# II.1
# Factor Models

## II.1.1 INTRODUCTION

This chapter describes the factor models that are applied by portfolio managers to analyse the potential returns on a portfolio of risky assets, to choose the optimal allocation of their funds to different assets and to measure portfolio risk. The theory of linear regression-based factor models applies to most portfolios of risky assets, excluding options portfolios but including alternative investments such as real estate, hedge funds and volatility, as well as traditional assets such as commodities, stocks and bonds. Stocks and bonds are the major categories of risky assets, and whilst bond portfolios could be analysed using regression-based factor models a much more powerful factor analysis for bond portfolios is based on principal component analysis (see Chapter II.2).

An understanding of both multiple linear regression and matrix algebra is necessary for the analysis of multi-factor models. Therefore, we assume that readers are already familiar with matrix theory from Chapter I.2 and the theory of linear regression from Chapter I.4. We also assume that readers are familiar with the theory of asset pricing and the optimal capital allocation techniques that were introduced in Chapter I.6.

Regression-based factor models are used to forecast the expected return and the risk of a portfolio. The expected return on each asset in the portfolio is approximated as a weighted sum of the expected returns to several market risk factors. The weights are called *factor sensitivities* or, more specifically, *factor betas* and are estimated by regression. If the portfolio only has cash positions on securities in the same country then market risk factors could include broad market indices, industry factors, style factors (e.g. value, growth, momentum, size), economic factors (e.g. interest rates, inflation) or statistical factors (e.g. principal components).[1] By inputting scenarios and stress tests on the expected returns and the volatilities and correlations of these risk factors, the factor model representation allows the portfolio manager to examine expected returns under different market scenarios.

Factor models also allow the market risk manager to quantify the systematic and specific risk of the portfolio:

- The market risk management of portfolios has traditionally focused only on the *undiversifiable risk* of a portfolio. This is the risk that cannot be reduced to zero by holding a large and diversified portfolio. In the context of a factor model, which aims to relate the distribution of a portfolio's return to the distributions of its risk factor returns, we also call the undiversifiable risk the *systematic risk*. A multi-factor model, i.e. a factor model with more than one risk factor, would normally be estimated using a multiple linear regression where the dependent variable is the return on an individual asset and the

[1] But for international portfolios exchange rates also affect the returns, with a beta of one. And if the portfolio contains futures then zero coupon rates should also be included in the market risk factors.

independent variables are the returns on different risk factors. Then the systematic risk is identified with the risk of the factor returns and the net portfolio sensitivities to each risk factor.

- The *specific risk*, also called the *idiosyncratic risk* or *residual risk*, is the risk that is not associated with the risk factor returns. In a linear regression model of the asset return on risk factor returns, it is the risk arising from the variance of the residuals. The specific risk on an individual asset may be high, especially when the model has only a few factors to explain the asset's returns. But in a sufficiently large and diversified portfolio the specific risk may be reduced to almost zero, since the specific risks on a large number of assets in different sectors of the economy, or in different countries, tend to cancel each other out.

The outline of the chapter is as follows. Section II.1.2 explains how a single-factor model is estimated. We compare two methods for estimating factor betas and show how the total risk of the portfolio can be decomposed into the systematic risk due to risk of the factors, and the specific risk that may be diversified away by holding a sufficiently large portfolio. Section II.1.3 describes the general theory of multi-factor models and explains how they are used in style attribution analysis. We explain how multi-factor models may be applied to different types of portfolios and to decompose the total risk into components related to broad classes of risk factors. Then in Section II.1.4 we present an empirical example which shows how to estimate a fundamental factor model using time series data on the portfolio returns and the risk factor returns. We suggest a remedy for the problem of multicollinearity that arises here and indeed plagues the estimation of most fundamental factor models in practice.

Then Section II.1.5 analyses the Barra model, which is a specific multi-factor model that is widely used in portfolio management. Following on from the Barra model, we analyse the way some portfolio managers use factor models to quantify *active risk,* i.e. the risk of a fund relative to its benchmark. The focus here is to explain why it is a mistake to use *tracking error*, i.e. the volatility of the active returns, as a measure of active risk. Tracking error is a metric for active risk only when the portfolio is tracking the benchmark. Otherwise, an increase in tracking error does not indicate that active risk is increased and a decrease in tracking error does not indicate that active risk has been reduced. The active risk of actively managed funds which by design do not track a benchmark cannot be measured by tracking error. However, we show how it is possible to adjust the tracking error into a correct, but basic active risk metric. Section II.1.6 summarizes and concludes.

## II.1.2 SINGLE FACTOR MODELS

This section describes how single factor models are applied to analyse the expected return on an asset, to find a portfolio of assets to suit the investor's requirements, and to measure the risk of an existing portfolio. We also interpret the meaning of a factor beta and derive a fundamental result on portfolio risk decomposition.

### II.1.2.1 Single Index Model

The capital asset pricing model (CAPM) was introduced in Section I.6.4. It hypothesizes the following relationship between the expected excess return on any single risky asset and the expected excess return on the market portfolio:

$$E(R_i) - R_f = \beta_i(E(R_M) - R_f),$$

where $R_i$ is the return on the $i$th risky asset, $R_f$ is the return on the *risk free* asset, $R_M$ is the return on the *market portfolio* and $\beta_i$ is the beta of the $i$th risky asset. The CAPM implies the following linear model for the relationship between ordinary returns rather than excess returns:

$$E(R_i) = \alpha_i + \beta_i E(R_M), \tag{II.1.1}$$

where $\alpha_i \neq 0$ unless $\beta_i = 1$.

The single index model is based on the expected return relationship (II.1.1) where the return $X$ on a factor such as a broad market index is used as a proxy for the market portfolio return $R_M$. Thus the single index model allows one to investigate the risk and return characteristics of assets *relative* to the broad market index. More generally, if the performance of a portfolio is measured relative to a benchmark other than a broad market index, then the benchmark return is used for the factor return $X$.

We can express the single index model in the form

$$R_{it} = \alpha_i + \beta_i X_t + \varepsilon_{it}, \quad \varepsilon_{it} \sim \text{i.i.d.}(0, \sigma_i^2). \tag{II.1.2}$$

Here $\alpha_i$ measures the asset's expected return relative to the benchmark or index (a positive value indicates an expected outperformance and a negative value indicates an expected underperformance); $\beta_i$ is the *risk factor sensitivity* of the asset; $\beta_i\sigma_X$ is the *systematic volatility* of the asset, $\sigma_X$ being the volatility of the index returns; and $\sigma_i$ is the *specific volatility* of the asset.

Consider a portfolio containing $m$ risky assets with portfolio weights $\mathbf{w} = (w_1, w_2, \ldots, w_m)'$, and suppose that each asset has a returns representation (II.1.2). Then the portfolio return may be written

$$Y_t = \alpha + \beta X_t + \varepsilon_t \quad t = 1, \ldots, T, \tag{II.1.3}$$

where each characteristic of the portfolio (i.e. its alpha and beta and its specific return) is a weighted sum of the individual assets' characteristics, i.e.

$$\alpha = \sum_{i=1}^{m} w_i\alpha_i, \quad \beta = \sum_{i=1}^{m} w_i\beta_i, \quad \varepsilon_t = \sum_{i=1}^{m} w_i\varepsilon_{it}. \tag{II.1.4}$$

Now the portfolio's characteristics can be estimated in two different ways:

- Assume some portfolio weights $\mathbf{w}$ and use estimates of the alpha, beta and residuals for each asset in (II.1.4) to infer the characteristics of this hypothetical portfolio. This way an *asset manager* can compare many different portfolios for recommendation to his investors.
- A *risk manager*, on the other hand, will apply the weights $\mathbf{w}$ of an existing portfolio that is held by an investor to construct a constant weighted artificial returns history for the portfolio. This series is used for $Y_t$ in (II.1.3) to assess the relative performance, the systematic risk and the specific risk of an existing portfolio.[2]

Thus risk managers and asset managers apply the same factor model in different ways, because they have different objectives. Asset managers need estimates of (II.1.2) for every

[2] The reconstructed 'constant weight' series for the portfolio returns will not be the same as the actual historical returns series for the portfolio, unless the portfolio was rebalanced continually so as to maintain the weights constant. The reason for using current weights is that the risk manager needs to represent the portfolio as it is now, not as it was last week or last year, and to use this representation to forecast its risk over a future risk horizon of a few days, weeks or months.

asset in the investor's universe in order to forecast the performance of many different portfolios and hence construct an optimal portfolio; by contrast, a risk manager takes an existing portfolio and uses (II.1.3) to forecast its risk characteristics. The next section explains how risk managers and asset managers also use different data and different statistical techniques to estimate the factor models that they use.

### II.1.2.2 Estimating Portfolio Characteristics using OLS

The main lesson to learn from this section is that risk managers and asset managers require quite different techniques to estimate the parameters of factor models because they have different objectives:

- When asset managers employ a factor model of the form (II.1.2) they commonly use long histories of asset prices and benchmark values, measuring returns at a weekly or monthly frequency and assuming that the true parameters are constant. In this case, the *ordinary least squares* (OLS) estimation technique is appropriate and the more data used to estimate them the better, as the sampling error will be smaller. Three to five years of monthly or weekly data is typical.
- When risk managers employ a factor model of the form (II.1.3) they commonly use shorter histories of portfolio and benchmark values than the asset manager, measuring returns daily and not assuming that the true values of the parameters are constant. In this case, a time varying estimation technique such as exponentially weighted moving averages or generalized autoregressive conditional heteroscedasticity is appropriate.

We shall now describe how to estimate (II.1.2) and (II.1.3) using the techniques that are appropriate for their different applications. For model (II.1.2) the OLS parameter estimates based on a sample of size $T$ are given by the formulae[3]

$$\hat{\beta}_i = \frac{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)\left(R_{it} - \overline{R}_i\right)}{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)^2} \quad \text{and} \quad \hat{\alpha}_i = \overline{R}_i - \hat{\beta}_i \overline{X}, \tag{II.1.5}$$

where $\overline{X}$ denotes the sample mean of the factor returns and $\overline{R}_i$ denotes the sample mean of the $i$th asset returns. The OLS estimate of the specific risk of the $i$th asset is the estimated standard error of the model, given by

$$s_i = \sqrt{\frac{RSS_i}{T-2}}, \tag{II.1.6}$$

where $RSS_i$ is the residual sum of squares in the $i$th regression. See Section I.4.2 for further details. The following example illustrates the use of these formulae to estimate model (II.1.2) for two US stocks, using the S&P 500 index as the risk factor.

Example II.1.1: OLS estimates of alpha and beta for two stocks

Use weekly data from 3 January 2000 until 27 August 2007 to estimate a single factor model for the Microsoft Corporation (MSFT) stock and the National Western Life Insurance Company (NWL) stock using the S&P 500 index as the risk factor.[4]

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3 See Section I.4.2.2.

4 Dividend adjusted data were downloaded from Yahoo! Finance.

(a) What do you conclude about the stocks' characteristics?
(b) Assuming the stocks' specific returns are uncorrelated, what are the characteristics of a portfolio with 70% of its funds invested in NWL and 30% invested in MSFT?

SOLUTION The spreadsheet for this example computes the weekly returns on the index and on each of the stocks and then uses the Excel regression data analysis tool as explained in Section I.4.2.7. The results are

$$\begin{aligned} R_{\mathrm{NWL}} &= \underset{(2.224)}{0.00358} + \underset{(7.129)}{0.50596}\, R_{\mathrm{SPX}}, \quad s_{\mathrm{NWL}} = 0.03212, \\ R_{\mathrm{MSFT}} &= \underset{(-0.3699)}{-0.00066} + \underset{(14.002)}{1.10421}\, R_{\mathrm{SPX}}, \quad s_{\mathrm{MSFT}} = 0.03569, \end{aligned} \tag{II.1.7}$$

where the figures in parentheses are the $t$ ratios. We conclude the following:

- Since $\hat{\alpha}_{\mathrm{NWL}} = 0.00358$ and this is equivalent to an average outperformance of 18.6% per annum, NWL is a stock with a significant alpha. It also has a low systematic risk because $\hat{\beta}_{\mathrm{NWL}} = 0.50596$, which is much less than 1. Its specific risk, expressed as an annual volatility, is $0.03212 \times \sqrt{52} = 23.17\%$.
- Since the $t$ ratio on $\hat{\alpha}_{\mathrm{MSFT}}$ is very small, MSFT has no significant outperformance or underperformance of the index. It also has a high systematic risk because the beta is slightly greater than 1 and a specific risk of $0.03569 \times \sqrt{52} = 25.74\%$, which is greater than the specific risk of NWL.

Now applying (II.1.4) gives a portfolio with the following characteristics:

$$\begin{aligned} \hat{\alpha} &= 0.7 \times 0.00358 - 0.3 \times 0.00066 = 0.00231, \\ \hat{\beta} &= 0.7 \times 0.50596 + 0.3 \times 1.10421 = 0.68543, \end{aligned}$$

and assuming the specific returns are uncorrelated implies that we can estimate the specific risk of the portfolio as

$$s = \sqrt{0.7^2 \times 23.17^2 + 0.3^2 \times 25.74^2} = 17.96\%.$$

The next example shows that it makes no difference to the portfolio alpha and beta estimates whether we estimate them:

- from the OLS regressions for the stocks, applying the portfolio weights to the stocks alphas and betas using (II.1.4) as we did above;
- by using an OLS regression of the form (II.1.3) on the constant weighted portfolio returns.

However, it does make a difference to our estimate of the specific risk on the portfolio!

EXAMPLE II.1.2: OLS ESTIMATES OF PORTFOLIO ALPHA AND BETA

A portfolio has 60% invested in American Express (AXP) stock and 40% invested in Cisco Systems (CSCO). Use daily data from 3 January 2000 to 31 December 2007 on the prices of these stocks and on the S&P 100 index (OEX) to estimate the portfolio's characteristics by:[5]

[5] Data were downloaded from Yahoo! Finance. The reason we use log returns in this example is explained in Section I.1.4.4.

(a) applying the same method as in Example II.1.1;
(b) regressing the constant weighted returns series {0.6 × Amex Return + 0.4 × Cisco Return} on the index returns.

SOLUTION The results are computed using an OLS regression of each stock return and of the constant weighted portfolio returns, and the alpha and beta estimates are summarized in Table II.1.1. Note that for the first two rows the last column is a weighted sum of the first two. That is, the portfolio's alpha could equally well have been calculated by just taking the weighted sum of the stocks' alphas, and similarly for the beta. However, if we compute the specific risk of the portfolio using the two methods we obtain, using method (a),

$$s_P = \sqrt{0.6^2 \times 0.01416^2 + 0.4^2 \times 0.02337^2} \times \sqrt{250} = 19.98\%.$$

But using method (b), we have

$$s_P = 0.01150 \times \sqrt{250} = 18.19\%.$$

The problem is that the specific risks are *not* uncorrelated, even though we made this assumption when we applied method (a).

**Table II.1.1** OLS alpha, beta and specific risk for two stocks and a 60:40 portfolio

| | Amex | Cisco | Portfolio |
|---|---|---|---|
| Alpha | 0.00018 | −0.00022 | 0.00002 |
| Beta | 1.24001 | 1.76155 | 1.44863 |
| Regression standard error | 0.01416 | 0.02337 | 0.01150 |
| Specific risk | 22.39 % | 36.96 % | 18.19 % |

We conclude that to estimate the specific risk of a portfolio we need to apply method (b). That is, we need to reconstruct a constant weighted portfolio series and calculate the specific risk from that regression. Alternatively and equivalently, we can save the residuals from the OLS regressions for each stock return and calculate the covariance matrix of these residuals. More details are given in Section II.1.3.3 below.

### II.1.2.3 Estimating Portfolio Risk using EWMA

Whilst OLS may be adequate for asset managers, it is not appropriate to use a long price history of monthly or weekly data for the risk management of portfolios. Market risks require monitoring on a frequent basis – daily and even intra-daily – and the parameter estimates given by OLS will not reflect current market conditions. They merely represent an *average* value over the time period covered by the sample used in the regression model.

So, for the purpose of mapping a portfolio and assessing its risks, higher frequency data (e.g. daily) could be used to estimate a time varying portfolio beta for the model

$$Y_t = \alpha_t + \beta_t X_t + \varepsilon_t, \tag{II.1.8}$$

where $X_t$ and $Y_t$ denote the returns on the market factor and on the stock (or portfolio), respectively, at time $t$. In this model the systematic and specific risks are no longer assumed constant over time. The time varying beta estimates in (II.1.8) better reflect the current risk factor sensitivity for daily risk management purposes. To estimate time varying betas we cannot apply OLS so that it covers only the recent past. This approach will lead to very significant problems, as demonstrated in Section II.3.6. Instead, a simple time varying model for the covariance and variance may be applied to estimate the parameters of (II.1.8). The simplest possible time varying parameter estimates are based on an *exponentially weighted moving average* (EWMA) model. However the EWMA model is based on a very simple assumption, that returns are i.i.d. The EWMA beta estimates vary over time, even though the model specifies only a constant, unconditional covariance and variance. More advanced techniques include the class of *generalized autoregressive conditional heteroscedasticity* (GARCH) models, where we model the *conditional* covariance and variance and so the true parameters as well as the parameter estimates change over time.[6]

A time varying beta is estimated as the covariance of the asset and factor returns divided by the variance of the factor returns. Denoting the EWMA smoothing constant by $\lambda$, the EWMA estimate of beta that is made at time $t$ is

$$\hat{\beta}_t^{\lambda} = \frac{\mathrm{Cov}_{\lambda}(X_t, Y_t)}{V_{\lambda}(X_t)}. \tag{II.1.9}$$

That is, the EWMA beta estimate is the ratio of the EWMA covariance estimate to the EWMA variance estimate with the *same* smoothing constant. The modeller must choose a value for $\lambda$ between 0 and 1, and values are normally in the region of 0.9–0.975. The decision about the value of $\lambda$ is discussed in Section II.3.7.2.

We now provide an example of calculating the time varying EWMA betas for the portfolio in Example II.1.2. Later on, in Section II.4.8.3 we shall compare this beta with the beta that is obtained using a simple bivariate GARCH model. We assume $\lambda = 0.95$, which corresponds to a half-life of approximately 25 days (or 1 month, in trading days) and compare the EWMA betas with the OLS beta of the portfolio that was derived in Example II.1.2. These are shown in Figure II.1.1, with the OLS beta of 1.448 indicated by a horizontal grey line. The EWMA beta, measured on the left-hand scale, is the time varying black line. The OLS beta is the average of the EWMA betas over the sample. Also shown in the figure is the EWMA estimate of the systematic risk of the portfolio, given by

$$\text{Systematic Risk} = \hat{\beta}_t^{\lambda}\sqrt{V_{\lambda}(X_t)} \times \sqrt{h}, \tag{II.1.10}$$

where $h$ denotes the number of returns per year, assumed to be 250 in this example.

During 2001 the portfolio had a beta much greater than 1.448, and sometimes greater than 2. The opposite is the case during the latter part of the sample. But note that this remark does depend on the choice of $\lambda$: the greater the value of $\lambda$ the smoother the resulting series, and when $\lambda = 1$ the EWMA estimate coincides with the OLS estimate. However, when $\lambda < 1$ the single value of beta, equal to 1.448, that is obtained using OLS does not reflect the day-to-day variation in the portfolio's beta as measured by the EWMA estimate.

A time varying estimate of the systematic risk is also shown in Figure II.1.1. The portfolio's systematic risk is depicted in the figure as an annualized percentage, measured on the right-hand scale. There are two components of the systematic risk, the beta and the volatility of the market factor, and the systematic risk is the *product* of these. Hence the systematic risk was relatively low, at around 10% for most of the latter part of the sample even though the

[6] EWMA and GARCH models are explained in detail in Chapters II.3 and II.4.

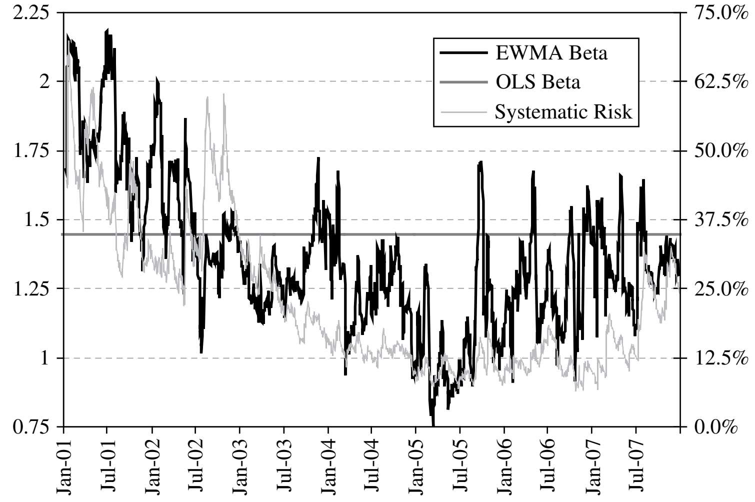


**Figure II.1.1** EWMA beta and systematic risk of the two-stock portfolio

portfolio's beta was greater than 1, because the S&P 100 index had a very low volatility during this period. On the other hand, in August and October 2002 the portfolio had a high systematic risk, not because it had a high beta but because the market was particularly volatile then. By contrast, the OLS estimate of systematic risk is unable to reflect such time variation. The average volatility of the S&P 100 over the entire sample was 18.3% and so OLS produces the single estimate of $18.3\% \times 1.448 = 26.6\%$ for systematic risk. This figure represents only an average of the systematic risk over the sample period.

### II.1.2.4 Relationship between Beta, Correlation and Relative Volatility

In the single index model the beta, market correlation and relative volatility of an asset or a portfolio with return $Y$ when the market return is $X$ are defined as

$$\beta = \frac{\mathrm{Cov}(X, Y)}{V(X)}, \quad \varrho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{V(X)V(Y)}}, \quad \nu = \sqrt{\frac{V(Y)}{V(X)}} \tag{II.1.11}$$

Hence,

$$\beta = \varrho\nu, \tag{II.1.12}$$

i.e. the *equity beta* is the product of the *market correlation* $\varrho$ and the *relative volatility* $\nu$ of the portfolio with respect to the index or benchmark.

The correlation is bounded above and below by $+1$ and $-1$ and the relative volatility is always positive. So the portfolio beta can be very large and negative if the portfolio is negatively correlated with the market, which happens especially when short positions are held. On the other hand, very high values of beta can be experienced for portfolios containing many risky stocks that are also highly correlated with the market.

In Figures II.1.2 and II.1.3 we show the daily EWMA estimates of beta, relative volatility and correlation (on the right-hand scale) of the Amex and Cisco stocks between

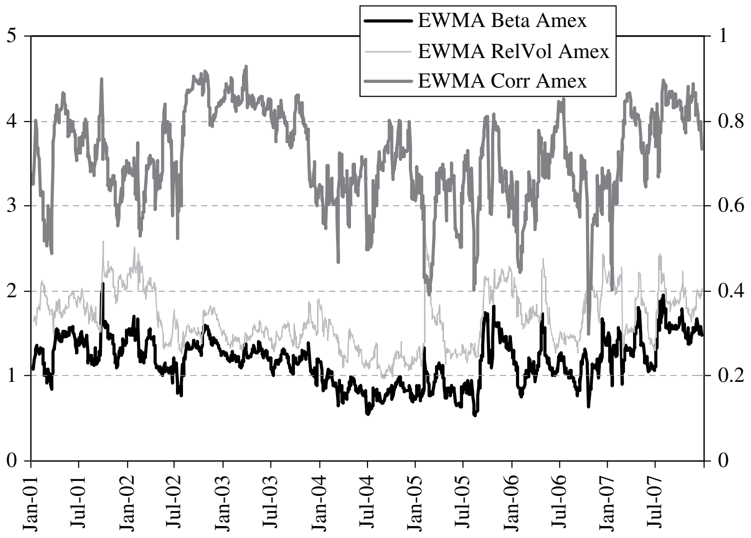


**Figure II.1.2** EWMA beta, relative volatility and correlation of Amex ($\lambda = 0.95$)

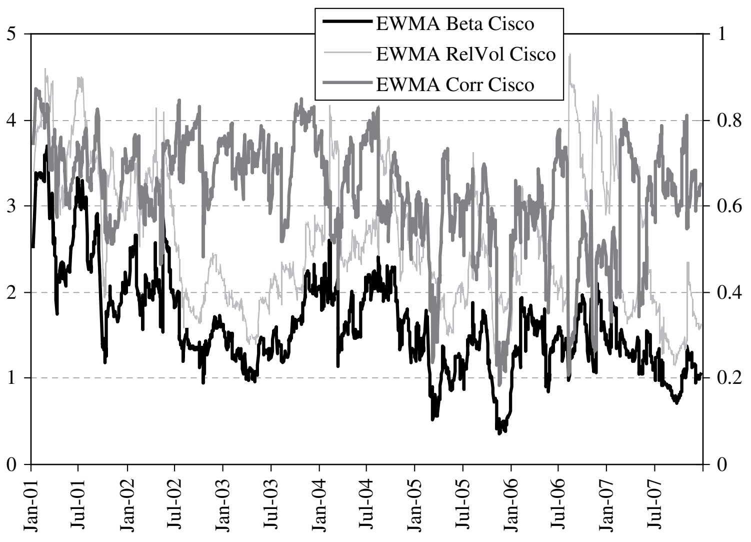


**Figure II.1.3** EWMA beta, relative volatility and correlation of Cisco ($\lambda = 0.95$)

January 2001 and December 2007.[7] The same scales are used in both graphs, and it is clear that Cisco has a greater systematic risk than Amex. The average market correlation of both stocks is higher for Amex (0.713 for Amex and 0.658 for Cisco) but Cisco is much more volatile than Amex, relative to the market. Hence, EWMA correlation is more unstable and its EWMA beta is usually considerably higher than the beta on Amex.

[7] As before, $\lambda = 0.95$.

### II.1.2.5 Risk Decomposition in a Single Factor Model

The principle of portfolio diversification implies that asset managers can reduce the specific risk of their portfolio by diversifying their investments into a large number of assets that have low correlation – and/or by holding long and short positions on highly correlated assets. This way the portfolio's specific risk can become insignificant. *Passive managers*, traditionally seeking only to track the market index, should aim for a net $\alpha = 0$ and a net portfolio $\beta = 1$ whilst simultaneously reducing the portfolio's specific risk as much as possible. *Active managers*, on the other hand, may have betas that are somewhat greater than 1 if they are willing to accept an increased systematic risk for an incremental return above the index.

Taking the expectation and variance of (II.1.3) gives

$$E(Y) = \alpha + \beta E(X). \tag{II.1.13}$$

If we assume $\mathrm{Cov}(X, \varepsilon) = 0$,

$$V(Y) = \beta^2 V(X) + V(\varepsilon). \tag{II.1.14}$$

It is very important to recognize that the total portfolio variance (II.1.14) represents the variance of portfolio returns around the expected return (II.1.13). It does not represent the variance about any other value! This is a common mistake and so I stress it here: it is statistical nonsense to measure the portfolio variance using a factor model and then to assume this figure represents the dispersion of portfolio returns around a mean that is anything other than (II.1.13). For example, the variance of a portfolio that is estimated from a factor model does *not* represent the variance about the target returns, except in the unlikely case that the expected return that is estimated by the model is equal to this target return.

The first term in (II.1.14) represents the systematic risk of the portfolio and the second represents the specific risk. When risk is measured as standard deviation the systematic risk component is $\beta\sqrt{V(X)}$ and the specific risk component is $\sqrt{V(\varepsilon)}$. These are normally quoted as an annualized percentage, as in the estimates given in the examples above.

From (II.1.14) we see that the volatility of the portfolio return – about the expected return given by the factor model – can be decomposed into three sources:

- the sensitivity to the market factor beta,
- the volatility of the market factor, and
- the specific risk.

One of the limitations of the equity beta as a risk measure is that it ignores the other two sources of risk: it says nothing about the risk of the market factor itself or about the specific risk of the portfolio.

We may express (II.1.14) in words as

$$\text{Total Variance} = \text{Systematic Variance} + \text{Specific Variance} \tag{II.1.15}$$

or, since risk is normally identified with standard deviation (or annualized standard deviation, i.e. volatility),

$$\text{Total Risk} = (\text{Systematic Risk}^2 + \text{Specific Risk}^2)^{1/2}. \tag{II.1.16}$$

Thus the components of risk are *not* additive. Only variance is additive, and then only under the assumption that the covariance between each risk factor's return and the specific return is 0.

## II.1.3 MULTI-FACTOR MODELS

The risk decomposition (II.1.14) rests on an assumption that the benchmark or index is uncorrelated with the specific returns on a portfolio. That is, we assumed in the above that $\mathrm{Cov}(X, \varepsilon) = 0$. But this is a very strong assumption that would not hold if there were important risk factors for the portfolio, other than the benchmark or index, that have some correlation with the benchmark or index. For this reason single factor models are usually generalized to include more than one risk factor, as assumed in the *arbitrage pricing theory* developed by Ross (1976). By generalizing the single factor model to include many risk factors, it becomes more reasonable to assume that the specific return is not correlated with the risk factors and hence the risk decomposition (II.1.16) is more likely to hold.

The success of multi-factor models in predicting returns in financial asset markets and analysing risk depends on both the choice of risk factors and the method for estimating factor sensitivities. Factors may be chosen according to fundamentals (price–earning ratios, dividend yields, style factors, etc.), economics (interest rates, inflation, gross domestic product, etc.), finance (such as market indices, yield curves and exchange rates) or statistics (e.g. principal component analysis or factor analysis). The factor sensitivity estimates for fundamental factor models are sometimes based on cross-sectional regression; economic or financial factor model betas are usually estimated via time series regression; and statistical factor betas are estimated using statistical techniques based on the analysis of the eigenvectors and eigenvalues of the asset returns covariance or correlation matrix. These specific types of multi-factor models are discussed in Sections II.1.4–II.1.6 below. In this section we present the general theory of multi-factor models and provide several empirical examples.

### II.1.3.1 Multi-factor Models of Asset or Portfolio Returns

Consider a set of $k$ risk factors with returns $X_1, \ldots, X_k$ and let us express the systematic return of the asset or the portfolio as a weighted sum of these. In a multi-factor model for an asset return or a portfolio return, the return $Y$ is expressed as a sum of the systematic component and an idiosyncratic or specific component $\varepsilon$ that is not captured by the risk factors. In other words, a multi-factor model is a multiple regression model of the form[8]

$$Y_t = \alpha + \beta_1 X_{1t} + \ldots + \beta_k X_{kt} + \varepsilon_t. \tag{II.1.17}$$

In the above we have used a subscript $t$ to denote the time at which an observation is made. However, some multi-factor models are estimated using cross-sectional data, in which case the subscript $i$ would be used instead.

#### *Matrix Form*

It is convenient to express (II.1.17) using matrix notation, but here we use a slightly different notation from that which we introduced for multivariate regression in Section I.4.4.2. For reasons that will become clear later, and in particular when we analyse the Barra model, it helps to isolate the constant term alpha in the matrix notation. Thus we write

$$\mathbf{y} = \boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2), \tag{II.1.18}$$

[8] In this chapter, since we are dealing with *alpha models*, it is convenient to separate the constant term alpha from the other coefficients. Hence we depart from the notation used for multiple regression models in Chapter I.4. There the total number of coefficients including the constant is denoted $k$, but here we have $k + 1$ coefficients in the model.

where the data may be cross-sectional or time series, $\mathbf{y}$ is the column of data on the asset or portfolio return, $\mathbf{X}$ is a matrix containing the data on the risk factor returns, $\boldsymbol{\alpha}$ is the vector $\alpha\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)'$, $\boldsymbol{\beta}$ is the vector $(\beta_1, \ldots, \beta_k)'$ of the asset or portfolio betas with respect to each risk factor, and $\boldsymbol{\varepsilon}$ is the vector of the asset's or portfolio's specific returns.

*OLS Estimation*

We remark that (II.1.18) is equivalent to

$$\mathbf{y} = \tilde{\mathbf{X}}\tilde{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \text{i.i.d.}\left(0, \sigma^2\mathbf{I}\right), \tag{II.1.19}$$

where $\mathbf{I}$ is the identity matrix and

$$\tilde{\mathbf{X}} = \begin{pmatrix} \mathbf{1} & \mathbf{X} \end{pmatrix} \text{ and } \tilde{\boldsymbol{\beta}} = \begin{pmatrix} \alpha \\ \boldsymbol{\beta} \end{pmatrix}.$$

To write down an expression for the OLS estimates of the portfolio alpha and betas, it is easier to use (II.1.19) than (II.1.18). Since (II.1.19) is the same matrix form as in Section I.4.4.2, the OLS estimator formula is

$$\hat{\tilde{\boldsymbol{\beta}}} = \left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-1}\tilde{\mathbf{X}}'\mathbf{y}. \tag{II.1.20}$$

*Expected Return and Variance Decomposition*

Applying the expectation and variance operators to (II.1.18) and assuming that the idiosyncratic return is uncorrelated with each of the risk factor returns, we have

$$E(Y) = \alpha + \boldsymbol{\beta}'E(\mathbf{X}) \tag{II.1.21}$$

and

$$V(Y) = \boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta} + V(\varepsilon), \tag{II.1.22}$$

where $E(\mathbf{X})$ is the vector of expected returns to each risk factor and $\boldsymbol{\Omega}$ is the covariance matrix of the risk factor returns. When OLS is used to estimate $\alpha$ and $\boldsymbol{\beta}$, then $E(\mathbf{X})$ is the vector of sample averages of each of the risk factor returns, and $\boldsymbol{\Omega}$ is the equally weighted covariance matrix.

Again I stress that the portfolio variance (II.1.22) represents the dispersion of asset or portfolio returns *about the expected return* (II.1.21); it does not represent dispersion about any other centre for the distribution.

Example II.1.3: Systematic and specific risk

Suppose the total volatility of returns on a stock is 25%. A linear model with two risk factors indicates that the stock has betas of 0.8 and 1.2 on the two risk factors. The factors have volatility 15% and 20% respectively and a correlation of −0.5. How much of the stock's volatility can be attributed to the risk factors, and how large is the stock's specific risk?

Solution The risk factor's annual covariance matrix is

$$\boldsymbol{\Omega} = \begin{pmatrix} 0.0225 & -0.015 \\ -0.015 & 0.04 \end{pmatrix},$$

and the stock's variance due to the risk factors is

$$\boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta} = \begin{pmatrix} 0.8 & 1.2 \end{pmatrix} \begin{pmatrix} 0.0225 & -0.015 \\ -0.015 & 0.04 \end{pmatrix} \begin{pmatrix} 0.8 \\ 1.2 \end{pmatrix} = 0.0432.$$

The volatility due to the risk factors is the square root of 0.0432, i.e. 20.78%. Now assuming that the covariance between the specific return and the systematic return is 0 and applying (II.1.15), we decompose the total variance of $0.25^2 = 0.0625$ as

$$0.0625 = 0.0432 + 0.0193.$$

Hence, the specific volatility of the stock is $\sqrt{0.0193} = 13.89\%$.

In summary, the stock's volatility of 25% can be decomposed into two portions, 20.78% due to the risk factors and 13.89% of idiosyncratic volatility (specific risk). Note that

$$25\% = (20.78\%^2 + 13.89\%^2)^{1/2},$$

in accordance with (II.1.16).

The example above illustrates some important facts:

- When the correlation between the specific return and the systematic return is zero, the variances are additive, not the volatilities.
- When the correlation between the specific return and the systematic return is non-zero, not even the variances are additive.
- The asset or portfolio's alpha does *not* affect the risk decomposition. The alpha does, however, have an important effect on the asset or portfolio's expected return.

### II.1.3.2 Style Attribution Analysis

In 1988 the Nobel Prize winner William F. Sharpe introduced a multi-factor regression of a portfolio's returns on the returns to standard factors as a method for attributing fund managers' investment decisions to different styles.[9] For equity portfolios these standard factors, which are called *style factors*, are constructed to reflect value stocks and growth stocks, and are further divided into large, small or medium cap stocks.[10]

- A *value stock* is one that trades at a lower price than the firm's financial situation would merit. That is, the asset value per share is high relative to the stock price and the *price–earnings ratio* of the stock will be lower than the market average. Value stocks are attractive investments because they appear to be undervalued according to traditional equity analysis.[11]
- A *growth stock* is one with a lower than average *price–earnings–growth ratio*, i.e. the rate of growth of the firm's earnings is high relative to its price–earnings ratio. Hence growth stocks appear attractive due to potential growth in the firm assets.

The aim of style analysis is to identify the styles that can be associated with the major risk factors in a portfolio. This allows the market risk analyst to determine whether a fund manager's performance is attributed to investing in a certain asset class, and within this class

[9] See Sharpe (1988, 1992).

[10] Cap is short for *capitalization* of the stock, being the total value of the firm's equity that is issued to the public. It is the market value of all outstanding shares and is computed by multiplying the market price per share by the number of shares outstanding.

[11] The price–earnings ratio is the ratio of the stock's price to the firm's annual earnings per share.

investing in the best performing style, or whether his success or failure was mainly due to market timing or stock picking. It also allows the analyst to select an appropriate benchmark against which to assess the fund manager's performance. Furthermore, investors seeking a fully diversified portfolio can use style analysis to ensure their investments are spread over both growth and value investments in both large and small cap funds.

*Style Indices*

A large number of value and growth style indices based on stocks of different market caps are available, including the value and growth indices from the S&P 500, Russell 1000, Russell 2000 and Wilshire 5000 indices. As the number of stocks in the index increases, their average market cap decreases. Hence, the S&P 500 value index contains value stocks with an average market cap that is much larger then the average market cap of the stock in the Wilshire 5000 value index. The criterion used to select the stocks in any index depends on their performance according to certain value and growth indicators. Value indicators may include the *book-to-price ratio* and the *dividend yield*, and growth indicators may include the *growth in earnings per share* and the *return on equity*.[12]

*Who Needs Style Analysis?*

Whilst style analysis can be applied to any portfolio, hedge funds are a main candidate for this analysis because their investment styles may be obscure. Information about hedge funds is often hard to come by and difficult to evaluate. Because of the diverse investment strategies used by hedge funds, style indices for hedge funds include factors such as option prices, volatility, credit spreads, or indices of hedge funds in a particular category or strategy.

*How to Attribute Investment Styles to a Portfolio*

Denote by $\mathbf{y}$ the vector of historical returns on the fund being analysed, and denote by $\mathbf{X}$ the matrix of historical data on the returns to the style factors that have been chosen. The selection of the set of style indices used in the analysis is very important. We should include enough indices to represent the basic asset classes which are relevant to the portfolio being analysed and are of interest to the investor; otherwise the results will be misleading. However, the risk–return characteristics for the selected indices should be significantly different, because including too many indices often results in severe multicollinearity.[13]

Style attribution analysis is based on a multiple regression of the form (II.1.18), but with some important constraints imposed. If we are to *fully* attribute the fund's returns to the styles then the constant $\alpha$ must be 0, and the regression coefficients $\boldsymbol{\beta}$ must be non-negative and sum to 1. Assuming the residuals are i.i.d., the usual OLS regression objective applies, and we may express the estimation procedure in the form of the following *constrained least squares* problem:

$$\min_{\boldsymbol{\beta}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^2 \text{ such that } \sum_{i=1}^{k}\beta_i = 1 \text{ and } \beta_i \geq 0, i = 1, \ldots, k. \tag{II.1.23}$$

[12] Up-to-date data on a large number of style indices are free to download from Kenneth French's homepage on http://mba.tuck.dartmouth.edu/pages/faculty/Ken.french/data_library.html. Daily returns since the 1960's and monthly and annual returns since the 1920's are available on nearly 30 US benchmark portfolios.

[13] Multicollinearity was introduced in Section I.4.4.8 and discussed further in Section II.1.3.6.

This is a quadratic programming problem that can be solved using specialist software.

For illustrative purposes *only* we now implement a style analysis using the Excel Solver. However, it should be emphasized that the optimizer for (II.1.23) should be carefully designed and using the Solver is not recommended in practice. See the excellent paper by Kim et al. (2005) for further details on estimating style attribution models.

Example II.1.4: Style attribution

Perform a style analysis on the following mutual funds:

- VIT – the Vanguard Index Trust 500 Index;
- FAA – the Fidelity Advisor Aggressive Fund;
- FID – the Fidelity Main Mutual Fund.

Use the following style factors:[14]

- Russell 1000 value: mid cap, value factor;
- Russell 1000 growth: mid cap, growth factor;
- Russell 2000 value: small cap, value factor;
- Russell 2000 growth: small cap, growth factor.

Solution Daily price data adjusted for dividends are downloaded from Yahoo! Finance from January 2003 to December 2006, and the results of the Excel Solver's optimization on (II.1.23) are reported in Table II.1.2, first for 2003–2004 and then for 2005–2006. This methodology allows one to compare the style differences between funds and to assess how the styles of a given fund evolve through time.

**Table II.1.2** Results of style analysis for Vanguard and Fidelity mutual funds

| 2003–2004 | R1000V | R1000G | R2000V | R2000G |
|---|---|---|---|---|
| VIT | 92.2% | 0.0% | 0.0% | 7.8% |
| FAA | 43.7% | 5.0% | 0.0% | 51.3% |
| FID | 94.1% | 0.0% | 0.0% | 5.9% |
| 2005–2006 | R1000V | R1000G | R2000V | R2000G |
| VIT | 90.7% | 1.7% | 0.0% | 7.6% |
| FAA | 22.5% | 7.0% | 0.0% | 70.5% |
| FID | 76.8% | 3.9% | 0.0% | 19.3% |

For example, during the period 2003–2004 the FAA appears to be a fairly balanced fund between value and growth and small and mid cap stocks. Its returns could be attributed 43.7% to mid cap value stocks, 5% to mid cap growth stocks and 51.3% to small cap growth stocks. However, during the period 2005–2006 the balance shifted significantly toward small cap growth stocks, because only 22.5% of its returns were attributed to mid cap value stocks, and 7% to mid cap growth stocks, whereas 70.5% of its returns were attributed to small cap growth stocks.

[14] To reflect cash positions in the portfolio Treasury bills should be added to the list of style factors, but since our aim is simply to illustrate the methodology, we have omitted them.

### II.1.3.3 General Formulation of Multi-factor Model

We start with the assumption of a multi-factor model of the form (II.1.18) for each asset in the investment universe. Each asset is assumed to have the same set of risk factors in the theoretical model, although in the estimated models it is typical that only a few of the risk factors will be significant for any single asset. Thus we have a linear factor model,

$$Y_{jt} = \alpha_j + \beta_{j1}X_{1t} + \ldots + \beta_{jk}X_{kt} + \varepsilon_{jt}, \quad \varepsilon_{jt} \sim \text{i.i.d.}(0, \sigma_j^2), \tag{II.1.24}$$

for each asset $j = 1, \ldots, m$. The equivalent matrix form of (II.1.24) is

$$\mathbf{y}_j = \boldsymbol{\alpha}_j + \mathbf{X}\boldsymbol{\beta}_j + \boldsymbol{\varepsilon}_j, \quad \boldsymbol{\varepsilon}_j \sim \text{i.i.d.}(0, \sigma_j^2\mathbf{I}), \tag{II.1.25}$$

where $T$ is the number of observations in the estimation sample; $\mathbf{y}_j$ is the $T \times 1$ vector of data on the asset returns; $\mathbf{X}$ is the same as in (II.1.18), i.e. a $T \times k$ matrix containing the data on the risk factor returns; $\boldsymbol{\alpha}_j$ is the $T \times 1$ vector $(\alpha_j, \ldots, \alpha_j)'$; $\boldsymbol{\beta}_j$ is the $k \times 1$ vector $\left(\beta_{j1}, \ldots, \beta_{jk}\right)'$ of the asset's betas with respect to each risk factor; and $\boldsymbol{\varepsilon}_j$ is the vector of the asset's specific returns.

We can even put all the models (II.1.25) into one big matrix model, although some care is needed with notation here so that we do not lose track![15] Placing the stock returns into a $T \times m$ matrix $\mathbf{Y}$, where each column represents data on one stock return, we can write

$$\mathbf{Y} = \mathbf{A} + \mathbf{XB} + \boldsymbol{\Psi}, \quad \boldsymbol{\Psi} \sim (\mathbf{0}, \boldsymbol{\Sigma}), \tag{II.1.26}$$

where $\mathbf{X}$ is the same as above, $\mathbf{A}$ is the $T \times m$ matrix whose $j$th column is the vector $\boldsymbol{\alpha}_j$, and $\mathbf{B}$ is the $k \times m$ matrix whose $j$th column is the vector $\boldsymbol{\beta}_j$. In other words, $\mathbf{B}$ is the matrix whose $i, j$th element is the sensitivity of the $j$th asset to the $i$th risk factor, $\boldsymbol{\Psi}$ is the $T \times m$ matrix of errors whose $j$th column is the vector $\boldsymbol{\varepsilon}_j$, and $\boldsymbol{\Sigma}$ is the covariance matrix of the errors, i.e.

$$V(\boldsymbol{\Psi}) = \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \cdots & \sigma_{1m} \\ \vdots & \vdots \ \vdots & \vdots \\ \sigma_{m1} & \sigma_{m2} \cdots & \sigma_m^2 \end{pmatrix},$$

where $\sigma_{ij}$ denotes the covariance between $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}_j$.

Now consider a portfolio with $m \times 1$ weights vector $\mathbf{w} = (w_1, \ldots, w_m)'$. The portfolio return at time $t$ as a weighted sum of asset returns, i.e.

$$Y_t = \sum_{j=1}^{m} w_j Y_{jt}.$$

In other words, the $T \times 1$ vector of data on the 'current weighted' portfolio returns is

$$\mathbf{y} = \mathbf{Yw}.$$

Hence, by (II.1.26),

$$\mathbf{y} = \mathbf{Aw} + \mathbf{XBw} + \boldsymbol{\Psi}\mathbf{w}. \tag{II.1.27}$$

[15] We only provide an intuitive representation here. The correct approach uses stacked variables and derives the covariance matrix of the errors as a Kronecker product of $\boldsymbol{\Sigma}$ with the $T \times T$ identity matrix. See, for example, Greene (2007) or Gross (2003) for further details.

But, of course, (II.1.27) must be identical to the model (II.1.18). Thus:

- the portfolio alpha vector is $\boldsymbol{\alpha} = \mathbf{A}\mathbf{w}$;
- the beta on the $j$th risk factor is the weighted sum of the asset betas on that risk factor, i.e. the portfolio beta vector is $\boldsymbol{\beta} = \mathbf{B}\mathbf{w}$;
- the portfolio's specific returns are $\boldsymbol{\varepsilon} = \boldsymbol{\Psi}\mathbf{w}$, i.e. the specific return at time $t$ is the weighted sum of the assets' specific returns at time $t$.

We remark that the expression of the portfolio's specific return in the form $\boldsymbol{\Psi}\mathbf{w}$ makes it clear that we must account for the correlation between asset specific returns when estimating the specific risk of the portfolio.

The above shows that, theoretically, we can estimate the portfolio's characteristics (alpha and beta and specific return) in two equivalent ways:

- find the portfolio weighted sum of the characteristics of each asset, or
- estimate the portfolio characteristics directly using the model (II.1.18).

However, whilst this is true for the theoretical model it will *not* be true for the estimated model unless there is only one factor. The reason is that because of the sampling error, weighting and summing the estimated asset characteristics as in (II.1.27) gives different results from those obtained by forming a current weighted historical series for the portfolio return and estimating the model (II.1.18).

Applying the variance operator to (II.1.27) and assuming that each asset's specific return is uncorrelated with each risk factor, gives an alternative to (II.1.22) in a form that makes the portfolio weights explicit, viz.

$$V(Y) = \boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta} + \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \qquad \text{(II.1.28)}$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of the assets' specific returns. So as in (II.1.14) one can again distinguish three sources of risk:

- the risks that are represented by the portfolio's factor sensitivities $\boldsymbol{\beta}$;
- the risks of the factors themselves, represented by the risk factor covariance matrix $\boldsymbol{\Omega}$;
- the idiosyncratic risks of the assets in the portfolio, represented by the variance of residual returns, $\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$.

EXAMPLE II.1.5: SYSTEMATIC RISK AT THE PORTFOLIO LEVEL

Suppose a portfolio is invested in only three assets, with weights −0.25, 0.75 and 0.5, respectively. Each asset has a factor model representation with the same two risk factors as in Example II.1.3 and the betas are: for asset 1, 0.2 for the first risk factor and 1.2 for the second risk factor; for asset 2, 0.9 for the first risk factor and 0.2 for the second risk factor; and for asset 3, 1.3 for the first risk factor and 0.7 for the second risk factor. What is the volatility due to the risk factors (i.e. the systematic risk) for this portfolio?

SOLUTION The net portfolio beta on each factor is given by the product $\mathbf{B}\mathbf{w}$. We have

$$\mathbf{B} = \begin{pmatrix} 0.2 & 0.9 & 1.3 \\ 1.2 & 0.2 & 0.7 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -0.25 \\ 0.75 \\ 0.5 \end{pmatrix}, \quad \text{so} \quad \boldsymbol{\beta} = \begin{pmatrix} 1.275 \\ 0.2 \end{pmatrix}.$$

With the same risk factor covariance matrix as in the previous example,

$$\boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta} = \begin{pmatrix} 1.275 & 0.2 \end{pmatrix} \begin{pmatrix} 0.0225 & -0.015 \\ -0.015 & 0.04 \end{pmatrix} \begin{pmatrix} 1.275 \\ 0.2 \end{pmatrix} = 0.0305,$$

so the portfolio volatility due to the risk factors is $\sqrt{0.0305} = 17.47\%$.

### II.1.3.4 Multi-factor Models of International Portfolios

In this text we always use the term *foreign exchange rate* (or *forex rate*) for the domestic value of a foreign unit of currency. International portfolios have an equivalent exposure to foreign exchange rates; for each nominal amount invested in a foreign security the same amount of foreign currency must be purchased. Put another way, for each country of investment the foreign exchange rate is a risk factor and the portfolio's sensitivity to the exchange rate risk factor is one. In addition to the exchange rate, for each country of exposure we have the usual (fundamental or statistical) market risk factors.

Consider an investment in a single foreign asset. The price of a foreign asset in domestic currency is the asset price in foreign currency multiplied by the foreign exchange rate. Hence the log return on a foreign asset in domestic currency terms is

$$R_D = R_F + X,$$

where $R_F$ is the asset return in foreign currency and $X$ is the forex return. We suppose the systematic return on the asset in foreign currency is related to a single *foreign market risk factor*, such as a broad market index, with return $R$ and factor beta $\beta$. Then the systematic return on the asset in domestic currency is $\beta R + X$. Hence, there are two risk factors affecting the return on the asset:

- the exchange rate (with a beta of 1); and
- the foreign market index (with a beta of $\beta$).

Thus the systematic variance of the asset return in domestic currency can be decomposed into three different components:

$$\text{Systematic Variance} = V(\beta R + X) = \beta^2 V(R) + V(X) + 2\beta \text{Cov}(R, X). \qquad \text{(II.1.29)}$$

For instance, if the asset is a stock, there are three components for systematic variance which are labelled:

- the *equity variance*, $\beta^2 V(R)$;
- the *forex variance*, $V(X)$;
- the *equity–forex covariance*, $2\beta \text{Cov}(R, X)$.

A portfolio of foreign assets in the same asset class with a single foreign market risk factor having return $R$ has the same variance decomposition as (II.1.29), but now $\beta$ denotes the *net* portfolio beta with respect to the market index, i.e. $\beta = \mathbf{w}'\boldsymbol{\beta}$, where $\mathbf{w}$ is the vector of portfolio weights and $\boldsymbol{\beta}$ is the vector of each asset's market beta.

We can generalize (II.1.29) to a large international portfolio with exposures in $k$ different countries. For simplicity we assume that there is a single market risk factor in each foreign market. Denote by $R_1, R_2, \ldots, R_k$ the returns on the market factors, by $\beta_1, \beta_2, \ldots, \beta_k$ the portfolio betas with respect to each market factor and by $X_1, X_2, \ldots, X_k$ the returns on the foreign exchange rates. Assuming $R_1$ is the domestic market factor, then $X_1 = 1$ and there are $k$ equity risk factors but only $k-1$ foreign exchange risk factors. Let $\mathbf{w} = (w_1, w_2, \ldots, w_k)'$ be the country portfolio weights, i.e. $w_i$ is the proportion of the portfolio's value that is invested in country $i$. Then the systematic return on the portfolio may be written as

$$w_1\beta_1 R_1 + w_2(\beta_2 R_2 + X_2) + \ldots + w_k(\beta_k R_k + X_k) = (\mathbf{Bw})'\mathbf{x} \qquad \text{(II.1.30)}$$

where $\mathbf{x}$ is the $2k-1 \times 1$ vector of equity and forex risk factor returns and $\mathbf{B}$ is the $(2k-1) \times k$ matrix of risk factor betas, i.e.

$$\mathbf{x} = (R_1, \ldots, R_k, X_2, \ldots, X_k)' \text{ and } \mathbf{B} = \begin{pmatrix} \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_k) \\ \mathbf{0} \quad \mathbf{I}_{(k-1)\times(k-1)} \end{pmatrix}.$$

Taking variances of (II.1.30) gives

$$\text{Systematic Variance} = (\mathbf{Bw})' \, \boldsymbol{\Omega} \, (\mathbf{Bw}) \,, \tag{II.1.31}$$

where

$$\boldsymbol{\Omega} = \begin{pmatrix} V(R_1) & \cdots & & \operatorname{Cov}(R_1, X_k) \\ \operatorname{Cov}(R_1, R_2) & V(R_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ \operatorname{Cov}(R_1, X_k) & \cdots & & V(X_k) \end{pmatrix}$$

is the covariance matrix of, the equity and forex risk factor returns.

We may partition the matrix $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_E & \boldsymbol{\Omega}_{EX} \\ \boldsymbol{\Omega}'_{EX} & \boldsymbol{\Omega}_X \end{pmatrix} \tag{II.1.32}$$

where $\boldsymbol{\Omega}_E$ is the $k \times k$ covariance matrix of the equity risk factor returns, $\boldsymbol{\Omega}_X$ is the $(k-1) \times (k-1)$ covariance matrix of the forex risk factor returns and $\boldsymbol{\Omega}_{EX}$ is the $k \times (k-1)$ 'quanto' covariance matrix containing the cross covariances between the equity risk factor returns and the forex risk factor returns. Substituting (II.1.32) into (II.1.31) gives the decomposition of systematic variance into equity, forex and equity–forex components as

$$\tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_E \tilde{\boldsymbol{\beta}} + \tilde{\mathbf{w}}' \boldsymbol{\Omega}_X \tilde{\mathbf{w}} + 2\tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{EX} \tilde{\mathbf{w}}, \tag{II.1.33}$$

where $\tilde{\mathbf{w}} = (w_2, \ldots, w_k)'$ and

$$\tilde{\boldsymbol{\beta}} = \operatorname{diag}(\beta_1, \ldots, \beta_k)\mathbf{w} = (w_1\beta_1, \ldots, w_k\beta_k)' .$$

Example II.1.6: Decomposition of systematic risk into equity and forex factors

A UK investor holds £2.5 million in UK stocks with a FTSE 100 market beta of 1.5, £1 million in US stocks with an S&P 500 market beta of 1.2, and £1.5 million in German stocks with a DAX 30 market beta of 0.8. The volatilities and correlations of the FTSE 100, S&P 500 and DAX 30 indices, and the USD/GBP and EUR/GBP exchange rates, are shown in Table II.1.3. Calculate the systematic risk of the portfolio and decompose it into equity, forex and equity–forex components.

**Table II.1.3** Risk factor correlations and volatilities

| Correlation | FTSE 100 | S&P 500 | DAX 30 | USD/GBP | EUR/GBP |
|---|---|---|---|---|---|
| FTSE 100 | 1 | | | | |
| S&P 500 | 0.8 | 1 | | | |
| DAX 30 | 0.7 | 0.6 | 1 | | |
| USD/GBP | 0.2 | −0.25 | 0.05 | 1 | |
| EUR/GBP | 0.3 | 0.05 | −0.15 | 0.6 | 1 |
| Volatilities | 20% | 22% | 25% | 10% | 12% |

Solution The covariance matrix of the risk factor returns is calculated from the information in Table II.1.3 in the spreadsheet, and this is given in Table II.1.4. The upper

**Table II.1.4** Risk factor covariance matrix

| | FTSE 100 | S&P 500 | DAX 30 | USD/GBP | EUR/GBP |
|---|---|---|---|---|---|
| FTSE 100 | 0.04 | 0.0352 | 0.035 | 0.004 | 0.0072 |
| S&P 500 | 0.0352 | 0.0484 | 0.033 | −0.0055 | 0.00132 |
| DAX 30 | 0.035 | 0.033 | 0.0625 | 0.00125 | −0.0045 |
| USD/GBP | 0.004 | −0.0055 | 0.00125 | 0.01 | 0.0072 |
| EUR/GBP | 0.0072 | 0.00132 | −0.0045 | 0.0072 | 0.0144 |

left shaded $3 \times 3$ matrix is the equity risk factor returns covariance matrix $\boldsymbol{\Omega}_E$, the lower right shaded $2 \times 2$ matrix is the forex factor returns covariance matrix $\boldsymbol{\Omega}_X$, and the upper right unshaded $3 \times 2$ matrix is the quanto covariance matrix $\boldsymbol{\Omega}_{EX}$. The risk factor beta matrix **B**, portfolio weights **w** and their product **Bw** are given as follows:

$$\mathbf{B} = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0.5 \\ 0.2 \\ 0.3 \end{pmatrix} \Rightarrow \mathbf{Bw} = \begin{pmatrix} 0.75 \\ 0.24 \\ 0.24 \\ 0.2 \\ 0.3 \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} 0.75 \\ 0.24 \\ 0.24 \end{pmatrix}.$$

Hence, the systematic variance is

$$(\mathbf{Bw})'\,\boldsymbol{\Omega}\,(\mathbf{Bw}) = \begin{pmatrix} 0.75 & 0.24 & 0.24 & 0.2 & 0.3 \end{pmatrix} \times \begin{pmatrix} 0.04 & 0.0352 & 0.035 & 0.004 & 0.0072 \\ 0.0352 & 0.0484 & 0.033 & -0.0055 & 0.00132 \\ 0.035 & 0.033 & 0.0625 & 0.00125 & -0.0045 \\ 0.004 & -0.0055 & 0.00125 & 0.01 & 0.0072 \\ 0.0072 & 0.00132 & -0.0045 & 0.0072 & 0.0144 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.24 \\ 0.24 \\ 0.20 \\ 0.30 \end{pmatrix} = 0.064096$$

and the systematic risk is $\sqrt{0.064096} = 25.32\%$.

The three terms in (II.1.33) are

$$\text{Equity Variance} = \tilde{\beta}'\boldsymbol{\Omega}_E\tilde{\beta} = \begin{pmatrix} 0.75 & 0.24 & 0.24 \end{pmatrix} \begin{pmatrix} 0.04 & 0.0352 & 0.035 \\ 0.0352 & 0.0484 & 0.033 \\ 0.035 & 0.033 & 0.0625 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.24 \\ 0.24 \end{pmatrix} = 0.05796,$$

so the equity risk component is $\sqrt{0.05796} = 24.08\%$;

$$\text{FX Variance} = \tilde{\mathbf{w}}'\boldsymbol{\Omega}_X\tilde{\mathbf{w}} = \begin{pmatrix} 0.2 & 0.3 \end{pmatrix} \begin{pmatrix} 0.01 & 0.0072 \\ 0.0072 & 0.0144 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} = 0.00256,$$

so the forex risk component is $\sqrt{0.00256} = 5.06\%$;

$$\text{Quanto Covariance} = \tilde{\beta}'\boldsymbol{\Omega}_{EX}\tilde{\mathbf{w}} = \begin{pmatrix} 0.75 & 0.24 & 0.24 \end{pmatrix} \begin{pmatrix} 0.004 & 0.0072 \\ -0.0055 & 0.00132 \\ 0.00125 & -0.0045 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} = 0.001787.$$

In accordance with (II.1.33) the three terms sum to the total systematic variance, i.e.

$$0.05796 + 0.00256 + 0.003574 = 0.064096.$$

Taking the square root gives the total systematic risk as 25.32%, which is identical to the result obtained by direct calculation above. The quanto covariance happened to be positive in this example, but it could be negative. In that case the total systematic variance will be less than the sum of the equity variance and the forex variance – and it could even be less than both of them!

When each stock in a portfolio has returns representation (II.1.25), the risk decomposition (II.1.28) shows how the portfolio's systematic risk is represented using the stock's factor betas $\mathbf{B}$ and the risk factor covariance matrix $\boldsymbol{\Omega}$. We can also decompose total risk into systematic risk and specific risk, using techniques that are similar to those used in the simple numerical example above.

## II.1.4 CASE STUDY: ESTIMATION OF FUNDAMENTAL FACTOR MODELS

In this section we provide an empirical case study of risk decomposition using historical prices of two stocks (Nokia and Vodafone) and four fundamental risk factors:[16]

(i) a broad market index, the New York Stock Exchange (NYSE) composite index;
(ii) an industry factor, the Old Mutual communications fund;
(iii) a growth style factor, the Riverside growth fund; and
(iv) a capitalization factor, the AFBA Five Star Large Cap fund.

Figure II.1.4 shows the prices of the two stocks and the four possible risk factors, with each series rebased to be 100 on 31 December 2000.

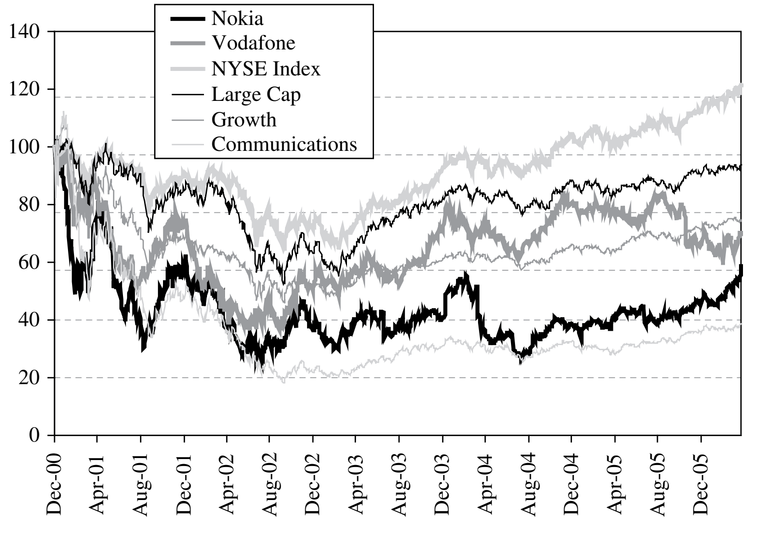


**Figure II.1.4** Two communications stocks and four possible risk factors

[16] All data were downloaded from Yahoo! Finance.

Using regression to build a multi-factor model with these four risk factors gives rise to some econometric problems, but these are not insurmountable as will be shown later in this section. The main problem with this factor model is with the selection of the risk factors. In general, the choice of risk factors to include in the regression factor model is based on the user's experience: there is no econometric theory to inform this choice.

### II.1.4.1 Estimating Systematic Risk for a Portfolio of US Stocks

The first example in this case study uses a factor model for each stock based on all four risk factors.

EXAMPLE II.1.7: TOTAL RISK AND SYSTEMATIC RISK

On 20 April 2006 a portfolio is currently holding \$3 million of Nokia stock and \$1 million of Vodafone stock. Using the daily closing prices since 31 December 2000 that are shown in Figure II.1.4:

(a) estimate the total risk of the portfolio volatility based on the historical returns on the two stocks;
(b) estimate the systematic risk of the portfolio using a four-factor regression model for each stock.

SOLUTION

(a) A current weighted daily returns series for the portfolio is constructed by taking $\{0.25 \times$ return on Vodafone $+ 0.75 \times$ return on Nokia$\}$. The standard deviation of these returns (over the whole data period) is 0.0269, hence the estimate of the portfolio volatility is $\sqrt{250} \times 0.0269 = 42.5\%$.[17]
(b) An OLS regression of the daily returns for each stock on the daily returns for the risk factors – again using the whole data period – produces the results shown in Table II.1.5. The $t$ statistics shown in the table are test statistics for the null hypothesis that the true factor beta is 0 against the two-sided alternative hypothesis that it is not equal to 0. The higher the absolute value of the $t$ statistic, the more likely we are to reject the null hypothesis and conclude that the factor does have a significant effect on the stock return. The $p$ value is the probability that the true factor beta is 0, so a high $t$ statistic gives a low probability value.

**Table II.1.5** Factor betas from regression model

| | Vodafone | | | Nokia | | |
|---|---|---|---|---|---|---|
| | est. beta | $t$ stat. | $p$ value | est. beta | $t$ stat. | $p$ value |
| Intercept | 0.000 | −0.467 | 0.640 | 0.000 | −0.118 | 0.906 |
| NYSE index | 0.857 | 5.835 | 0.000 | −0.267 | −1.545 | 0.123 |
| Communications | 0.137 | 2.676 | 0.008 | 0.271 | 4.471 | 0.000 |
| Growth | 0.224 | 1.885 | 0.060 | 0.200 | 1.432 | 0.152 |
| Large Cap | 0.009 | 0.068 | 0.946 | 1.146 | 7.193 | 0.000 |

[17] Nokia and Vodafone are both technology stocks, which were extremely volatile during this sample period.

Leaving aside the problems associated with this regression until the next subsection, we extract from this the sensitivity matrix

$$\mathbf{B} = \begin{pmatrix} 0.857 & -0.267 \\ 0.137 & 0.271 \\ 0.224 & 0.200 \\ 0.009 & 1.146 \end{pmatrix}.$$

Now, given the weights vector

$$\mathbf{w} = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix},$$

the net portfolio betas are

$$\boldsymbol{\beta} = \begin{pmatrix} 0.857 & -0.267 \\ 0.137 & 0.271 \\ 0.224 & 0.200 \\ 0.009 & 1.146 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0.0136 \\ 0.2372 \\ 0.2620 \\ 0.8618 \end{pmatrix}.$$

In the spreadsheet for this example we also calculate the risk factor returns covariance matrix as

$$\boldsymbol{\Omega} = \begin{pmatrix} 10.02 & 17.52 & 10.98 & 11.82 \\ 17.52 & 64.34 & 28.94 & 27.53 \\ 10.98 & 28.94 & 16.86 & 15.06 \\ 11.82 & 27.53 & 15.06 & 16.90 \end{pmatrix} \times 10^{-5}.$$

The portfolio variance attributable to the risk factors is $\boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta}$ and this is calculated in the spreadsheet as $36.78 \times 10^{-5}$. The systematic risk, expressed as an annual percentage, is the square root of this. It is calculated in the spreadsheet as 30.3%. The reason why this is much lower than the total risk of the portfolio that is estimated in part (a) is that the factor model does not explain the returns very well. The $R^2$ of the regression is the squared correlation between the stock return and the explained part of the model (i.e. the sum of the factor returns weighted by their betas). The correlation is 58.9% for the Vodafone regression and 67.9% for the Nokia regression. These are fairly high but not extremely high, so a significant fraction of the variability in each of the stock's returns is unaccounted for by the model. This variability remains in the model's residuals, so the specific risks of these models can be significant.

### II.1.4.2 Multicollinearity: A Problem with Fundamental Factor Models

Multicollinearity is defined in Section I.4.4.8. It refers to the correlation between the explanatory variables in a regression model: if one or more explanatory variables are highly correlated then it is difficult to estimate their regression coefficients. We say that a model has a high degree of multicollinearity if two or more explanatory variables are highly (positive or negatively) correlated. Then their regression coefficients cannot be estimated with much precision and, in technical terms, the efficiency of the OLS estimator is reduced. The multicollinearity problem becomes apparent when the estimated coefficients change considerably when adding another (collinear) variable to the regression. There is no statistical test for multicollinearity, but a useful rule of thumb is that a model will suffer from it if the square of the pairwise correlation between two explanatory variables is greater than the multiple $R^2$ of the regression.

A major problem with estimating fundamental factor models using time series data is that potential factors are very often highly correlated. In this case the factor betas cannot be estimated with precision. To understand the effect that multicollinearity has on the estimated factor betas, let us consider again the factor model of Example II.1.7. Table II.1.6 starts with an OLS estimation of a single factor for each stock (the returns on the NYSE composite index) and then adds one factor at a time. Each time we record the factor beta estimate, its $t$ statistic and probability value as explained in Example II.1.7. We exclude the intercept as it is always insignificantly different from zero in these regressions, but in each case we state the $R^2$ of the regression.

**Table II.1.6** Multicollinearity in time series factor models

| Vodafone | 1 Factor | | | 2 Factors | | | 3 Factors | | | 4 Factors | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | beta | $t$ stat. | p-value | beta | $t$ stat. | p-value | beta | $t$ stat. | p-value | beta | $t$ stat. | p-value |
| NYSE index | 1.352 | 25.024 | 0.000 | 0.996 | 13.580 | 0.000 | 0.864 | 8.539 | 0.000 | 0.857 | 5.835 | 0.000 |
| Communications | | | | 0.204 | 7.042 | 0.000 | 0.139 | 3.103 | 0.002 | 0.137 | 2.676 | 0.008 |
| Growth | | | | | | | 0.224 | 1.895 | 0.058 | 0.224 | 1.885 | 0.060 |
| Large Cap | | | | | | | | | | 0.009 | 0.068 | 0.946 |
| Multiple $R$ | | 0.566 | | | 0.587 | | | 0.589 | | | 0.589 | |

| Nokia | 1 Factor | | | 2 Factors | | | 3 Factors | | | 4 Factors | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | beta | t stat. | p-value | beta | t stat. | p-value | beta | t stat. | p-value | beta | t stat. | p-value |
| NYSE index | 1.777 | 25.475 | 0.000 | 0.795 | 9.022 | 0.000 | 0.635 | 5.218 | 0.000 | −0.267 | −1.545 | 0.123 |
| Communications | | | | 0.561 | 16.134 | 0.000 | 0.483 | 8.962 | 0.000 | 0.271 | 4.471 | 0.000 |
| Growth | | | | | | | 0.273 | 1.919 | 0.055 | 0.200 | 1.432 | 0.152 |
| Large Cap | | | | | | | | | | 1.146 | 7.193 | 0.000 |
| Multiple $R$ | | 0.573 | | | 0.662 | | | 0.663 | | | 0.679 | |

The one-factor model implies that both stocks are high risk, relative to the NYSE index: their estimated betas are 1.352 (Vodafone) and 1.777 (Nokia) and both are significantly greater than 1. The $R^2$ of 56.6% (Vodafone) and 57.3% (Nokia) indicates a reasonable fit, given there is only one factor. The two-factor model shows that the communications factor is also able to explain the returns on both stocks, and it is especially important for Nokia, with a $t$ statistic of 16.134.

Notice that the addition of this factor has dramatically changed the NYSE beta estimate: it is now below 1, for both stocks. In the three-factor model the NYSE beta estimate becomes even lower, and so does the communications beta. Yet the growth index is only marginally significant: it has a probability value of around 5%. The addition of the final 'large cap' factor in the four-factor model has little effect on Vodafone – except that the NYSE and communications beta estimates become even less precise (their $t$ statistics become smaller) – and the large cap factor does not seem to be important for Vodafone. But it is very important for Nokia: the $t$ statistic is 7.193 so the beta of 1.146 is very highly significantly different from 0. And now the NYSE and communications beta estimates change dramatically. Starting with a NYSE beta of 1.777 in the single factor model, we end up in the four-factor model with a beta estimate of −0.267!

So, what is going on here? Which, if any, of these is the correct beta estimate? Let us see whether multicollinearity could be affecting our results. It certainly seems to be the case, because our betas estimates are changing considerably when we add further factors. Table II.1.7 shows the factor correlation matrix for the sample period. All the factors are very highly correlated. The lowest correlation, of 69%, is between the NYSE Index and the communications factor. The square of this is lower than the multiple $R^2$ of the regressions. However, the other correlations shown in Table II.1.7 are very high, and their squares are higher than the multiple $R^2$ of the regressions. Obviously multicollinearity is causing problems in these models. The 'large cap' factor is the most highly correlated with the other factors and this explains why the model really fell apart when we added this factor.

**Table II.1.7** Factor correlation matrix

| | NYSE Index | Communications | Growth | Large Cap |
|---|---|---|---|---|
| NYSE index | 1 | | | |
| Communications | 0.690 | 1 | | |
| Growth | 0.845 | 0.879 | 1 | |
| Large Cap | 0.909 | 0.835 | 0.892 | 1 |

Because of the problem with multicollinearity the only reliable factor beta estimate is one where each factor is taken individually in its own single factor model. But no single factor model can explain the returns on a stock very well. A large part of the stock returns variation will be left to the residual and so the systematic risk will be low and the stock specific risk high. We cannot take these individual beta estimates into (II.1.24) with $k = 4$: they need to be estimated simultaneously. So how should we proceed? The next section describes the method that I recommend.

### II.1.4.3 Estimating Fundamental Factor Models by Orthogonal Regression

The best solution to a multicollinearity problem is to apply principal component analysis to all the potential factors and then use the principal components as explanatory variables, instead of the original financial or economic factors. Principal component analysis was introduced in Section I.2.6 and we summarize the important learning points about this analysis at the beginning of the next chapter. In the context of the present case study we shall illustrate how principal component analysis may be applied in orthogonal regression to mitigate the multicollinearity problem in our four-factor model.

We shall apply principal component analysis to the risk factor returns covariance matrix. Table II.1.8 displays the eigenvalues of this matrix, and the collinearity of the risk factor returns is evident since the first eigenvalue is relatively large. It indicates that the first principal component explains over 90% of the variation in the risk factors and hence it is capturing a strong common trend in the four risk factors. With just two principal components this proportion rises to 97.68%.[18]

[18] But note that the second and higher principal components do not have an intuitive interpretation because the system is not ordered, as it is in a term structure.

**Table II.1.8** Eigenvalues and eigenvectors of the risk factor covariance matrix

| Eigenvalues | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|
| Eigenvalue | 0.000976 | 0.000080 | 0.000017 | 0.0000078 |
| Variation explained | 90.25% | 7.44% | 1.60% | 0.72% |
| Cumulative variation | 90.25% | 97.68% | 99.28% | 100% |
| **Eigenvectors** | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
| NYSE index ($RF_1$) | 0.259987 | 0.609012 | 0.103850 | 0.742110 |
| Communications ($RF_2$) | 0.795271 | –0.566012 | 0.139657 | 0.166342 |
| Growth ($RF_3$) | 0.391886 | 0.271074 | –0.845368 | –0.241448 |
| Large cap ($RF_4$) | 0.382591 | 0.485030 | 0.505039 | –0.602749 |

Since the principal components are uncorrelated by design, a regression of the stock's returns on the principal components has no problem with multicollinearity – quite the opposite in fact, because the factors are orthogonal. Then the estimated coefficients in this regression can be used to recover the risk factor betas. To see how this is done, recall from Section I.2.6 that the $m$th principal component is related to the $m$th eigenvector $\mathbf{w}_m$ and the risk factor returns as follows:

$$PC_m = w_{1m}RF_1 + \ldots + w_{4m}RF_4, \quad \text{where } \mathbf{w}_m = (w_{1m}, w_{2m}, w_{3m}, w_{4m})'. \qquad \text{(II.1.34)}$$

Now suppose we estimate a regression of the stock's returns on the principal component factors, using OLS, and the estimated regression model is

$$\text{Vodafone return} = \sum_{i=1}^{k} \hat{\gamma}_i PC_i \quad (k \leq 4). \qquad \text{(II.1.35)}$$

Substituting (II.1.34) into (II.1.35) gives the representation of the stock's return in terms of the original factors:

$$\text{Vodafone return} = \sum_{i=1}^{4} \hat{\beta}_i RF_i, \quad \text{where} \quad \hat{\beta}_i = \sum_{j=1}^{k} \hat{\gamma}_j w_{ij}. \qquad \text{(II.1.36)}$$

Hence the net betas will be a weighted sum of the regression coefficients $\hat{\gamma}_i$ in (II.1.35).

Table II.1.9 shows these regression coefficients and their $t$ statistics, first with $k=4$ and then with $k=2$, and below this the corresponding risk factor betas obtained using (II.1.36). Note that when all four principal components are used the risk factor betas are identical to those shown in the last column of Table II.1.6, as is the regression $R^2$.

However, our problem is that the four-factor model estimates were seriously affected by multicollinearity. Of course there is no such problem in the regression of Table II.1.9, so this does not bias the $t$ statistics on the principal components. But we still cannot disentangle the separate effects of the risk factors on the stock returns. The solution is to use only the two main principal components as explanatory variables, as in the right-hand section of Table II.1.9 which corresponds to the results when $k=2$. Then the regression $R^2$ is not much less than it is when $k=4$, but the net betas on each risk factor are quite different from those shown in the right-hand column of Table II.1.6. We conclude that the estimates for the risk factor betas shown in the right-hand column of Table II.1.9 are more reliable than those in the right-hand column of Table II.1.6.

**Table II.1.9** Using orthogonal regression to obtain risk factor betas

| Vodafone | 4-Factor | | 2-Factor | |
|---|---|---|---|---|
| | Coefficients | *t* stat. | Coefficients | *t* stat. |
| PC1 | 0.4230 | 24.8809 | 0.4230 | 24.8044 |
| PC2 | 0.5090 | 8.5935 | 0.5090 | 8.5670 |
| PC3 | −0.0762 | −0.5971 | | |
| PC4 | 0.5989 | 3.1390 | | |
| *R* | 58.87% | | 58.44% | |
| Net betas | | | | |
| NYSE index | 0.8566 | | 0.4200 | |
| Communications | 0.1373 | | 0.0483 | |
| Growth | 0.2236 | | 0.3038 | |
| Large Cap | 0.0092 | | 0.4087 | |
| **Nokia** | **4-Factor** | | **2-Factor** | |
| | Coefficients | *t* stat. | Coefficients | *t* stat. |
| PC1 | 0.6626 | 33.0451 | 0.6626 | 32.7808 |
| PC2 | 0.2942 | 4.2113 | 0.2942 | 4.1776 |
| PC3 | 0.4194 | 2.7860 | | |
| PC4 | −0.8926 | −3.9669 | | |
| *R* | 67.88% | | 67.17% | |
| Net betas | | | | |
| NYSE index | −0.2674 | | 0.3514 | |
| Communications | 0.2705 | | 0.3604 | |
| Growth | 0.2003 | | 0.3394 | |
| Large Cap | 1.1461 | | 0.3962 | |

In Example II.1.7 we estimated the systematic risk that is due to the four risk factors as 24.7%. But there the risk factor beta matrix was affected by multicollinearity. Now we use the orthogonal regression estimates given in the right-hand column of Table II.1.9, i.e.

$$\mathbf{B} = \begin{pmatrix} 0.4200 & 0.3514 \\ 0.0483 & 0.3604 \\ 0.3038 & 0.3394 \\ 0.4087 & 0.3962 \end{pmatrix}.$$

This gives the portfolio beta vector as

$$\boldsymbol{\beta} = \begin{pmatrix} 0.3686 \\ 0.2824 \\ 0.3305 \\ 0.3993 \end{pmatrix},$$

and the systematic risk is now calculated as 30.17%, as shown in the spreadsheet for this example.

## II.1.5 ANALYSIS OF BARRA MODEL

The Barra model is a fundamental multi-factor regression model where a stock return is modelled using market and industry risk factor returns and certain fundamental factors called the *Barra risk indices*. The risk associated with a stock return is decomposed into the undiversifiable risk due to the market factor and two types of diversifiable risk: (a) the risk due to fundamental factors and industry risk factors, and (b) specific risk.

Barra has developed models for specific equity markets, starting with the US market in 1975, followed by the UK market in 1982, and since then many others. In each market Barra calculates a number of common risk indices and an industry classification to explain the diversifiable risks associated with a given stock. In the UK equity model there are 12 common risk indices and 38 industry indices.

The purpose of the Barra model is to analyse the relationship between a portfolio's return and the return on its benchmark. The difference between these two returns is called the *relative return*, also called the *active return*. A precise definition is given in Section II.1.5.2 below. The Barra model has two parts:

- an optimizer (ACTIVOPS) used to construct benchmark tracking portfolios with a required number of stocks and to design portfolios with maximum expected return given constraints on risk and weightings;
- a risk characterization tool (IPORCH) used to assess the *tracking error* (i.e. the standard deviation of the active returns) given a portfolio and benchmark.

With the help of the risk indices and industry indices, the Barra model explains the active return on a portfolio and the uncertainty about this active return in terms of:

- the *relative alpha* of the portfolio, i.e. the difference between the alpha of the portfolio and the benchmark alpha (note that if the benchmark is the market index then its alpha is 0);
- the *relative betas* of the portfolio, i.e. the difference between the beta of the portfolio and the benchmark beta, with respect to the market, industry factors and Barra risk indices (note that if the benchmark is the market index then its market beta is 1 and its other betas are 0).

### II.1.5.1 Risk Indices, Descriptors and Fundamental Betas

The Barra fundamental risk factors are also called *common risk indices* because they reflect common characteristics among different companies. The risk indices and their structure are different for every country. Each risk index is built from a number of subjectively chosen *descriptors*. For instance, the risk index 'Growth' in the UK model is given by the following descriptors:

- earnings growth over 5 years;
- asset growth;
- recent earnings change;
- change in capital structure;
- low yield indicator.

Each descriptor is standardized with respect to the stock universe: in the case of the UK model the universe is the FT All Share index. The standardization is applied so that the FT All Share index has zero sensitivity to each descriptor and so that the variance of descriptor values taken over all stocks in the universe is 1.

The *factor loading* on each descriptor is determined by a cross-sectional regression of all stocks in the universe, updated every month. That is, the factor loading is the estimated regression coefficient on the descriptor from the regression

$$Y_i = \beta_1 D_{i1} + \ldots + \beta_M D_{iM} + \varepsilon_i,$$

where $M$ is the number of descriptors, $D_{i1}, \ldots, D_{iM}$ are the descriptor values for stock $i$ and $i = 1, \ldots, N$ where $N$ is the number of stocks in the universe. Each risk index has a Barra *fundamental beta* which is calculated as the sum of the factor loadings on all the descriptors for that risk index.

The use of these descriptors allows the Barra model to analyse companies with very little history of returns, because the relevant descriptors for a stock can be allocated qualitatively. No history is required because the firm's descriptors may be allocated on the basis of the company profile, but historical data are useful for testing the judgement used. The chosen descriptors are then grouped into risk indices, so that the important determinants of the returns can be analysed. In the UK model the risk indices are:

- *earnings variability*, which also measures cash-flow fluctuations;
- *foreign exposure*, which depends on percentage of sales that are exports, and other descriptors related to tax and world markets;
- *growth*, which indicates the historical growth rate;
- *labour intensity*, which estimates the importance of labour costs, relative to capital;
- *leverage*, which depends on the debt–equity ratio and related descriptors;
- *non-FTA indicator*, which captures the behaviour of small firms not in the FTSE All Share index;
- *size*, which depends on market capitalization;
- *success*, which is related to earnings growth;
- *trading activity*, which is relative turnover as a percentage of total capitalization;
- *value to price*, which is determined by the ratio of book value to market price and other related descriptors;
- *variability*, a measure of the stock's systematic risk; and
- *yield*, a measure of current and historical dividend yield.

The *market portfolio* is the portfolio of all stocks in the universe with weights proportional to their capitalization. In the UK model the market portfolio is taken to be the FT All Share index. Each month descriptors are standardized so that the risk index sensitivities of the market portfolio are 0, and so that each risk index has a variance of 1 when averaged over all stocks in the universe. Hence, the covariance matrix of the descriptors equals the correlation matrix. Each month the risk index correlation matrix is obtained from the correlation matrix of the standardized descriptors for each stock in the universe.

Each stock in the universe is assigned to one or more of the industries. In the UK model this is done according to the Financial Times classification. The Barra handbook is not entirely clear about the method used to estimate the covariances of the industry factors and their factor betas. My own interpretation is that they use cross-sectional analysis, just as they do for the risk indices. Each month there are $N$ data points for each industry factor, where $N$ is the number of stocks in the industry. For instance, the industry 'Breweries' will have a vector such as $(0, 0, 1, 1, 0, \ldots, 1)$ where 1 in the $i$th place indicates that stock $i$ is included in the brewery industry. This way the industry data will have the same dimension as the descriptor and risk index data, and then the Barra model will be able to estimate, each month, a cross-correlation matrix between the risk indices and the industry factors, as per the results shown in the Barra handbook. The cross-correlation matrix – which is the same as the cross-covariance matrix because of the standardization described above – is important because it is used in the risk decomposition of a portfolio, as explained in the next subsection.

### II.1.5.2 Model Specification and Risk Decomposition

Consider a specific portfolio $P$ and its corresponding benchmark $B$. The multi-factor Barra model applied to this portfolio and its benchmark may be written

$$\begin{aligned} R_P &= \alpha_P + \beta_P X + \sum_{k=1}^{12} \beta_P^{F,k} R^{F,k} + \sum_{k=1}^{38} \beta_P^{I,k} R^{I,k} + \varepsilon_P, \\ R_B &= \alpha_B + \beta_B X + \sum_{k=1}^{12} \beta_B^{F,k} R^{F,k} + \sum_{k=1}^{38} \beta_B^{I,k} R^{I,k} + \varepsilon_B, \end{aligned} \qquad \text{(II.1.37)}$$

with the following notation:

$X$ : return on the market index
$R^{F,k}$ : return on the $k$th (standardized) risk index;
$R^{I,k}$ : return on the $k$th industry index;
$\alpha_P$ : portfolio alpha;
$\alpha_B$ : benchmark alpha ( $=0$ if benchmark is market index);
$\beta_P$ : portfolio market beta;
$\beta_B$ : benchmark market beta ( $=1$ if benchmark is market index);
$\beta_P^{F,k}$ : portfolio fundamental beta on the $k$th (standardized) risk index;
$\beta_B^{F,k}$ : benchmark fundamental beta ( $=0$ if benchmark is market index);
$\beta_P^{I,i}$ : portfolio beta on the $i$th industry index;
$\beta_B^{F,i}$ : benchmark beta on the $i$th industry index ( $=0$ if benchmark is market index);
$\varepsilon_P$ : portfolio specific return;
$\alpha_B$ : benchmark specific return ( $=0$ if benchmark is market index).

In more concise matrix notation the model (II.1.37) may be written

$$\begin{aligned} R_P &= \alpha_P + \beta_P X + \left(\beta_P^F\right)' \mathbf{R}^F + \left(\beta_P^I\right)' \mathbf{R}^I + \varepsilon_P, \\ R_B &= \alpha_B + \beta_B X + \left(\beta_B^F\right)' \mathbf{R}^F + \left(\beta_B^I\right)' \mathbf{R}^I + \varepsilon_B. \end{aligned} \qquad \text{(II.1.38)}$$

where $\left(\beta_P^F\right)' = \left(\beta_P^{F,1}, \ldots, \beta_P^{F,12}\right)$ and the other vector notation follows analogously.

The active return on the portfolio is then defined as[19]

$$Y = R_P - R_B = (\alpha_P - \alpha_B) + (\beta_P - \beta_B) X + \left(\beta_P^F - \beta_B^F\right)' \mathbf{R}^F + \left(\beta_P^I - \beta_B^I\right)' \mathbf{R}^I + (\varepsilon_P - \varepsilon_B)$$

Now defining the *relative alpha* as $\alpha = \alpha_P - \alpha_B$ and the *relative betas* as

$$\beta = \beta_P - \beta_B, \ \boldsymbol{\beta}^F = \beta_P^F - \beta_B^F \ \text{ and } \ \boldsymbol{\beta}^I = \beta_P^I - \beta_B^I$$

and setting $\varepsilon = \varepsilon_P - \varepsilon_B$ we may write the model in terms of the portfolio's active return as:

$$Y = \alpha + \beta X + \boldsymbol{\beta}^{F\prime} \mathbf{R}^F + \boldsymbol{\beta}^{I\prime} \mathbf{R}^I + \varepsilon. \qquad \text{(II.1.39)}$$

---

[19] This definition is based on the relationship between active, portfolio and benchmark *log* returns. But ordinary returns are used in the derivation of the factor model for the portfolio (because the portfolio return is the weighted sum of the stock returns, not log returns). Hence, the relationship (II.1.39) is based on the fact that returns and log returns are approximately equal if the return is small, even though this is the case only when returns are measured over a short time interval such as one day.

Taking expectations of the active return and noting that the Barra fundamental risk indices are standardized to have zero expectation gives

$$E(Y) = \alpha + \beta E(X) + \boldsymbol{\beta}^{I\prime} E(\mathbf{R}^{I}), \tag{II.1.40}$$

and taking variances of the active return gives:

$$V(Y) = \boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta} + V(\varepsilon). \tag{II.1.41}$$

where $\boldsymbol{\beta}$ is the column vector of all the betas in the model and $\boldsymbol{\Omega}$ is the covariance matrix of the market, risk index and industry factor returns.

The user of the Barra model defines a portfolio and a benchmark and then the IPORCH risk characterization tool estimates the portfolio's alpha and the vector of portfolio betas. It also outputs the *ex ante tracking error*, which is defined as the annualized square root of $V(Y)$ in (II.1.41). It is important to note that this ex ante tracking error represents uncertainty about the expected relative return (II.1.40) and not about any other relative return. In particular, the tracking error does *not* represent dispersion about a relative return of zero, unless the portfolio is tracking the benchmark. When a portfolio is designed to track a benchmark, stocks are selected in such a way that the expected relative return is zero. But in actively managed portfolios the alpha should not be zero, otherwise there is no justification for the manager's fees. In this case (II.1.40) will not be zero, unless by chance $\alpha + \beta E(X) + \boldsymbol{\beta}^{I\prime} E(\mathbf{R}^{I}) = 0$, which is very highly unlikely.

Further discussion of this very important point about the application of the Barra model to the measurement of active risk is given in the next section. It is important not to lose sight of the fact that the Barra model is essentially a model for *alpha* management, i.e. its primary use is to optimize active returns by designing portfolios with maximum expected return, given constraints on risk and weightings.[20] It is also useful for constructing benchmark tracking portfolios with a required number of stocks. It may also be used for estimating and forecasting portfolio *risk* but only if the user fully understands the risk that the Barra model measures.

Unfortunately, it is a common mistake to estimate the tracking error using the model and then to represent this figure as a measure of active risk when the expected active return is non-zero. In the next section we explain why it is *mathematical nonsense* to use the tracking error to measure active risk when the expected active return is non-zero. Using a series of pedagogical examples, we demonstrate that it is improper practice for active fund managers to represent the tracking error to their clients as a measure of active risk.

## II.1.6 TRACKING ERROR AND ACTIVE RISK

In this section we critically examine how the classical methods for estimating and forecasting volatility were applied to fund management during the 1990s. In the 1980s many institutional clients were content with passive fund management that sought merely to track an index or a benchmark. But during the 1990s more clients moved toward active fund management, seeking returns over and above the benchmark return and being willing to accept a small

[20] The advantage of using the Barra model as a risk assessment tool is that portfolio returns and risk are measured within the same model. However, its forecasting properties are limited because the parameters are estimated using cross-sectional data. This is especially true for short term risk forecasting over horizons of less than 1 month, because the model is only updated on a monthly basis.

amount of active risk in order to achieve this return. Hence the fund manager's performance was, and still is, assessed relative to a benchmark. This benchmark can be a traded asset itself, but many benchmarks are not necessarily tradable, such as the *London Interbank Offered Rate* (LIBOR).

Whatever the benchmark, it is standard to measure risk relative to the benchmark and to call this risk the *active risk* or the *relative risk* of the fund. We begin this section by demonstrating that the precise definition of active or relative risk is not at all straightforward. In fact, even the fundamental concept of 'measuring risk relative to a benchmark' has led to considerable confusion amongst risk managers of funds. The main aim of this section is to try to dispel this confusion, and so we begin by defining our terminology very carefully.

### II.1.6.1 Ex Post versus Ex Ante Measurement of Risk and Return

*Ex post* is Latin for 'from after', so ex post risk and return are measured directly from historical observations on the past evolution of returns. *Ex ante* is Latin for 'from before'. Ex ante risk and return are forward looking and when they are forecast, these forecasts are usually based on some model. In fund management the ex ante risk model is the same as the ex ante returns model. This is usually a regression-based factor model that portfolio managers use to select assets and allocate capital to these assets in an optimal manner. The model is defined by some prior beliefs about the future evolution of the portfolio and the benchmark. These beliefs may be, but need not be, based on historical data.

### II.1.6.2 Definition of Active Returns

Active return is also commonly called the *relative return*. It is the difference between the portfolio's return and the benchmark return. Hence, if a portfolio tracks the benchmark exactly its active returns are zero. In general, we model the active returns using a factor model framework, for instance using the Barra model that was described in the previous section.

The portfolio return is the change in a portfolio's value over a certain period expressed as a percentage of its current value. Thus if $V_P$ and $V_B$ denote the values of the portfolio and the benchmark respectively, then the one-period ex post return on the portfolio, measured at time $t$, is

$$R_{Pt} = \frac{V_{Pt} - V_{P,t-1}}{V_{P,t-1}} \tag{II.1.42}$$

and the one-period ex post return on the benchmark, measured at time $t$, is

$$R_{Bt} = \frac{V_{Bt} - V_{B,t-1}}{V_{B,t-1}}. \tag{II.1.43}$$

The one-period ex post active return measured at time $t$, denoted $R_t$, is defined by the relationship

$$(1 + R_t)(1 + R_{\mathrm{B}t}) = (1 + R_{\mathrm{P}t}). \tag{II.1.44}$$

A portfolio manager's performance is usually assessed over a period of several months, so for performance measurement it is not really appropriate to use the log approximation to returns. However, in an *ex ante* risk model it may be necessary to assess risks over a short horizon, in which case we may use the log return. The one-period ex post log returns are

$$r_{Pt} = \ln\left(\frac{V_{Pt}}{V_{P,t-1}}\right), \quad r_{Bt} = \ln\left(\frac{V_{Bt}}{V_{B,t-1}}\right), \tag{II.1.45}$$

and the ex ante log returns are

$$r_{Pt} = \ln\left(\frac{V_{P,t+1}}{V_{Pt}}\right), \quad r_{Bt} = \ln\left(\frac{V_{B,t+1}}{V_{Bt}}\right). \tag{II.1.46}$$

Now, either ex post or ex ante,

$$r_t = r_{Pt} - r_{Bt}. \tag{II.1.47}$$

That is, the active log return is the portfolio's log return minus the benchmark's log return.

Note that to measure the ex ante active returns we need a value for both the portfolio and the benchmark at time $t+1$. For this it is necessary to use a model, such as the Barra model, that aims to forecast future values of all the assets in the investment universe.

### II.1.6.3 Definition of Active Weights

In Section I.1.4 we proved that

$$R_P = \sum_{i=1}^{k} w_i R_i \tag{II.1.48}$$

where $R_P$ is the return on a portfolio, $R_i$ is the one-period return on asset $i$, $k$ is the number of assets in the portfolio and $w_i$ is the portfolio weight on asset $i$ at the beginning of the period, defined as the value of the portfolio's holding in asset $i$ at time $t$ divided by the total value of the portfolio at time $t$.

Log returns are very convenient analytically and, over short time periods the log return is approximately equal to the return, as shown in Section I.1.4. Using this approximation, the log return on the portfolio and the benchmark may also be written as a weighted sum of the asset log returns:

$$r_{Pt} = \sum_{i=1}^{k} w_{Pit} r_{it}, \quad r_{Bt} = \sum_{i=1}^{k} w_{Bit} r_{it}, \tag{II.1.49}$$

where $r_{it}$ is the log return on asset $i$ at time $t$, $w_{Pit}$ is the portfolio's weight on asset $i$ at time $t$, and $w_{Bit}$ is the benchmark's weight on asset $i$ at time $t$. From (II.1.47) and (II.1.49) we have

$$r_t = \sum_{i=1}^{k} (w_{Pit} - w_{Bit})\, r_{it} = \sum_{i=1}^{k} w_{it} r_{it}, \tag{II.1.50}$$

and $w_{it} = w_{\mathrm{P}it} - w_{\mathrm{B}it}$ is called the portfolio's *active weight* on asset $i$ at time $t$.

That is, the active weight on an asset in the benchmark is just the difference between the portfolio's weight and the benchmark's weight on that asset.

### II.1.6.4 Ex Post Tracking Error

Suppose that we measure risk ex post, using a time series of $T$ active returns. Denote the active return at time $t$ by $R_t$ and the average active return over the sample by $\overline{R}$. Then the ex post tracking error ($TE$) is estimated as

$$TE = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_t - \overline{R}\right)^2}. \tag{II.1.51}$$

Thus the tracking error is the standard deviation of active returns. It is usually quoted in annual terms, like volatility.

### Example II.1.8: Tracking error of an underperforming fund

An ex post tracking error is estimated from a sample of monthly returns on the fund and the benchmark. The fund returns exactly 1% less than the benchmark during every month in the sample. More precisely, the active return on the fund is exactly −1% each month. What is the tracking error on this fund?

Solution Since the active return is constant, it has zero standard deviation. Hence the tracking error is zero.

The above example is extreme, but illustrative. A zero tracking error would also result if we assumed that the active return was exactly +1% each month. More generally, the tracking error of an underperforming fund – or indeed an overperforming fund – can be very small when the performance is stable. But the fund need not be tracking the benchmark: it may be very far from the benchmark. The following example illustrates this point in a more realistic framework, where the fund does not have a constant active return. Instead we just assume that the fund consistently underperforms the benchmark.

### Example II.1.9: Why tracking error only applies to tracking funds

A fund's values and its benchmark values between 1990 and 2006 are shown in Table II.1.10. The data cover a period of 16 years and for comparison the value of the benchmark and of the funds are set to 100 at the beginning of the period. What is the ex post tracking error of the fund measured from these data? How risky is this fund?

**Table II.1.10** Values of a fund and a benchmark[a]

| Date | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Benchmark | 100 | 120 | 138 | 145 | 159 | 159 | 175 | 210 | 200 | 210 | 262 | 249 | 249 | 299 | 284 | 290 | 319 |
| Fund | 100 | 115 | 129 | 128 | 135 | 129 | 136 | 155 | 144 | 147 | 178 | 161 | 156 | 179 | 162 | 157 | 164 |

[a] The prices shown have been rounded – see the spreadsheet for this example for the precise figures.

Solution The spreadsheet for this example shows how the ex post $TE$ is calculated. In fact the prices of the fund and benchmark were rounded in Table II.1.10 and using their exact values we obtain $TE = 1\%$. But this is not at all representative of the risk of the fund. The fund's value in 2006 was half the value of the benchmark! Figure II.1.5 illustrates the values of the fund and the benchmark to emphasize this point.

We see that the only thing that affects the ex post tracking error is the variability of the active returns. It does not matter what the level of the mean active return is because this mean is taken out of the calculation: only the *mean deviations* of the active returns are used.

These examples show that there is a real problem with ex post tracking error if risk managers try to apply this metric to active funds, or indeed any fund that has a non-zero mean active return. Tracking error only measures the 'risk of relative returns'. It does not measure the risk of the fund relative to the benchmark. Indeed, the benchmark is *irrelevant* to the calculation of ex post tracking error, as the next example shows.

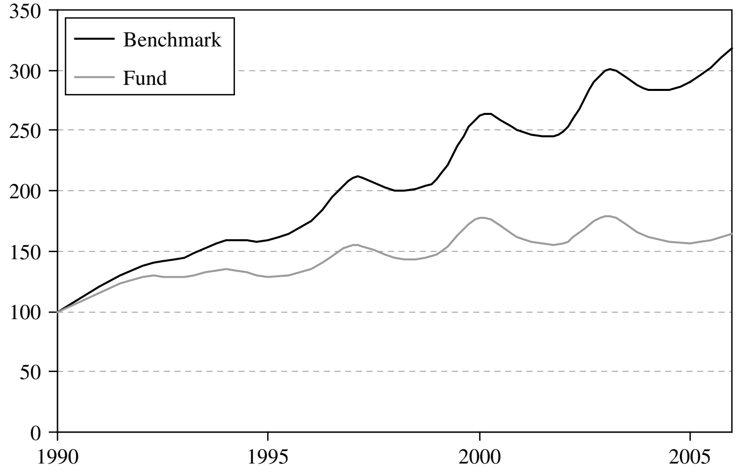


**Figure II.1.5** A fund with ex post tracking error of only 1%

EXAMPLE II.1.10: IRRELEVANCE OF THE BENCHMARK FOR TRACKING ERROR

Consider one fund and two possible benchmarks, whose values are shown in Table II.1.11. What is the ex post tracking error of the fund measured relative to each benchmark based on these data?

**Table II.1.11** Values of a fund and two benchmarks[a]

| Date | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Benchmark 1 | 100 | 90 | 104 | 124 | 161 | 186 | 204 | 235 | 258 | 271 | 339 | 254 | 216 | 216 | 238 | 262 | 275 |
| Benchmark 2 | 100 | 93 | 110 | 136 | 182 | 216 | 245 | 291 | 330 | 357 | 460 | 355 | 311 | 321 | 364 | 413 | 447 |
| Fund | 100 | 91 | 104 | 127 | 167 | 190 | 206 | 234 | 260 | 271 | 346 | 256 | 221 | 223 | 243 | 262 | 273 |

[a] The prices shown have been rounded – see the spreadsheet for this example for the precise figures.

SOLUTION The spreadsheet calculates the ex post $TE$ relative to each benchmark and it is 1.38% relative to *both* benchmarks. But the fund is tracking benchmark 1 and substantially underperforming benchmark 2 as we can see from the time series of their values illustrated in Figure II.1.6. The fund has the *same* tracking error relative to both benchmarks. But surely, if the risk is being measured *relative* to the benchmark then the result should be different depending on the benchmark. Indeed, given the past performance shown above, the fund has a very high risk relative to benchmark 2 but a very small risk relative to benchmark 1.

In summary, the name ‘tracking error’ derives from the fact that *tracking* funds may use (II.1.51) as a risk metric. However, we have demonstrated why ex post tracking error is not a suitable risk metric for actively managed funds. It is only when a fund tracks a benchmark closely that ex post tracking error is a suitable choice of risk metric.

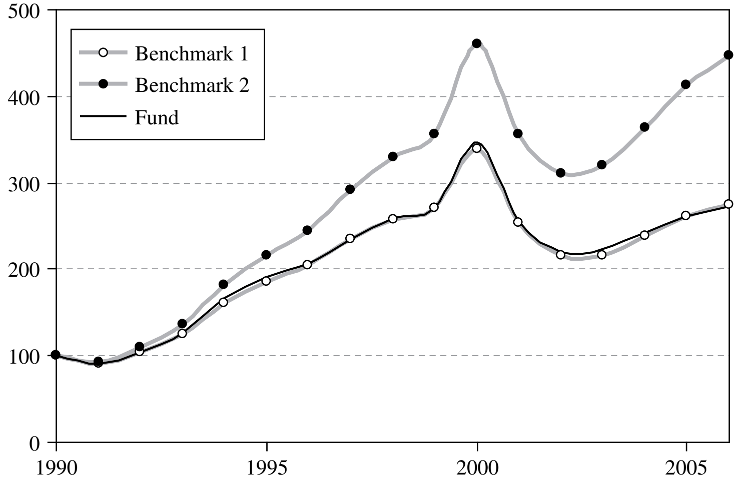


**Figure II.1.6** Irrelevance of the benchmark for tracking error

### II.1.6.5 Ex Post Mean-Adjusted Tracking Error

We call the square root of the average squared active return the ex post *mean-adjusted tracking error,* i.e.

$$MATE = \sqrt{\frac{1}{T}\sum_{t=1}^{T} R_t^2} \tag{II.1.52}$$

Straightforward calculations show that

$$(MATE)^2 = \frac{T-1}{T}\left(TE^2\right) + \overline{R}^2 \tag{II.1.53}$$

Hence, the mean-adjusted tracking error will be larger than the tracking error when the mean active return is quite different from zero:

$$TE \approx MATE \text{ if } \overline{R} \approx 0 \text{ and, for large } T, TE < MATE \text{ when } \overline{R} \neq 0.$$

Earlier we saw that when volatility is estimated from a set of historical *daily* returns it is standard to assume that the mean return is very close to zero. In fact, we have assumed this throughout the chapter. However, in active fund management it should not be assumed that the mean active return is zero for two reasons. Firstly, returns are often measured at the monthly, not the daily frequency, and over a period of 1 month an assumption of zero mean is not usually justified for any market. Secondly, we are dealing with an active return here, not just an ordinary return, and since the fund manager's mandate is to *outperform* the benchmark their client would be very disappointed if $\overline{R} \approx 0$. It is only in a passive fund, which aims merely to track a benchmark, that the average active return should be very close to zero.

EXAMPLE II.1.11: INTERPRETATION OF MEAN-ADJUSTED TRACKING ERROR

Calculate the ex post mean-adjusted tracking error for:

(a) the fund in Example II.1.9 relative to its benchmark; and
(b) the fund in Example II.1.10 relative to both benchmarks.

What can you infer from your results?

SOLUTION The mean-adjusted tracking error can be calculated directly on the squared active returns using (II.1.52) and this is done in the spreadsheet for this example. Alternatively, since we already know the ex post $TE$, we may calculate the mean active return and use (II.1.53).

(a) For the fund in Example II.1.9 we have $T = 16$, $TE = 1\%$ and $\overline{R} = -4.06\%$. Hence,

$$MATE = \sqrt{0.01^2 \times \frac{15}{16} + 0.0406^2} = 4.18\%.$$

The $MATE$ is much greater than $TE$ because it captures the fact that the fund deviated considerably from the benchmark.

(b) For the fund in Example II.1.10 we again have $T = 16$, and,

relative to benchmark 1, $TE = 1.38\%$ and $\overline{R} = -0.04\%$;

relative to benchmark 2, $TE = 1.38\%$ and $\overline{R} = -3.04\%$.

Hence, using (II.1.53) we have,

relative to benchmark 1, $MATE = \sqrt{0.0138^2 \times \frac{15}{16} + 0.0004^2} = 1.34\%$;

relative to benchmark 2, $MATE = \sqrt{0.0138^2 \times \frac{15}{16} + 0.0304^2} = 3.32\%$.

Relative to benchmark 1, where the mean active return is very near zero, the mean-adjusted tracking error is approximately the same as the tracking error. In fact $MATE$ is less than $TE$, which is only possible when both $T$ and $\overline{R}$ are relatively small. Relative to benchmark 2, the mean active return is far from zero and the mean-adjusted tracking error is much larger than the tracking error.

We have already observed that the fund's risk should be much higher relative to benchmark 2, because it substantially underperformed that benchmark, yet the tracking error could not distinguish between the risks relative to either benchmark. However, the mean-adjusted tracking error does capture the difference in mean active returns: it is substantially higher relative to benchmark 2 than benchmark 1.

EXAMPLE II.1.12: COMPARISON OF $TE$ AND $MATE$

Figure II.1.7 shows a benchmark and two funds whose risk is assessed relative to that benchmark. Fund A is a passive fund that tracks the benchmark closely, and fund B is an active fund that has been allowed to deviate substantially from the benchmark allocations. As a result of poor investment decisions it has underperformed the benchmark disastrously. Which fund has more risk relative to the benchmark?

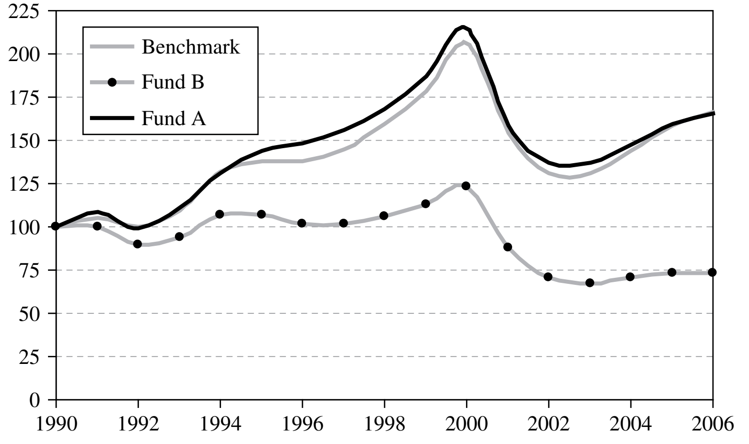


**Figure II.1.7** Which fund has an ex post tracking error of zero?

SOLUTION Fund B has a lower tracking error than fund A. In fact, the tracking error of fund B (the underperforming fund) is zero! So according to $TE$ fund A has more risk! However the real difference between the two funds is in their average active return: it is 0 for fund A but $-5\%$ for fund B.

Table II.1.12 shows the annual returns on the benchmark and on both of the funds, and the active return on each fund in each year, calculated using (II.1.44). From the active returns, their mean and their squares, formulae (II.1.51) and (II.1.53) have been used to calculate the $TE$ and $MATE$ for each fund. Only the $MATE$ identifies that fund B is more risky than fund A.

**Table II.1.12** $TE$ and $MATE$ for the funds in Figure II.1.7

| Year | Benchmark | Fund A | Fund B | Active A | Active B |
|---|---|---|---|---|---|
| 1990 | 5% | 9% | 0% | 3.81% | −5.00% |
| 1991 | −5% | −9% | −10% | −4.21% | −5.00% |
| 1992 | 10% | 12% | 4% | 1.82% | −5.00% |
| 1993 | 20% | 18% | 14% | −1.67% | −5.00% |
| 1994 | 5% | 10% | 0% | 4.76% | −5.00% |
| 1995 | 0% | 3% | −5% | 3.00% | −5.00% |
| 1996 | 5% | 5% | 0% | 0.00% | −5.00% |
| 1997 | 10% | 8% | 4% | −1.82% | −5.00% |
| 1998 | 12% | 11% | 6% | −0.89% | −5.00% |
| 1999 | 15% | 15% | 9% | −0.06% | −5.00% |
| 2000 | −25% | −26% | −29% | −1.33% | −5.00% |
| 2001 | −15% | −14% | −19% | 1.18% | −5.00% |
| 2002 | 0% | 0% | −5% | 0.00% | −5.00% |
| 2003 | 10% | 8% | 4% | −1.82% | −5.00% |
| 2004 | 10% | 8% | 4% | −1.82% | −5.00% |
| 2005 | 5% | 4% | 0% | −0.95% | −5.00% |
| | | Average | | 0.00% | −5.00% |
| | | $TE$ | | 2.38% | 0.00% |
| | | $MATE$ | | 2.30% | 5.00% |

To summarize the lessons learned from the above examples, the ex post tracking error does not measure the risk of a fund deviating from a benchmark; it only measures the variability of active returns. The level of the benchmark is irrelevant to tracking error – only the variability in benchmark returns and the variability in the fund's returns matter for the tracking error. In short, a fund with a stable active return will always have a low tracking error, irrespective of the level of active returns. However, the mean-adjusted tracking error includes a measure of the fund's deviation from the benchmark as well as a measure of the variability in active returns. Here it is not only the stability of active returns that matters for the risk metric; their general level is also taken into account.

### II.1.6.6 Ex Ante Tracking Error

For the definition of an ex ante forecast of $TE$ and of $MATE$ we need to use a model for expected returns, and the most usual type of model to employ for this is regression based on a factor model. In Section II.1.3.1 we wrote the general multi-factor regression model in matrix form as

$$\mathbf{y} = \boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{II.1.54}$$

and hence we derived the following expression for the expected return:

$$E(Y) = \alpha + \beta' E(\mathbf{X}), \tag{II.1.55}$$

where $E(\mathbf{X})$ is the vector of expected returns to each risk factor. Similarly, the variance of the return about this expected value is

$$V(Y) = \beta' \boldsymbol{\Omega} \beta + V(\varepsilon), \tag{II.1.56}$$

where $\boldsymbol{\Omega}$ is the covariance matrix of the factor returns.

To define the ex ante tracking error we suppose that $Y$ represents not the ordinary return but the *active return* on a fund. Likewise, the alpha and betas above are the *relative alpha* and *relative betas* of the fund. These are the difference between the fund's ordinary alpha and factor betas and the benchmark's alpha and factor betas. Now, given the relative alpha and betas in (II.1.54), then (II.1.55) yields the expected active return in terms of the relative alpha and betas and $E(\mathbf{X})$, the vector of expected returns to each risk factor. Similarly, (II.1.56) gives the variance of active returns in terms of $\boldsymbol{\beta}$ and $\boldsymbol{\Omega}$, the covariance matrix of the factor returns.

The ex ante tracking error is the square root of the variance of active returns given by (II.1.56), quoted in annualized terms. If the covariance matrix $\boldsymbol{\Omega}$ contains forecasts of the volatilities and correlations of the risk factor returns then (II.1.56) represents a *forecast* of the risk of active returns, i.e. the standard deviation of active returns. In other words, the ex ante tracking error measures variance *about the expected active return* (II.1.55).

It is very important to stress that (II.1.56) is a variance about (II.1.55), i.e. the expected active return that is estimated by the factor model and *only* about this expected active return. Thus the square root of (II.1.56), i.e. the tracking error, is a measure of risk relative to the expected active return (II.1.55).

Suppose we target an active return that is different from (II.1.55). For instance, we might target an outperformance of the benchmark by 2% per annum. Then it would be mathematically incorrect to represent the square root of (II.1.56), i.e. the tracking error, as the risk relative to the target active return of 2%. However, during the 1990s it was standard practice, at least by some leading fund managers, to forecast a tracking error in a factor model framework and then, somehow, to interpret this tracking error as representing the potential for a fund to deviate from its *target* active return. Suppose the target active return is 2% per annum and the expected active return based on their risk model is also 2% per annum. Then there is nothing incorrect about this interpretation. But if the expected active return based on their risk model is *not* 2%, then it is misleading to interpret the ex ante tracking error as the potential deviation from the target return.

### II.1.6.7 Ex Ante Mean-Adjusted Tracking Error

A forecast active return is a distribution. An expected active return is just one point in this distribution, i.e. its expected value, but the returns model also forecasts the entire distribution, albeit often rather crudely. Indeed, *any* forecast from a statistical model is a distribution. We may choose to focus on a single *point forecast,* usually of the expectation of this distribution, but the model still forecasts an entire distribution and this distribution is specific to the estimated model. If the point forecast of the expected return changes, so does the whole distribution, and usually it does not just 'shift' with a different expected return; the variance of the return about this expectation also changes! In short, there is only *one* distribution of active returns in the future that is forecast by any statistical model and it is inconsistent with the model to change one of its parameters, leaving the other parameters unchanged. One may as well throw away the model and base forecasts entirely on subjective beliefs.

Consider Figure II.1.8, which shows an active return forecast depicted as a normal distribution where the mean of that distribution – the expected active return $E(Y)$ – is assumed to be less than the target active return. Now, if the target active return is *not* equal to $E(Y)$, which is very often the case, then there are *two* sources of risk relative to the benchmark: the risk arising from dispersion about the mean return (i.e. tracking error) and the risk that the mean return differs from the target return. The tracking error *ignores* the second source of active risk.

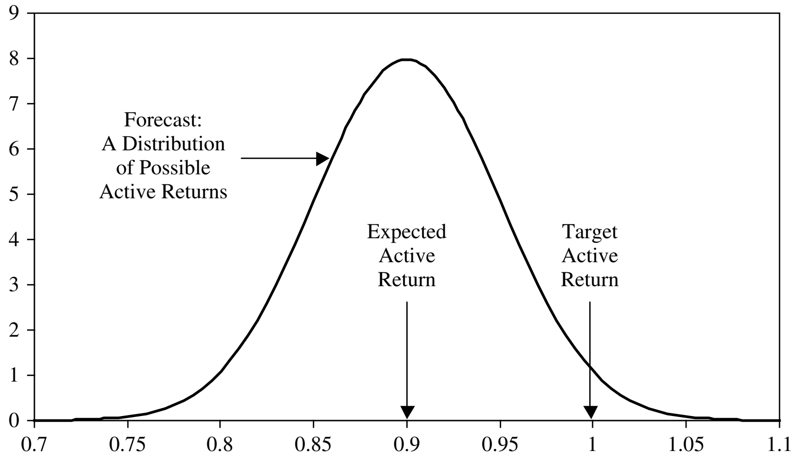


**Figure II.1.8** Forecast and target active returns

However, the mean-adjusted ex ante tracking error *does* take account of model predictions for active returns that may differ from the target active return. We define

$$MATE = \sqrt{V(Y) + (E(Y) - Y^*)^2}, \tag{II.1.57}$$

where $Y^*$ is the target active return and $E(Y)$ and $V(Y)$ are forecast by the risk model.

EXAMPLE II.1.13: WHICH FUND IS MORE RISKY (1)?

A risk model is used to forecast the ex ante tracking errors for two funds. Both funds have the same ex ante tracking error of 4%. However, the model gives different predictions for the expected active return on each fund: it is 0% for fund A and 1% for fund B. The target active return is 2%. Which fund is more risky relative to this target?

SOLUTION Since both funds have the same tracking error ($TE$), they have the same risk according to the $TE$ metric. But $TE$ does not measure risk relative to the target active return. The mean-adjusted tracking error ($MATE$) is 4.47% for fund A and 4.12% for fund B. Hence, according to the $MATE$ metric, fund A is more risky. This is intuitive, since the expected active return on fund A is further from the target active return than the expected active return on fund B.

This example has shown that if two index tracking funds have the same tracking error, the fund that has the highest absolute value for expected active return will have the greatest mean-adjusted tracking error.

EXAMPLE II.1.14: WHICH FUND IS MORE RISKY (2)?

A risk model is used to forecast the ex ante tracking error for two funds. The predictions are $TE = 2\%$ for fund A and $TE = 5\%$ for fund B. The funds have the same expected active return. Which fund is more risky?

SOLUTION Fund B has a larger ex ante tracking error than fund A and so is more risky than fund A according to this risk metric. It does not matter what the target active return is, because this has no effect on the ex ante tracking error. Fund B also has the larger mean-adjusted tracking error, because the funds have the same expected active return. For instance, if the expected active return is either $+1\%$ or $-1\%$ then $MATE = 2.24\%$ for fund A and $MATE = 5.10\%$ for fund B.

Hence both the $TE$ and the mean-adjusted $TE$ agree that fund B is more risky. If two funds have the same expected active return then the fund that has the highest tracking error will have the greatest mean-adjusted tracking error.

But this is not the whole story about active risk. Figure II.1.9 depicts the ordinary returns distributions for the two funds considered in Example II.1.14. Now we make the further assumption that the predicted returns are normally distributed, and that the two funds have the same expected return of 1%. Two different target returns, of 0% and 2%, are depicted on the figure using vertical dotted and solid lines, respectively. We note:

- There is a 42% chance that fund B returns less than 0%, but only a 31% chance that fund A returns less than 0%. So, fund B is more risky than fund A relative to a target of 0%.

- There is a 69% chance that fund A returns less than 2% but only a 58% chance that fund B returns less than 2%. So, fund A is more risky than fund B relative to a target of 2%.

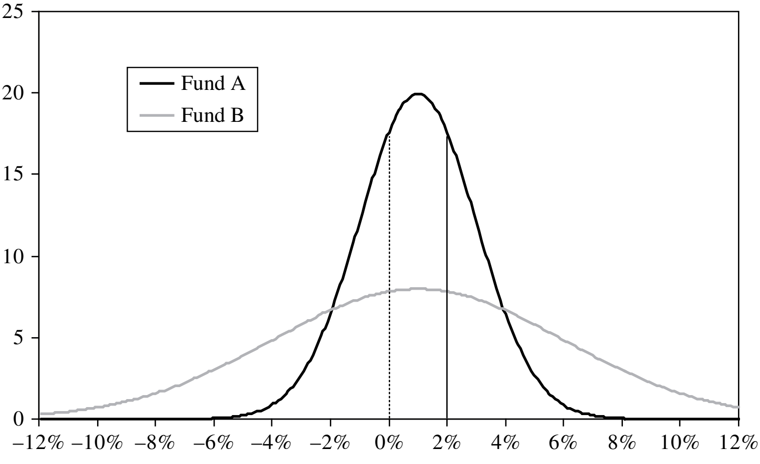


**Figure II.1.9** Returns distributions for two funds

However, both $TE$ and $MATE$ rank fund B as the riskier fund relative to *both* benchmarks. Although $MATE$ does capture the risk that the expected return will deviate from the target return, it cannot capture the difference between a good forecast, where the expected return is greater than target, and a bad forecast, where the expected return is less than target.[21] $MATE$ penalizes *any* deviation between the expected return and the target and it does not matter whether the deviation is positive or negative.

This example shows that when the expected active return derived from the risk model is different from the target return $Y^*$ then the potential for the expected return to deviate from the target return usually represents *much* the largest element of active risk as perceived by the clients. Yet this part of the risk is commonly ignored by mathematically inept and ill-informed fund managers.

Another lesson to be learned from the above example is that if $E(Y) < Y^*$, i.e. if the expected active return is less than the target active return, then the worst case occurs when the tracking error is *small*. In other words, if the model predicts an active return that is less than the target it is *better* for the investors if the tracking error is large!

### II.1.6.8 Clarification of the Definition of Active Risk

In the 1980s and early 1990s the decisions made by active fund managers were usually controlled through strict imposition of control ranges. That is, the active weights were not allowed to become too great. However, since then some fund managers have dropped control ranges in favour of metrics such as tracking error that could (if used properly) provide

[21] This is because their difference is squared in the formula (II.1.57) for this risk metric.

a better description of active risk. Various definitions of active risk can be found in the literature. One of the most complete definitions is given by Wikipedia.[22]

> *Active risk* refers to that segment of risk in an investment portfolio that is due to active management decisions made by the portfolio manager. It does not include any risk (return) that is merely a function of the market's movement. In addition to risk (return) from specific stock selection or industry and factor 'bets', it can also include risk (return) from market timing decisions. *A portfolio's active risk, then, is defined as the annualized standard deviation of the monthly difference between portfolio return and benchmark return.*

The last sentence makes it abundantly clear that, according to this (incorrect) definition, 'active risk' is measured by the tracking error. However, using our series of pedagogical examples above, we have demonstrated that measuring active risk using this metric is mathematically incorrect, except when the expected active return is zero, which is only the case for *passive*, benchmark-tracking funds. The definition of active risk given above is therefore contradictory, because the first sentence states that active risk is the 'risk in an investment portfolio that is due to *active* management decisions'. All risk averse clients and fund managers would agree that the risk 'due to active management decisions' should include the risk that an actively managed portfolio underperforms the benchmark. But we have proved that tracking error, i.e. the annualized standard deviation of the monthly difference between portfolio return and benchmark return, does not include this risk.

The Wikipedia definition is one of numerous other contradictory and confusing definitions of active risk. A myth – that tracking error equates to active risk – is still being perpetuated. In fact, at the time of writing (and I sincerely hope these will be corrected soon) virtually *all* the definitions of active risk available on the internet that also define a way to measure it fall into the trap of assuming tracking error is a suitable metric for active risk. Many simply define active risk as the standard deviation of the active returns, and leave it at that!

Active risk was originally a term applied to *passive* management where the fund manager's objective is to track an index as closely as possible. There is very little scope to deviate from the index because the fund aims for a zero active return. In other words, the *expected* active return is zero for a passive fund and, as shown above, it is only in this case that tracking error *is* synonymous with active risk. But actively managed funds have a mandate to outperform an index, so by definition their expected active return is not zero. Hence the active risk of actively managed funds *cannot* be measured by tracking error.

If nothing else, I hope that this section has made clear to active fund managers that it is extremely important to define one's terms very carefully. The enormously ambiguous phrase *risk of returns relative to the benchmark,* which is often used to define active risk, could be interpreted as the risk [of returns] relative to the benchmark, i.e. the risk of deviating from the benchmark. But it could also be interpreted as the risk of [returns relative to the benchmark], i.e. the standard deviation of active returns, and this is different from the first interpretation! Measuring *returns* relative to a benchmark does not go hand in hand with measuring *risk* relative to a benchmark, unless the expected active return is zero. So the tracking error metric is fine for funds that actually track the benchmark, i.e. for passive funds. Indeed, it is from this that the name derives. But for funds that have a mandate *not*

[22] See http://en.wikipedia.org/wiki/Active_risk. This is the definition at the time of going to press, but I shall be adding a discussion to this page with a reference to this chapter when the book is in print.

to track a benchmark, i.e. for actively managed funds, the tracking error cannot be used to measure the active risk. It measures the risk of [returns relative to the benchmark] but says nothing *at all* about the real risk that active managers take, which is the risk that the fund will underperform the benchmark.

## II.1.7 SUMMARY AND CONCLUSIONS

In this chapter we have described the use of factor models for analysing the risk and return on portfolios of risky assets. Even though the returns distribution of a portfolio could be modelled without using a factor model, the advantages of factor models include the ability to:

- attribute total risk to different sources, which is useful for performance analysis, benchmark selection and risk capital allocation; and
- evaluate portfolio risk under 'what if' scenarios, i.e. when risk factor values are stressed to extreme levels.

Many factor models are estimated using historical time series data. Such models may be used to forecast the risk and expected returns of portfolios of risky assets. Basic measures of risk may be based purely on a fund's historical returns, but the analyst will gain further insight into the risk characteristics of the portfolio by employing stress tests and scenario analysis. This is the main reason for using factor models to capture portfolio risk. If all that we wanted was a risk measurement, we could just use historical data on stock returns to form a 'current weighted' portfolio and measure its volatility – this is much easier than building a good factor model. But the factor model is a great tool for value-at-risk modelling, especially for the stress tests and scenario analysis that form part of the day-to-day work of a risk analyst.

Factor models are also used for *style analysis*, i.e. to attribute funds' returns to value, growth and other style factors. This helps investors to identify the sources of returns knowing only the funds returns and no details about the fund's strategies. Style analysis can be used to select appropriate benchmarks against which to measure performance and as a guide for portfolio diversification. In one of the empirical examples in this chapter we have implemented a style analysis for a simple portfolio, and the results were based on a constrained quadratic programming problem.

The examples developed in the Excel workbook for this chapter take the reader through many different factor models. In some cases we have decomposed total risk into *systematic risk* and *specific risk* components. We also showed how the total systematic risk of international stock portfolios may be decomposed into *equity risk* and *foreign exchange risk* components. In other examples we estimated *fundamental factor models* whose risk factors are market and style indices, estimating their betas using regression analysis. But there was a very high correlation between the different risk factor returns, as so often happens with these models, and this necessitated the use of *orthogonal regression* techniques to properly identify the factor betas.

We also provided a detailed analysis of the *Barra model*, which employs time series and cross-sectional data to analyse the return (and also the risk) on both active and passive portfolios. For the benefit of users of the Barra model, we have carefully explained the correct way to measure the risk of active portfolios that are optimized using this model. Then we provided a critical discussion of the way that *active risk* has been, and may continue to be, measured by many fund managers. The definition of active risk is fraught with difficulty and ambiguous terms. *Active risk is the risk that an actively managed investment portfolio deviates from the benchmark*. Beware of other definitions, and there are many! In the 1990s many fund managers assessed active risk using the *tracking error*, i.e. the volatility of the active returns. Even nowadays many practitioners regard active risk and tracking error as synonymous. But we have demonstrated that this is a mistake – and potentially a very costly one! It is a common fallacy that tracking error can be used as an active risk metric. Using many pedagogical examples, we have carefully explained why *tracking error says nothing at all about the risk relative to a benchmark*. Tracking error only measures the volatility of relative returns.

Desirable properties for a good active risk metric include:

(a) if the active risk measure falls then the fund moves closer to the benchmark; and
(b) if the fund moves closer to the benchmark then the active risk measure falls.

However, tracking error has neither of these properties. The examples in Section II.1.6 have shown that a reduction in tracking error does not imply that the fund moves closer to the benchmark. It only implies that the active returns have become more stable. Also, moving closer to the benchmark does not imply that tracking error will be reduced and moving away from the benchmark does not imply that tracking error will increase.

Tracking error is *not* a suitable metric for measuring active risk, either ex post or ex ante. It is fine for passive funds, as its name suggests. In passive funds the expected future active return is zero and the ex post mean active return is likely to be very close to zero. Then tracking error measures the volatility around the benchmark. But more generally, tracking error measures volatility around the expected active return in the model – not the volatility around a zero active return, and not the volatility around the target outperformance, nor around any other value! In active fund management the aim is to *outperform* a benchmark by taking positions that may deviate markedly from those in the benchmark. Hence, the expected active return should not be zero; it should be equal to the target outperformance set by the client. The *mean-adjusted* tracking error *is* an active risk metric, but it is not a very good one. It penalizes returns that are greater than the benchmark return as much as it penalizes returns that are less than the benchmark return. That is, it is not a *downside* risk metric.