Neumann. Because the unknown velocity is a vector, one needs to generalize the boundary conditions to the vector case. The essential and natural boundary conditions for Equation (1.11) are

$$u_i = g_i \quad \text{on} \quad (\Gamma_t)_{gi}, \tag{1.19}$$

$$\sigma_{ij} n_j = h_i \quad \text{on} \quad (\Gamma_t)_{hi}, \tag{1.20}$$

where, for every velocity component i, $(\Gamma_t)_{gi}$ and $(\Gamma_t)_{hi}$ are the complementary subsets of the domain boundary Γ_t, n_i's are components of the unit outward normal vector $\mathbf{n}$, and g_i and h_i are given functions.

REMARK 1.4 *Equations (1.19) and (1.20) pertain to boundary condition specification for individual Cartesian components of the velocity and traction vectors. This is sufficient for many cases of interest. However, more general boundary conditions are possible and will be discussed in the later sections of this book.*

In the case where the fluid velocity vector is specified on the entire boundary of the fluid domain, the pressure is determined up to an arbitrary constant (that is, if p satisfies Equations (1.6) and (1.7), then so does $p + C$, where C is an arbitrary constant over Ω_t) and

$$\int_{\Gamma_t} \mathbf{g} \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_t} g_i n_i \, d\Gamma = 0, \tag{1.21}$$

which is a consequence of the incompressibility constraint. Equation (1.21) is often the source of difficulty for FSI coupling in the case of flows in enclosed domains.

1.1.2 Model Differential Equations

The Navier–Stokes equations of incompressible flows describe a wide range of behavior in viscous incompressible flows. Several simplifications of these equations are considered in the literature (and in practice) to better understand and model the physical phenomena involved. The two important special cases of the Navier–Stokes equations are the Stokes and Euler equations of incompressible flows.

Stokes equations. The Stokes equations are obtained by neglecting the convective terms in Equation (1.6), that is,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} - \mathbf{f}\right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{1.22}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{1.23}$$

The above model is used for describing very slow (e.g., "creeping") flows. Note that the Stokes equations are linear with respect to both velocity and pressure, while the Navier–Stokes equations are not.

Euler equations. The other special case corresponds to inviscid flows described by the Euler equations of incompressible flows, namely

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{f}\right) - \boldsymbol{\nabla} p = \mathbf{0}, \tag{1.24}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{1.25}$$

The Euler equations retain the quadratic nonlinearity of the convective term.

Advection–diffusion equation. The *linear* advection–diffusion equation can be seen as an equation obtained by relaxing the incompressibility constraint (and, as a result, neglecting the pressure, which is also the Lagrange multiplier that enforces the incompressibility constraint) and "freezing" the advective velocity $\mathbf{u}$, namely

$$\rho\left(\frac{\partial \boldsymbol{\phi}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\boldsymbol{\phi} - \mathbf{f}\right) - \mu \Delta \boldsymbol{\phi} = \mathbf{0}, \tag{1.26}$$

where we replaced $\mathbf{u}$ with a vector $\boldsymbol{\phi}$. Dividing Equation (1.26) by the density and realizing that the vector components of $\boldsymbol{\phi}$ are decoupled, we obtain a classical form of the scalar, time-dependent advection–diffusion equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\phi - \nu \Delta \phi - f = 0, \tag{1.27}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The advection–diffusion equation (Equation (1.27)) generally models transport of species, with concentration denoted by ϕ, in the presence of molecular diffusion. In this case ν is replaced with molecular diffusivity κ. Equation (1.27) is often employed as a starting point for the development of numerical formulations in fluid mechanics.

1.1.3 Nondimensional Equations and Numbers

To nondimensionalize the equations of fluid mechanics, we select a characteristic flow speed U and length scale L. We define

$$\mathbf{u} = \mathbf{u}^* U, \tag{1.28}$$

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}^* \frac{1}{L}, \tag{1.29}$$

where $\mathbf{u}^*$ is the nondimensional flow velocity and $\boldsymbol{\nabla}^*$ is the nondimensional gradient operator. Equation (1.29) implies

$$\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{1}{L^2}(\boldsymbol{\nabla}^*) \cdot (\boldsymbol{\nabla}^*) = \frac{1}{L^2}\Delta^*, \tag{1.30}$$

where Δ^* is the nondimensional Laplace operator. Starting from the steady version of the advection–diffusion equation (Equation (1.27)), assuming zero forcing, and introducing the definitions given by Equations (1.28)–(1.30), we obtain

$$\frac{U}{L}(\mathbf{u}^* \cdot \boldsymbol{\nabla}^*)\phi - \frac{\nu}{L^2}(\Delta^*)\phi = 0. \tag{1.31}$$

Rearranging the terms in the above equation gives

$$(\mathbf{u}^* \cdot \boldsymbol{\nabla}^*)\,\phi - \frac{1}{\mathrm{Pe}}\,(\Delta^*)\,\phi = 0, \tag{1.32}$$

where

$$\mathrm{Pe} = \frac{UL}{\nu} \tag{1.33}$$

is the *Peclet number* that represents the significance of advection relative to diffusion. For large Pe advection dominates, while for small Pe diffusion dominates. The case $\mathrm{Pe} = \infty$ corresponds to pure advection. In the case of pure advection ϕ may only be set on the inflow part of the boundary Γ_t^-, which is defined as

$$\Gamma_t^- = \{\mathbf{x} \mid \mathbf{u} \cdot \mathbf{n} \leq 0 \; \forall \mathbf{x} \subset \Gamma_t\}. \tag{1.34}$$

The advection-dominant case gives rise to interior and boundary layers in ϕ, which causes difficulties in the numerical approximation of advection–diffusion equations. Diffusion dominance precludes the formation of thin layers in the solution and presents very little challenge to the numerical approximation. This situation is illustrated in Figure 1.2.

Performing the same analysis for the Navier–Stokes equations, we obtain the analog of the Peclet number, the *Reynolds number* Re, given by

$$\mathrm{Re} = \frac{UL}{\nu} = \frac{\rho UL}{\mu}. \tag{1.35}$$

The limit $\mathrm{Re} \to 0$ yields the Stokes flow, while large Re values give rise to turbulent solutions to the Navier–Stokes equations. Turbulence may be characterized as a continuous spectrum of spatial and temporal scales in the velocity and pressure fields, and also presents a significant challenge for accurate approximation of the solutions to the Navier–Stokes equations in this regime.

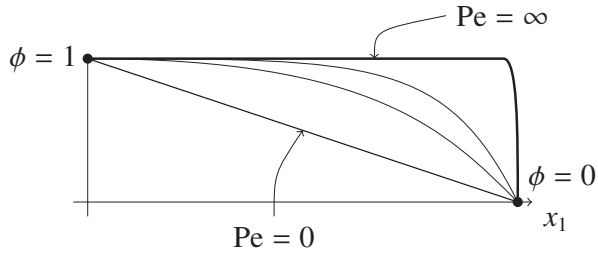

Figure 1.2 Illustration of the solution behavior for the advection–diffusion equation at Pe number ranging from zero to nearly infinity. In the figure, 1D setup is assumed, advective velocity is constant and points from left to right, and ϕ is set to unity on the left side of the interval and zero on the right side. For Pe = 0 the analytical solution is a straight line connecting the prescribed boundary values. As the Pe number increases, the solution forms a thin boundary layer on the right side of the domain. Such boundary layers present a source of difficulty for numerical approximation of the advection–diffusion equation

1.1.4 Some Specific Boundary Conditions

In this section we give a detailed account of the boundary conditions that are most often used in fluid mechanics simulations.

Solid surface. At a solid surface it is convenient to split the velocity vector into its normal and tangential components. For this, along with the normal vector $\mathbf{n}$, we define in 3D two orthonormal tangential vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ (see Figure 1.3). In this new basis the velocity vector components become

$$u_n = \mathbf{u} \cdot \mathbf{n}, \tag{1.36}$$

$$u_{t_1} = \mathbf{u} \cdot \mathbf{t}_1, \tag{1.37}$$

$$u_{t_2} = \mathbf{u} \cdot \mathbf{t}_2. \tag{1.38}$$

Independent of whether the flow is viscous or inviscid, the no-penetration boundary condition becomes

$$u_n = g_n, \tag{1.39}$$

where g_n is the normal velocity of the solid surface. In the case of viscous flows, the remaining velocity components are set to

$$u_{t_1} = g_{t_1}, \tag{1.40}$$

$$u_{t_2} = g_{t_2}, \tag{1.41}$$

where g_t's are the tangential velocities of the solid surface. This results in the so-called no-slip boundary condition. However, in the case of turbulent boundary layers, or in the presence of "rough" surfaces, the tangential traction boundary conditions are adopted in place of no-slip boundary conditions, leading to so-called wall function formulations (see, e.g., Wilcox, 1998).

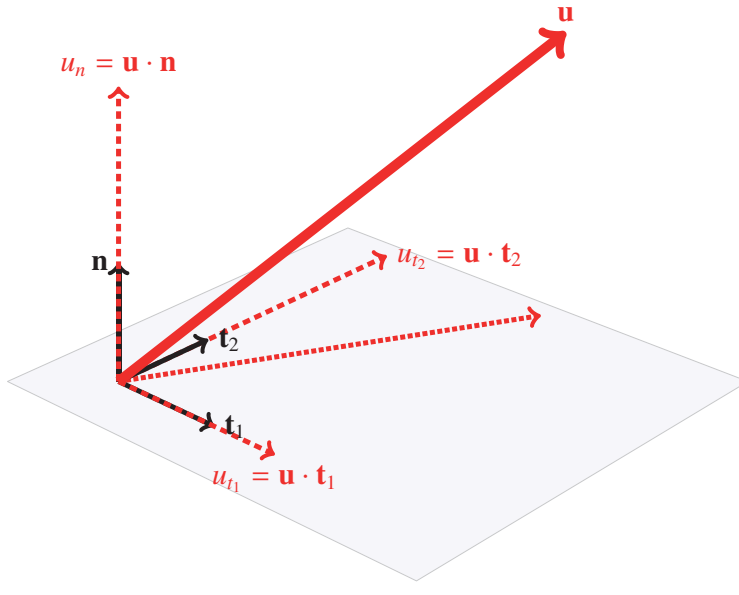

Figure 1.3 The normal vector $\mathbf{n}$ and two orthonormal vectors $\mathbf{t}_1$ and $\mathbf{t}_2$

Free surface. Another case of interest are the boundary conditions at a fluid surface that is free to deform, or the free surface (see, e.g., Tezduyar, 1992; Tezduyar *et al.*, 1993, 1996; Johnson and Tezduyar, 1994; Guler *et al.*, 1999; Akin *et al.*, 2007; Takizawa *et al.*, 2007a,b; Akkerman *et al.*, 2011). In this case, the following traction boundary condition holds:

$$\boldsymbol{\sigma}\mathbf{n} = -p_{\text{atm}}\mathbf{n}, \tag{1.42}$$

where p_{atm} is the atmospheric pressure. The pressure may be scaled such that $p_{\text{atm}} = 0$, in which case a homogeneous traction boundary condition is enforced.

External boundaries. This situation presents the most commonly encountered setup in computational fluid mechanics. One is interested in computing the flow over an object placed in a free stream. A truncated problem domain is created, which encloses the object and contains external boundaries. Typically, the external boundaries contain the inflow, outflow, and lateral (or side) boundaries (see Figure 1.4). No-slip condition is applied on the object. The external boundaries are placed sufficiently far from the object to approximate the free-stream conditions. The free-stream conditions are

$$\mathbf{u} = \mathbf{u}_\infty, \tag{1.43}$$

or

$$\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\sigma}_\infty\mathbf{n}, \tag{1.44}$$

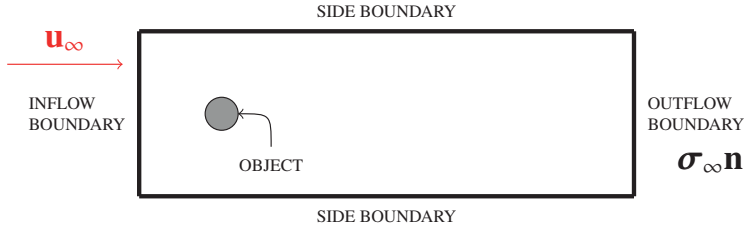

Figure 1.4 External boundaries (inflow, outflow, and side)

where the subscript ∞ is used to indicate quantities far from the object. Using index notation, the free-stream boundary conditions may be expressed as

$$u_i = (u_i)_\infty \quad \text{on} \quad (\Gamma_t)_{gi}, \tag{1.45}$$

$$\sigma_{ij} n_j = \left(\sigma_{ij}\right)_\infty n_j \quad \text{on} \quad (\Gamma_t)_{hi}. \tag{1.46}$$

In most cases,

$$\mathbf{u}_\infty = \begin{Bmatrix} U \\ 0 \\ 0 \end{Bmatrix}, \tag{1.47}$$

which leads to

$$\boldsymbol{\sigma}_\infty = -p_\infty \mathbf{I}, \tag{1.48}$$

and, consequently, to

$$\boldsymbol{\sigma}_\infty \mathbf{n} = -p_\infty \mathbf{n}. \tag{1.49}$$

It is often possible to scale the pressure so that $p_\infty = 0$. With this, the recommended boundary conditions at the external boundaries become:

- At the inflow boundary the entire velocity vector is prescribed:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} U \\ 0 \\ 0 \end{Bmatrix}. \tag{1.50}$$

- At the outflow boundary, free-stream traction boundary conditions are prescribed:

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}. \tag{1.51}$$

  Assuming the normal vector is $\mathbf{n} = (1, 0, 0)^T$, we obtain, in component form,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{1.52}$$

Outflow boundary conditions given by Equation (1.51) are referred to as traction-free or "do-nothing" boundary conditions (see, e.g., Gresho and Sani, 2000). The second phrase originates from the fact that in the finite element method zero-stress boundary conditions are satisfied naturally and require no additional computer implementation.

- At the lateral boundaries zero normal velocity and zero tangential traction are prescribed, namely

$$\mathbf{u} \cdot \mathbf{n} = 0, \tag{1.53}$$

$$\mathbf{t}_1 \cdot \boldsymbol{\sigma}\mathbf{n} = 0, \tag{1.54}$$

$$\mathbf{t}_2 \cdot \boldsymbol{\sigma}\mathbf{n} = 0. \tag{1.55}$$

For the top and bottom lateral boundaries, where the normal vector is $\mathbf{n} = (0, \pm 1, 0)^T$, and the tangential vectors are $\mathbf{t}_1 = (1, 0, 0)^T$ and $\mathbf{t}_2 = (0, 0, 1)^T$, we obtain

$$\begin{Bmatrix} \sigma_{12} \\ u_2 \\ \sigma_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{1.56}$$

For the span-wise lateral boundaries, where the normal vector is $\mathbf{n} = (0, 0, \pm 1)^T$, and the tangential vectors are $\mathbf{t}_1 = (1, 0, 0)^T$ and $\mathbf{t}_2 = (0, 1, 0)^T$, we obtain

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{1.57}$$

We will consider other, more specialized, boundary conditions in the chapters on specific applications of FSI.

1.1.5 Weak Form of the Navier–Stokes Equations

We denote by $\mathcal{S}_u$ and $\mathcal{S}_p$ the sets of infinite-dimensional trial functions for the velocity and pressure. The function sets $\mathcal{S}_u$ and $\mathcal{S}_p$ are defined as

$$\mathcal{S}_u = \left\{ \mathbf{u} \;\middle|\; \mathbf{u}(\cdot, t) \in (H^1(\Omega_t))^{n_{\text{sd}}},\ u_i = g_i \ \text{ on } \ (\Gamma_t)_{gi} \right\} \tag{1.58}$$

and

$$\mathcal{S}_p = \left\{ p \;\middle|\; p(\cdot) \in L^2(\Omega_t),\ \int_{\Omega_t} p \, \mathrm{d}\Omega = 0 \ \text{ if } \ \Gamma_t = (\Gamma_t)_{\text{g}} \right\}. \tag{1.59}$$

Here $L^2(\Omega_t)$ denotes the space of scalar-valued functions that are square-integrable on Ω_t, and $(H^1(\Omega_t))^{n_{\text{sd}}}$ denotes the space of vector-valued functions with square-integrable derivatives on Ω_t. The functions in $\mathcal{S}_u$ satisfy the essential boundary conditions of the fluid mechanics problem. In the case when the essential boundary conditions are set on all of Γ_t, we require that the average of the pressure field over Ω_t is zero, which is built into the definition of $\mathcal{S}_p$.

In conjunction with $\mathcal{S}_u$ and $\mathcal{S}_p$, we define the sets of test functions (also called "weighting functions" in this book) for the linear-momentum and continuity equations, denoted by $\mathcal{V}_u$ and $\mathcal{V}_p$ as

$$\mathcal{V}_u = \left\{ \mathbf{w} \;\middle|\; \mathbf{w}(\cdot) \in (H^1(\Omega_t))^{n_{\text{sd}}},\ w_i = 0 \ \text{ on } \ (\Gamma_t)_{\text{g}i} \right\} \tag{1.60}$$

and

$$\mathcal{V}_p = \mathcal{S}_p. \tag{1.61}$$

Note that the function sets $\mathcal{S}_u$ and $\mathcal{V}_u$ only differ in the definition of boundary conditions, that is, the test functions for the linear-momentum balance equations vanish on the parts of the boundary where the fluid velocity is prescribed. The sets of pressure trial functions and the continuity equation test functions are coincident.

To derive the weak form of the fluid mechanics equations, following the standard approach, we multiply Equations (1.6) and (1.7) by the linear-momentum and continuity equation test functions respectively, integrate over Ω_t, and add the equations to obtain

$$\int_{\Omega_t} \mathbf{w} \cdot \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} - \mathbf{f} \right) \, \mathrm{d}\Omega - \int_{\Omega_t} \mathbf{w} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{u}, p)) \, \mathrm{d}\Omega + \int_{\Omega_t} q \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}\Omega = 0. \tag{1.62}$$

We integrate by parts the Cauchy stress terms in Equation (1.62) and apply the homogeneous form of the essential boundary conditions on $\mathbf{w}$ to get

$$- \int_{\Omega_t} \mathbf{w} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{u}, p)) \, \mathrm{d}\Omega = \int_{\Omega_t} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\sigma}(\mathbf{u}, p) \, \mathrm{d}\Omega - \int_{(\Gamma_t)_{\text{h}}} \mathbf{w} \cdot \boldsymbol{\sigma}(\mathbf{u}, p) \, \mathbf{n} \, \mathrm{d}\Gamma, \tag{1.63}$$

where $(\Gamma_t)_{\text{h}}$ is the abstract representation of the natural boundary of the fluid mechanics domain. We replace the traction vector $\boldsymbol{\sigma}(\mathbf{u}, p)\,\mathbf{n}$ with its prescribed value $\mathbf{h}$ on $(\Gamma_t)_{\text{h}}$ in the last term on the right-hand-side of Equation (1.63) to obtain

$$- \int_{\Omega_t} \mathbf{w} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{u}, p)) \, \mathrm{d}\Omega = \int_{\Omega_t} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\sigma}(\mathbf{u}, p) \, \mathrm{d}\Omega - \int_{(\Gamma_t)_{\text{h}}} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}\Gamma. \tag{1.64}$$

Combining Equations (1.62) and (1.64) results in the weak form of the Navier–Stokes equations: find $\mathbf{u} \in \mathcal{S}_u$ and $p \in \mathcal{S}_p$, such that $\forall\ \mathbf{w} \in \mathcal{V}_u$ and $q \in \mathcal{V}_p$:

$$\begin{aligned} \int_{\Omega_t} \mathbf{w} \cdot \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} - \mathbf{f} \right) \, \mathrm{d}\Omega + \int_{\Omega_t} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\sigma}(\mathbf{u}, p) \, \mathrm{d}\Omega \\ - \int_{(\Gamma_t)_{\text{h}}} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}\Gamma + \int_{\Omega_t} q \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}\Omega = 0. \end{aligned} \tag{1.65}$$

The weak formulation given by Equation (1.65) is the point of departure for the finite element formulations of the fluid mechanics problem. The details of the finite element method will be presented in later chapters. Note that while the velocity boundary conditions for the weak formulation of the fluid mechanics equations are built into the corresponding function sets, the traction boundary conditions are imposed weakly as a consequence of the integration-by-parts procedure described by Equations (1.63) and (1.64).

1.2 Governing Equations of Structural Mechanics

In this section we present the governing equations of structural mechanics. The equations are derived on the basis of 3D continuum modeling. We conclude this section with a presentation of the governing equations for shell, membrane, and cable structures.

1.2.1 Kinematics

Let $\Omega_0 \in \mathbb{R}^{n_{\text{sd}}}$ be the material domain of a structure in the reference configuration, and let Γ_0 be its boundary. Let $\Omega_t \in \mathbb{R}^{n_{\text{sd}}}$, $t \in (0, T)$, be the material domain of a structure in the current configuration, and let Γ_t be its boundary. For the upcoming developments we assume that the reference configuration coincides with the initial configuration or the configuration of the structure taken at $t = 0$. Let $\mathbf{X}$ be the coordinates of the initial or reference configuration, and let $\mathbf{y}$ be the displacement with respect to the initial configuration. We think of $\mathbf{y} = \mathbf{y}(\mathbf{X}, t)$ as a time-varying vector field over Ω_0 and define a mapping

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{y}(\mathbf{X}, t), \tag{1.66}$$

which maps the coordinates of material points in the reference configuration to their counterparts in the current configuration. We also denote by $\mathbf{x}$ the coordinates of the current configuration. Because this "abuse of notation" is standard practice in continuum mechanics, we adopt it here as well. The setup is illustrated in Figure 1.5.

The velocity $\mathbf{u}$ and acceleration $\mathbf{a}$ of the structure are obtained by differentiating the displacement $\mathbf{y}$ with respect to time holding the material coordinate $\mathbf{X}$ fixed, namely

$$\mathbf{u} = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} \tag{1.67}$$

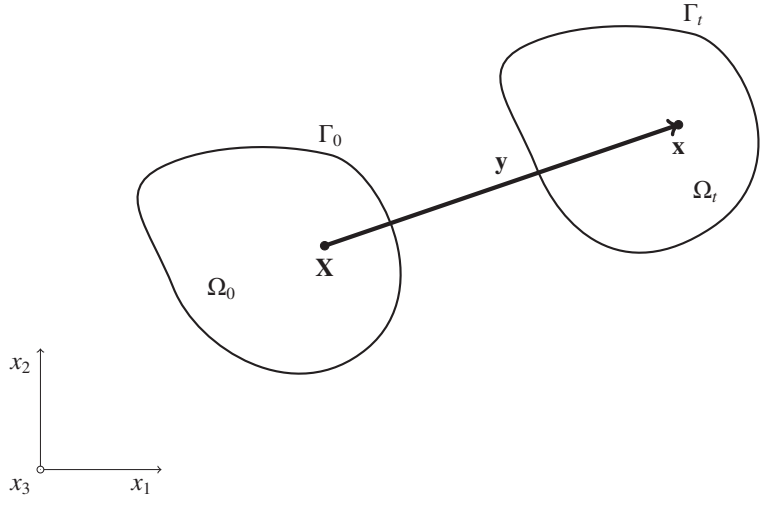

Figure 1.5 Reference and current configurations

and

$$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}t^2}. \tag{1.68}$$

REMARK 1.5 *Here, and everywhere in the book,* $\frac{\mathrm{d}}{\mathrm{d}t}$ *will denote the total time derivative, or the time derivative taken holding the material coordinate* $\mathbf{X}$ *fixed.*

The deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{I} + \frac{\partial \mathbf{y}}{\partial \mathbf{X}}, \tag{1.69}$$

which we use to define the Cauchy–Green deformation tensor $\mathbf{C}$ as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \tag{1.70}$$

and the Green–Lagrange strain tensor $\mathbf{E}$ as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right). \tag{1.71}$$

The determinant of the deformation gradient J is given by

$$J = \det \mathbf{F}. \tag{1.72}$$

We now introduce the index notation for the structural mechanics. Due to the presence of the reference and current configurations, the quantities referring to the reference configuration are typically subscripted with upper-case indices (e.g., I, J, K), and those referring to the current configuration with lower-case indices (e.g., i, j, k). However, despite the use of different index types, we assume that the vector and tensor components in the reference and current configurations are referred to a fixed Cartesian basis. The summation convention applies to the upper- and lower-case indices *separately*, and all indices take on the values $1, \ldots, n_{\mathrm{sd}}$.

The Cartesian components of the deformation gradient become

$$F_{iI} = \frac{\partial x_i}{\partial X_I} = \delta_{iI} + \frac{\partial y_i}{\partial X_I}, \tag{1.73}$$

which means that one "leg" of the deformation gradient tensor is in the reference and another is in the current configuration. The components of the Cauchy–Green deformation tensor and the Green–Lagrange strain tensor are given by

$$C_{IJ} = F_{iI}F_{iJ} \tag{1.74}$$

and

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}), \tag{1.75}$$

respectively. Note that the tensors $\mathbf{C}$ and $\mathbf{E}$ are completely defined in the reference or undeformed configuration.

1.2.2 Principle of Virtual Work and Variational Formulation of Structural Mechanics

The starting point for the structural mechanics formulations is the *principle of virtual work* (see, e.g., Belytschko *et al.*, 2000):

$$\delta W = \delta W_{\text{int}} + \delta W_{\text{ext}} = 0, \tag{1.76}$$

where W, W_{int}, and W_{ext} are the total, internal, and external work, respectively, and δ denotes their variation with respect to the virtual displacement $\mathbf{w}$. Given the structural displacement $\mathbf{y}$, δW is computed by taking the directional derivative of W as

$$\delta W = \frac{\mathrm{d}}{\mathrm{d}\epsilon} W(\mathbf{y} + \epsilon \mathbf{w})\bigg|_{\epsilon=0}. \tag{1.77}$$

Here δW_{ext} includes the virtual work done by the inertial and body forces and surface tractions, and is given by

$$\delta W_{\text{ext}} = \int_{\Omega_t} \mathbf{w} \cdot \rho\,(\mathbf{f} - \mathbf{a})\ \mathrm{d}\Omega + \int_{(\Gamma_t)_{\text{h}}} \mathbf{w} \cdot \mathbf{h}\ \mathrm{d}\Gamma, \tag{1.78}$$

where ρ is the mass density of the structure in the current configuration, $\mathbf{f}$ is the body force per unit mass, and $\mathbf{h}$ is the external traction vector applied on the subset $(\Gamma_t)_{\text{h}}$ of the total boundary Γ_t.

The virtual work done by the internal stresses, δW_{int}, may be computed as

$$\delta W_{\text{int}} = -\int_{\Omega_0} \delta\mathbf{E} : \mathbf{S}\ \mathrm{d}\Omega. \tag{1.79}$$

Here $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor, which is symmetric and work-conjugate to $\mathbf{E}$. Although $\mathbf{S}$ is not possible to measure experimentally, it plays a prominent role in constitutive modeling of materials.

In Equation (1.79), $\delta\mathbf{E}$ is the variation of the Green–Lagrange strain tensor, which is also referred to as the virtual strain. Putting Equations (1.76)–(1.79) together, and recognizing that $\mathbf{w}$ is arbitrary, we arrive at the variational formulation of the structural mechanics problem: find the structural displacement $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\ \mathbf{w} \in \mathcal{V}_y$:

$$\int_{\Omega_t} \mathbf{w} \cdot \rho\mathbf{a}\ \mathrm{d}\Omega + \int_{\Omega_0} \delta\mathbf{E} : \mathbf{S}\ \mathrm{d}\Omega - \int_{\Omega_t} \mathbf{w} \cdot \rho\mathbf{f}\ \mathrm{d}\Omega - \int_{(\Gamma_t)_{\text{h}}} \mathbf{w} \cdot \mathbf{h}\ \mathrm{d}\Gamma = 0. \tag{1.80}$$

Here $\mathcal{S}_y$ and $\mathcal{V}_y$ are the sets of trial and test functions for the structural mechanics problem, defined as

$$\mathcal{S}_y = \left\{ \mathbf{y} \;\middle|\; \mathbf{y}(\cdot, t) \in (H^1(\Omega_t))^{n_{\text{sd}}},\ y_i = g_i \ \text{ on } \ (\Gamma_t)_{\text{g}i} \right\}, \tag{1.81}$$

and

$$\mathcal{V}_y = \left\{ \mathbf{w} \;\middle|\; \mathbf{w}(\cdot) \in (H^1(\Omega_t))^{n_{\text{sd}}},\ w_i = 0 \ \text{ on } \ (\Gamma_t)_{\text{g}i} \right\}. \tag{1.82}$$

Here, for each i, $(\Gamma_t)_{\text{g}i}$ and $(\Gamma_t)_{\text{h}i}$ are the complementary subsets of the domain boundary Γ_t, and g_i is a given function. Note that the essential boundary conditions are built into the definition of the function sets of the structural mechanics problem. The variational formulation given by Equation (1.80) is the point of departure for the structural mechanics finite element formulations.

1.2.3 Conservation of Mass

In Equation (1.80) the structural density ρ in the current configuration is not known a priori. To derive the dependence of the structural density on the structural displacement, we first define the structural mass m as

$$m = \int_{\Omega_t} \rho \, \mathrm{d}\Omega_t. \tag{1.83}$$

We assume that the structural mass is conserved at all times, which may be expressed as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 0. \tag{1.84}$$

Introducing Equation (1.83) into Equation (1.84), changing variables to the reference configuration, and taking the time derivative inside the integral, we obtain

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho \, \mathrm{d}\Omega_t = \int_{\Omega_0} \frac{\mathrm{d}\rho J}{\mathrm{d}t} \, \mathrm{d}\Omega_0 = 0. \tag{1.85}$$

Because Ω_0 is arbitrary, we can localize the results to any material point in the structure as

$$\frac{\mathrm{d}\rho J}{\mathrm{d}t} = 0. \tag{1.86}$$

This, in turn, means that the product ρJ is only a function of the material point, namely $\rho J = \rho J(\mathbf{X})$. At $t = 0$ the structure is undeformed, meaning $J = 1$. Defining $\rho_0 = \rho_0(\mathbf{X})$ to be the structural mass density in the undeformed configuration, we obtain the following point-wise statement of the conservation of mass:

$$\rho_0 = \rho J. \tag{1.87}$$

Because ρ_0 is considered to be known, given the structural displacement field, Equation (1.87) may be used to obtain the density at a material point in the current configuration using this simple algebraic expression. The relationship given by Equation (1.87) is known as the Lagrangian description of mass conservation.

1.2.4 Structural Mechanics Formulation in the Current Configuration

In the variational formulation given by Equation (1.80), the stress terms are written with respect to the reference configuration, while the remaining terms are expressed in the current configuration. In order to have a formulation that is written purely in the current configuration, we proceed as follows. We first make explicit the dependence of the virtual strain $\delta\mathbf{E}$ on the virtual displacement $\mathbf{w}$. Starting with the definition of $\mathbf{E}$ in Equation (1.71) and taking the variation as in Equation (1.77) gives

$$\delta\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\boldsymbol{\nabla}_X\mathbf{w} + \boldsymbol{\nabla}_X\mathbf{w}^T\mathbf{F}\right), \tag{1.88}$$

where $\boldsymbol{\nabla}_X$ denotes the gradient taken with respect to the spatial coordinates of the reference configuration. Due to the symmetry of $\mathbf{S}$, the scalar product $\delta\mathbf{E} : \mathbf{S}$ simplifies to

$$\delta\mathbf{E} : \mathbf{S} = \boldsymbol{\nabla}_X\mathbf{w} : \mathbf{P}, \tag{1.89}$$

where

$$\mathbf{P} = \mathbf{FS} \tag{1.90}$$

is the first Piola–Kirchhoff stress tensor, which is nonsymmetric. With these definitions the variational formulation of the structural mechanics problem becomes: find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\ \mathbf{w} \in \mathcal{V}_y$:

$$\int_{\Omega_t} \mathbf{w} \cdot \rho \mathbf{a} \mathrm{d}\Omega + \int_{\Omega_0} \boldsymbol{\nabla}_X \mathbf{w} : \mathbf{P}\ \mathrm{d}\Omega - \int_{\Omega_t} \mathbf{w} \cdot \rho \mathbf{f}\ \mathrm{d}\Omega - \int_{(\Gamma_t)_\mathrm{h}} \mathbf{w} \cdot \mathbf{h}\ \mathrm{d}\Gamma = 0. \tag{1.91}$$

We now change variables in the stress terms in Equation (1.91) to obtain

$$\int_{\Omega_0} \boldsymbol{\nabla}_X \mathbf{w} : \mathbf{P}\ \mathrm{d}\Omega = \int_{\Omega_0} \boldsymbol{\nabla}_X \mathbf{w} : (\mathbf{FS})\ \mathrm{d}\Omega = \int_{\Omega_0} \frac{\partial w_i}{\partial X_J} \frac{\partial x_i}{\partial X_I} S_{IJ}\ \mathrm{d}\Omega \tag{1.92}$$

$$= \int_{\Omega_t} \frac{\partial w_i}{\partial x_j} \left(\frac{\partial x_i}{\partial X_I} S_{IJ} \frac{\partial x_j}{\partial X_J} J^{-1} \right) \mathrm{d}\Omega \tag{1.93}$$

$$= \int_{\Omega_t} \boldsymbol{\nabla} \mathbf{w} : \left(J^{-1} \mathbf{FSF}^T \right) \mathrm{d}\Omega, \tag{1.94}$$

where in the last term we recognize the Cauchy stress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = J^{-1} \mathbf{FSF}^T. \tag{1.95}$$

Using index notation, the component form of Equation (1.95) may be written as

$$\sigma_{ij} = J^{-1} F_{iI} S_{IJ} F_{jJ}. \tag{1.96}$$

The Cauchy stress, unlike the second Piola–Kirchhoff stress, may be measured experimentally. Due to the the symmetry of the Cauchy stress tensor, we can write

$$\int_{\Omega_t} \boldsymbol{\nabla} \mathbf{w} : \boldsymbol{\sigma}\ \mathrm{d}\Omega = \int_{\Omega_t} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\sigma}\ \mathrm{d}\Omega, \tag{1.97}$$

where, as before,

$$\boldsymbol{\varepsilon}(\mathbf{w}) = \frac{1}{2} \left(\boldsymbol{\nabla} \mathbf{w} + \boldsymbol{\nabla} \mathbf{w}^T \right). \tag{1.98}$$

Combining Equations (1.91), (1.92), (1.95), and (1.97), we obtain the structural mechanics variational formulation in the current configuration: find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\ \mathbf{w} \in \mathcal{V}_y$:

$$\int_{\Omega_t} \mathbf{w} \cdot \rho \mathbf{a}\ \mathrm{d}\Omega + \int_{\Omega_t} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\sigma}\ \mathrm{d}\Omega - \int_{\Omega_t} \mathbf{w} \cdot \rho \mathbf{f}\ \mathrm{d}\Omega - \int_{(\Gamma_t)_\mathrm{h}} \mathbf{w} \cdot \mathbf{h}\ \mathrm{d}\Gamma = 0. \tag{1.99}$$

An equivalent variational formulation of the structural mechanics problem may be developed in the reference configuration. We present that in the next section.

To infer the strong formulation of the structural problem from Equation (1.99), we integrate by parts the stress terms, apply the homogeneous form of the essential boundary conditions on $\mathbf{w}$, and group the interior and boundary integral terms to obtain

$$\int_{\Omega_t} \mathbf{w} \cdot \left(\rho \left(\mathbf{a} - \mathbf{f} \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \right)\ \mathrm{d}\Omega + \int_{(\Gamma_t)_\mathrm{h}} \mathbf{w} \cdot \left(\boldsymbol{\sigma} \mathbf{n} - \mathbf{h} \right)\ \mathrm{d}\Gamma = 0. \tag{1.100}$$

Because Equation (1.100) holds for all admissible $\mathbf{w}$, we conclude that

$$\rho\,(\mathbf{a}-\mathbf{f})-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=\mathbf{0} \tag{1.101}$$

at every point inside Ω_t and

$$\boldsymbol{\sigma}\mathbf{n}-\mathbf{h}=\mathbf{0} \tag{1.102}$$

at every point on the traction boundary $(\Gamma_t)_{\mathrm{h}}$. Equations (1.101) and (1.102) constitute the point-wise balance of linear momentum and traction boundary condition, respectively, in the current configuration.

Using index notation, the strong form of the structural mechanics equations may be written as

$$\rho\,(a_i-f_i)-\sigma_{ij,j}=0 \ \text{ in } \ \Omega_t, \tag{1.103}$$

$$y_i=g_i \ \text{ on } \ (\Gamma_t)_{\mathrm{g}i}, \tag{1.104}$$

$$\sigma_{ij}n_j=h_i \ \text{ on } \ (\Gamma_t)_{\mathrm{h}i}. \tag{1.105}$$

Here, n_i's are components of the unit outward normal vector $\mathbf{n}$ in the current configuration, and h_i's are given functions. Note that the boundary conditions for the displacement lead to the boundary conditions for the velocity and acceleration, namely

$$u_i=\frac{\mathrm{d}g_i}{\mathrm{d}t} \ \text{ on } \ (\Gamma_t)_{\mathrm{g}i}, \tag{1.106}$$

and

$$a_i=\frac{\mathrm{d}^2g_i}{\mathrm{d}t^2} \ \text{ on } \ (\Gamma_t)_{\mathrm{g}i}. \tag{1.107}$$

1.2.5 Structural Mechanics Formulation in the Reference Configuration

To infer the weak form of the structural mechanics equations in the reference configuration Ω_0, we again start with the variational formulation given by Equation (1.80), and change variables in the inertial and body force terms as

$$\int_{\Omega_t}\mathbf{w}\cdot\rho\,(\mathbf{a}-\mathbf{f})\ \mathrm{d}\Omega=\int_{\Omega_0}\mathbf{w}\cdot\rho_0\,(\mathbf{a}-\mathbf{f})\ \mathrm{d}\Omega. \tag{1.108}$$

Mass conservation given by Equation (1.87) is also employed to arrive at the above result. Combining Equations (1.80), (1.88), (1.89), and (1.108), we obtain the following variational formulation of the structural mechanics problem posed in Ω_0: find $\mathbf{y}\in\mathcal{S}_y$, such that $\forall\ \mathbf{w}\in\mathcal{V}_y$:

$$\int_{\Omega_0}\mathbf{w}\cdot\rho_0\mathbf{a}\ \mathrm{d}\Omega+\int_{\Omega_0}\boldsymbol{\nabla}_X\mathbf{w}:\mathbf{P}\ \mathrm{d}\Omega-\int_{\Omega_0}\mathbf{w}\cdot\rho_0\mathbf{f}\ \mathrm{d}\Omega-\int_{(\Gamma_0)_{\mathrm{h}}}\mathbf{w}\cdot\hat{\mathbf{h}}\ \mathrm{d}\Gamma=0, \tag{1.109}$$

where $\hat{\mathbf{h}}$, is the traction vector acting in the reference configuration. Integrating by parts the stress terms in Equation (1.109), applying the homogeneous form of the essential boundary conditions on $\mathbf{w}$, and grouping the interior and boundary integrals, we obtain

$$\int_{\Omega_0}\mathbf{w}\cdot\left(\rho_0\,(\mathbf{a}-\mathbf{f})-\boldsymbol{\nabla}_X\cdot\mathbf{P}\right)\ \mathrm{d}\Omega+\int_{(\Gamma_0)_{\mathrm{h}}}\mathbf{w}\cdot\left(\mathbf{P}\hat{\mathbf{n}}-\hat{\mathbf{h}}\right)\ \mathrm{d}\Gamma=0, \tag{1.110}$$

which holds for all admissible $\mathbf{w}$, and where $\hat{\mathbf{n}}$ is the unit normal in the reference configuration. From Equation (1.110) we infer the point-wise balance of linear momentum in Ω_0:

$$\rho_0 (\mathbf{a} - \mathbf{f}) - \boldsymbol{\nabla}_X \cdot \mathbf{P} = \mathbf{0}, \tag{1.111}$$

and the traction boundary condition on $(\Gamma_0)_{\mathrm{h}}$:

$$\mathbf{P}\hat{\mathbf{n}} - \hat{\mathbf{h}} = \mathbf{0}. \tag{1.112}$$

Using index notation, the strong form of the structural mechanics boundary value problem may be written as

$$\rho_0 (a_i - f_i) - P_{iI,I} = 0 \ \text{ in } \ \Omega_0, \tag{1.113}$$

$$y_i = g_i \ \text{ on } \ (\Gamma_0)_{gi}, \tag{1.114}$$

$$P_{iI}\hat{n}_I = \hat{h}_i \ \text{ on } \ (\Gamma_0)_{\mathrm{h}i}. \tag{1.115}$$

REMARK 1.6 *The variational statement given by Equation (1.99) is sometimes called the updated Lagrangian formulation of structural mechanics, while Equation (1.109) corresponds to the so-called total Lagrangian formulation of structural mechanics (see, e.g., Belytschko et al., 2000). Both formulations are equivalent in that they produce identical solutions for the same input. The choice of the formulation is often dictated by constitutive modeling, boundary conditions, ease of computer implementation, and other factors.*

1.2.6 Additional Boundary Conditions of Practical Interest

In this section we briefly describe two cases of structural mechanics boundary conditions that are often employed in practice. These are the follower pressure load and elastic-foundation boundary conditions.

Follower pressure load. This case presents a situation where the structural deformation is driven by the external applied pressure load on $(\Gamma_t)_{\mathrm{h}}$. In this case, the stress vector $\mathbf{h}$ becomes

$$\mathbf{h} = -p\mathbf{n}, \tag{1.116}$$

where p is the magnitude of the applied pressure. Because the pressure is applied to the part of the domain boundary that is in motion, this boundary condition leads to a nonlinearity that needs to be handled in the computation. Changing variables to the reference configuration and using Nanson's formula (see. e.g., Holzapfel, 2000)

$$\mathbf{n}\, \mathrm{d}\Gamma_t = J\mathbf{F}^{-T}\hat{\mathbf{n}}\, \mathrm{d}\Gamma_0, \tag{1.117}$$

where $\mathrm{d}\Gamma_t$ and $\mathrm{d}\Gamma_0$ are the differential surface area elements in the current and reference configurations, respectively, we obtain

$$\int_{(\Gamma_t)_{\mathrm{h}}} \mathbf{w} \cdot \mathbf{h}\, \mathrm{d}\Gamma_t = -\int_{(\Gamma_t)_{\mathrm{h}}} \mathbf{w} \cdot p\mathbf{n}\, \mathrm{d}\Gamma_t = -\int_{(\Gamma_0)_{\mathrm{h}}} \mathbf{w} \cdot pJ\mathbf{F}^{-T}\hat{\mathbf{n}}\, \mathrm{d}\Gamma_0. \tag{1.118}$$

From Equation (1.118) we conclude that the resultant reference configuration traction vector $\hat{\mathbf{h}}$ for the follower pressure load is

$$\hat{\mathbf{h}} = -pJ\mathbf{F}^{-T}\hat{\mathbf{n}}, \tag{1.119}$$

and the corresponding boundary condition becomes

$$\mathbf{P}\hat{\mathbf{n}} + pJ\mathbf{F}^{-T}\hat{\mathbf{n}} = \mathbf{0}, \tag{1.120}$$

or, using index notation,

$$P_{iI}\hat{n}_I + pJF^{-1}_{Ii}\hat{n}_I = 0. \tag{1.121}$$

Elastic foundation. In this case, the structure is assumed to be supported by an elastic-foundation, which is modeled using the spring analogy. The traction vector is made proportional to the structural displacement as

$$\mathbf{h} = -k\mathbf{y}, \tag{1.122}$$

where $k > 0$ is the spring constant. In this case, the traction term in the variational equations of structural mechanics is replaced with

$$\int_{(\Gamma_t)_\mathrm{h}} \mathbf{w}\cdot\mathbf{h}\,\mathrm{d}\Gamma = -\int_{(\Gamma_t)_\mathrm{h}} \mathbf{w}\cdot k\mathbf{y}\,\mathrm{d}\Gamma. \tag{1.123}$$

Note that the limit $k \to \infty$ represents a rigid foundation and the limit $k \to 0$ gives a zero-traction boundary condition. In some cases it is desirable to only employ the spring analogy in the direction normal to the boundary. In this case, the boundary condition (1.122) is replaced by

$$\mathbf{h} = -k\,(\mathbf{y}\cdot\mathbf{n})\;\mathbf{n}, \tag{1.124}$$

which yields the following traction term:

$$\int_{(\Gamma_t)_\mathrm{h}} \mathbf{w}\cdot\mathbf{h}\,\mathrm{d}\Gamma = -\int_{(\Gamma_t)_\mathrm{h}} (\mathbf{w}\cdot\mathbf{n})\,k\,(\mathbf{y}\cdot\mathbf{n})\;\mathrm{d}\Gamma. \tag{1.125}$$

1.2.7 Some Constitutive Models

To present the constitutive models in structural mechanics, we restrict the presentation to the class of hyperelastic materials. More complicated cases, such as inelastic materials, may be found in Simo and Hughes (1998). The theory of hyperelasticity assumes the existence of a stored elastic-energy density per unit volume of the undeformed configuration, φ, expressed as a function of the strain as

$$\varphi = \varphi(\mathbf{E}). \tag{1.126}$$

The second Piola–Kirchhoff stress $\mathbf{S}$ is obtained by differentiating φ with respect to $\mathbf{E}$ as

$$\mathbf{S}(\mathbf{E}) = \frac{\partial\varphi(\mathbf{E})}{\partial\mathbf{E}}. \tag{1.127}$$

Given the second Piola–Kirchhoff stress, the Cauchy stress is computed according to Equation (1.95) or (1.96). The tensor of elastic moduli, which plays an important role in the linearization of the structural mechanics equations, is defined as the second derivative of φ with respect to $\mathbf{E}$, namely,

$$\mathbb{C}(\mathbf{E}) = \frac{\partial^2 \varphi(\mathbf{E})}{\partial \mathbf{E} \partial \mathbf{E}}. \tag{1.128}$$

Different forms of $\varphi(\mathbf{E})$ in Equation (1.126) lead to different constitutive relationships between stress and strain. Here we present a few important cases.

St. Venant–Kirchhoff model. This model is characterized by $\varphi(\mathbf{E})$ defined as

$$\varphi(\mathbf{E}) = \frac{1}{2}\mathbf{E} : \mathbb{C}\mathbf{E}, \tag{1.129}$$

where $\mathbb{C}$ is a fourth-rank tensor of elastic moduli that is *independent of the state of deformation*. Using Equation (1.127), it is easy to see that this choice of the elastic-energy density leads to a linear relationship between the Green–Lagrange strain and the second Piola–Kirchhoff stress:

$$\mathbf{S}(\mathbf{E}) = \mathbb{C}\mathbf{E}. \tag{1.130}$$

Because $\mathbf{E}$ respects the principle of objectivity (see e.g., Holzapfel, 2000), this constitutive relationship is applicable to modeling structures in the regime of large displacements in that all rigid-body motions (i.e., translations and large rotations) produce no strains. However, the model is only physically valid in the small-strain regime as many materials deviate from a linear stress-strain relationship, even for very modest strain levels.

The constitutive tensor $\mathbb{C}$ may be designed to represent various types of material anisotropy (e.g., composite materials). The simplest case is the isotropic material for which the constitutive tensor, in the component form, becomes

$$\mathbb{C}_{IJKL} = \left(\kappa - \frac{2}{3}\mu\right)\delta_{IJ}\delta_{KL} + \mu\left(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}\right), \tag{1.131}$$

where κ and μ are the material bulk and shear moduli, respectively. These are related to the material Young's modulus, E, and Poisson's ratio, ν, as

$$\kappa = \lambda + \frac{2}{3}\mu, \tag{1.132}$$

$$\mu = \frac{E}{2(1+\nu)}, \tag{1.133}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{1.134}$$

where μ and λ are the well-known Lamé constants of the linear-elasticity model.

Neo-Hookean model with dilatational penalty. It is well known that the St. Venant–Kirchhoff model is unstable in the regime of strong compression (see Holzapfel, 2000).

A constitutive model that addresses this drawback was proposed in Simo and Hughes (1998). The model is isotropic, and the elastic-energy density takes the form

$$\varphi(\mathbf{C}, J) = \frac{1}{2}\mu\left(J^{-2/3}\mathrm{tr}\mathbf{C} - 3\right) + \frac{1}{2}\kappa\left(\frac{1}{2}\left(J^2 - 1\right) - \ln J\right). \tag{1.135}$$

Note that the elastic-energy density is written with respect to the Cauchy–Green stress tensor $\mathbf{C}$ and the determinant of the Jacobian of the deformation gradient J. The $J^2 - 1$ term penalizes the deviation of J from unity and the $\ln J$ term stabilizes the formulation for the regime of strong compression. With this definition of the elastic-energy density, the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the tensor of elastic moduli $\mathbb{C}$ may be explicitly computed, and are given by

$$\mathbf{S} = \mu J^{-2/3}\left(\mathbf{I} - \frac{1}{3}\mathrm{tr}\mathbf{C}\,\mathbf{C}^{-1}\right) + \frac{1}{2}\kappa\left(J^2 - 1\right)\mathbf{C}^{-1} \tag{1.136}$$

and

$$\begin{aligned}\mathbb{C} = &\left(\frac{2}{9}\mu J^{-2/3}\mathrm{tr}\mathbf{C} + \kappa J^2\right)\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\\ &+\left(\frac{2}{3}\mu J^{-2/3}\mathrm{tr}\mathbf{C} - \kappa\left(J^2 - 1\right)\right)\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\\ &-\frac{2}{3}\mu J^{-2/3}\left(\mathbf{I}\otimes\mathbf{C}^{-1} + \mathbf{C}^{-1}\otimes\mathbf{I}\right).\end{aligned} \tag{1.137}$$

In (1.137), the symbols $\otimes$ and $\odot$ are defined as

$$(\mathbf{A}\otimes\mathbf{B})_{IJKL} = (\mathbf{A})_{IJ}\,(\mathbf{B})_{KL}\,, \tag{1.138}$$

$$\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right)_{IJKL} = \frac{\left(\mathbf{C}^{-1}\right)_{IK}\left(\mathbf{C}^{-1}\right)_{JL} + \left(\mathbf{C}^{-1}\right)_{IL}\left(\mathbf{C}^{-1}\right)_{JK}}{2}. \tag{1.139}$$

In this model, κ and μ are also interpreted as the material bulk and shear moduli, respectively. One can see this by evaluating the material constitutive tensor $\mathbb{C}$ from Equation (1.137) for the case when the reference and current configurations coincide. In this case, $\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{C} = \mathbf{I}$, and the material modulus from Equation (1.137) reduces to the form given by Equation (1.131).

Mooney–Rivlin model. For the compressible Mooney–Rivlin material, the expression for $\mathbf{S}$, in component form, is given as

$$S_{IJ} = 2\left(C_1 + C_2 C_{KK}\right)\delta_{IJ} - 2C_2 C_{IJ} + \left(K_{\mathrm{PEN}}\ln J - 2\left(C_1 + 2C_2\right)\right)C^{-1}_{IJ}, \tag{1.140}$$

where C_1 and C_2 are the Mooney–Rivlin material constants. The near-incompressibility is enforced with the penalty term $K_{\mathrm{PEN}}\ln J$ (see Betsch *et al.*, 1996), where K_{PEN} is a penalty parameter determined based on the expression given in Stuparu (2002) for the bulk modulus:

$$K_{\mathrm{PEN}} = \frac{2\left(C_1 + C_2\right)}{\left(1 - 2\nu_{\mathrm{PEN}}\right)}. \tag{1.141}$$

Here ν_{PEN} (with a value close to 0.50) is the "penalty" Poisson's ratio used in the expression in place of the actual Poisson's ratio.

Fung model. For the Fung material, the expression for $\mathbf{S}$ is given as

$$S_{IJ} = 2D_1D_2(e^{D_2(C_{KK}-3)}\delta_{IJ} - C_{IJ}^{-1}) + K_{\text{PEN}} \ln J\, C_{IJ}^{-1}, \tag{1.142}$$

where D_1 and D_2 are the Fung material constants, and K_{PEN} is defined as

$$K_{\text{PEN}} = \frac{2D_1D_2}{(1-2\nu_{\text{PEN}})}. \tag{1.143}$$

1.2.8 *Linearization of the Structural Mechanics Equations: Tangent Stiffness and Equations of Linear Elasticity*

In this section we linearize the structural mechanics equations. Linearization gives a tangent stiffness operator that is used in the implementation of the Newton–Raphson method to solve the nonlinear structural equations. Linearization also gives rise to the equations of linear elastodynamics, which are often used in structural modeling.

To arrive at the linearized structural mechanics problem, we "perturb" the structure around its deformed state and only keep the terms that are linear in the displacement perturbation. Namely, we set

$$\delta W(\mathbf{w}, \bar{\mathbf{y}}) + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta W(\mathbf{w}, \bar{\mathbf{y}} + \epsilon\mathbf{y})\bigg|_{\epsilon=0} = 0, \tag{1.144}$$

where $\bar{\mathbf{y}}$ is the structural displacement that defines its deformed state, $\mathbf{y}$ now plays the role of a small displacement perturbation, and

$$\delta W(\mathbf{w}, \bar{\mathbf{y}}) = \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \bar{\mathbf{a}}\, \mathrm{d}\Omega + \int_{\Omega_0} \boldsymbol{\nabla}_X \mathbf{w} : \bar{\mathbf{P}}\, \mathrm{d}\Omega - \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \bar{\mathbf{f}}\, \mathrm{d}\Omega - \int_{(\Gamma_0)_{\text{h}}} \mathbf{w} \cdot \bar{\hat{\mathbf{h}}}\, \mathrm{d}\Gamma \tag{1.145}$$

are the structural mechanics variational equations evaluated at $\bar{\mathbf{y}}$. The superimposed bar denotes quantities evaluated at the deformed state.

Linearization of the internal virtual-work terms gives

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta W_{\text{int}}(\mathbf{w}, \bar{\mathbf{y}} + \epsilon\mathbf{y})\bigg|_{\epsilon=0} = \delta \int_{\Omega_0} \mathbf{F}^T \boldsymbol{\nabla}_X \mathbf{w} : \mathbf{S}\, \mathrm{d}\Omega \tag{1.146}$$

$$= \int_{\Omega_0} \left(\delta\mathbf{F}^T \boldsymbol{\nabla}_X \mathbf{w} : \bar{\mathbf{S}} + \bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{w} : \delta\mathbf{S}\right) \mathrm{d}\Omega \tag{1.147}$$

$$= \int_{\Omega_0} \left(\boldsymbol{\nabla}_X \mathbf{w} : \boldsymbol{\nabla}_X \mathbf{y} \bar{\mathbf{S}} + \bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{w} : \left(\overline{\frac{\partial \mathbf{S}}{\partial \mathbf{E}}}\right) \frac{1}{2}\left(\bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{y} + \boldsymbol{\nabla}_X \mathbf{y}^T \bar{\mathbf{F}}\right)\right) \mathrm{d}\Omega. \tag{1.148}$$

Recognizing that

$$\frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 \varphi}{\partial \mathbf{E} \partial \mathbf{E}} = \mathbb{C}, \tag{1.149}$$

and using the minor symmetry of $\mathbb{C}$, Equation (1.148) may be written as

$$\int_{\Omega_0} \left(\bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{w} : \bar{\mathbb{C}} \bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{y} + \boldsymbol{\nabla}_X \mathbf{w} : \boldsymbol{\nabla}_X \mathbf{y} \bar{\mathbf{S}} \right) \mathrm{d}\Omega. \tag{1.150}$$

Using index notation, Equation (1.150) may be expressed as

$$\int_{\Omega_0} w_{i,J} \bar{D}_{iJkL} y_{k,L} \, \mathrm{d}\Omega, \tag{1.151}$$

where $\bar{D}_{iJkL}$'s are the components of the *tangent stiffness* tensor given by

$$\bar{D}_{iJkL} = \bar{F}_{iI} \bar{\mathbb{C}}_{IJKL} \bar{F}_{kK} + \delta_{ik} \bar{S}_{JL}. \tag{1.152}$$

The first term on the right-hand-side of Equation (1.152) is the *material stiffness*, while the second term is the *geometric stiffness* contribution to the tangent stiffness tensor.

Linearization of the external virtual-work gives

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \delta W_{\mathrm{ext}}(\mathbf{w}, \bar{\mathbf{y}} + \epsilon \mathbf{y}) \Big|_{\epsilon=0} = \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \mathbf{a} \, \mathrm{d}\Omega - \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \mathbf{f} \, \mathrm{d}\Omega - \int_{(\Gamma_0)_{\mathrm{h}}} \mathbf{w} \cdot \hat{\mathbf{h}} \, \mathrm{d}\Gamma, \tag{1.153}$$

where $\mathbf{a}$, $\mathbf{f}$, and $\hat{\mathbf{h}}$ are now the increments of acceleration, body force, and surface traction, respectively.

Using Equation (1.144) and the above derivations, we arrive at the variational statement of a complete linearized problem: given the structural displacement state $\bar{\mathbf{y}}$, find the displacement perturbation $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\, \mathbf{w} \in \mathcal{V}_y$:

$$\begin{aligned} &\int_{\Omega_0} \mathbf{w} \cdot \rho_0 \bar{\mathbf{a}} \, \mathrm{d}\Omega + \int_{\Omega_0} \boldsymbol{\nabla}_X \mathbf{w} : \bar{\mathbf{P}} \, \mathrm{d}\Omega - \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \bar{\mathbf{f}} \, \mathrm{d}\Omega - \int_{(\Gamma_0)_{\mathrm{h}}} \mathbf{w} \cdot \bar{\hat{\mathbf{h}}} \, \mathrm{d}\Gamma \\ &\quad + \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \mathbf{a} \, \mathrm{d}\Omega + \int_{\Omega_0} \left(\bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{w} : \bar{\mathbb{C}} \bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{y} + \boldsymbol{\nabla}_X \mathbf{w} : \boldsymbol{\nabla}_X \mathbf{y} \bar{\mathbf{S}} \right) \mathrm{d}\Omega \\ &\quad - \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \mathbf{f} \, \mathrm{d}\Omega - \int_{(\Gamma_0)_{\mathrm{h}}} \mathbf{w} \cdot \hat{\mathbf{h}} \, \mathrm{d}\Gamma = 0. \end{aligned} \tag{1.154}$$

If the structural mechanics equations are linearized around an equilibrium configuration (i.e., the displacement state $\bar{\mathbf{y}}$ satisfies the variational equations), then the linearized problem given by Equation (1.154) reduces to: given $\bar{\mathbf{y}}$, find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\, \mathbf{w} \in \mathcal{V}_y$:

$$\begin{aligned} &\int_{\Omega_0} \mathbf{w} \cdot \rho_0 \mathbf{a} \, \mathrm{d}\Omega + \int_{\Omega_0} \left(\bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{w} : \bar{\mathbb{C}} \bar{\mathbf{F}}^T \boldsymbol{\nabla}_X \mathbf{y} + \boldsymbol{\nabla}_X \mathbf{w} : \boldsymbol{\nabla}_X \mathbf{y} \bar{\mathbf{S}} \right) \mathrm{d}\Omega \\ &\quad - \int_{\Omega_0} \mathbf{w} \cdot \rho_0 \mathbf{f} \, \mathrm{d}\Omega - \int_{(\Gamma_0)_{\mathrm{h}}} \mathbf{w} \cdot \hat{\mathbf{h}} \, \mathrm{d}\Gamma = 0. \end{aligned} \tag{1.155}$$

An important special case of the formulation given by Equation (1.155) is obtained when the virtual-work equations are linearized around the stress-free undisplaced configuration. In this case, $\bar{\mathbf{y}} = \mathbf{0}$, $\bar{\mathbf{F}} = \mathbf{I}$, $\bar{\mathbf{E}} = \mathbf{0}$, $\bar{\mathbf{S}} = \mathbf{0}$, and we obtain

$$\int_{\Omega} \mathbf{w} \cdot \rho \mathbf{a} \, \mathrm{d}\Omega + \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathbb{C} \boldsymbol{\varepsilon}(\mathbf{y}) \, \mathrm{d}\Omega - \int_{\Omega} \mathbf{w} \cdot \rho \mathbf{f} \, \mathrm{d}\Omega - \int_{\Gamma_{\mathrm{h}}} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}\Gamma = 0, \tag{1.156}$$

where $\boldsymbol{\varepsilon}(\mathbf{y})$ is the linear, or infinitesimal, strain, which is a linearization of the Green–Lagrange strain about an undeformed configuration. Infinitesimal strain vanishes for the rigid-body translation, however, it does not vanish for the rigid-body rotation. As a result, the formulation given by Equation (1.156) is not suitable for structural mechanics problems where large deformations are expected (see Figure 1.6). Equation (1.156) represents the well-known variational formulation of linear elastodynamics. The equations of linear elastostatics are obtained by omitting the inertial term in Equation (1.156).

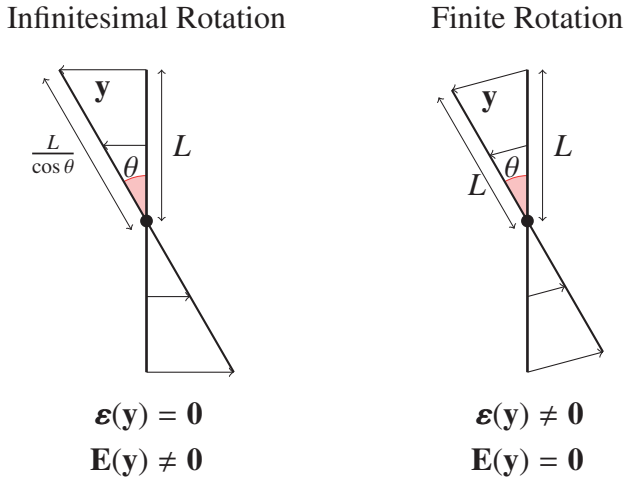

Figure 1.6 Elastic bar undergoing infinitesimal and finite rotations. In the case of infinitesimal rotation, linear-elastic analysis produces zero strain and, as a result, zero stress, which is nonphysical since the bar elongates. In the case of finite rotation, linear-elastic analysis produces a nonzero strain and stress, which is also nonphysical. In general, structural analysis with large displacements requires an objective strain measure to produce physically correct results

REMARK 1.7 *Note that because the reference and current configurations coincide, we no longer distinguish between* $\mathbf{x}$ *and* $\mathbf{X}$ *in Equation (1.156). As a result, we remove the subscript from* $\boldsymbol{\nabla}_X$*, and set* $\Omega = \Omega_0 = \Omega_t$ *and* $\Gamma = \Gamma_0 = \Gamma_t$.

We conclude this section with the linearization of the follower pressure load. For convenience, we employ index notation. We begin with the expression for the follower-pressure-load boundary condition in the reference configuration and first compute

$$\delta\left[JF_{Ii}^{-1}\right] = \delta J F_{Ii}^{-1} + J\delta F_{Ii}^{-1} = JF_{Jj}^{-1}y_{j,J}F_{Ii}^{-1} - JF_{Ij}^{-1}y_{j,J}F_{Ji}^{-1}. \tag{1.157}$$

Introducing the above variation into Equation (1.118), we obtain

$$-\int_{(\Gamma_0)_h} w_i p\delta\left[JF_{Ii}\right]\hat{n}_I \,\mathrm{d}\Gamma = -\int_{(\Gamma_0)_h} w_i pJ\left(F_{Jj}^{-1}F_{Ii}^{-1} - F_{Ij}^{-1}F_{Ji}^{-1}\right)\hat{n}_I y_{j,J}\,\mathrm{d}\Gamma. \tag{1.158}$$

Changing variables in the above expression to the current configuration, we obtain

$$-\int_{(\Gamma_t)_h}\left(w_i p n_i y_{j,j} - w_i p n_j y_{j,i}\right)\mathrm{d}\Gamma, \tag{1.159}$$

which is somewhat simpler than Equation (1.158). The above linearization is employed in the implementation of consistent tangent stiffness matrices in nonlinear structural analysis. An alternative linearization of the follower pressure load, which uses the parametric coordinates of the boundary surface, may be found in Wriggers (2008).

1.2.9 Thin Structures: Shell, Membrane, and Cable Models

1.2.9.1 Kirchhoff–Love Shell Model

In this section we follow the developments of Kiendl *et al.* (2009, 2010) and Bazilevs *et al.* (2011c) that present the governing equations of the Kirchhoff–Love shell theory. The theory is appropriate for thin-shell structures and, when discretized using smooth basis functions, requires no rotational degrees of freedom.

In the case of shells, the 3D continuum description is reduced to that of the shell midsurface, and the transverse normal stress is neglected. Furthermore, the Kirchhoff–Love theory assumes that the shell director remains normal to its middle surface during the deformation, which implies that the transverse shear strains are zero. As a result, only in-plane stress and strain tensors are considered, and the indices $\alpha = 1, 2$ and $\beta = 1, 2$ are employed to denote their components. We denote by Γ_0^s and Γ_t^s the shell midsurface in the reference and deformed configurations, respectively. Furthermore, h_{th} is the (variable) shell thickness, and $\xi_3 \in [-h_{\text{th}}/2, h_{\text{th}}/2]$ is the through-thickness coordinate.

We introduce the following standard shell kinematic quantities and relationships (see Bischoff *et al.*, 2004; Kiendl *et al.*, 2009 for more details):

$$E_{\alpha\beta} = \varepsilon_{\alpha\beta} + \xi_3 \kappa_{\alpha\beta}, \tag{1.160}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\mathbf{g}_\alpha \cdot \mathbf{g}_\beta - \mathbf{G}_\alpha \cdot \mathbf{G}_\beta\right), \tag{1.161}$$

$$\kappa_{\alpha\beta} = -\frac{\partial \mathbf{g}_\alpha}{\partial \xi_\beta} \cdot \mathbf{g}_3 - \left(-\frac{\partial \mathbf{G}_\alpha}{\partial \xi_\beta} \cdot \mathbf{G}_3\right), \tag{1.162}$$

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{x}}{\partial \xi_\alpha}, \tag{1.163}$$

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{X}}{\partial \xi_\alpha}, \tag{1.164}$$

$$\mathbf{g}_3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\|\mathbf{g}_1 \times \mathbf{g}_2\|}, \tag{1.165}$$

$$\mathbf{G}_3 = \frac{\mathbf{G}_1 \times \mathbf{G}_2}{\|\mathbf{G}_1 \times \mathbf{G}_2\|}, \tag{1.166}$$

$$\mathbf{G}^\alpha = (\mathbf{G}_\alpha \cdot \mathbf{G}_\beta)^{-1} \mathbf{G}_\beta. \tag{1.167}$$

Here, $E_{\alpha\beta}$, $\varepsilon_{\alpha\beta}$, and $\kappa_{\alpha\beta}$ are the contravariant components of the in-plane Green–Lagrange strain, membrane strain, and curvature tensors, respectively. The spatial coordinates of the *shell midsurface* in the current and reference configurations are $\mathbf{x} = \mathbf{x}(\xi_1, \xi_2)$ and $\mathbf{X} = \mathbf{X}(\xi_1, \xi_2)$, parameterized by ξ_1 and ξ_2. The covariant surface basis vectors in the current and reference configurations are $\mathbf{g}_\alpha$ and $\mathbf{G}_\alpha$. The unit outward normal vectors to the shell midsurface in the current and reference configurations are $\mathbf{g}_3$ and $\mathbf{G}_3$. The contravariant surface basis vectors in the reference configuration are denoted by $\mathbf{G}^\alpha$ (see Figure 1.7).

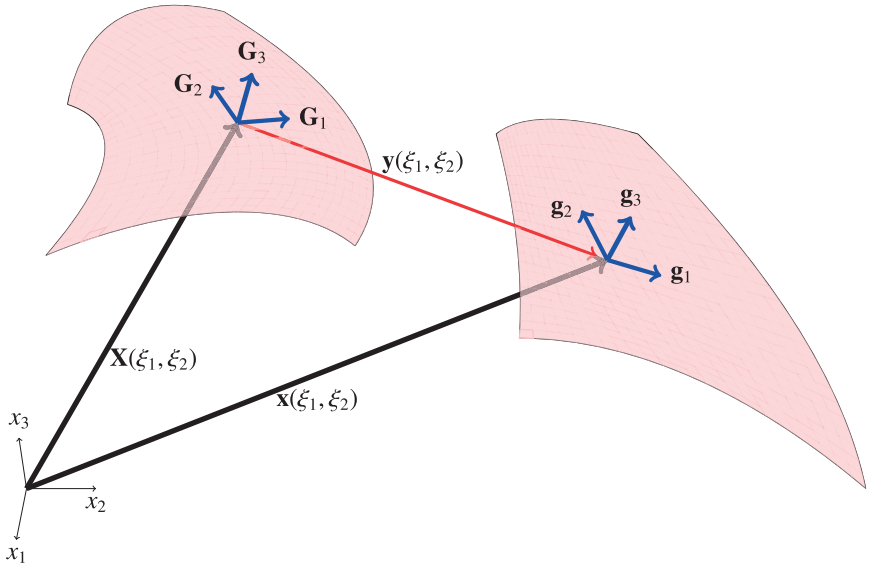

Figure 1.7 Shell kinematics

We select the *local Cartesian basis* vectors as follows:

$$\overline{\mathbf{e}}_1 = \frac{\mathbf{G}_1}{\|\mathbf{G}_1\|}, \tag{1.168}$$

$$\overline{\mathbf{e}}_2 = \frac{\mathbf{G}_2 - (\mathbf{G}_2 \cdot \overline{\mathbf{e}}_1)\overline{\mathbf{e}}_1}{\|\overline{\mathbf{G}}_2 - (\mathbf{G}_2 \cdot \overline{\mathbf{e}}_1)\overline{\mathbf{e}}_1\|}, \tag{1.169}$$

that is, the first local basis vector is the normalized first covariant basis vector in the reference configuration. The local Cartesian basis vectors $\overline{\mathbf{e}}_\alpha$ are used in expressing a constitutive relationship for the shell. Because the local basis is orthonormal, we make no distinction between covariant and contravariant quantities, which are expressed with respect to it.

With the above definitions, we calculate the components of the Green–Lagrange strain tensor and its variation in the local coordinate system as

$$\overline{E}_{\alpha\beta} = \overline{\varepsilon}_{\alpha\beta} + \xi_3 \overline{\kappa}_{\alpha\beta}, \tag{1.170}$$

$$\delta\overline{E}_{\alpha\beta} = \delta\overline{\varepsilon}_{\alpha\beta} + \xi_3 \delta\overline{\kappa}_{\alpha\beta}, \tag{1.171}$$

$$\overline{\varepsilon}_{\alpha\beta} = \varepsilon_{\gamma\delta}(\mathbf{G}^\gamma \cdot \overline{\mathbf{e}}_\alpha)(\mathbf{G}^\delta \cdot \overline{\mathbf{e}}_\beta), \tag{1.172}$$

$$\overline{\kappa}_{\alpha\beta} = \kappa_{\gamma\delta}(\mathbf{G}^\gamma \cdot \overline{\mathbf{e}}_\alpha)(\mathbf{G}^\delta \cdot \overline{\mathbf{e}}_\beta), \tag{1.173}$$

$$\delta\overline{\varepsilon}_{\alpha\beta} = \delta\varepsilon_{\gamma\delta}(\mathbf{G}^\gamma \cdot \overline{\mathbf{e}}_\alpha)(\mathbf{G}^\delta \cdot \overline{\mathbf{e}}_\beta), \tag{1.174}$$

$$\delta\overline{\kappa}_{\alpha\beta} = \delta\kappa_{\gamma\delta}(\mathbf{G}^\gamma \cdot \overline{\mathbf{e}}_\alpha)(\mathbf{G}^\delta \cdot \overline{\mathbf{e}}_\beta). \tag{1.175}$$

The variations $\delta\varepsilon_{\gamma\delta}$ and $\delta\kappa_{\gamma\delta}$ may be computed directly by taking the variational derivatives of the expressions given by Equations (1.161) and (1.162) with respect to the displacement vector.

We define the vectors of membrane strain and curvature components in the local coordinate system as

$$\overline{\boldsymbol{\varepsilon}} = \begin{bmatrix} \overline{\varepsilon}_{11} \\ \overline{\varepsilon}_{22} \\ \overline{\varepsilon}_{12} \end{bmatrix} \tag{1.176}$$

and

$$\overline{\boldsymbol{\kappa}} = \begin{bmatrix} \overline{\kappa}_{11} \\ \overline{\kappa}_{22} \\ \overline{\kappa}_{12} \end{bmatrix}, \tag{1.177}$$

together with a Green–Lagrange strain vector

$$\overline{\mathbf{E}} = \overline{\boldsymbol{\varepsilon}} + \xi_3 \overline{\boldsymbol{\kappa}}. \tag{1.178}$$

We assume St. Venant–Kirchhoff material law and write the following stress–strain relationship in the local coordinate system:

$$\overline{\mathbf{S}} = \overline{\mathbb{C}}\,\overline{\mathbf{E}}, \tag{1.179}$$

where $\overline{\mathbf{S}}$ is a vector of components of the second Piola–Kirchhoff stress tensor in the local coordinate system, and $\overline{\mathbb{C}}$ is a constitutive material matrix, which is symmetric. Introducing Equations (1.178) and (1.179) into the expression for the internal virtual work given by Equation (1.79), we obtain

$$\delta W_{\text{int}} = -\int_{\Omega_0} \delta\overline{\mathbf{E}} \cdot \overline{\mathbf{S}}\, \mathrm{d}\Omega \tag{1.180}$$

$$= -\int_{\Gamma_0^s} \left(\int_{h_{\text{th}}} \delta\overline{\mathbf{E}} \cdot \overline{\mathbb{C}}\,\overline{\mathbf{E}}\, \mathrm{d}\xi_3 \right) \mathrm{d}\Gamma \tag{1.181}$$

$$= -\int_{\Gamma_0^s} \delta\overline{\boldsymbol{\varepsilon}} \cdot \left(\left(\int_{h_{\text{th}}} \overline{\mathbb{C}}\, \mathrm{d}\xi_3 \right) \overline{\boldsymbol{\varepsilon}} + \left(\int_{h_{\text{th}}} \xi_3 \overline{\mathbb{C}}\, \mathrm{d}\xi_3 \right) \overline{\boldsymbol{\kappa}} \right) \mathrm{d}\Gamma$$

$$- \int_{\Gamma_0^s} \delta\overline{\boldsymbol{\kappa}} \cdot \left(\left(\int_{h_{\text{th}}} \xi_3 \overline{\mathbb{C}}\, \mathrm{d}\xi_3 \right) \overline{\boldsymbol{\varepsilon}} + \left(\int_{h_{\text{th}}} {\xi_3}^2 \overline{\mathbb{C}}\, \mathrm{d}\xi_3 \right) \overline{\boldsymbol{\kappa}} \right) \mathrm{d}\Gamma. \tag{1.182}$$

For a general orthotropic material,

$$\overline{\mathbb{C}}_{\text{ort}} = \begin{bmatrix} \dfrac{E_1}{(1-\nu_{12}\nu_{21})} & \dfrac{\nu_{21}E_1}{(1-\nu_{12}\nu_{21})} & 0 \\ \dfrac{\nu_{12}E_2}{(1-\nu_{12}\nu_{21})} & \dfrac{E_2}{(1-\nu_{12}\nu_{21})} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}. \tag{1.183}$$

In Equation (1.183), E_1 and E_2 are the Young's moduli in the directions defined by the local basis vectors, ν_{12} and ν_{21} are the Poisson's ratios, G_{12} is the shear modulus, and $\nu_{21}E_1 = \nu_{12}E_2$

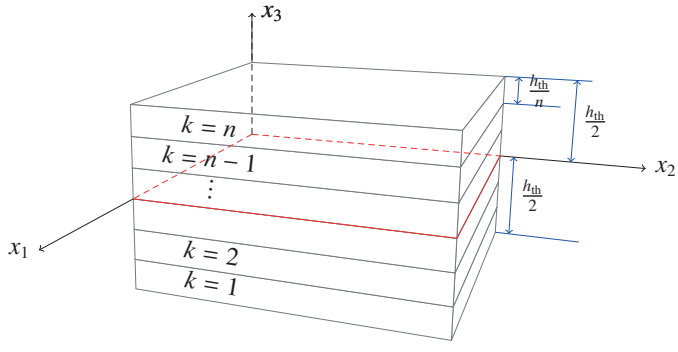

Figure 1.8 Schematic of a composite laminate

to ensure the symmetry of the constitutive material matrix $\overline{\mathbb{C}}_{\text{ort}}$. In the case of an isotropic material, $E_1 = E_2 = E$, $\nu_{21} = \nu_{12} = \nu$, and $G_{12} = E/(2(1+\nu))$.

In the case of composite materials, we assume that the structure is composed of a set of plies, each modeled as an orthotropic material. We use the classical laminated-plate theory Reddy, 2004, and homogenize the material through-thickness constitutive behavior for a given composite ply layout. Let k denotes the k^{th} ply (or lamina) and let n be the total number of plies (see Figure 1.8). We assume each ply has the same thickness h_{th}/n. Pre-integrating through the shell thickness in Equation (1.182), the extensional stiffness $\mathbf{K}_{\text{exte}}$, coupling stiffness $\mathbf{K}_{\text{coup}}$, and bending stiffness $\mathbf{K}_{\text{bend}}$ are given by

$$\mathbf{K}_{\text{exte}} = \int_{h_{\text{th}}} \overline{\mathbb{C}}\, \mathrm{d}\xi_3 = \frac{h_{\text{th}}}{n} \sum_{k=1}^{n} \overline{\mathbb{C}}_k, \tag{1.184}$$

$$\mathbf{K}_{\text{coup}} = \int_{h_{\text{th}}} \xi_3 \overline{\mathbb{C}}\, \mathrm{d}\xi_3 = \frac{h_{\text{th}}^2}{n^2} \sum_{k=1}^{n} \overline{\mathbb{C}}_k \left(k - \frac{n}{2} - \frac{1}{2}\right), \tag{1.185}$$

$$\mathbf{K}_{\text{bend}} = \int_{h_{\text{th}}} {\xi_3}^2 \overline{\mathbb{C}}\, \mathrm{d}\xi_3 = \frac{h_{\text{th}}^3}{n^3} \sum_{k=1}^{n} \overline{\mathbb{C}}_k \left(\left(k - \frac{n}{2} - \frac{1}{2}\right)^2 + \frac{1}{12}\right), \tag{1.186}$$

where

$$\overline{\mathbb{C}}_k = \mathbf{T}^T(\phi_k)\, \overline{\mathbb{C}}_{\text{ort}}\, \mathbf{T}(\phi_k), \tag{1.187}$$

$$\mathbf{T}(\phi) = \begin{bmatrix} \cos^2\phi & \sin^2\phi & \sin\phi\cos\phi \\ \sin^2\phi & \cos^2\phi & -\sin\phi\cos\phi \\ -2\sin\phi\cos\phi & 2\sin\phi\cos\phi & \cos^2\phi - \sin^2\phi \end{bmatrix}. \tag{1.188}$$

In the above equations, ϕ is the fiber orientation angle in each ply, Equation (1.187) transforms $\overline{\mathbb{C}}_{\text{ort}}$ from the principal material coordinates to the laminate coordinates (defined by the local Cartesian basis) for each ply, and $\overline{\mathbb{C}}_k$ is constant within each ply. Note that, setting $n = 1$ and $\overline{\mathbb{C}}_k = \overline{\mathbb{C}}_{\text{ort}}$ in Equations (1.184)–(1.186), we get $\mathbf{K}_{\text{coup}} = \mathbf{0}$ and

$$\mathbf{K}_{\text{exte}} = h_{\text{th}} \overline{\mathbb{C}}_{\text{ort}}, \tag{1.189}$$

$$\mathbf{K}_{\text{bend}} = \frac{h_{\text{th}}^3}{12}\overline{\mathbb{C}}_{\text{ort}}, \tag{1.190}$$

which are the classical membrane and bending stiffnesses for an orthotropic shell.

With the above definitions, the expression for the internal virtual work for a composite shell may now be compactly written as

$$\delta W_{\text{int}} = -\int_{\Gamma_0^s} \delta\overline{\boldsymbol{\varepsilon}} \cdot \left(\mathbf{K}_{\text{exte}}\overline{\boldsymbol{\varepsilon}} + \mathbf{K}_{\text{coup}}\overline{\boldsymbol{\kappa}}\right) \, \mathrm{d}\Gamma - \int_{\Gamma_0^s} \delta\overline{\boldsymbol{\kappa}} \cdot \left(\mathbf{K}_{\text{coup}}\overline{\boldsymbol{\varepsilon}} + \mathbf{K}_{\text{bend}}\overline{\boldsymbol{\kappa}}\right) \, \mathrm{d}\Gamma. \tag{1.191}$$

The complete variational formulation of the Kirchhoff–Love shell is given by: find the displacement of the shell midsurface $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\ \mathbf{w} \in \mathcal{V}_y$:

$$\begin{aligned}
&\int_{\Gamma_0^s} \mathbf{w} \cdot h_{\text{th}}\overline{\rho}_0 \, (\mathbf{a} - \mathbf{f}) \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_0^s} \delta\overline{\boldsymbol{\varepsilon}} \cdot \left(\mathbf{K}_{\text{exte}}\overline{\boldsymbol{\varepsilon}} + \mathbf{K}_{\text{coup}}\overline{\boldsymbol{\kappa}}\right) \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_0^s} \delta\overline{\boldsymbol{\kappa}} \cdot \left(\mathbf{K}_{\text{coup}}\overline{\boldsymbol{\varepsilon}} + \mathbf{K}_{\text{bend}}\overline{\boldsymbol{\kappa}}\right) \, \mathrm{d}\Gamma - \int_{(\Gamma_t^s)_{\text{h}}} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}\Gamma = 0,
\end{aligned} \tag{1.192}$$

where $(\Gamma_t^s)_{\text{h}}$ is the shell subdomain with a prescribed traction boundary condition, and $\overline{\rho}_0$ is the through-thickness-averaged shell density given by

$$\overline{\rho}_0 = \frac{1}{h_{\text{th}}} \int_{h_{\text{th}}} \rho_0 \, \mathrm{d}\xi_3. \tag{1.193}$$

Note that, for simplicity of the exposition, in Equation (1.192), we omitted the terms corresponding to the prescribed traction on the edges of the shell. Although not presented here, such terms are implemented in our structural analysis programs.

1.2.9.2 Membrane Model

The membrane formulation is obtained by neglecting the curvature tensor in the definition of the in-plane Green–Lagrange strain (see Equation (1.160)) in the equations of the Kirchhoff–Love shell. This results in a simplified structural model, in which the bending effects are neglected. The variational formulation for membrane structures may be stated as follows: find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\ \mathbf{w} \in \mathcal{V}_y$:

$$\int_{\Gamma_0^s} \mathbf{w} \cdot h_{\text{th}}\overline{\rho}_0 \, (\mathbf{a} - \mathbf{f}) \, \mathrm{d}\Gamma + \int_{\Gamma_0^s} \delta\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{K}_{\text{exte}}\overline{\boldsymbol{\varepsilon}} \, \mathrm{d}\Gamma - \int_{(\Gamma_t^s)_{\text{h}}} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}\Gamma = 0. \tag{1.194}$$

Since only the first derivatives with respect to the parametric domain coordinates are employed in this variational formulation, C^0-continuous basis functions may be used to discretize the membrane equations.

The membrane formulation, in the case of an isotropic material, may be written without using the local coordinate system. In this case, the variational formulation of the membrane model becomes: find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\ \mathbf{w} \in \mathcal{V}_y$:

$$\int_{\Gamma_0^s} \mathbf{w} \cdot h_{\text{th}}\overline{\rho}_0 \, (\mathbf{a} - \mathbf{f}) \, \mathrm{d}\Gamma + \int_{\Gamma_0^s} \delta\varepsilon_{\alpha\beta} h_{\text{th}} S^{\alpha\beta} \, \mathrm{d}\Gamma - \int_{(\Gamma_t^s)_{\text{h}}} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}\Gamma = 0, \tag{1.195}$$

where

$$S^{\alpha\beta} = \left(\bar{\lambda} G^{\alpha\beta} G^{\gamma\delta} + \mu \left(G^{\alpha\gamma} G^{\beta\delta} + G^{\alpha\delta} G^{\beta\gamma}\right)\right) \varepsilon_{\gamma\delta}, \tag{1.196}$$

$\varepsilon_{\gamma\delta}$ are the in-plane components of the Green–Lagrange strain tensor given by Equation (1.161), $\bar{\lambda} = 2\lambda\mu/(\lambda + 2\mu)$, and $G^{\alpha\beta}$ are the contravariant metric tensor components in the undeformed configuration. Just as in the case of the shell model, for simplicity of the exposition, we omitted the edge-traction terms from Equations (1.194) and (1.195).

1.2.9.3 Cable Model

For cables, under the assumption of uniaxial tension, the parametric domain reduces to a line (see Figure 1.9), the indices $\alpha = \beta = 1$, and the bending effects are also neglected in the developments in Section 1.2.9.1. Furthermore, the Poisson's effect is also neglected, which leaves the Young's modulus E_c as the only material parameter in the cable constitutive model. The variational formulation for cable structures may be stated as follows: find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\, \mathbf{w} \in \mathcal{V}_y$:

$$\int_{S_0} \mathbf{w} \cdot A_c \overline{\rho}_0 \left(\mathbf{a} - \mathbf{f}\right) \, \mathrm{d}S + \int_{S_0} \delta\overline{\varepsilon}_{11} A_c E_c \overline{\varepsilon}_{11} \, \mathrm{d}S - \int_{(S_t)_h} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}S = 0, \tag{1.197}$$

where S_0 and S_t are the curves that define the cable axis in the reference and deformed configuration, respectively, $(S_t)_h$ is the part of S_t with a prescribed traction boundary condition, and, in this case, $\mathbf{h}$ has the dimensions of force per unit length.

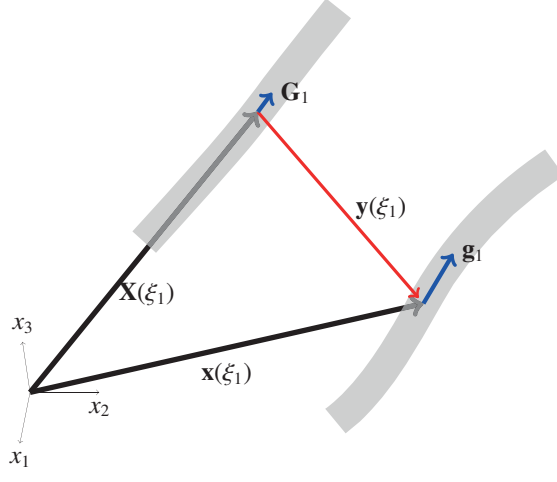

Figure 1.9 Cable kinematics

The cable formulation may be stated without using the local coordinate system as follows: find $\mathbf{y} \in \mathcal{S}_y$, such that $\forall\, \mathbf{w} \in \mathcal{V}_y$:

$$\int_{S_0} \mathbf{w} \cdot A_c \overline{\rho}_0 \left(\mathbf{a} - \mathbf{f}\right) \, \mathrm{d}S + \int_{S_0} \delta\varepsilon_{11} A_c E_c G^{11} G^{11} \varepsilon_{11} \, \mathrm{d}S - \int_{(S_t)_h} \mathbf{w} \cdot \mathbf{h} \, \mathrm{d}S = 0. \tag{1.198}$$

Although the terms representing the traction boundary conditions at the end points are not explicitly included in Equation (1.198), they are implemented in our analysis programs.

1.3 Governing Equations of Fluid Mechanics in Moving Domains

In this section we revisit the fluid mechanics governing equations and recast them in the ALE framework. The ALE form of the fluid mechanics equations is often used to simulate flows on moving domains, including FSI.

1.3.1 Kinematics of ALE and Space–Time Descriptions

The ALE description also makes use of a reference domain. However, the major difference from the Lagrangian approach, which is typically adopted for structural mechanics, is that the motion of the fluid mechanics problem reference domain does not follow the motion of the fluid itself. For this reason, we denote this reference domain by $\hat{\Omega} \in \mathbb{R}^{n_{\text{sd}}}$ and the coordinates in this reference domain by $\hat{\mathbf{x}}$. See Figure 1.10 for an illustration. The fluid spatial domain Ω_t is given by

$$\Omega_t = \left\{ \mathbf{x} \;\middle|\; \mathbf{x} = \boldsymbol{\phi}(\hat{\mathbf{x}}, t) \;\; \forall \hat{\mathbf{x}} \in \hat{\Omega}, t \in (0, T) \right\}. \tag{1.199}$$

The mapping given by Equation (1.199) takes the form

$$\boldsymbol{\phi}(\hat{\mathbf{x}}, t) = \hat{\mathbf{x}} + \hat{\mathbf{y}}(\hat{\mathbf{x}}, t), \tag{1.200}$$

where $\hat{\mathbf{y}}$ is the time-dependent displacement of the reference fluid domain. With this definition of the ALE map, the fluid domain velocity is given by

$$\hat{\mathbf{u}} = \left. \frac{\partial \hat{\mathbf{y}}}{\partial t} \right|_{\hat{x}}, \tag{1.201}$$

where $\left|_{\hat{x}}\right.$ denotes the time derivative taken holding $\hat{\mathbf{x}}$ fixed, the deformation gradient is defined as

$$\hat{\mathbf{F}} = \frac{\partial \mathbf{x}}{\partial \hat{\mathbf{x}}} = \mathbf{I} + \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{x}}}, \tag{1.202}$$

and $\hat{J} = \det \hat{\mathbf{F}}$ is the Jacobian of the deformation gradient.

We recall the Piola transformation, a classical result in continuum mechanics (see, e.g., Wriggers, 2008). Given an arbitrary vector field $\boldsymbol{\gamma}$ defined on the spatial domain, we define a vector field on the reference domain as

$$\hat{\boldsymbol{\gamma}} = \hat{J} \hat{\mathbf{F}}^{-1} \boldsymbol{\gamma}. \tag{1.203}$$

In this case, the following equality holds:

$$\int_{\Omega_t} \boldsymbol{\nabla} \cdot \boldsymbol{\gamma} \, \mathrm{d}\Omega = \int_{\hat{\Omega}} \boldsymbol{\nabla}_{\hat{x}} \cdot \hat{\boldsymbol{\gamma}} \, \mathrm{d}\hat{\Omega}. \tag{1.204}$$

Relationship (1.203) is the Piola transformation, which preserves the divergence, or conservation, structure of the vector field in the reference configuration (see Equation (1.204)). The Piola transformation also applies to tensor-valued quantities.

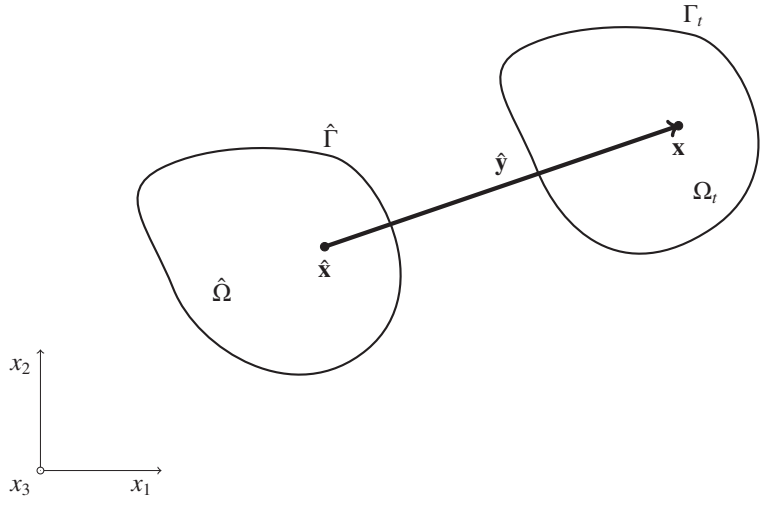

Figure 1.10 Reference and fluid spatial domains

To derive the ALE equations of fluid mechanics, we introduce the notion of a space–time domain, which we will also use in the later sections when discussing the space–time method for moving-domain problems. We begin with the fluid domain reference configuration $\hat{\Omega}$ and define its space–time counterpart $\hat{Q}$ by extruding $\hat{\Omega}$ along the time axis as

$$\hat{Q} = \hat{\Omega} \times (0, T) = \left\{ (\hat{\mathbf{x}}, t) \;\middle|\; \forall \hat{\mathbf{x}} \in \hat{\Omega}, t \in (0, T) \right\}. \tag{1.205}$$

The space–time domain in the current configuration Q_t is defined by

$$Q_t = \left\{ (\mathbf{x}, t) \;\middle|\; \begin{Bmatrix} t \\ \mathbf{x} \end{Bmatrix} = \begin{Bmatrix} t \\ \boldsymbol{\phi}(\hat{\mathbf{x}}, t) \end{Bmatrix} \; \forall (\hat{\mathbf{x}}, t) \in \hat{Q} \right\}. \tag{1.206}$$

Note that time is synchronized in both configurations. The deformation gradient from $\hat{Q}$ to Q_t is given directly by

$$\begin{Bmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial \hat{\mathbf{x}}} \\ \frac{\partial \mathbf{x}}{\partial t} & \frac{\partial \mathbf{x}}{\partial \hat{\mathbf{x}}} \end{Bmatrix} = \begin{Bmatrix} 1 & \mathbf{0}^T \\ \hat{\mathbf{u}} & \hat{\mathbf{F}} \end{Bmatrix}, \tag{1.207}$$

and its determinant is coincident with that of the spatial mapping

$$\det \begin{Bmatrix} 1 & \mathbf{0}^T \\ \hat{\mathbf{u}} & \hat{\mathbf{F}} \end{Bmatrix} = \hat{J}. \tag{1.208}$$

Just as in the spatial case (see Equation (1.203)), there is an analog of the Piola transformation for space–time domains (see Bazilevs *et al.*, 2008). Given a vector field, $(\gamma_0, \boldsymbol{\gamma})^T : Q_t \to \mathbb{R}^{n_{\text{sd}}+1}$, we define a vector field, $(\hat{\gamma}_0, \hat{\boldsymbol{\gamma}})^T : \hat{Q} \to \mathbb{R}^{n_{\text{sd}}+1}$ as

$$\begin{Bmatrix} \hat{\gamma}_0 \\ \hat{\boldsymbol{\gamma}} \end{Bmatrix} = \hat{J} \begin{Bmatrix} 1 & \mathbf{0}^T \\ \hat{\mathbf{u}} & \hat{\mathbf{F}} \end{Bmatrix}^{-1} \begin{Bmatrix} \gamma_0 \\ \boldsymbol{\gamma} \end{Bmatrix} = \hat{J} \begin{Bmatrix} 1 & \mathbf{0}^T \\ -\hat{\mathbf{F}}^{-1}\hat{\mathbf{u}} & \hat{\mathbf{F}}^{-1} \end{Bmatrix} \begin{Bmatrix} \gamma_0 \\ \boldsymbol{\gamma} \end{Bmatrix} = \begin{Bmatrix} \hat{J}\gamma_0 \\ \hat{J}\hat{\mathbf{F}}^{-1}(\boldsymbol{\gamma} - \gamma_0 \hat{\mathbf{u}}) \end{Bmatrix}. \tag{1.209}$$

In this case, the following space–time integral relationship holds:

$$\int_{Q_t} \left(\frac{\partial \gamma_0}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\gamma} \right) \mathrm{d}Q = \int_{\hat{Q}} \left(\left. \frac{\partial \hat{\gamma}_0}{\partial t} \right|_{\hat{x}} + \boldsymbol{\nabla}_{\hat{x}} \cdot \hat{\boldsymbol{\gamma}} \right) \mathrm{d}\hat{Q}, \tag{1.210}$$

which shows that the space–time Piola transformation preserves the conservation structure of a vector field in space–time.

Using index notation, the space–time Piola transformation given by Equation (1.209) becomes

$$\begin{Bmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_I \end{Bmatrix} = \begin{Bmatrix} \hat{J}\gamma_0 \\ \hat{J}\hat{F}^{-1}_{Ii} \left(\gamma_i - \gamma_0 \hat{u}_i \right) \end{Bmatrix}, \tag{1.211}$$

where the upper- and lower-case indices refer to the referential and spatial quantities, respectively, and the integral equality (1.210) may be written as

$$\int_{Q_t} \left(\gamma_{0,t} + \gamma_{i,i} \right) \mathrm{d}Q = \int_{\hat{Q}} \left(\left. \hat{\gamma}_{0,t} \right|_{\hat{x}} + \hat{\gamma}_{I,I} \right) \mathrm{d}\hat{Q}. \tag{1.212}$$

1.3.2 ALE Formulation of Fluid Mechanics

We begin with the conservative form of the linear-momentum equation written on the spatial domain Ω_t, $t \in (0, T)$ (see Equation (1.1)):

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma}) - \rho \mathbf{f} = \mathbf{0}. \tag{1.213}$$

This equation is a suitable starting point for space–time finite element discretizations, which approximate both the space and time behavior using basis functions. However, if one wants to use a more standard semi-discrete approach, in which the space part is handled with finite elements and the time part is handled with a finite-difference-like time integration method, Equation (1.213) is not a convenient starting point. To arrive at a form of the differential equations that is suitable for a semi-discrete approach, we first integrate Equation (1.213) over Q_t:

$$\int_{Q_t} \left(\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma}) - \rho \mathbf{f} \right) \mathrm{d}Q = \mathbf{0}, \tag{1.214}$$

and then rewrite the result using index notation as

$$\int_{Q_t} \left((\rho u_i)_{,t} + \left(\rho u_i u_j - \sigma_{ij} \right)_{,j} - \rho f_i \right) \mathrm{d}Q = 0. \tag{1.215}$$

Changing variables $Q_t \to \hat{Q}$ in Equation (1.215) and applying the space–time Piola transformation given by Equation (1.209) (with $\gamma_0 = \rho u_i$ and $\gamma_j = \rho u_i u_j - \sigma_{ij}$ for each component i), we obtain

$$\int_{\hat{Q}} \left(\left(\hat{J} \rho u_i \right)_{,t} \Big|_{\hat{x}} + \left(\hat{J} \left(\rho u_i (u_j - \hat{u}_j) - \sigma_{ij} \right) \hat{F}^{-1}_{Jj} \right)_{,J} - \hat{J} \rho f_i \right) \mathrm{d}\hat{Q} = 0. \tag{1.216}$$

Because $\int_{\hat{Q}} = \int_0^T \int_{\hat{\Omega}}$, and $\hat{\mathbf{x}}$ and t are independent variables, we can interchange the order of space and time integrations, namely $\int_{\hat{Q}} = \int_{\hat{\Omega}} \int_0^T$. We note that this cannot be done for $\int_{Q_t}$, because $\mathbf{x}$ and t are not independent. Furthermore, because the time interval is arbitrary, we can "localize" Equation (1.216) in time to obtain

$$\int_{\hat{\Omega}} \left(\left(\hat{J}\rho u_i \right)_{,t}\Big|_{\hat{x}} + \left(\hat{J} \left(\rho u_i \left(u_j - \hat{u}_j \right) - \sigma_{ij} \right) \hat{F}^{-1}_{Jj} \right)_{,J} - \hat{J}\rho f_i \right) \mathrm{d}\hat{\Omega} = 0. \tag{1.217}$$

Changing variables $\hat{\Omega} \to \Omega_t$ in Equation (1.217) and applying the spatial Piola transformation given by Equation (1.203) gives

$$\int_{\Omega_t} \left(\frac{1}{\hat{J}} \left(\hat{J}\rho u_i \right)_{,t}\Big|_{\hat{x}} + \left(\rho u_i (u_j - \hat{u}_j) - \sigma_{ij} \right)_{,j} - \rho f_i \right) \mathrm{d}\Omega = 0. \tag{1.218}$$

Localizing Equation (1.218) in space gives a point-wise balance of linear momentum:

$$\frac{1}{\hat{J}} \left(\hat{J}\rho u_i \right)_{,t}\Big|_{\hat{x}} + \left(\rho u_i (u_j - \hat{u}_j) - \sigma_{ij} \right)_{,j} - \rho f_i = 0, \tag{1.219}$$

which may be rewritten using vector notation as

$$\frac{1}{\hat{J}} \frac{\partial \hat{J}\rho \mathbf{u}}{\partial t}\Big|_{\hat{x}} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes (\mathbf{u} - \hat{\mathbf{u}}) - \boldsymbol{\sigma}) - \rho \mathbf{f} = \mathbf{0}. \tag{1.220}$$

Equation (1.220) above is the *conservative form* of the linear-momentum balance equation of fluid mechanics in the ALE description. This form is often taken as the starting point of the numerical formulations of fluid mechanics on moving domains (see, e.g., Le Tallec and Mouro, 2001).

A so-called *convective form* of the ALE equations may be obtained from the conservative form as follows. We first differentiate through the time derivative and convective terms in Equation (1.220) to obtain

$$\rho \left(\frac{1}{\hat{J}} \frac{\partial \hat{J}}{\partial t}\Big|_{\hat{x}} \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t}\Big|_{\hat{x}} \right) + \rho \left(\mathbf{u}\boldsymbol{\nabla} \cdot (\mathbf{u} - \hat{\mathbf{u}}) + (\mathbf{u} - \hat{\mathbf{u}}) \cdot \boldsymbol{\nabla}\mathbf{u} \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho \mathbf{f} = \mathbf{0}, \tag{1.221}$$

where we also assume that the density ρ is constant. Using a well-known identity in continuum mechanics (see, e.g., Wriggers, 2008)

$$\frac{\partial \hat{J}}{\partial t}\Big|_{\hat{x}} = \hat{J}\boldsymbol{\nabla} \cdot \hat{\mathbf{u}} \tag{1.222}$$

and the incompressibility constraint $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ in Equation (1.221), we obtain

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t}\Big|_{\hat{x}} + (\mathbf{u} - \hat{\mathbf{u}}) \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{f} \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0}. \tag{1.223}$$

This is the convective form of the linear-momentum balance equation of incompressible flows in the ALE description. Note that Equation (1.223) is a substantially simplified version of

Equation (1.220). While in the fully continuous setting the conservative and convective forms of the fluid mechanics equations are equivalent, this is not always the case in the discrete setting. See the remarks below for further elaboration.

REMARK 1.8 *The discrete geometric conservation law is satisfied if the numerical formulation preserves a constant fluid velocity in space and time when there are no body forces and the stress tensor is self-equilibrating (see, e.g., Farhat et al., 2001 and references therein). If the constant fluid velocity is assumed, it is easily seen that Equation (1.223), corresponding to the convective form of the linear-momentum equations, is identically satisfied. Furthermore, assuming that the time integration method "respects" a constant solution, that is, if the velocity field is constant in time the discrete approximation to the time derivative is zero, then the formulation satisfies the geometric conservation law at the fully discrete level. Any reasonable time integration method should satisfy this condition.*

REMARK 1.9 *The satisfaction of the discrete geometric conservation law for the ALE formulation based on the conservative form of the linear-momentum balance given by Equation (1.220) depends on whether the identity given by Equation (1.222) holds at the fully discrete level. Due to the different treatment of space and time discretizations in ALE methods, Equation (1.222) may not be satisfied in the fully discrete case.*

REMARK 1.10 *The situation is reversed for the global conservation of linear momentum. The conservative form of the linear-momentum equations typically leads to fully discrete formulations that are globally momentum-conserving. Global momentum conservation for ALE formulations based on the convective form of the equations typically holds only up to the time discretization (see, e.g., Bazilevs et al., 2008).*

REMARK 1.11 *Because in space–time formulations the basis functions depend on both space and time, Equation (1.222) holds at the fully discrete level. As a result, space–time formulations naturally satisfy the discrete geometric conservation law and the global conservation of linear momentum.*