1
COMPLEX ALGEBRAIC VARIETIES

An *algebraic variety* is defined to be the set of complex zeros of homogeneous polynomials in projective space and may be viewed a priori as an analytic subvariety of $\mathbb{P}^n$. In case the variety is smooth, we may consider the associated abstract compact complex manifold, whose properties will be intrinsic to—i.e., not depending on the particular embedding of—the variety. Broadly speaking, we will approach algebraic geometry as the study of the interplay between the intrinsic and extrinsic or projective properties of algebraic varieties.

In Section 1 we introduce the notion of divisors and line bundles; the material here is central for all that follows. Since a compact complex manifold admits no global holomorphic functions, we might rather expect its structure to be reflected in the global meromorphic functions and related linear systems of divisors on the manifold; this notion is a basic one in classical algebraic geometry. Associated to a divisor is a holomorphic line bundle, to a meromorphic function a line bundle together with a holomorphic section, and to a line bundle its Chern class. The subsequent formalism, developed by Kodaira and Spencer and others in the early 1950s, gives an extremely useful technique for dealing with codimension-one subvarieties (points on a curve, curves on a surface, etc.) on an algebraic variety.

The basic question of constructing meromorphic functions with prescribed properties—e.g., the principal parts on a Riemann surface—is a problem admitting local solutions where the obstruction to patching these together globally may be measured by a sheaf cohomology group. The *Kodaira vanishing theorem* provides the most useful condition under which these higher groups are zero. It is a remarkable result, one which is proved by potential theory and differential geometry, but which in the end turns out to be equivalent to the Lefschetz theorem concerning the topological position of a hyperplane section of a complex algebraic variety. Explaining these matters occupies Section 2.

In Section 3 we began the transition

$$\left\{\begin{matrix}\text{abstract compact}\\ \text{complex manifold}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{algebraic variety}\\ \text{in projective space}\end{matrix}\right\}$$

The intermediate step is an analytic variety in projective space; the *Chow theorem* asserts that this must be an algebraic variety. The essential philosophical point here is illustrated by the identity of the two objects "global meromorphic function on the Riemann sphere" and "rational function of one complex variable." The practical consequence is that we may work either locally complex analytically or globally algebraically with the same end result. Our approach at this stage is analytic, as this ties in more readily with the topological and metric properties of an algebraic variety, but the understanding that in the end we are talking about solutions of polynomial equations is fundamental.

In Section 4 we state and prove Kodaria's characterization of those compact complex manifolds which are derived from algebraic varieties, thus providing the essential link between the intrinsic and extrinsic properties of a variety. This embedding theorem and Chow's theorem are existence theorems—they do not by themselves provide a constructive method for finding the equations defining the image of a variety under a projective embedding—but together they form the philosophical cornerstone for our analytic treatment of algebraic geometry.

In the final section of this chapter we explain in some detail the Grassmannian, a variety whose points parametrize the linear subspaces of some fixed dimension in projective space and whose internal structure reflects the nongeneric intersections of a variable linear space with a fixed one. One reason for placing this discussion here is that the Grassmannian illustrates quite nicely the general structure theorems of this chapter. Another is that extensive use will be made in the following chapters of the Schubert calculus, a quantitative expression of the nongeneric incidence relations among linear spaces that is inherent in the structure of the Grassmannian.

1. DIVISORS AND LINE BUNDLES

Divisors

Let M be a complex manifold of dimension n, not necessarily compact. We recall from Section 1 of Chapter 0 some facts about analytic hypersurfaces in M:

Any analytic subvariety $V \subset M$ of dimension $n-1$ is an analytic hypersurface, i.e., for any point $p \in V \subset M$, V can be given in a neighborhood of p as the zeros of a single holomorphic function f. Moreover, any holomorphic function g defined at p and vanishing on V is divisible by f in a neighborhood of p. f is called a *local defining function* for V near p, and is unique up to multiplication by a function nonzero at p.

If V_i^* is a connected component of $V^* = V - V_s$, then $\overline{V_i^*}$ is an analytic subvariety in M. Thus V can be expressed uniquely as the union of irreducible analytic hypersurfaces

$$V = V_1 \cup \cdots \cup V_m,$$

where the V_i's are the closures of the connected components of V^*. In particular, V is irreducible if and only if V^* is connected.

Now we define:

DEFINITION. A *divisor* D on M is a locally finite formal linear combination

$$D = \sum a_i \cdot V_i$$

of irreducible analytic hypersurfaces of M.

"Locally finite" here means that for any $p \in M$, there exists a neighborhood of p meeting only a finite number of the V_i's appearing in D; of course, if M is compact, this just means the sum is finite. The set of divisors in M is naturally an additive group, denoted $\operatorname{Div}(M)$.

A divisor $D = \sum a_i V_i$ is called *effective* if $a_i \geqslant 0$ for all i; we write $D \geqslant 0$ for D effective. An analytic hypersurface V will usually be identified with the divisor $\sum V_i$ where the V_i's are the irreducible components of V.

Let $V \subset M$ be an irreducible analytic hypersurface, $p \in V$ any point, and f a local defining function for V near p. For any holomorphic function g defined near p, we define the *order* $\operatorname{ord}_{V,p}(g)$ *of* g *along* V *at* p to be the largest integer a such that in the local ring $\mathcal{O}_{M,p}$,

$$g = f^a \cdot h.$$

By the result from p. 10 that relatively prime elements of $\mathcal{O}_{M,p}$ stay relatively prime in nearby local rings, we see that for g a holomorphic function on M, $\operatorname{ord}_{V,p}(g)$ *is independent of* p. Thus we can define the *order* $\operatorname{ord}_V(g)$ *of* g *along* V to be simply the order of g along V at any point $p \in V$. Note that for g, h any holomorphic functions, V any irreducible hypersurface,

$$\operatorname{ord}_V(gh) = \operatorname{ord}_V(g) + \operatorname{ord}_V(h).$$

Now let f be a meromorphic function on M, written locally as

$$f = \frac{g}{h}$$

with g,h holomorphic and relatively prime. For V an irreducible hypersurface, we define

$$\operatorname{ord}_V(f) = \operatorname{ord}_V(g) - \operatorname{ord}_V(h).$$

We usually say that *f has a zero of order* a *along* V if $\operatorname{ord}_V(f) = a > 0$, and that f has a *pole of order* a *along* V if $\operatorname{ord}_V(f) = -a < 0$.

We define the *divisor* (f) *of the meromorphic function* f by

$$(f) = \sum_V \operatorname{ord}_V(f) \cdot V.$$

If f is written locally as g/h, we take the *divisor of zeros* $(f)_0$ of f to be

$$(f)_0 = \sum_V \operatorname{ord}_V(g) \cdot V$$

and the *divisor of poles* $(f)_\infty$ to be

$$(f)_\infty = \sum_V \operatorname{ord}_V(h) \cdot V.$$

Clearly these are well-defined as long as we require g and h to be relatively prime, and

$$(f) = (f)_0 - (f)_\infty.$$

Divisors can also be described in sheaf-theoretic terms, as follows: Let $\mathfrak{M}^*$ denote the multiplicative sheaf of meromorphic functions on M not identically 0, and $\mathcal{O}^*$ the subsheaf of nonzero holomorphic functions. Then *a divisor* D *on* M *is simply a global section of the quotient sheaf* $\mathfrak{M}^*/\mathcal{O}^*$. On the one hand, a global section $\{f\}$ of $\mathfrak{M}^*/\mathcal{O}^*$ is given by an open cover $\{U_\alpha\}$ of M and meromorphic functions $f_\alpha \not\equiv 0$ in U_α with

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta);$$

for any $V \subset M$, then,

$$\operatorname{ord}_V(f_\alpha) = \operatorname{ord}_V(f_\beta),$$

and we can associate to $\{f\}$ the divisor

$$D = \sum_V \operatorname{ord}_V(f_\alpha) \cdot V,$$

where for each V we choose α such that $V \cap U_\alpha \neq \varnothing$. On the other hand, given

$$D = \sum_{V_i} a_i V_i,$$

we can find an open cover $\{U_\alpha\}$ of M such that in each U_α, every V_i appearing in D has a local defining function $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. We can then set

$$f_\alpha = \prod_i g_{i\alpha}^{a_i} \in \mathfrak{M}^*(U_\alpha)$$

to obtain a global section of $\mathfrak{M}^*/\mathcal{O}^*$. The f_α's are called *local defining functions* for D. It follows immediately from the definitions that the identification

$$H^0(M, \mathfrak{M}^*/\mathcal{O}^*) = \mathrm{Div}(M)$$

is in fact a homomorphism.

Given a holomorphic map $\pi: M \to N$ of complex manifolds, we define a map

$$\pi^*: \mathrm{Div}(N) \to \mathrm{Div}(M)$$

by associating to every divisor $D = (\{U_\alpha\}, \{f_\alpha\})$ on N the *pullback divisor* $\pi^* D = (\{\pi^{-1} U_\alpha\}, \{\pi^* f_\alpha\})$ on M; this is well-defined as long as $\pi(M) \not\subset D$. Note that for a divisor on N given by an analytic hypersurface $V \subset N$, the pullback divisor $\pi^* V$ on M lies over V but need not coincide with the analytic hypersurface $\pi^{-1}(V) \subset M$—multiplicities may occur.

We want to make one more remark before going on to consider line bundles. On a Riemann surface M, any point is an irreducible analytic hypersurface, and so clearly $\mathrm{Div}(M)$ is always large. This is, in a sense, misleading: *a complex manifold* M *of dimension greater than one need not have any nonzero divisors on it at all.* If, however M is embedded in projective space $\mathbb{P}^N$, the intersections of M with hyperplanes in $\mathbb{P}^N$ generate a large number of divisors. In fact, among all compact complex manifolds those which are embeddable in projective space can be characterized by having "sufficiently many" divisors, in a sense that we shall make precise in later sections.

Line Bundles

All line bundles discussed in this section are taken to be holomorphic. Recall that for any holomorphic line bundle $L \xrightarrow{\pi} M$ on the complex manifold M, we can find an open cover $\{U_\alpha\}$ of M and trivializations

$$\varphi_\alpha: L_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}$$

of $L_{U_\alpha} = \pi^{-1}(U_\alpha)$. We define the transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$ for L relative to the trivializations $\{\varphi_\alpha\}$ by

$$g_{\alpha\beta}(z) = \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)|_{L_z} \in \mathbb{C}^*.$$

The functions $g_{\alpha\beta}$ are clearly holomorphic, nonvanishing, and satisfy

$$(*) \qquad \begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} = 1, \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1; \end{cases}$$

conversely, given a collection of functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ satisfying these identities, we can construct a line bundle L with transition functions

$\{g_{\alpha\beta}\}$ by taking the union of $U_\alpha \times \mathbb{C}$ over all α and identifying $\{z\} \times \mathbb{C}$ in $U_\alpha \times \mathbb{C}$ and $U_\beta \times \mathbb{C}$ via multiplication by $g_{\alpha\beta}(z)$.

Now, given L as above, for any collection of nonzero holomorphic functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ we can define alternate trivializations of L over $\{U_\alpha\}$ by

$$\varphi'_\alpha = f_\alpha \cdot \varphi_\alpha;$$

transition functions $g'_{\alpha\beta}$ for L relative to $\{\varphi'_\alpha\}$ will then be given by

$$(**) \qquad g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta}.$$

On the other hand, any other trivialization of L over $\{U_\alpha\}$ can be obtained in this way, and so we see that collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition functions define the same line bundle if and only if there exist functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying (**).

The description of line bundles by transition functions lends itself well to a sheaf-theoretic interpretation. First, the transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ for a line bundle $L \to M$ represent a Čech 1-cochain on M with coefficients in $\mathcal{O}^*$; the relation (*) simply asserts that $\delta(\{g_{\alpha\beta}\}) = 0$, i.e., $\{g_{\alpha\beta}\}$ is a Čech cocycle. Moreover, by the last paragraph, two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same line bundle if and only if their difference $\{g_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta}\}$ is a Čech coboundary; consequently *the set of line bundles on* M *is just* $H^1(M, \mathcal{O}^*)$.

We can give the set of line bundles on M the structure of a group, multiplication being given by tensor product and inverses by dual bundles. If L is given by data $\{g_{\alpha\beta}\}$, L' by $\{g'_{\alpha\beta}\}$, we have seen that

$$L \otimes L' \sim \{g_{\alpha\beta} g'_{\alpha\beta}\}, \qquad L^* \sim \{g^{-1}_{\alpha\beta}\},$$

and so the group structure on the set of line bundles is the same as the group structure on $H^1(M, \mathcal{O}^*)$. The group $H^1(M, \mathcal{O}^*)$ is called the *Picard group* of M, denoted $\operatorname{Pic}(M)$.

We now describe the basic correspondence between divisors and line bundles. Let D be a divisor on M, with local defining functions $f_\alpha \in \mathcal{M}^*(U_\alpha)$ over some open cover $\{U_\alpha\}$ of M. Then the functions

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

are holomorphic and nonzero in $U_\alpha \cap U_\beta$, and in $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \frac{f_\alpha}{f_\beta} \cdot \frac{f_\beta}{f_\gamma} \cdot \frac{f_\gamma}{f_\alpha} = 1.$$

The line bundle given by the transition functions $\{g_{\alpha\beta}=f_\alpha/f_\beta\}$ is called the *associated line bundle* of D, and written $[D]$. We check that it is well-defined: if $\{f'_\alpha\}$ are alternate local data for D, then $h_\alpha=f_\alpha/f'_\alpha\in\mathcal{O}^*(U_\alpha)$, and

$$g'_{\alpha\beta}=\frac{f'_\alpha}{f'_\beta}=g_{\alpha\beta}\cdot\frac{h_\beta}{h_\alpha}$$

for each α,β.

The correspondence $[\quad]$ has these immediate properties: First, if D and D' are two divisors given by local data $\{f_\alpha\}$ and $\{f'_\alpha\}$, respectively, then $D+D'$ is given by $\{f_\alpha\cdot f'_\alpha\}$; it follows that

$$[D+D']=[D]\otimes[D']$$

so the map

$$[\quad]:\ \mathrm{Div}(M)\to\mathrm{Pic}(M)$$

is a homomorphism. Second, if $D=(f)$ for some meromorphic function f on M, we may take as local data for D over any cover $\{U_\alpha\}$ the functions $f_\alpha=f|_{U_\alpha}$; then $f_\alpha/f_\beta=1$ and so $[D]$ is trivial. Conversely, if D is given by local data $\{f_\alpha\}$ and the line bundle $[D]$ is trivial, then there exist functions $h_\alpha\in\mathcal{O}^*(U_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta}=g_{\alpha\beta}=\frac{h_\alpha}{h_\beta};$$

$f=f_\alpha\cdot h_\alpha^{-1}=f_\beta\cdot h_\beta^{-1}$ is then a global meromorphic function on M with divisor D. Thus *the line bundle* [D] *associated to a divisor* D *on* M *is trivial if and only if* D *is the divisor of a meromorphic function*. We say that two divisors D,D' on M are *linearly equivalent* and write $D\sim D'$ if $D=D'+(f)$ for some $f\in\mathfrak{M}^*(M)$, or equivalently if $[D]=[D']$.

Also, note that $[\quad]$ is functorial: that is, if $f\colon M\to N$ is a holomorphic map of complex manifolds, it is easy to check that for any $D\in\mathrm{Div}(N)$,

$$\pi^*([D])=[\pi^*(D)].$$

All these assertions are implicit in the following cohomological interpretation of the correspondence $[\quad]$. The exact sheaf sequence

$$0\to\mathcal{O}^*\xrightarrow{i}\mathfrak{M}^*\xrightarrow{j}\mathfrak{M}^*/\mathcal{O}^*\to 0$$

on M gives us, in part, the exact sequence

$$H^0(M,\mathfrak{M}^*)\xrightarrow{j_*}H^0(M,\mathfrak{M}^*/\mathcal{O}^*)\xrightarrow{\delta}H^1(M,\mathcal{O}^*)$$

of cohomology groups. The reader may easily verify that under the natural identifications

$$\mathrm{Div}(M)=H^0(M,\mathfrak{M}^*/\mathcal{O}^*)\quad\text{and}\quad\mathrm{Pic}(M)=H^1(M,\mathcal{O}^*)$$

for any meromorphic function f on M,

$$j_* f = (f),$$

and for any divisor D on M,

$$\delta D = [D].$$

Indeed, we will generally violate the previous multiplicative notation and write $L + L'$ for the tensor product of two line bundles or mL for the mth tensor power $L^{\otimes m}$ of L.

We now wish to discuss holomorphic and meromorphic sections of line bundles. Let $L \to M$ be a holomorphic line bundle, with trivializations $\varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$ over an open cover $\{U_\alpha\}$ of M and transition functions $\{g_{\alpha\beta}\}$ relative to $\{\varphi_\alpha\}$. As we have seen, the trivializations φ_α induce isomorphisms

$$\varphi_\alpha^* : \ \mathcal{O}(L)(U_\alpha) \longrightarrow \mathcal{O}(U_\alpha);$$

we see via the correspondence

$$s \in \mathcal{O}(L)(U) \to \{s_\alpha = \varphi_\alpha^*(s) \in \mathcal{O}(U \cap U_\alpha)\}$$

that a section of L over $U \subset M$ is given exactly by a collection of functions $s_\alpha \in \mathcal{O}(U \cap U_\alpha)$ satisfying

$$s_\alpha = g_{\alpha\beta} \cdot s_\beta$$

in $U \cap U_\alpha \cap U_\beta$.

In the same way, a *meromorphic* section s of L over U—defined to be a section of the sheaf $\mathcal{O}(L) \otimes_{\mathcal{O}} \mathfrak{M}$—is given by a collection of meromorphic functions $s_\alpha \in \mathfrak{M}(U \cap U_\alpha)$ satisfying $s_\alpha = g_{\alpha\beta} \cdot s_\beta$ in $U \cap U_\alpha \cap U_\beta$. Note that the quotient of two meromorphic sections $s, s' \not\equiv 0$ of L is a well-defined meromorphic function.

If s is a global meromorphic section of L, $s_\alpha / s_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, and so for any irreducible hypersurface $V \subset M$,

$$\operatorname{ord}_V(s_\alpha) = \operatorname{ord}_V(s_\beta).$$

Thus we can define the *order* of s along V by

$$\operatorname{ord}_V(s) = \operatorname{ord}_V(s_\alpha)$$

for any α such that $U_\alpha \cap V \neq \emptyset$; we take the *divisor* (s) *of the meromorphic section* s to be given by

$$(s) = \sum_V \operatorname{ord}_V(s) \cdot V.$$

With this convention s is holomorphic if and only if (s) is effective.

Now if $D \in \operatorname{Div}(M)$ is given by local data $f_\alpha \in \mathfrak{M}(U_\alpha)$, then the functions f_α clearly give a meromorphic section s_f of $[D]$ with $(s_f) = D$.

Conversely, if L is given by trivializations φ_α with transition functions $g_{\alpha\beta}$ and s is any global meromorphic section of L, we see that

$$\frac{s_\alpha}{s_\beta} = g_{\alpha\beta},$$

i.e., $L=[(s)]$. Thus if *D is any divisor such that* $[D]=L$, *there exists a meromorphic section* s *of* L *with* $(s)=D$, *and for any meromorphic section* s *of* L, $L=[(s)]$. In particular, we see that L is the line bundle associated to some divisor D on M if and only if it has a global meromorphic section not identically zero; it is the line bundle of an effective divisor if and only if it has a nontrivial global holomorphic section.

We can also view this correspondence as follows: Given a divisor

$$D = \sum a_i V_i$$

on M, let $\mathcal{L}(D)$ denote the space of meromorphic functions f on M such that

$$D + (f) \geqslant 0,$$

i.e., that are holomorphic on $M - \cup V_i$ with

$$\operatorname{ord}_{V_i}(f) \geqslant -a_i.$$

We denote by $|D| \subset \operatorname{Div}(M)$ the set of all effective divisors linearly equivalent to D; if $L=[D]$, we write $|L|$ for $|D|$. Let s_0 be a global meromorphic section of $[D]$ with $(s_0)=D$. Then for any global holomorphic section s of $[D]$, the quotient

$$f_s = \frac{s}{s_0}$$

is a meromorphic function on M with

$$(f_s) = (s) - (s_0) \geqslant -D,$$

i.e.,

$$f_s \in \mathcal{L}(D)$$

and

$$(s) = D + (f_s) \in |D|.$$

On the other hand, for any $f \in \mathcal{L}(D)$ the section $s = f \cdot s_0$ of $[D]$ is holomorphic. Thus *multiplication by* s_0 *gives an identification*

$$\mathcal{L}(D) \xrightarrow{\otimes s_0} H^0(M, \mathcal{O}([D])).$$

Now suppose M is compact. For every $D' \in |D|$, there exists $f \in \mathcal{L}(D)$ such that

$$D' = D + (f),$$

and conversely any two such functions f, f' differ by a nonzero constant. Thus we have the additional correspondence

$$|D| \cong \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}\big(H^0(M, \mathcal{O}([D]))\big).$$

In general, the family of effective divisors on M corresponding to a linear subspace of $\mathbb{P}(H^0(M, \mathcal{O}(L)))$ for some $L \to M$ is called a *linear system* of divisors; a linear system is called *complete* if it is of the form $|D|$, i.e., if it contains every effective divisor linearly equivalent to any of its members. When we speak of the *dimension* of a linear system, we will refer to the dimension of the projective space parametrizing it; thus, when we write dim $|D|$ for the dimension of the complete linear system associated to a divisor D, we have

$$\dim |D| = h^0(M, \mathcal{O}(D)) - 1.$$

A linear system of dimension 1 is called a *pencil*, of dimension 2 a *net*, and of dimension 3 a *web*.

We will mention here two special properties of linear systems. The first is elementary: if $E = \{D_\lambda\}_{\lambda \in \mathbb{P}^n}$ is a linear system, then for any $\lambda_0, \dots, \lambda_n$ linearly independent in $\mathbb{P}^n$,

$$D_{\lambda_0} \cap \cdots \cap D_{\lambda_n} = \bigcap_{\lambda \in \mathbb{P}^n} D_\lambda.$$

The common intersection of the divisors in a linear system is called the *base locus* of the system; in particular, a divisor F in the base locus—that is, such that $D_\lambda - F \geqslant 0$ for all λ—is called a *fixed component* of E.

The second property is more remarkable; like the first, it is peculiar to linear systems and is not the case for general families of divisors, even general families of linearly equivalent divisors. This is

Bertini's Theorem. *The generic element of a linear system is smooth away from the base locus of the system.*

Proof. If the generic element of a linear system is singular away from the base locus of the system, then the same will be true for a generic pencil contained in the system; thus it suffices to prove Bertini for a pencil.

Suppose $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a pencil, given in a polydisc Δ contained in M by

$$D_\lambda = (f(z_1, \dots, z_n) + \lambda \cdot g(z_1, \dots, z_n) = 0)$$

and suppose P_λ is a singular point of the divisor D_λ $(\lambda \neq 0, \infty)$ but not in the base locus B of the pencil. We have then

$$f(P_\lambda) + \lambda g(P_\lambda) = 0$$

and

$$\frac{\partial f}{\partial z_i}(P_\lambda) + \lambda \frac{\partial g}{\partial z_i}(P_\lambda) = 0, \qquad i = 1, \dots, n.$$

Since P_λ is not a base point of $\{D_\lambda\}$, f and g cannot both vanish at P_λ and so neither one can; thus

$$\lambda = -\frac{f(P_\lambda)}{g(P_\lambda)}$$

and

$$\frac{\partial f}{\partial z_i}(P_\lambda) - \frac{f(P_\lambda)}{g(P_\lambda)} \cdot \frac{\partial g}{\partial z_i}(P_\lambda) = 0.$$

Then

$$\frac{\partial}{\partial z_i}\left(\frac{f}{g}\right)(P_\lambda) = \frac{(\partial f/\partial z_i)(P_\lambda) - [f(P_\lambda)/g(P_\lambda)] \cdot (\partial g/\partial z_i)(P_\lambda)}{g(P_\lambda)} = 0.$$

Now the locus V of singular points of the divisors D_λ, being locally the image in Δ of the variety $S \subset \Delta \times \mathbb{P}^1_\lambda$ cut out by the equations $\{f + \lambda g = 0, \partial f/\partial z_i + \lambda \partial g/\partial z_i = 0\}$, is an analytic subvariety of Δ. But by the calculation above *the ratio* f/g *is constant on every connected component of* $V - B$ and so V can meet only finitely many divisors D_λ away from the base locus of $\{D_\lambda\}$. Q.E.D.

The essential point here is that a pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ with base locus B gives a holomorphic mapping

$$M - B \to \mathbb{P}^1$$

since by linearity every $p \in M - B$ is on a *unique* D_λ. The Bertini theorem is a refinement of Sard's theorem for this mapping.

We make one final remark about sections of line bundles, which will be used repeatedly throughout the book. Recall that if $D = \sum a_i V_i$ is any effective divisor on the complex manifold M, $s_0 \in H^0(M, \mathcal{O}([D]))$ a section of $[D]$ with divisor D, then tensoring with s_0 gives an identification between the meromorphic functions on M with poles of order $\leqslant a_i$ on V_i and holomorphic sections of $[D]$. More generally, if E is any holomorphic vector bundle on M, $\mathcal{E}$ its sheaf of holomorphic sections, we write $\mathcal{E}(D)$ for the sheaf of meromorphic sections of E with poles of order $\leqslant a_i$ on V_i, $\mathcal{E}(-D)$ for the sheaf of sections of E vanishing to order $\geqslant a_i$ along V_i. Again, *tensoring with* s_0 *or* s_0^{-1} *gives identifications*

$$(*) \qquad \begin{aligned} \mathcal{E}(D) &\xrightarrow{\otimes s_0} \mathcal{O}(E \otimes [D]), \\ \mathcal{E}(-D) &\xrightarrow{\otimes s_0^{-1}} \mathcal{O}(E \otimes [-D]). \end{aligned}$$

Thus in particular if D is a smooth analytic hypersurface, the sequence of sheaves

$$0 \to \mathcal{O}_M(E \otimes [-D]) \xrightarrow{\otimes s_0} \mathcal{O}_M(E) \xrightarrow{r} \mathcal{O}_D(E|_D) \to 0,$$

where r is the restriction map, is exact. Henceforth, *we shall make the identification* (*) *implicitly and write* $\mathcal{O}(D)$ *for* $\mathcal{O}([D])$.

Chern Classes of Line Bundles

Let M now be a compact complex manifold of dimension n. The exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

gives a boundary map in cohomology

$$H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}).$$

For a line bundle $L \in \operatorname{Pic}(M) = H^1(M, \mathcal{O}^*)$, we define the *first Chern class* $c_1(L)$ of L (or simply *Chern class*) to be $\delta(L) \in H^2(M, \mathbb{Z})$; for D a divisor on M, we define the Chern class of D to be $c_1([D])$. By a slight abuse of language, we will sometimes write $c_1(L) \in H^2_{\mathrm{DR}}(M)$ for the image of $c_1(L)$ under the natural map $H^2(M, \mathbb{Z}) \to H^2_{\mathrm{DR}}(M)$.

As an immediate consequence of the definition, note that

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

and

$$c_1(L^*) = -c_1(L).$$

Also, if $f: M \to N$ is a holomorphic map of complex manifolds, the diagram

$$\begin{array}{ccc} H^1(M, \mathcal{O}^*) & \longrightarrow & H^2(M, \mathbb{Z}) \\ \uparrow f^* & & \uparrow f^* \\ H^1(N, \mathcal{O}^*) & \longrightarrow & H^2(N, \mathbb{Z}) \end{array}$$

commutes, so that for $L \to N$ any line bundle,

$$c_1(f^*L) = f^*c_1(L).$$

We will be concerned in this subsection with giving two alternate interpretations of the Chern class; first, however, we want to make one observation:

Let $\mathcal{C}^\infty$ and $\mathcal{C}^{\infty *}$ denote the sheaves of C^∞ functions and nonzero C^∞ functions, respectively. The transition functions of a C^∞ complex line bundle L then give a Čech cocycle

$$\{ g_{\alpha\beta} \} \in C^1(M, \mathcal{C}^*),$$

and by the same argument as for holomorphic bundles, the bundle L is determined, up to C^∞ isomorphism, by the cohomology class $[\{ g_{\alpha\beta}\}] \in H^1(M, \mathcal{C}^*)$. Now we have an exact sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{C} \xrightarrow{\exp} \mathcal{C}^* \to 0$$

and since the long exact sequence in Čech cohomology is functorial, the inclusion maps $\mathcal{O} \to \mathcal{C}$ and $\mathcal{O}^* \to \mathcal{C}^*$ give a commutative diagram

$$\begin{array}{ccccc} H^1(M, \mathcal{C}) & \longrightarrow & H^1(M, \mathcal{C}^*) & \xrightarrow{\delta'} & H^2(M, \mathbb{Z}) \\ \uparrow & & \uparrow & & \| \\ H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{O}^*) & \xrightarrow{\delta} & H^2(M, \mathbb{Z}) \end{array}$$

with both rows exact. Thus we can define the Chern class $c_1(L)$ of a C^∞ line bundle to be $\delta'(L)$, and this definition agrees with the one above for holomorphic bundles. But in the upper row we have $H^1(M, \mathcal{C}) = 0$, since the sheaf $\mathcal{C}$ is fine; the conclusion is that *a complex line bundle is determined up to* C^∞ *isomorphism by its Chern class.*

Recall now that for any vector bundle $E \to M$ of rank k and any connection D on E, the curvature operator D^2 is represented, in terms of a trivialization φ_α of E over U_α, by a $k \times k$ matrix Θ_α of 2-forms; if φ_β is another trivialization, we have

$$\Theta_\alpha = g_{\alpha\beta} \cdot \Theta_\beta \cdot g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k$ is the transition function relative to φ_α and φ_β. In particular, if E is a line bundle, since $GL_1 = \mathbb{C}^*$ is commutative $\Theta = \Theta_\alpha = \Theta_\beta$ is a closed, globally defined differential form of degree 2, called the *curvature form* of E.

Recall also that for any analytic subvariety $V \subset M$ of dimension k, we have defined the *fundamental class* $(V) \in H_{2k}(M, \mathbb{R})$ to be given by the linear functional

$$\varphi \to \int_V \varphi$$

on $H^{2k}_{DR}(M)$; we denote its Poincaré dual by η_V. In particular, we take the fundamental class of a divisor $D = \sum a_i V_i$ on M to be $\sum a_i (V_i)$; we denote its Poincaré dual by

$$\eta_D = \sum a_i \cdot \eta_{V_i}.$$

This subsection will be devoted to proving the

Proposition. 1. *For any line bundle* L *with curvature form* Θ,

$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Theta\right] \in H^2_{\rm DR}(M).$$

2. *If* L=[D] *for some* D∈Div(M),

$$c_1(L) = \eta_D \in H^2_{\rm DR}(M).$$

Proof. First, we unwind the definition of $c_1(L)$ for $L \to M$ a line bundle with trivializations φ_α and transition functions $g_{\alpha\beta}$ relative to a cover $\underline{U} = \{U_\alpha\}$ of M. We may assume the open sets U_α are simply connected and set

$$h_{\alpha\beta} = \frac{1}{2\pi\sqrt{-1}}\log g_{\alpha\beta}.$$

By the definition of δ, if we set

$$\begin{aligned} z_{\alpha\beta\gamma} &= h_{\alpha\beta} + h_{\beta\gamma} - h_{\alpha\gamma} \\ &= \frac{1}{2\pi\sqrt{-1}}\left(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma}\right), \end{aligned}$$

then $\{z_{\alpha\beta\gamma}\} \in Z^2(\underline{U}, \mathbb{Z})$ is a cocycle representing $c_1(L)$.

Now choose any connection D on L. In terms of the frame $e_\alpha(z) = \varphi_\alpha^{-1}(z, 1)$ on U_α, D is given by its connection matrix, which in this case is a 1-form θ_α. As was worked out in Section 5 of Chapter 0, in $U_\alpha \cap U_\beta$

$$\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} + dg_{\alpha\beta}\cdot g_{\alpha\beta}^{-1},$$

i.e.,

$$\theta_\beta - \theta_\alpha = -g_{\alpha\beta}^{-1}dg_{\alpha\beta} = -d(\log g_{\alpha\beta}),$$

and the curvature matrix is the global 2-form

$$\Theta = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha = d\theta_\beta.$$

Since Θ is given as a closed 2-form and $c_1(L)$ is given as a Čech cocycle, we must now look at the explicit form of the de Rham isomorphism. From the proof of de Rham's theorem, we have exact sequences of sheaves

$$0 \to \mathbb{R} \to \mathcal{Q}^0 \to \mathcal{Z}^1_d \to 0, \qquad 0 \to \mathcal{Z}^1_d \to \mathcal{Q}^1 \to \mathcal{Z}^2_d \to 0,$$

giving us boundary isomorphisms

$$\frac{H^0(\mathcal{Z}^2_d)}{dH^0(\mathcal{Q}^1)} \xrightarrow{\delta_1} H^1(\mathcal{Z}^1_d), \qquad H^1(\mathcal{Z}^1_d) \xrightarrow{\delta_2} H^2(\mathbb{R}).$$

To calculate $\delta_1(\Theta)$, we write Θ locally as $d\theta_\alpha$; we see from the definition of δ_1 that

$$\delta_1(\Theta) = \{\theta_\beta - \theta_\alpha\} \in Z^1(\mathcal{Z}^1_d).$$

Now $\theta_\beta - \theta_\alpha = -d\log g_{\alpha\beta}$, so

$$\begin{aligned}\delta_2\delta_1(\Theta) &= \delta_2(\{\theta_\beta - \theta_\alpha\})\\ &= \{-(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma})\}\\ &= -2\pi\sqrt{-1}\cdot c_1(L).\end{aligned}$$

To prove assertion 2 we have to show that, for Θ a curvature matrix for the bundle $[D]$, the cohomology class $[(\sqrt{-1}/2\pi)\Theta]$ is the Poincaré dual of $(D)=\sum a_i(V_i)$—i.e., that for every real, closed form $\psi \in A^{2n-2}(M)$,

$$\frac{\sqrt{-1}}{2\pi}\int_M \Theta\wedge\psi = \sum a_i\int_{V_i}\psi.$$

Since both $D \mapsto c_1([D])$ and $D \to \eta_D$ are homomorphisms from $\operatorname{Div}(M)$ to $H^2_{\mathrm{DR}}(M)$, we may take $D=V$ an irreducible subvariety.

First, we compute the curvature form of a metric connection on $[D]$. To do this, let e be a local nonzero holomorphic section of $[V]$ and write

$$|e(z)|^2 = h(z).$$

Then for any section $s=\lambda\cdot e$, the connection matrix θ for the metric connection D in terms of the frame e must satisfy

$$\theta = \theta^{1,0}$$

and

$$\begin{aligned}d(|s|^2) &= (Ds,s)+(s,Ds)\\ &= ((d\lambda+\theta\lambda)e,\lambda e)+(\lambda e,(d\lambda+\theta\lambda)e)\\ &= h\cdot\bar\lambda\cdot d\lambda + h\cdot\lambda\cdot d\bar\lambda + h\cdot|\lambda|^2(\theta+\bar\theta).\end{aligned}$$

Now

$$\begin{aligned}d(|s|^2) &= d(\lambda\cdot\bar\lambda\cdot h)\\ &= h\cdot\bar\lambda\cdot d\lambda + h\cdot\lambda\cdot d\bar\lambda + |\lambda|^2\cdot dh.\end{aligned}$$

So we have

$$\theta+\bar\theta = \frac{dh}{h},$$

i.e., $\theta = \partial\log h = \partial\log|e|^2$, and

$$\begin{aligned}\Theta &= d\theta - \theta\wedge\theta = d\theta\\ &= \bar\partial\partial\log|e|^2\\ &= 2\pi\sqrt{-1}\,dd^c\log|e|^2.\end{aligned}$$

Note that this holds for any nonzero holomorphic section e.

Now let $D=V$ be given by local data f_α and let s be a global section $\{f_\alpha\}$ of $[D]$ vanishing exactly on V. Set

$$D(\varepsilon) = (|s(z)|<\varepsilon)\subset M.$$

For small ε, $D(\varepsilon)$ is just a tubular neighborhood around V in M, and

$$\begin{aligned}\int_M \Theta\wedge\psi &= \lim_{\varepsilon\to 0} 2\pi\sqrt{-1}\int_{M-D(\varepsilon)} dd^c\log|s|^2\wedge\psi\\ &= \lim_{\varepsilon\to 0}\left(\frac{2\pi}{\sqrt{-1}}\right)\int_{\partial D(\varepsilon)} d^c\log|s|^2\wedge\psi\end{aligned}$$

by Stokes' theorem. In $U_\alpha\cap D(\varepsilon)$, write

$$|s|^2=|f_\alpha|^2\cdot h_\alpha=f_\alpha\cdot\bar f_\alpha\cdot h_\alpha$$

with $h_\alpha>0$; we have

$$\begin{aligned}d^c\log|s|^2 &= d^c\log\left(f_\alpha\cdot\bar f_\alpha\cdot h_\alpha\right)\\ &= \frac{\sqrt{-1}}{4\pi}\left(\bar\partial\log\bar f_\alpha-\partial\log f_\alpha+(\bar\partial-\partial)\log h_\alpha\right).\end{aligned}$$

Since $d^c\log h_\alpha$ is bounded and, as we have seen in the proof of Stokes' theorem for analytic varieties, $\text{vol}(\partial D(\varepsilon))\to 0$ as $\varepsilon\to 0$, we deduce that

$$\lim_{\varepsilon\to 0}\int_{\partial D(\varepsilon)} d^c\log h_\alpha\wedge\psi=0.$$

Moreover, $\bar\partial\log\bar f_\alpha=\overline{\partial\log f_\alpha}$ and, since ψ is real, this implies

$$\int_{\partial D(\varepsilon)}\bar\partial\log\bar f_\alpha\wedge\psi=\overline{\int_{\partial D(\varepsilon)}\partial\log f_\alpha\wedge\psi}.$$

Thus in U_α,

$$\lim_{\varepsilon\to 0}\frac{2\pi}{\sqrt{-1}}\int_{\partial D(\varepsilon)} d^c\log|s|^2=\lim_{\varepsilon\to 0}-\sqrt{-1}\cdot\text{Im}\int_{\partial D(\varepsilon)}\partial\log f_\alpha\wedge\psi.$$

Now in the neighborhood of any smooth point $z_0\in V\cap U_\alpha$, we can find a holomorphic coordinate system $w=(w_1,\dots,w_n)$ with $w_1=f_\alpha$. Write $\psi=\psi(w)\,dw'\wedge d\bar w'+\varphi$, where $w'=(w_2,\dots,w_n)$ and all terms of φ contain either dw_1 or $d\bar w_1$; then in any polydisc Δ around z_0,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial D(\varepsilon)\cap\Delta}\partial\log f_\alpha\wedge\psi &= \lim_{\varepsilon\to 0}\int_{|w_1|=\varepsilon}\frac{dw_1}{w_1}\cdot\psi(w)\cdot dw'\wedge d\bar w'\\ &= 2\pi\sqrt{-1}\int_{w'}\psi(0,w')\cdot dw'\wedge d\bar w'\\ &= 2\pi\sqrt{-1}\int_{V\cap\Delta}\psi\end{aligned}$$

and so

$$\begin{aligned}\int_M\Theta\wedge\psi &= -\sqrt{-1}\cdot\text{Im}\left(2\pi\sqrt{-1}\int_V\psi\right)\\ &= \frac{2\pi}{\sqrt{-1}}\int_V\psi.\end{aligned}$$

Q.E.D.

The conclusion that *the Chern class* $c_1([D])$ *represents, on the one hand, the Poincaré dual of the fundamental homology cycle carried by a divisor* D, *and on the other hand is given in de Rham cohomology by* $(\sqrt{-1}/2\pi)$ *times the curvature of any connection in the line bundle* [D], is of fundamental importance for what follows. The method of proof of this proposition, i.e., applying Stokes' theorem to a differential form with singularities—is likewise ubiquitous, and will be systematized in Chapter 3.

The simplest consequence of this proposition is the fact that the divisor (f) of a meromorphic function is homologous to zero. This is intuitively clear: drawing an arc γ from $\lambda_0=\infty$ to $\lambda_1=\infty$ on the Riemann sphere P^1_λ, the divisors

$$\{(\lambda_0 f+\lambda_1)\}_{[\lambda_0,\lambda_1]\in\gamma}$$

trace out a chain with boundary $(f)_0-(f)_\infty$.

Examples

1. In case M is a compact connected Riemann surface, a divisor D on M is just a finite sum

$$D=\sum n_i p_i$$

of points $p_i\in M$ with multiplicities n_i. The *degree* of D is defined to be its fundamental class $(D)\in H_0(M,\mathbb{Z})\cong\mathbb{Z}$; clearly

$$\deg D=\sum n_i.$$

By the above proposition, if Θ is the curvature form of a connection in the line bundle $[D]$,

$$\frac{\sqrt{-1}}{2\pi}\int_M\Theta=\langle c_1([D]),[M]\rangle=\deg D.$$

In general, we define the *degree* of a line bundle on M by

$$\deg(L)=\langle c_1(L),[M]\rangle,$$

or in other words $\deg(L)=c_1(L)$ under the isomorphism $H^2(M,\mathbb{Z})\cong\mathbb{Z}$ given by the natural orientation on M.

Note that by the relation proved on page 77 between the curvature form Θ of a metric connection on the tangent bundle of a Riemann surface and the ordinary Gaussian curvature K_M the classical Gauss-Bonnet theorem gives

$$\deg T'(M)=\frac{1}{4\pi}\int_M K_M\cdot\Phi=\chi(M).$$

2. By the exact cohomology sequence

$$H^1(\mathbb{P}^n,\mathcal{O})\to H^1(\mathbb{P}^n,\mathcal{O}^*)\xrightarrow{c_1}H^2(\mathbb{P}^n,\mathbb{Z})$$

arising from the exponential sheaf sequence on $\mathbb{P}^n$ and by the vanishing of $H^1(\mathbb{P}^n, \mathcal{O})$ (Section 7 of Chapter 1), we see that *every line bundle on* $\mathbb{P}^n$ *is determined by its Chern class*, i.e.,

$$\operatorname{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}.$$

In other words, *every divisor on* $\mathbb{P}^n$ *is linearly equivalent to a multiple of the hyperplane divisor* $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$. The bundle $[H]$ associated to a hyperplane in $\mathbb{P}^n$ is called *the hyperplane bundle*; its inverse, $J = [H]^* = [-H]$, is called *the universal bundle* on $\mathbb{P}^n$.

We can give a direct geometric construction of the universal bundle J on $\mathbb{P}^n$ as follows. Let $\mathbb{P}^n \times \mathbb{C}^{n+1}$ be the trivial bundle of rank $n+1$ on $\mathbb{P}^n$, with all fibers identified to $\mathbb{C}^{n+1}$. Then the universal bundle is just the subbundle J of $\mathbb{P}^n \times \mathbb{C}^{n+1}$ whose fiber at each point $Z \in \mathbb{P}^n$ is the line in $\mathbb{C}^{n+1}$ represented by Z, i.e.,

$$J_Z = \{\lambda(Z_0, \ldots, Z_n), \lambda \in \mathbb{C}\}.$$

To see that in fact $J = [-H]$, consider the section e_0 of J over $U_0 = (Z_0 \neq 0) \subset \mathbb{P}^n$ given by

$$e_0(Z) = \left(1, \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right).$$

e_0 is clearly holomorphic and nonzero in U_0 and extends to a global meromorphic section of J with a pole of order 1 along the hyperplane $(Z_0 = 0) \subset \mathbb{P}^n$. Thus $J = [(e_0)] = [-H]$.

If $M \subset \mathbb{P}^n$ is a submanifold of projective space, we usually call the restriction of $[H] \to \mathbb{P}^n$ to M simply the *hyperplane bundle on* M; by functoriality, it is the line bundle associated to a generic hyperplane section $\mathbb{P}^{n-1} \cap M$ of M.

3. Let M be a compact complex manifold, $V \subset M$ a smooth analytic hypersurface. Recall that we defined the *normal bundle* N_V on V to be the quotient line bundle

$$N_V = \frac{T'_{M,V}}{T'_V}.$$

We defined the *conormal bundle* N_V^* to be the dual of N_V; it is the subbundle of $T^{*\prime}_{M,V}$ consisting of cotangent vectors to M that are zero on $T'_V \subset T'_{M,V}$.

There is an easy formula for the conormal bundle of a smooth hypersurface V, which we now derive: Suppose V is given locally by functions $f_\alpha \in \mathcal{O}(U_\alpha)$; the line bundle $[V]$ on M is then given by transition functions $\{g_{\alpha\beta} = f_\alpha / f_\beta\}$. Now since $f_\alpha \equiv 0$ on $V \cap U_\alpha$, the differential df_α is a section of the conormal bundle N_V^* of V; since V is smooth, df_α is everywhere nonzero. On $U_\alpha \cap U_\beta \cap V$, moreover, we have

$$\begin{aligned} df_\alpha &= d(g_{\alpha\beta} f_\beta) \\ &= dg_{\alpha\beta} \cdot f_\beta + g_{\alpha\beta} \cdot df_\beta \\ &= g_{\alpha\beta} \cdot df_\beta, \end{aligned}$$

i.e., *the sections* $df_\alpha \in \Gamma(U_\alpha, \mathcal{O}(N_V^*))$ *together give a nonzero global section of* $N_V^* \otimes [V]$. Thus $N_V^* \otimes [V]$ is the trivial line bundle; this is the

Adjunction Formula I

$$N_V^* = [-V]|_V.$$

4. One of the most important line bundles in general is the highest exterior power of the holomorphic cotangent bundle

$$K_M = \wedge^n T_M^{*\prime},$$

called the *canonical bundle* of the n-dimensional complex manifold M. Holomorphic sections of K_M are holomorphic n-forms, i.e., $\mathcal{O}(K_M) = \Omega_M^n$.

We will compute the canonical bundle $K_{\mathbb{P}^n}$ of projective space: Let $Z_0, \ldots, Z_n$ be homogeneous coordinates on $\mathbb{P}^n$, $w_i = Z_i / Z_0$ Euclidean coordinates on $U_0 = (Z_0 \neq 0)$, and consider the meromorphic n-form

$$\omega = \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \cdots \wedge \frac{dw_n}{w_n}.$$

ω is clearly nonzero in U_0 with a single pole along each hyperplane $(Z_i = 0)$, $i = 1, \ldots, n$. Now if $w_i' = Z_i / Z_j$, $i = 0, \ldots, \hat{j}, \ldots, n$ are Euclidean coordinates on $U_j = (Z_j \neq 0)$, then

$$w_i = \frac{w_i'}{w_0'}, \quad i \neq j; \qquad w_j = \frac{1}{w_0'},$$

which gives

$$\frac{dw_i}{w_i} = \frac{dw_i'}{w_i'} - \frac{dw_0'}{w_0'}, \quad i \neq j; \qquad \frac{dw_j}{w_j} = \frac{-dw_0'}{w_0'},$$

and so in terms of $\{w_i'\}$,

$$\omega = (-1)^j \cdot \frac{dw_0'}{w_0'} \wedge \cdots \wedge \widehat{\frac{dw_j'}{w_j'}} \wedge \cdots \wedge \frac{dw_n'}{w_n'}.$$

Thus we see that ω has likewise a single pole along the hyperplane $(Z_0 = 0)$, and consequently

$$K_{\mathbb{P}^n} = [(\omega)] = [-(n+1)H].$$

In general, we can compute the canonical bundle K_V of a smooth analytic hypersurface V in a manifold M in terms of K_M as follows. We have an exact sequence of vector bundles on V

$$0 \to N_V^* \to T_M^{*\prime}|_V \to T_V^{*\prime} \to 0.$$

By simple linear algebra,

$$(\wedge^n T_M^{*\prime})|_V \cong \wedge^{n-1} T_V^{*\prime} \otimes N_V^*,$$

i.e.,

$$K_V = K_M|_V \otimes N_V.$$

Combining this with the adjunction formula I above, we have the

Adjunction Formula II

$$(*) \qquad K_V = \left(K_M \otimes [V]\right)|_V.$$

We can give the corresponding map on sections

$$\Omega_M^n(V) \xrightarrow{\text{P.R.}} \Omega_V^{n-1}$$

as follows: Considering a section ω of $\Omega_M^n(V)$ as a meromorphic n-form with a single pole along V and holomorphic elsewhere, we write

$$\omega = \frac{g(z)\,dz_1 \wedge \cdots \wedge dz_n}{f(z)},$$

where $z = (z_1, \ldots, z_n)$ are local coordinates on M and V is given locally by $f(z)$. Under the isomorphism $(*)$, then, ω corresponds to the form ω' such that

$$\omega = \frac{df}{f} \wedge \omega'$$

Explicitly,

$$df = \sum \frac{\partial f}{\partial z_i} \cdot dz_i,$$

and so we can take

$$\omega' = (-1)^{i-1} \frac{g(z)\,dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{\partial f / \partial z_i}$$

for any i such that $\partial f / \partial z_i \neq 0$. The map

$$\frac{g(z) dz_1 \wedge \cdots \wedge dz_n}{f(z)} \longrightarrow (-1)^{i-1} \left. \frac{g(z)\,dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{\partial f / \partial z_i} \right|_{f=0}$$

is called the *Poincaré residue map*, denoted P.R.

Note that the kernel of the Poincaré residue map consists simply of the holomorphic n-forms on M. The exact sheaf sequence

$$0 \to \Omega_M^n \longrightarrow \Omega_M^n(V) \xrightarrow{\text{P.R.}} \Omega_V^{n-1} \to 0$$

then gives us, in part, the exact sequence

$$H^0(M,\Omega_M^n(V)) \xrightarrow{\text{P.R.}} H^0(V,\Omega_V^{n-1}) \xrightarrow{\delta} H^1(M,\Omega_M^n),$$

i.e., the Poincaré residue map is surjective on global sections if $H^1(M,\Omega_M^n) = H^{n,1}(M)=0$. For example, since $H^{n,1}(\mathbb{P}^n)=0$ for $n>1$, *every holomorphic form of top degree on a hypersurface* V *in* $\mathbb{P}^n$ *is the Poincaré residue of a meromorphic form on* $\mathbb{P}^n$. We will see later that the meromorphic n-forms on $\mathbb{P}^n$ are easy to describe, so that we can readily write down the holomorphic $(n-1)$-forms on V.

2. SOME VANISHING THEOREMS AND COROLLARIES

The Kodaira Vanishing Theorem

Let M be a compact Kähler manifold.

DEFINITION. A line bundle $L\to M$ is *positive* if there exists a metric on L with curvature form Θ such that $(\sqrt{-1}/2\pi)\Theta$ is a positive $(1,1)$-form; L is *negative* if L^* is positive. A divisor D on M is positive if the line bundle $[D]$ is.

The positivity of a line bundle is a topological property, as we see from the

Proposition. *If* ω *is any real, closed* $(1,1)$*-form with*

$$[\omega] = c_1(L) \in H^2_{\mathrm{DR}}(M),$$

then there exists a metric connection on L *with curvature form* $\Theta = (\sqrt{-1}/2\pi)\omega$. *Thus* L *is positive if and only if its Chern class may be represented by a positive form in* $\mathrm{H}^2_{\mathrm{DR}}(\mathrm{M})$.

Proof. Let $|s|^2$ be a metric on L with curvature form Θ. If $\varphi: L_U\to U\times\mathbb{C}$ is a trivialization of L over an open set U, s a section of L over U and s_U the corresponding holomorphic function, then

$$|s|^2 = h(z)\cdot |s_U|^2$$

for some positive function $h(z)$. The curvature form and Chern class are given by

$$\Theta = -\partial\bar\partial \log h(z),$$
$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Theta\right] \in H^2_{\mathrm{DR}}(M).$$

Now let $|s|'^2$ be another metric on L with curvature form Θ'. Then $|s|'^2/|s|^2=e^{\rho}$ for some real C^∞ function ρ on M, and from the local formula

$$h'(z) = e^{\rho(z)}h(z)$$

it follows that

$$\Theta = \partial\bar{\partial}\rho + \Theta'.$$

In particular,

$$\left[\frac{\sqrt{-1}}{2\pi}\Theta\right] = \left[\frac{\sqrt{-1}}{2\pi}\Theta'\right].$$

Working in the other direction, suppose that $(\sqrt{-1}/2\pi)\varphi$ is a real, closed (1,1)-form representing $c_1(L)$ in $H^2_{\mathrm{DR}}(M)$. If we can solve the equation

$$\Theta = \partial\bar{\partial}\rho + \varphi$$

for a real C^∞ function ρ, then the metric $e^{\rho}|s|^2$ on L will have curvature form φ. Our proposition therefore follows from the

Lemma. *If η is any* (p, q)*-form on a compact Kähler manifold, and η is* d-, ∂-, *or* $\bar{\partial}$*-exact, then*

$$\eta = \partial\bar{\partial}\gamma$$

for some (p − 1, q − 1)*-form* γ. *If* p = q *and η is real, then we may take* $\sqrt{-1}\,\gamma$ *also to be real.*

Proof. Let G_d denote the Green's operator associated to the Laplacian Δ_d, and similarly for G_∂ and $G_{\bar{\partial}}$. From the basic identity of page 115

$$\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar{\partial}}$$

it follows first that

$$2G_d = G_\partial = G_{\bar{\partial}},$$

and then that all the operators d, ∂, $\bar{\partial}$, d^*, ∂^*, and $\bar{\partial}^*$ commute with the Green's operators.

Now, since η is d-, ∂-, or $\bar{\partial}$-exact, its harmonic projection under any of the above Laplacians is zero. By the Hodge decomposition for $\bar{\partial}$,

$$\eta = \bar{\partial}\bar{\partial}^* G_{\bar{\partial}}\eta.$$

But $\bar{\partial}^* G_{\partial}\eta$ has pure type $(p, q-1)$ and so

$$\partial\left(\bar{\partial}^* G_{\bar{\partial}}\eta\right) = \pm\bar{\partial}^* G_{\bar{\partial}}(\partial\eta) = 0.$$

Since the harmonic space for ∂ is the same as the harmonic space for $\bar{\partial}$ and hence is orthogonal to the range of $\bar{\partial}^*$, we deduce by the Hodge decomposition for ∂ that

$$\bar{\partial}^* G_{\bar{\partial}}\eta = \partial\partial^* G_\partial\left(\bar{\partial}^* G_{\bar{\partial}}\eta\right).$$

By commuting the various operators,

$$\eta = \pm\bar{\partial}\partial\left(\partial^*\bar{\partial}^* G_{\bar{\partial}}^2 \eta\right),$$

which implies the lemma. Q.E.D.

The basic example of a positive line bundle is the hyperplane bundle $[H]$ on $\mathbb{P}^n$. Recall that the dual of the hyperplane bundle is the bundle J whose fiber at $Z \in \mathbb{P}^n$ is the line $\{\lambda Z\} \subset \mathbb{C}^{n+1}$; we can put a metric on J by setting $|(Z_0, \ldots, Z_n)|^2 = \Sigma |Z_i|^2$. If Z is any nonzero section of J—i.e., a local lifting $U \subset \mathbb{P}^n \to \mathbb{C}^{n+1} - \{0\}$—then the curvature form in J is given by

$$\Theta^* = \bar{\partial}\partial \log \|Z\|^2 = 2\pi\sqrt{-1}\; dd^c \log \|Z\|^2.$$

The curvature form Θ for the dual metric in $[H]$ is then $-\Theta^*$, and consequently

$$\frac{\sqrt{-1}}{2\pi}\,\Theta = dd^c \log \|Z\|^2,$$

i.e., $(\sqrt{-1}/2\pi)\Theta$ is just the associated $(1,1)$-form ω of the Fubini-Study metric on $\mathbb{P}^n$, which we have seen is positive. As a corollary, we see again that the Poincaré dual of $[\omega] \in H^2_{\mathrm{DR}}(\mathbb{P}^n)$ is the fundamental class (H) of a hyperplane.

Note that since the restriction to a submanifold $V \subset M$ of a positive form is again positive, $L_{|V} \to V$ will be positive if $L \to M$ is. In particular, the hyperplane bundle on any complex submanifold of $\mathbb{P}^n$ is positive.

Our aim in this section is to prove that certain Čech cohomology groups $H^q(M, \Omega^p(L))$ associated to a positive line bundle $L \to M$ are zero. To begin with, we transpose the problem into one involving $\bar{\partial}$-cohomology and harmonic forms by a technique that will be familiar from the previous discussion.

Recall that for any holomorphic vector bundle $E \to M$, the $\bar{\partial}$-operator

$$\bar{\partial}:\; A^{p,q}(E) \to A^{p,q+1}(E)$$

is defined for global C^∞ E-valued differential forms, and satisifes $\bar{\partial}^2 = 0$. We let $Z^{p,q}_{\bar{\partial}}(E)$ denote the space of $\bar{\partial}$-closed E-valued differential forms of type (p,q), and we define the *Dolbeault cohomology groups* $H^{p,q}_{\bar{\partial}}(E)$ *of* E to be

$$H^{p,q}_{\bar{\partial}}(E) = \frac{Z^{p,q}_{\bar{\partial}}(E)}{\bar{\partial}A^{p,q-1}(E)}.$$

Let $\mathcal{Z}^{p,q}_{\bar{\partial}}(E)$ denote the sheaf of $\bar{\partial}$-closed E-valued (p,q)-forms. The exact sheaf sequences

$$0 \to \mathcal{Z}^{p,q}_{\bar{\partial}}(E) \to \mathcal{Q}^{p,q}(E) \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,q+1}_{\bar{\partial}}(E) \to 0$$

give us isomorphisms

$$H^i\big(M,\mathcal{Z}_{\bar\partial}^{p,q+1}(E)\big) \xrightarrow{\delta} H^{i+1}\big(M,\mathcal{Z}_{\bar\partial}^{p,q}(E)\big),$$

since the sheaves $\mathcal{Q}^{p,q}(E)$ admit partitions of unity and hence have no Čech cohomology. Thus, repeating the reasoning from the proof of de Rham's theorem,

$$H^q(M,\Omega^p(E)) \cong H_{\bar\partial}^{p,q}(E).$$

Next we want to discuss harmonic theory in holomorphic vector bundles. Suppose we have metrics given on M and E; we have then induced metrics on all tangential tensor bundles of M tensored with E or E^*. In particular, if $\{\varphi_i\}$ is a local coframe for the metric on $T_M^{*\prime}$ and $\{e_\alpha\}$ a unitary frame for E, any section η of $A^{p,q}(E)$ can be written locally as

$$\eta(z) = \frac{1}{p!q!}\sum_{I,J,\alpha}\eta_{I,J,\alpha}(z)\varphi_I\wedge\bar\varphi_J\otimes e_\alpha;$$

for $\eta,\psi\in A^{p,q}(E)$,

$$(\eta(z),\psi(z)) = \frac{2^{p+q-n}}{p!q!}\sum_{I,J,\alpha}\eta_{I,J,\alpha}(z)\cdot\overline{\psi_{I,J,\alpha}(z)}\,.$$

Again, we define an inner product on $A^{p,q}(E)$ by setting

$$(\eta,\psi) = \int_M(\eta(z),\psi(z))\Phi,$$

where Φ is the volume form on M.

We have a "wedge product"

$$\wedge:\ A^{p,q}(E)\otimes A^{p',q'}(E^*)\mapsto A^{p+p',q+q'}(M)$$

defined by

$$(\eta\otimes s)\wedge(\eta'\otimes s') = \langle s,s'\rangle\cdot\eta\wedge\eta';$$

we define an operator

$$*_E:\ A^{p,q}(E)\to A^{n-p,n-q}(E^*)$$

by requiring, for $\eta,\psi\in A^{p,q}(E)$,

$$(\eta,\psi) = \int_M\eta\wedge *_E\psi.$$

Explicitly, if $\{e_\alpha\}$ and $\{e_\alpha^*\}$ are dual unitary frames for E and E^*, then for $\eta\in A^{p,q}(E)$ written as

$$\eta = \sum\eta_\alpha\otimes e_\alpha,\qquad \eta_\alpha\in A^{p,q}(M),$$
$$*_E\eta = \sum *\eta_\alpha\otimes e_\alpha^*,$$

where $*$ is the usual star operator on $A^{p,q}(M)$.

We take

$$\bar{\partial}^*: A^{p,q}(E) \to A^{p,q-1}(E)$$

to be given by

$$\bar{\partial}^* = -*_E \cdot \bar{\partial} \cdot *_E;$$

as before, $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$, i.e., for all $\varphi \in A^{p,q-1}(E)$ and $\psi \in A^{p,q}(E)$,

$$(\bar{\partial}\varphi, \psi) = (\varphi, \bar{\partial}^*\psi).$$

Finally, the $\bar{\partial}$-Laplacian on E is defined by

$$\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}: A^{p,q}(E) \to A^{p,q}(E).$$

An E-valued form φ is called *harmonic* if $\Delta\varphi = 0$. (Again, harmonic forms φ are exactly the forms of smallest norm in their Dolbeault cohomology class $\varphi + \bar{\partial}A^{p,q-1}(E)$.) We let

$$\mathcal{H}^{p,q}(E) = \operatorname{Ker} \Delta$$

be the *harmonic space*.

Now, the analytic part of the proof of the Hodge theorem for the $\bar{\partial}$-Laplacian on ordinary differential forms on M is essentially local: we can always find appropriate solutions of $\Delta\varphi = 0$ in the completion of $A^{p,q}(M)$ in the L_2-norm; the problem is to show that these solutions are in fact C^∞. Writing out E-valued forms in terms of a frame for E, all the local estimates used in the proof of the Hodge theorem for $A^*(M)$ go over to $A^{p,q}(E)$—the only difference is that in each estimate we will get lower-order terms involving the coefficient functions for the metric on E as well as the metric on $T_M{}^*$, and these can be estimated out as before. Thus the Hodge theorem holds for the $\bar{\partial}$-Laplacian on E, that is:

1. $\mathcal{H}^{p,q}(E)$ is finite dimensional, and

2. If $\mathcal{H}$ denotes the orthogonal projection $A^{p,q}(E) \to \mathcal{H}^{p,q}(E)$, there exists an operator

$$G: A^{p,q}(E) \to A^{p,q}(E)$$

such that

$$G(\mathcal{H}^{p,q}(E)) = 0,$$

$$[G, \bar{\partial}] = [G, \bar{\partial}^*] = 0,$$

and

$$I = \mathcal{H} + \Delta G.$$

3. Consequently, there is an isomorphism

$$\mathcal{H}^{p,q}(E) \longrightarrow H^{p,q}_{\bar{\partial}}(E),$$

and

4. The $*$-operator gives an isomorphism

$$H^q(M,\Omega^p(E)) \simeq H^{n-q}(M,\Omega^{n-p}(E^*)).^*$$

For $p=0$, this last result reads

$$H^q(M,\mathcal{O}(E)) \simeq H^{n-q}(M,\mathcal{O}(E^*\otimes K_M)).^*$$

This isomorphism is called *Kodaira-Serre duality*.

Now if M is Kähler with associated (1,1)-form ω, we define the operator

$$L: A^{p,q}(E) \to A^{p+1,q+1}(E)$$

by setting, for $\eta \in A^{p,q}(M)$ and $s \in A^0(E)$,

$$L(\eta\otimes s) = \omega \wedge \eta \otimes s;$$

let $\Lambda = L^*$ be the adjoint of L. If $D = D' + D''$ $(D'' = \bar\partial)$ is the metric connection on E, then we have the *basic identity*

$$[\Lambda,\bar\partial] = -\frac{\sqrt{-1}}{2} D'^*.$$

This identity follows from the analogous identity $[\Lambda,\bar\partial] = -(\sqrt{-1}/2)\partial^*$ on scalar forms $A^{p,q}(M)$, which we have already proved. To see this, pick a local frame $\{e_\alpha\}$ for E; if $\theta = \theta' + \theta''$ is the connection matrix for D in terms of $\{e_\alpha\}$, we can write, for $\eta \in A^{p,q}(E)$,

$$\eta = \sum_\alpha \eta_\alpha \otimes e_\alpha, \qquad \eta_\alpha \in A^{p,q}(M),$$

$$\bar\partial\eta = \sum_\alpha \bar\partial\eta_\alpha \otimes e_\alpha + \sum_{\alpha,\beta} (\eta_\alpha \wedge \theta''_{\alpha\beta}) \otimes e_\beta,$$

$$\Lambda\eta = \sum_\alpha \Lambda(\eta_\alpha) \otimes e_\alpha,$$

so

$$\begin{aligned}[\Lambda,\bar\partial]\eta &= \sum [\Lambda,\bar\partial]\eta_\alpha \otimes e_\alpha + [\Lambda,\theta'']\eta \\ &= \sum -\frac{\sqrt{-1}}{2}\partial^*\eta_\alpha \otimes e_\alpha + [\Lambda,\theta'']\eta.\end{aligned}$$

Similarly,

$$D'\eta = \sum_\alpha \partial\eta_\alpha \otimes e_\alpha + \sum_{\alpha,\beta} (\eta_\alpha \wedge \theta'_{\alpha\beta}) \otimes e_\beta,$$

i.e.

$$D'^*\eta = \sum_i \partial^*\eta_\alpha \otimes e_\alpha + \theta'^*\eta.$$

The difference

$$[\Lambda,\bar\partial] + \frac{\sqrt{-1}}{2} D'^* = [\Lambda,\theta''] + \frac{\sqrt{-1}}{2}\theta'^*$$

is consequently an *intrinsically defined algebraic operator*; since we can choose at each $z_0 \subset M$ a frame for E in a neighborhood of z_0 for which $\theta(z_0)$ vanishes, we see that $[\Lambda,\bar\partial]+(\sqrt{-1}/2)D'^*=0$.

We will use the representation of Čech cohomology by harmonic forms to prove our first main result on the cohomology of vector bundles, the

Kodaira-Nakano Vanishing Theorem. *If* $L\to M$ *is a positive line bundle, then*

$$H^q(M,\Omega^p(L))=0 \qquad \text{for } p+q>n.$$

Proof.* By hypothesis we can find a metric in L whose curvature form Θ is $2\pi/\sqrt{-1}$ times the associated (1,1)-form of a Kähler metric; let the metric on M be the one given by $\omega=(\sqrt{-1}/2\pi)\Theta$. Now by harmonic theory

$$H^q(M,\Omega^p(L))\cong \mathcal{H}^{p,q}(L).$$

To prove the result, we will show that there are no nonzero harmonic L-valued forms of degree larger than n. We do this by interpreting the curvature operator $\Theta\eta=\Theta\wedge\eta$ alternately as $(2\pi/\sqrt{-1})L(\eta)$, and as $D^2\eta$, where D is the metric connection on L, and using the basic identity above.

Let $\eta\in\mathcal{H}^{p,q}(L)$ be a harmonic form. Then

$$\Theta=D^2=\bar\partial D'+D'\bar\partial,$$

so from $\bar\partial\eta=0$

$$\Theta\eta=\bar\partial D'\eta,$$

and

$$\begin{aligned}2\sqrt{-1}\,(\Lambda\Theta\eta,\eta)&=2\sqrt{-1}\,(\Lambda\bar\partial D'\eta,\eta)\\&=2\sqrt{-1}\left(\left(\bar\partial\Lambda-\frac{\sqrt{-1}}{2}D'^*\right)D'\eta,\eta\right)\\&=(D'^*D'\eta,\eta)=(D'\eta,D'\eta)\geqslant 0,\end{aligned}$$

since $(\bar\partial\Lambda D'\eta,\eta)=(\Lambda D'\eta,\bar\partial^*\eta)=0$. Similarly,

$$\begin{aligned}2\sqrt{-1}\,(\Theta\Lambda\eta,\eta)&=2\sqrt{-1}\,(D'\bar\partial\Lambda\eta,\eta)\\&=2\sqrt{-1}\left(D'\left(\Lambda\bar\partial+\frac{\sqrt{-1}}{2}D'^*\right)\eta,\eta\right)\\&=-(D'D'^*\eta,\eta)=-(D'^*\eta,D'^*\eta)\leqslant 0.\end{aligned}$$

*This proof is due to Y. Akizuki and S. Nakano, Note on Kodaira-Spencer's proof of Lefschetz's theorems, *Proc. Japan Acad.*, Vol. 30 (1954).

Combining,

$$2\sqrt{-1}\,([\Lambda,\Theta]\eta,\eta) \geqslant 0.$$

But $\Theta=(2\pi/\sqrt{-1}\,)L$, and so

$$\begin{aligned} 2\sqrt{-1}\,([\Lambda,\Theta]\eta,\eta) &= 4\pi([\Lambda,L]\eta,\eta) \\ &= 4\pi(n-p-q)\|\eta\|^2 \geqslant 0. \end{aligned}$$

Thus $p+q>n \rightarrow \eta=0$. Q.E.D.

As was suggested when we first introduced cohomology, the groups $H^q(M,\Omega^p(E))$ $(q \geqslant 1)$ most frequently arise as obstructions to globally solving analytic problems —this is especially true for $q=1$ as in the Mittag-Leffler problem, but once one admits H^1's, then all the rest become involved. The Kodaira vanishing theorem—together with its variants to be discussed later—is the best general method for eliminating cohomology.

Dualizing the Kodaira vanishing theorem, we obtain:

$H^q(M,\Omega^p(L)) = 0$ *for* $p+q<n$ *in case* $L \rightarrow M$ *is a negative line bundle.*

The special case when $p=q=0$ can be proved by elementary methods as follows: What we have to show is that

$$(*) \qquad H^0(M,\mathcal{O}(L)) = 0$$

in case $L \rightarrow M$ has a metric with curvature form equal to $2/\sqrt{-1}$ times a negative (1,1)-form. Suppose $s \neq 0 \in H^0(M,\mathcal{O}(L))$, and let $x_0 \in M$ be a point where $|s|^2$ attains a maximum. By hypothesis, if we write $z_i = x_i + \sqrt{-1}\,y_i$ the coefficient matrix for the curvature form

$$\begin{aligned} \left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}\log\left(\frac{1}{|s|^2}\right)\right) = \frac{1}{4}\Bigg(\left(\frac{\partial^2}{\partial x_i \partial x_j}+\frac{\partial^2}{\partial y_i \partial y_j}\right) \\ +\sqrt{-1}\left(\frac{\partial^2}{\partial x_j \partial y_i}-\frac{\partial^2}{\partial y_j \partial x_i}\right)\Bigg)\left(\log\frac{1}{|s|^2}\right) \end{aligned}$$

is negative definite hermitian, and in particular the real symmetric matrix

$$\left(\frac{\partial^2}{\partial x_i \partial x_j}+\frac{\partial^2}{\partial y_i \partial y_j}\right)\log\frac{1}{|s|^2}$$

is negative definite. But $\log(1/|s|^2)$ attains a minimum at x_0, and by the maximum principle, the matrices

$$\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)\log\frac{1}{|s|^2} \quad \text{and} \quad \left(\frac{\partial^2}{\partial y_i \partial y_j}\right)\log\frac{1}{|s|^2}$$

must both be positive semidefinite a contradiction.

In case M is a Riemann surface, the special case (∗) is the general case, since $p+q<1 \Rightarrow p=q=0$. The theorem then is even more elementary: if Θ is a curvature form for L with $(\sqrt{-1}\,/2\pi)\Theta$ negative, we have

$$c_1(L) = \int_M \frac{\sqrt{-1}}{2\pi}\Theta < 0.$$

But if $s \neq 0 \in H^0(M, \mathcal{O}(L))$, then L is the line bundle associated to the effective divisor $D=(s)$, and we have

$$c_1(L) = \deg D \geqslant 0,$$

a contradiction.

As an immediate consequence of the vanishing theorem, we see that

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) = 0 \qquad \text{for } 1 \leqslant q \leqslant n-1, \quad \text{all } k.$$

This follows directly from the dualized version of the vanishing theorem in case k is negative; if k is nonnegative,

$$\begin{aligned} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) &= H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(kH - K_{\mathbb{P}^n})) \\ &= H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((k+n+1)H)) \\ &= 0 \end{aligned}$$

by the original version of the theorem.

The Lefschetz Theorem on Hyperplane Sections

Using the Kodaira vanishing theorem, we can give a proof of the famous Lefschetz theorem relating the homology of a projective variety to that of its hyperplane sections.

Let M be an n-dimensional compact, complex manifold and $V \subset M$ a smooth hypersurface with $L=[V]$ positive—e.g., $M \subset \mathbb{P}^N$ a submanifold of projective space and $V = M \cap H$ a hyperplane section of M. Then we have the

Lefschetz Hyperplane Theorem. *The map*

$$H^q(M, \mathbb{Q}) \to H^q(V, \mathbb{Q})$$

induced by the inclusion $i: V \hookrightarrow M$ *is an isomorphism for* $q \leqslant n-2$ *and injective for* $q = n-1$.

Proof. It will suffice to prove the result over $\mathbb{C}$. By the Hodge decomposition

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M),$$

and by Dolbeault

$$H^{p,q}(M) \cong H^{p,q}_{\bar\partial}(M) \cong H^q(M, \Omega^p_M).$$

The same holding for V, it is sufficient to prove that the map

$$H^p(M,\Omega_M^q)\to H^p(V,\Omega_V^q)$$

is an isomorphism for $p+q\leqslant n-2$, and injective for $p+q=n-1$.

To see this, we factor the restriction map $\Omega_M^p\to\Omega_V^p$ by

$$\Omega_M^p\xrightarrow{r}\Omega_M^p|_V\xrightarrow{i}\Omega_V^p,$$

where $\Omega_M^p|_V$ is the sheaf of sections of $(\Lambda^pT_M^{*\prime})|_V$—considered either as a sheaf on V or, by extension, as a sheaf on M —r is the restriction map, and i is the pullback map induced by the natural projection $(\Lambda^pT_M^{*\prime})|_V\to\Lambda^pT_V^{*\prime}$.

The kernel of the restriction map r is clearly just the sheaf of holomorphic p-forms on M vanishing along V, so we have an exact sequence of sheaves on M

$$(*)\qquad 0\to\Omega_M^p(-V)\to\Omega_M^p\xrightarrow{r}\Omega_M^p|_V\to 0.$$

We can likewise fit the map i into an exact sequence: for $p\in V$, the sequence

$$0\to N_{V,p}^*\to T_p^{*\prime}(M)\to T_p^{*\prime}(V)\to 0,$$

yields, by linear algebra,

$$0\to N_{V,p}^*\otimes\wedge^{p-1}T_p^{*\prime}(V)\to\wedge^pT_p^{*\prime}(M)\to\wedge^pT_p^{*\prime}(V)\to 0,$$

and consequently an exact sequence of sheaves on V

$$0\to\Omega_V^{p-1}(N_V^*)\to\Omega_M^p|_V\xrightarrow{i}\Omega_V^p\to 0.$$

But by the adjunction formula I, $N_V^*=[-V]|_V$; we can thus rewrite this last sequence as

$$(**)\qquad 0\to\Omega_V^{p-1}(-V)\to\Omega_M^p|_V\to\Omega_V^p\to 0.$$

Now $[-V]$ is negative on M, and likewise $[-V]|_V$ is negative on V. The Kodaira vanishing theorem accordingly gives

$$H^q(M,\Omega_M^p(-V))=0,\qquad p+q<n,$$
$$H^q(V,\Omega_V^{p-1}(-V))=0,\qquad p+q<n.$$

By the exact cohomology sequences associated to the sheaf sequences $(*)$ and $(**)$, recalling that $H^*(M,\Omega_M^p|_V)=H^*(V,\Omega_M^p|_V)$,

$$H^q(M,\Omega_M^p)\overset{r^*}{\cong}H^q(M,\Omega_M^p|_V)\overset{i^*}{\cong}H^q(V,\Omega_V^p)$$

for $p+q\leqslant n-2$, and with both maps injective for $p+q=n-1$. Q.E.D.

The Lefschetz theorem on hyperplane sections is, of course, purely topological. There is another proof using a little Morse theory; we will give here a sketch of the argument:*

*Due to R. Bott, On a theorem of Lefschetz, *Mich. Math. J.*, Vol. 6 (1959), pp. 211–216.

To begin with, suppose that A is a compact manifold, $B\subset A$ a smooth submanifold, and $\varphi: A\to\mathbb{R}^+$ a C^∞ function such that $\varphi^{-1}(0)=B$. A *critical point* $x_\nu\in A$ of φ is a point such that $d\varphi(x_\nu)=0$; $\varphi(x_\nu)$ is called a *critical value* of φ. At each critical point the *Hessian* $\partial^2\varphi/(\partial u_i\partial u_j)=H(\varphi)$ is a well-defined quadratic form in the tangent space $T_{x_\nu}(A)$; the critical point is *nondegenerate* in case $H(\varphi)$ is nonsingular. The function φ is called a *Morse function* if all critical points of φ are nondegenerate; according to a standard approximation theorem, such functions are dense in the C^2-topology. By the main lemma of Morse theory, if φ is a Morse function and the Hessian $H(\varphi)$ is nonsingular in the normal bundle to B in A, then the homotopy type of

$$A_t=\{x\in A:\ \varphi(x)\leqslant t\},$$

remains the same as long as t does not cross a critical value (this is obvious; we just retract along the gradient vector field of φ), and changes by attaching a cell of dimension k when we cross a critical value whose Hessian has exactly k negative eigenvalues. (This requires a local analysis of the Morse function ψ around the critical point x_ν, and is the main step.)

Now let M be a compact, complex manifold, $L\to M$ a positive holomorphic line bundle, and $s\in H^0(M,\mathcal{O}(L))$ a holomorphic section whose zero divisor $V=(s)$ is a smooth hypersurface. Choose a metric for $L\to M$ such that $(\sqrt{-1}\ /2\pi)\Theta=(\sqrt{-1}\ /2\pi)\partial\bar\partial\log|s|^{-2}$ is positive and set

$$\varphi(x)=\log|s|^2.$$

φ—or a function near φ in the C^2 topology—may be used as a Morse function (the fact that $\varphi: M\to[-\infty,\infty)$ with $\varphi^{-1}(-\infty)=V$ causes no essential difficulty; what is important is that $d(|s|)\neq 0$ along V). Now for any critical point $x\in M$ of φ, the matrix

$$\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial\bar z_j}\right)\log\frac{1}{|s|^2}=\left(\frac{1}{4}\left(\frac{\partial^2}{\partial x_i\partial x_j}+\frac{\partial^2}{\partial y_i\partial y_j}\right)+\frac{\sqrt{-1}}{4}\left(\frac{\partial^2}{\partial y_i\partial x_j}-\frac{\partial^2}{\partial x_j\partial y_i}\right)\right)\log|s|^2$$

is negative definite hermitian, and consequently the Hessian

$$H(\varphi)=\begin{bmatrix}\dfrac{\partial^2}{\partial x_i\partial x_j} & \dfrac{\partial^2}{\partial x_i\partial y_j}\\[2ex] \dfrac{\partial^2}{\partial x_j\partial y_i} & \dfrac{\partial^2}{\partial y_i\partial y_j}\end{bmatrix}\log|s|^2$$

of φ has at least n negative eigenvalues. Clearly, this will also be true for functions ψ sufficiently close to φ in the C^2-topology. Thus, by Morse theory, as far as homotopy type is concerned M is obtained from V by attaching cells of dimension at least n, and this gives the Lefschetz theorem on the homotopy level and for homology with $\mathbb{Z}$-coefficients. Q.E.D.

When $n=1$, the theorem doesn't say anything. However, when $n=2$—i.e., M is a (connected and compact) complex surface—and $V \subset M$ is a Riemann surface embedded as a positive divisor, then the Lefschetz theorem gives

$$H_0(V,\mathbb{Z}) \cong H_0(M,\mathbb{Z}) = \mathbb{Z},$$
$$H_1(V,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to 0,$$

i.e., all of the first homology of the 4-manifold M lies on the irreducible embedded Riemann surface V.

We may also apply it to hypersurfaces of projective space: since any effective nonzero divisor on $\mathbb{P}^n$ is positive, the theorem tells us that if V is any smooth hypersurface in $\mathbb{P}^n$, then $H^{2k-1}(V)=0$ for $k \neq n/2$, while $H^{2k}(V)$ is generated by the class of a k-plane section of V for $k<n/2$. In particular any smooth hypersurface of dimension 2 or more is connected and simply connected. The same results apply, for an appropriate range of k, to any submanifold of projective space given as the transverse intersection of hypersurfaces.

A final remark on the Lefschetz theorem: Lefschetz's method was insofar as possible to study the topology of an algebraic variety M inductively, reducing questions about the homology of M to questions about the homology of a smaller-dimensional variety. His original proof of the last theorem asserted that for a hyperplane section V of M, the map $H_q(V,\mathbb{Z}) \to H_q(M,\mathbb{Z})$ is an isomorphism for $q<n-1$ and surjective in dimension $n-1$. By the hard Lefschetz theorem, the homology of M in dimension above n is mirrored in dimensions less than n, and by the Lefschetz decomposition, any nonprimitive cycle in dimension n can be obtained by intersecting a cycle in dimension greater than n with hyperplanes. Thus, the Lefschetz theorems together assert that the only "new" rational homology in varieties in each dimension is the primitive homology of the middle dimension.

Theorem B

Our second vanishing theorem for the cohomology of holomorphic vector bundles is less precise but broader in scope than the Kodaira Vanishing Theorem:

Theorem B. *Let* M *be a compact, complex manifold and* L→M *a positive line bundle. Then for any holomorphic vector bundle* E, *there exists* μ *such that*

$$H^q(M,\mathcal{O}(L^\mu \otimes E)) = 0 \qquad \text{for } q>0, \mu \geqslant \mu_0.$$

Proof. Before we prove this, note that in case E is a line bundle the result is already implied by the Kodaira theorem: just take μ_0 such that $L^\mu \otimes E \otimes$

K_M^* is positive for $\mu \geq \mu_0$; then since $c_1(L^\mu \otimes E) = \mu c_1(L) + c_1(E)$

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) = H^q(M, \Omega^n(L^\mu \otimes E \otimes K_M^*)) = 0$$

for $q > 0$, $\mu \geq \mu_0$. Indeed, the proof of Theorem B is essentially the same as that of Kodaira's theorem, the only difference being that now we must associate a definite sign to the curvature operator on a general vector bundle.

First, by Kodaira-Serre duality,

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) \simeq H^{n-q}(M, \mathcal{O}(L^{-\mu} \otimes E^* \otimes K_M)),$$

so it will be sufficient to prove that for any E, there exists μ_0 such that

$$H_{\bar\partial}^{0,p}(M, L^{-\mu} \otimes E) \simeq H^p(M, \mathcal{O}(L^{-\mu} \otimes E)) = 0$$

for $\mu \geq \mu_0$, $p < n$.

Choose a metric in L such that $\omega = (\sqrt{-1}/2\pi)\Theta_L$ is positive, where Θ_L is the curvature form associated to the metric; let the metric on M be the one given by ω. Now we have seen that if E, E' are two hermitian vector bundles and if we give $E \otimes E'$ the induced metric, then

$$D_{E \otimes E'} = D_E \otimes 1 + 1 \otimes D_{E'}$$

and so

$$\Theta_{E \otimes E'} = \Theta_E \otimes 1 + 1 \otimes \Theta_{E'}$$

where D, Θ always refer to the metric connection and curvature. In particular, for L and E as above with any metric on E,

$$\Theta_{L^\mu \otimes E} = \frac{2\pi\mu}{\sqrt{-1}}\,\omega \otimes 1_E + \Theta_E.$$

Let $\eta \in \mathcal{H}^{0,p}(L^{-\mu} \otimes E)$ be harmonic. Writing Θ for $\Theta_{L^{-\mu} \otimes E}$, D for $D_{L^{-\mu} \otimes E}$, we have

$$\Theta = D^2 = D'\bar\partial + \bar\partial D',$$

so

$$\Theta\eta = \bar\partial D'\eta,$$

and by the Kähler identity

$$[\Lambda, \bar\partial] = -\frac{\sqrt{-1}}{2}\, D'^*$$

we see that

$$\begin{aligned} 2\sqrt{-1}\,(\Lambda\Theta\eta, \eta) &= 2\sqrt{-1}\,(\Lambda\bar\partial D'\eta, \eta) \\ &= 2\sqrt{-1}\left(\left(\bar\partial\Lambda + \frac{1}{2\sqrt{-1}}\,D'^*\right)D'\eta, \eta\right) \\ &= (D'^* D'\eta, \eta) = (D'\eta, D'\eta) \geq 0, \end{aligned}$$

since $(\bar\partial\Lambda D'\eta,\eta)=(\Lambda D'\eta,\bar\partial^*\eta)=0$. On the other hand,

$$\begin{aligned} 2\sqrt{-1}\,(\Theta\Lambda\eta,\eta) &= 2\sqrt{-1}\,\big(D'\bar\partial\Lambda\eta,\eta\big) \\ &= 2\sqrt{-1}\left(\left(\Lambda\bar\partial - \frac{1}{2\sqrt{-1}}D'^*\right)\eta, D'^*\eta\right) \\ &= -\,(D'^*\eta, D'^*\eta) \leqslant 0. \end{aligned}$$

Thus we have

$$2\sqrt{-1}\,([\Lambda,\Theta]\eta,\eta) \geqslant 0.$$

But now

$$\Theta = \Theta_{L^{-\mu}\otimes E} = \Theta_E - \frac{2\pi}{\sqrt{-1}}\mu\omega,$$

and so

$$\begin{aligned} 2\sqrt{-1}\,([\Lambda,\Theta]\eta,\eta) &= 2\sqrt{-1}\,([\Lambda,\Theta_E]\eta,\eta) - 4\pi\mu([\Lambda,L]\eta,\eta) \\ &= 2\sqrt{-1}\,([\Lambda,\Theta_E]\eta,\eta) - 4\pi\mu(n-p)\,\|\eta\|^2. \end{aligned}$$

Now $[\Lambda,\Theta_E]$ is bounded on $A^{0,*}(L^{-\mu}\otimes E)$, so we can write

$$|([\Lambda,\Theta_E]\eta,\eta)| \leqslant C\,\|\eta\|^2,$$

and consequently for $p<n$,

$$\mu > \frac{C}{2\pi} \Rightarrow \eta = 0$$

i.e.,

$$\mathcal{H}^{0,p}(L^{-\mu}\otimes E) = 0 \qquad \text{for } \mu > \frac{C}{2\pi}, \qquad p<n. \qquad \text{Q.E.D.}$$

The Lefschetz Theorem on (1, 1)-classes

As an application of Theorem B, we will complete our picture of the correspondences among divisors, line bundles, and Chern classes on a complex submanifold of projective space. First, we have the

Proposition. *Let* $M\subset\mathbb{P}^N$ *be a submanifold. Then every line bundle on* M *is of the form* $L=[D]$ *for some divisor* D; *i.e.,*

$$\operatorname{Pic}(M) \cong \frac{\operatorname{Div}(M)}{\textit{linear equivalence}}.$$

Proof. To prove this, we have to show that every line bundle on M has a global meromorphic section. To find such a section, let H denote the restriction to M of the hyperplane bundle on $\mathbb{P}^N$. We will show that for $\mu \gg 0$, $L+\mu H$ has a nontrivial global holomorphic section s; then if t is any global holomorphic section of $[H]$ over M, s/t^{μ} will be a global meromorphic section of L as desired.

We proceed by induction on $n=\dim M$: assume that for every submanifold $V\subset \mathbb{P}^N$ of dimension less than n and every line bundle $L\to V$, $H^0(V,\mathcal{O}(L+\mu H)\neq 0$ for $\mu\gg 0$. By Bertini's theorem we can find a hyperplane $\mathbb{P}^{N-1}\subset\mathbb{P}^N$ with $V=\mathbb{P}^{N-1}\cap M$ smooth; we consider the exact sheaf sequence

$$0\to \mathcal{O}_M(L+(\mu-1)H)\xrightarrow{\otimes s}\mathcal{O}_M(L+\mu H)\xrightarrow{r}\mathcal{O}_V(L+\mu H)\to 0,$$

where s is the section of H vanishing exactly on H and r is the restriction map. For $\mu\gg 0$ we have both

$$H^0(V,\mathcal{O}(L+\mu H))\neq 0$$

by induction and

$$H^0(M,\mathcal{O}(L+\mu H))\to H^0(V,\mathcal{O}(L+\mu H))\to 0,$$

since

$$H^1(M,\mathcal{O}(L+(\mu-1)H))=0$$

by Theorem B. Thus $H^0(M,\mathcal{O}(L+\mu H))\neq 0$, and the result is proved. Q.E.D.

We now consider for a moment the general problem of analytic cycles. On a compact Kähler manifold M, the Hodge decomposition

$$H^n(M,\mathbb{C})=\bigoplus_{p+q=n}H^{p,q}(M)$$

on complex cohomology gives a slightly coarser decomposition of real cohomology

$$H^n(M,\mathbb{R})=\bigoplus_{\substack{p+q=n\\ p\leq q}}(H^{p,q}(M)\oplus H^{q,p}(M))\cap H^n(M,\mathbb{R}).$$

A natural question to ask is whether we can characterize geometrically the classes in homology that are Poincaré dual to classes in one of these factors. For example, we say a homology class $\gamma\in H_{2p}(M,\mathbb{Z})$ is *analytic* if it is a rational linear combination of fundamental classes of analytic subvarieties of M; dually, we say a cohomology class is *analytic* if its Poincaré dual is. Now, we have seen for purely local reasons that if $V\subset M$ is an analytic subvariety of dimension p and ψ any differential form on M,

$$\int_V\psi=\int_V\psi^{n-p,n-p}.$$

Thus if η is the harmonic form on M representing the cohomology class η_V

and ψ any harmonic form,

$$\int_M \psi \wedge \eta = \int_V \psi = \int_V \psi^{n-p,n-p} = \int_M \psi \wedge \eta^{p,p}$$

i.e., $\eta = \eta^{p,p}$, and so we see that *any analytic cohomology class of degree* 2p *is of pure type* (p,p). The famous *Hodge Conjecture* asserts that the converse is also true: On $M \subset \mathbb{P}^N$ a submanifold of projective space every rational cohomology class of type (p,p) is analytic. Whether the Hodge conjecture is true or false is at present unknown; it is a very beautiful and very difficult problem. The only case which has been proved in general is the case $p = 1$; this is the

Lefschetz Theorem on (1,1)-Classes. *For* $M \subset \mathbb{P}^N$ *a submanifold, every cohomology class*

$$\gamma \in H^{1,1}(M) \cap H^2(M,\mathbb{Z})$$

is analytic; in fact

$$\gamma = \eta_D$$

for some divisor D *on* M.

Here, of course, we are writing $H^2(M,\mathbb{Z})$ for its image under the natural inclusion in $H^2(M,\mathbb{R})$.

Proof. Consider again the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

and the associated cohomology sequence

$$H^1(M,\mathcal{O}^*) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \xrightarrow{i_*} H^2(M,\mathcal{O}) \cong H^{0,2}(M).$$

We claim that the map i_* is given by first mapping $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})$ and then projecting onto the $(0,2)$-factor of $H^2(M,\mathbb{C})$ in the Hodge decomposition; i.e., that the diagram

$$\begin{array}{ccc} H^2(M,\mathbb{Z}) & \xrightarrow{i_*} & H^2(M,\mathcal{O}) \\ \downarrow & & \\ H^2(M,\mathbb{C}) & & \downarrow \wr \ \text{Dolbeault} \\ \text{de Rham} \downarrow \wr & & \\ H^2_{\mathrm{DR}}(M,\mathbb{C}) & \xrightarrow{\pi^{0,2}} & H^{0,2}_{\bar{\partial}}(M) \end{array}$$

commutes. (The map $\pi^{0,2}$ is defined on the form level, since for $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2} \in Z^2_d(M)$, $\bar{\partial}\omega^{0,2} = (d\omega)^{0,3} = 0$). To see this, let $z = (z_{\alpha\beta\gamma}) \in Z^2(M,\mathbb{Z})$; to find the image of z under the de Rham isomorphism, we take

$f_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$ such that

$$z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} \qquad \text{in } U_\alpha \cap U_\beta \cap U_\gamma;$$

since $z_{\alpha\beta\gamma}$ is constant, $df_{\alpha\beta} + df_{\beta\gamma} - df_{\alpha\gamma} = 0$, so $(df_{\alpha\beta}) \in Z^1(M, \mathcal{Q}_k^1)$ and we can find $\omega_\alpha \in A^1(U_\alpha)$ such that

$$df_{\alpha\beta} = \omega_\beta - \omega_\alpha \qquad \text{in } U_\alpha \cap U_\beta.$$

The global 2-form $d\omega_\alpha = d\omega_\beta$ then represents the image of z in $H^2_{\mathrm{DR}}(M, \mathbb{C})$. On the other hand, take the image of $i_* z$ under the Dolbeault isomorphism: we write

$$z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma},$$
$$\bar{\partial} f_{\alpha\beta} = \omega_\beta^{0,1} - \omega_\alpha^{0,1},$$

and we see that $\bar{\partial}\omega_\alpha^{0,1} = (d\omega_\alpha)^{0,2}$ represents z in $H^{0,2}_{\bar{\partial}}(M)$.

Now we are just about done: given $\gamma \in H^{1,1}(M) \cap H^*(M, \mathbb{Z})$, we have $i_*(\gamma) = 0$, and hence $\gamma = c_1(L)$ is the Chern class of some line bundle $L \in H^1(M, \mathcal{O}^*)$. Writing $L = [D]$ for some divisor $D = \sum n_i V_i$,

$$\gamma = c_1([D]) = \eta_D. \qquad \text{Q.E.D.}$$

Note that since the isomorphism

$$L^{n-1}: H^{1,1}(M, \mathbb{Q}) \longrightarrow H^{n-1,n-1}(M, \mathbb{Q})$$

of the hard Lefschetz theorem is given by intersection with $n-1$ hyperplanes, it takes analytic classes to analytic classes; thus the Lefschetz (1, 1) theorem also implies the Hodge conjecture for $H^{2n-2}(M, \mathbb{Q}) \cap H^{n-1,n-1}(M)$. In particular, we see that *the intersection pairing between divisors and curves on a submanifold of projective space is nondegenerate.*

3. ALGEBRAIC VARIETIES

Analytic and Algebraic Varieties

Let $X_0, \ldots, X_n$ denote Euclidean coordinates on $\mathbb{C}^{n+1}$ and also the corresponding homogeneous coordinates on $\mathbb{P}^n$. Recall that the *universal bundle* $J \to \mathbb{P}^n$ is the subbundle of the trivial bundle $\mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$ whose fiber over a point $X \in \mathbb{P}^n$ is simply the line $\{\lambda X\}_\lambda \subset \mathbb{C}^{n+1}$ corresponding to X. The *hyperplane bundle* $H \to \mathbb{P}^n$ is the dual of J, i.e., it is the bundle whose fiber over $X \in \mathbb{P}^n$ corresponds to the space of linear functionals on the line $\{\lambda X\}$. As we saw in Section 1 of this chapter, the Chern class of H is the fundamental class ω of a hyperplane in $\mathbb{P}^n$—that is, a generator of $H^2(\mathbb{P}^n, \mathbb{Z})$—and it follows from $H^1(\mathbb{P}^n, \mathcal{O}) = 0$ that every line bundle on $\mathbb{P}^n$ is a multiple H^d of H.

Consider now the global sections of the bundle H. First, we note that any linear functional L on $\mathbb{C}^{n+1}$ induces a section σ_L of H by restriction, i.e., by setting

$$\sigma_L(X) = L|_{\{\lambda X\}}.$$

Clearly σ_L is identically zero only if L is, so we have an injection

$$\mathbb{C}^{n+1*} \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(H)).$$

In fact, all of $H^0(\mathbb{P}^n, \mathcal{O}(H))$ is obtained in this way: if σ is any section of $H, D=(\sigma)$ its zero divisor, then the fundamental class η_D is given by

$$\eta_D = c_1(H) = \omega$$

and by the argument of Section 4, Chapter 0, it follows that D is a hyperplane in $\mathbb{P}^n$. If we let $L \in \mathbb{C}^{n+1*}$ be any linear functional vanishing on the hyperplane $\pi^{-1}D \subset \mathbb{C}^{n+1}$, then, the meromorphic function σ/σ_L will be holomorphic on all of $\mathbb{P}^n$, hence constant.

In general, the fiber of a power H^d of H over a point X corresponds to the space of d-linear forms on the line $\{\lambda X\} \subset \mathbb{C}^{n+1}$, and so as before any d-linear form F on $\mathbb{C}^{n+1}$ induces by restriction a global section

$$\sigma_F(X) = F|_{\{\lambda X\}}$$

of H^d. Since we are restricting F to one line at a time, we see that $\sigma_F = 0$ if F is alternating in any two factors, and so we have a map

$$\mathrm{Sym}^d(\mathbb{C}^{n+1*}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(H^d))$$

from the space of symmetric d-linear forms on $\mathbb{C}^{n+1}$ —that is, homogeneous polynomials $F(X_0,\ldots,X_n)$ of degree d in $X_0,\ldots,X_n$ — to the space of global sections of H^d. Again, the map is injective, and the zero divisor of the section σ_F is just the image in $\mathbb{P}^n$ of the zero locus of $F(X_0,\ldots,X_n)$ in $\mathbb{C}^{n+1}$.

We claim now that *these are all the global sections* of H^d. To show this, let σ be any global section of H^d, and denote by σ_F be the section of H^d corresponding to an arbitrary homogeneous polynomial $F(X_0,\ldots,X_n)$. The quotient σ/σ_F is then a meromorphic function on $\mathbb{P}^n$; let

$$G' = \pi^*\left(\frac{\sigma}{\sigma_F}\right)$$

be its pullback to $\mathbb{C}^{n+1} - \{0\}$. G' has a simple pole along the divisor $F=0$ in $\mathbb{C}^{n+1} - \{0\}$ and is holomorphic elsewhere, so the function

$$G = G' \cdot F$$

is holomorphic everywhere in $\mathbb{C}^{n+1} - \{0\}$ *and hence by Hartogs' theorem extends to an entire holomorphic function on* $\mathbb{C}^{n+1}$. Now since $G'(\lambda X) = G'(X)$ for all $X \in \mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}$, and $F(\lambda X) = \lambda^d F(X)$,

$$G(\lambda X) = \lambda^d G(X),$$

i.e., G is homogeneous of degree d. Thus if $\iota: t \to (\mu_0 t,\ldots,\mu_n t)$ is any line through the origin in $\mathbb{C}^{n+1}$, the pullback $\iota^* G$ either is identically zero or has a zero of order d at $t=0$ and a pole of order d at $t=\infty$, i.e.,

$$\iota^* G = \mu \cdot t^d$$

for some μ. It follows that the power series expansion

$$G(X_0,\ldots,X_n) = \sum a_{i_0,\ldots,i_n} X_0^{i_0}\cdots X_n^{i_n}$$

for G around the origin in $\mathbb{C}^{n+1}$ contains no terms of degree other than d, i.e., that G *is a homogeneous polynomial of degree* d *in* $X_0,\ldots,X_n$. Thus $\sigma=\sigma_G$ is of the desired form, and we have shown that every global section of H^d is given by a homogeneous polynomial in $X_0,\ldots,X_n$.

We note in passing that there is a useful formula for the dimension $h^0(\mathbb{P}^n,\mathcal{O}(H^d))$ of the space of global sections of H^d, that is, the number of monomials $X_0^{i_0},\ldots,X_n^{i_n}$ of degree d in $(n+1)$ variables. We associate to any sequence $i_0,\ldots,i_n$ of integers with $\Sigma i_k = d$ the set of n integers

$$\{i_0+1, i_0+i_1+2,\ldots,i_0+\cdots+i_{n-1}+n\}\subset\{1,\ldots,d+n\}.$$

This subset of $\{1,\ldots,d+n\}$ determines the sequence i_k, and conversely any subset of n distinct numbers between 1 and $d+n$ corresponds to such a sequence. Thus the number of monomials of degree d in $X_0,\ldots,X_n$ is just the number $\binom{d+n}{n}$ of subsets of order n in a set of order $d+n$, and so

$$h^0(\mathbb{P}^n,\mathcal{O}(H^d)) = \binom{d+n}{n}.$$

Note that the locus of a homogeneous polynomial $F(X_0,\ldots,X_n)$ of degree d in the homogeneous coordinates X_i may also be given in terms of Euclidean coordinates $x_i = X_i/X_0$, $i=1,\ldots,n$ in $(X_0\neq 0)$ by the inhomogeneous polynomial of degree $\leqslant d$

$$f(x_1,\ldots,x_n) = F(1,x_1,\ldots,x_n) = \frac{1}{X_0^d}F(X_0,\ldots,X_n),$$

and conversely any such polynomial

$$f(x_1,\ldots,x_n) = \sum a_{i_1\ldots i_n} x_1^{i_1}\cdots x_n^{i_n}$$

corresponds to a homogeneous polynomial

$$F(X_0,\ldots,X_n) = \sum a_{i_1,\ldots,i_n} X_0^{d-\Sigma i_k}\cdot X_1^{i_1}\cdots X_n^{i_n}.$$

f is called the *affine*, or *inhomogeneous* form of F.

We now make the

DEFINITION. An *algebraic variety* $V\subset\mathbb{P}^n$ is the locus in $\mathbb{P}^n$ of a collection of homogeneous polynomials $\{F_\alpha(X_0,\ldots,X_n)\}$.

An algebraic variety is clearly an analytic subvariety of $\mathbb{P}^n$ and will be considered primarily as such (i.e., an algebraic variety $V\subset\mathbb{P}^n$ is called smooth, irreducible, connected, etc. if it has these properties as an analytic subvariety of $\mathbb{P}^n$). Conversely, we will show that any analytic subvariety of projective space is expressible as the locus of homogeneous polynomials. We have already done this in essence for hypersurfaces: if $V \subset \mathbb{P}^n$ is any divisor, the line bundle $[V]$ is of the form H^d for some d, and V is the zero locus of some section σ of $[V]$. But all sections σ of H^d are of the form σ_F, and so

$$V = (\sigma_F) = (F(X_0, \ldots, X_n) = 0)$$

is algebraic. In general, suppose $V \subset \mathbb{P}^n$ is a k-dimensional variety, $p \in \mathbb{P}^n$ any point not lying on V. We can find an $(n-k-1)$-plane $\mathbb{P}^{n-k-1}$ in $\mathbb{P}^n$ through p and missing V; let $\mathbb{P}^{n-k-2}$ be an $(n-k-2)$-plane in $\mathbb{P}^{n-k-1}$ disjoint from p. Let π denote the projection from $\mathbb{P}^{n-k-2}$ onto a complementary $(k+1)$-plane $\mathbb{P}^{k+1}$; choose coordinates $X_0, \ldots, X_n$ on $\mathbb{P}^n$ so that

$$\mathbb{P}^{k+1} = (X_{k+2} = \cdots = X_n = 0)$$
$$\mathbb{P}^{n-k-2} = (X_0 = \cdots = X_{k+1} = 0)$$

and

$$\pi([X_0, \ldots, X_n]) = [X_0, \ldots, X_{k+1}].$$

By the proper mapping theorem the image $\pi(V)$ of V in $\mathbb{P}^{k+1}$ is an analytic hypersurface in $\mathbb{P}^{k+1}$, and by the hypothesis that $\mathbb{P}^{n-k-1} = \overline{\mathbb{P}^{n-k-2}, p}$ misses V, $\pi(p)$ will lie outside $\pi(V)$. By what we have seen, we can find a homogeneous polynomial $F(X_0, \ldots, X_{k+1})$ vanishing along $\pi(V)$ but not at $\pi(p)$; correspondingly, the polynomial

$$\tilde{F}(X_0, \ldots, X_n) = F(X_0, \ldots, X_{k+1})$$

vanishes on V but not at p. We can thus find, for any point $p \not\in V$, a polynomial vanishing identically on V but not at p, and so we have

Chow's Theorem. *Any analytic subvariety of projective space is algebraic.*

If $F(X_0, \ldots, X_n)$ and $G(X_0, \ldots, X_n) \not\equiv 0$ are two homogeneous polynomials of the same degree d in the homogeneous coordinates X on $\mathbb{P}^n$, the quotient

$$\varphi(X) = \frac{F(X)}{G(X)}$$

is a well-defined meromorphic function on $\mathbb{P}^n$; such a mermorphic function is called a *rational function*. Note that after dividing top and bottom by powers of X_0, we may write the function φ as

$$\varphi(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)},$$

where f and g are polynomials (not necessarily both of degree d) in the Euclidean coordinates x_i. Thus the field $K(\mathbb{P}^n)$ of rational functions on $\mathbb{P}^n$ is isomorphic to $\mathbb{C}(x_1, \ldots, x_n)$.

It is not hard to see that any meromorphic function on $\mathbb{P}^n$ is rational. By Chow's theorem, both the zero-divisor $(\varphi)_0$ and the polar divisor $(\varphi)_\infty$ of φ are expressible as the loci of homogeneous polynomials $F(X)$ and $G(X)$. Since moreover the divisor (φ) is homologous to zero, F and G have the same degree, so F/G is a well-defined rational function on $\mathbb{P}^n$; then from

$$(F/G) = (\varphi)$$

it follows that

$$\varphi = \lambda F/G$$

for some $\lambda \in \mathbb{C}$.

Now if $V \subset \mathbb{P}^n$ is any smooth variety, a meromorphic function on V is called rational if it is the restriction to V of a rational function on $\mathbb{P}^n$. The rational functions of V a priori form a subfield of the field $\mathfrak{M}(V)$ of meromorphic functions; in fact,

Every meromorphic function on an algebraic variety $V \subset \mathbb{P}^n$ *is rational.*

The proof of this assertion is in two stages: first, we express V as a branched cover of a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$ by projection, and deduce from this representation that the pullback $\pi^* K(\mathbb{P}^k)$ to V of the field of rational functions on $\mathbb{P}^k$ has index at most $d = \deg(V)$ in the field $\mathfrak{M}(V)$; we then show that the field $K(V)$ is an extension of degree at least d over $\pi^* K(\mathbb{P}^k)$.

For the first part, choose a generic $(n-k-1)$-plane $\mathbb{P}^{n-k-1}$ in $\mathbb{P}^n$; at this stage we require only that $\mathbb{P}^{n-k-1}$ be disjoint from V. Let $\mathbb{P}^k$ be a complementary k-plane, and $\pi: V \to \mathbb{P}^k$ the projection from $\mathbb{P}^{n-k-1}$. For each point p of $\mathbb{P}^k$, the inverse image $\pi^{-1}(p)$ is just the intersection of V with the $(n-k)$-plane $\overline{\mathbb{P}^{n-k-1}, p}$; since $\overline{\mathbb{P}^{n-k-1}, p}$ will generically intersect V in $d = \deg(V)$ points, π expresses V as a d-sheeted branched cover of $\mathbb{P}^k$ almost everywhere. In fact, π must be everywhere finite: if for any point p in $\mathbb{P}^k$ the $(n-k)$-plane $\overline{\mathbb{P}^{n-k-1}, p}$ intersected V in a curve, that curve would necessarily meet the hyperplane $\mathbb{P}^{n-k-1} \subset \overline{\mathbb{P}^{n-k-1}, p}$, contrary to the hypothesis that $\mathbb{P}^{n-k-1}$ is disjoint from V.

Note that if we choose homogeneous coordinates $X = [X_0, \ldots, X_n]$ on $\mathbb{P}^n$ such that $\mathbb{P}^{n-k-1}$ is given as $(X_0 = \cdots = X_k = 0)$ and $\mathbb{P}^k$ as $(X_{k+1} = \cdots = X_n = 0)$, the map π is given by

$$\pi([X_0, \ldots, X_n]) = [X_0, \ldots, X_k].$$

In particular, the pullback $\pi^* f$ to V of any rational function f on $\mathbb{P}^k$ is clearly rational, so that on V we have inclusions

$$\pi^* K(\mathbb{P}^k) \subset K(V) \subset \mathfrak{M}(V).$$

Now, to see that $\pi^* K(\mathbb{P}^k)$ has index at most d in $M(V)$, let φ be any meromorphic function on V and let $D = (\varphi)_\infty$. Let $B \subset \mathbb{P}^k$ be the branch locus of π; on $\mathbb{P}^k - B$ we can define functions ψ_i by

$$\psi_1(p) = \sum_{q \in \pi^{-1}(p)} \varphi(q),$$

$$\psi_2(p) = \sum_{q \neq q' \in \pi^{-1}(p)} \varphi(q)\cdot\varphi(q'),$$

$$\vdots$$

$$\psi_d(p) = \prod_{q \in \pi^{-1}(p)} \varphi(q),$$

i.e., we let $\psi_i(p)$ be the ith symmetric polynomial in the values of φ at the d points of $\pi^{-1}(p)$ in V. ψ_i is then a holomorphic function on $\mathbb{P}^k - B - \pi(D)$, and being bounded away from $\pi(D)$ it extends by the Riemann extension theorem to a holomorphic function on $\mathbb{P}^k - \pi(D)$. We claim that ψ_i extends to a meromorphic function on all of $\mathbb{P}^k$. If $p \in \pi(D)$ is any point and $f(X)$ a local defining function for $\pi(D)$ in a neighborhood Δ of p, then for m sufficiently large, the function

$$\varphi' = \varphi \cdot \pi^* f^m$$

will be holomorphic in $\pi^{-1}(\Delta)$. For $q \in \Delta - B$, then, let

$$\psi_i'(q) = \sum_{\alpha_1, \ldots, \alpha_i} \left(\prod \varphi'(p_{\alpha_1}) \cdots \varphi'(p_{\alpha_i}) \right)$$

be the ith symmetric function of the values of φ' at the points of $\pi^{-1}(q)$; being bounded in any compact subset of Δ, ψ_i' likewise extends to a holomorphic function on Δ. Writing

$$\psi_i = \frac{\psi_i'}{f^{i-m}},$$

we see that ψ_i extends to a meromorphic function in Δ, and hence in all of $\mathbb{P}^k$. Thus the functions ψ_i are rational functions. But now on V we have

$$\varphi^d - \pi^*\psi_1 \cdot \varphi^{d-1} + \pi^*\psi_2 \cdot \varphi^{d-2} - \cdots + (-1)^d \pi^*\psi = 0,$$

i.e., every meromorphic function $\varphi \in \mathfrak{M}(V)$ satisfies a polynomial relation of degree d over $\pi^* K(\mathbb{P}^k)$. By the primitive element theorem, then, the field extension $\mathfrak{M}(V) \supset \pi^* K(\mathbb{P}^k)$ is finite of degree at most d.

To complete the proof of our assertion, we want to exhibit a rational function on V which satisfies no polynomial relation of degree less than d over the field $\pi^* K(\mathbb{P}^k)$. To do this, we factor the projection map π: choose generic planes $\mathbb{P}^{n-k-2} \subset \mathbb{P}^{n-k-1}$ and $\mathbb{P}^{k+1} \subset \mathbb{P}^k$, and let $\pi' : V \to \mathbb{P}^{k+1}$ be projection from $\mathbb{P}^{n-k-2}$. We may take homogeneous coordinates $X = [X_0, \ldots, X_n]$ on $\mathbb{P}^n$ such that

$$\mathbb{P}^{n-k-1} = (X_0 = \cdots = X_k = 0), \qquad \mathbb{P}^k = (X_{k+1} = \cdots = X_n = 0),$$

$$\mathbb{P}^{n-k-2} = (X_0 = \cdots = X_{k+1} = 0), \qquad \mathbb{P}^{k+1} = (X_{k+2} = \cdots = X_n = 0);$$

in terms of these coordinates, π is given as before and

$$\pi'([X_0,\ldots,X_n]) = [X_0,\ldots,X_{k+1}]$$

so that π is just the composition of π' with projection from the point $(X_0 = \cdots = X_k = X_{k+2} = \cdots = X_n = 0)$ in $\mathbb{P}^{k+1}$ onto $\mathbb{P}^k$. Note that, $\mathbb{P}^{n-k-2}$ having been chosen generically, the map π' will be one-to-one over an open set in its image: this will be the case as long as for some point p in V the $(n-k-1)$-plane $\overline{\mathbb{P}^{n-k-2},p}$ meets V only at p—but for any p in V, the generic $(n-k-1)$-plane through p meets V only at p.

Now, consider the rational function

$$x_{k+1} = \frac{X_{k+1}}{X_0}$$

on V. Suppose that x_{k+1} satisfied an equation of the form

$$x_{k+1}^{d'} + \psi_1(x_1,\ldots,x_k)\cdot x_{k+1}^{d'-1} + \cdots + \psi_{d'}(x_1,\ldots,x_k) \equiv 0,\ d' < d.$$

Then for a generic point $p = [\alpha_0,\ldots,\alpha_k]$ in $\mathbb{P}^k$, the inverse image of p in $\pi'(V) \subset \mathbb{P}^{k+1}$ would consist of at most of the d' points $\{[\alpha_0,\ldots,\alpha_k,\beta]\}$, where

$$\beta^{d'} + \psi_1\left(\frac{\alpha_1}{\alpha_0},\ldots,\frac{\alpha_k}{\alpha_0}\right)\beta^{d'-1} + \cdots + \psi_{d'}\left(\frac{\alpha_1}{\alpha_0},\ldots,\frac{\alpha_k}{\alpha_0}\right) = 0.$$

But since the projection $\pi': V \to \mathbb{P}^{k+1}$ is generically one-to-one onto its image and the fibers of π generically consist of $d > d'$ points, this is impossible. Q.E.D.

Note, as a consequence, that the field of rational functions on an algebraic variety V is independent of the embedding. Thus, the *sheaf of germs of polynomial functions on* V, which associates to every open set U on V the ring of rational functions on V finite in U, is intrinsically associated to V. This sheaf, the basic structure sheaf in algebraic treatments of the subject, is also denoted by $\mathcal{O}_V$.

It is not hard to see by the same sort of argument that

1. Any meromorphic differential form on a smooth variety is algebraic, that is, expressible in terms of rational functions and their differentials.
2. Any holomorphic map between smooth varieties may be given by rational functions.
3. Any holomorphic vector bundle on a smooth variety is algebraic, that is, may be given by rational transition functions.

The first assertion we can prove now: clearly the differentials $d\varphi$ of the rational functions on V span the cotangent space to V at every point, and so a finite number of them do; any meromorphic form on V is then a linear combination of wedge products of these forms with meromorphic, hence rational, coefficient functions. The second assertion will follow once we see in the following section that the product $V \times W$ of two algebraic varieties is again a variety; by Chow's theorem the graph $\Gamma \subset V \times W \subset \mathbb{P}^m$ is then cut out by polynomials. The third assertion will be clear once we have discussed the Grassmannian manifold and proved an embedding theorem for vector bundles on algebraic varieties in Sections 5 and 6 of this chapter.

All these results are special instances of the general *G.A.G.A. principle** that *any global analytic object on an algebraic variety is algebraic*. The importance of Chow's theorem and the G.A.G.A. principle is, in this treatment, primarily philosophical rather than practical. While we shall not use them as tools in our study—most of our techniques apply uniformly to all analytic phenomena on a variety, so it will not be useful for us to know, for instance, that a given meromorphic function or map is rational—they assure us that, in treating varieties as analytic rather than algebraic entities, we are still dealing with the same class of objects.

Degree of a Variety

The fundamental projective invariant of an algebraic variety $V \subset \mathbb{P}^n$ is its degree, defined as follows: Taking the class of a k-plane $\mathbb{P}^k \subset \mathbb{P}^n$ as generator, we have an isomorphism

$$H_{2k}(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}.$$

The *degree* of a k-dimensional variety $V \subset \mathbb{P}^n$ is its fundamental class in $H_{2k}(\mathbb{P}^n, \mathbb{Z})$ via this identification.

Alternative definitions abound. First, by Bertini applied to the smooth locus of V the generic $(n-k)$-plane $\mathbb{P}^{n-k} \subset \mathbb{P}^n$ will intersect V transversely, and so will meet V in exactly

$$\#(\mathbb{P}^{n-k} \cdot V) = \text{degree}(V)$$

points; thus we may define the degree of a variety to be the number of points of intersections of V with a generic linear subspace of complementary dimension. On the other hand, if ω is the standard Kähler form on $\mathbb{P}^n$,

$$\int_V \omega^k = \text{degree}(V) \cdot \int_{\mathbb{P}^k} \omega^k = \text{degree}(V),$$

so we may define the degree of V to be simply its volume divided by $k!$. (This is sometimes called the Wirtinger theorem.) In case $V \subset \mathbb{P}^n$ is a hypersurface, we have seen that it may be given in terms of homogeneous

*So named after J. P. Serre's paper, Geometrie Algebrique et Geometrie Analytique, *Annals of the Institute Fourier*, Vol. 6.

coordinates $X_0,\ldots,X_n$ on $\mathbb{P}^n$ as the locus

$$V = (F(X_0,\ldots,X_n)=0)$$

of a homogeneous polynomial F. If F has degree d, then the fundamental class of $V=(\sigma_F)$ is $\eta_V = c_1(H^d)$—that is, d times the class of a hyperplane – so V has degree d. Alternatively, if

$$[Y_0, Y_1] \xrightarrow{\mu} [a_0Y_0+b_0Y_1,\ldots,a_nY_0+b_nY_1]$$

is a generic line in $\mathbb{P}^n$, the pullback μ^*F of F to $\mathbb{P}^1$ will be homogeneous of degree d in Y_0 and Y_1, and so by the fundamental theorem of algebra will have exactly d roots. The degree of V is thus the degree d of the polynomial F.

A basic fact about degree is that it is multiplicative with respect to intersections. Since a $\mathbb{P}^{n-k_1}$ and a $\mathbb{P}^{n-k_2}$ intersect transversely in a $\mathbb{P}^{n-k_1-k_2}$, the degree of the intersection of two varieties meeting transversely almost everywhere is the product of their degrees. More generally, if V and W are varieties of degrees d_1 and d_2 in $\mathbb{P}^n$ intersecting in a variety of the appropriate dimension, $\{Z_i\}$ the irreducible components of $V\cap W$, then

$$d_1\cdot d_2 = \sum_i \operatorname{mult}_{Z_i}(V\cdot W)\cdot \operatorname{degree}(Z_i)$$

with $\operatorname{mult}_{Z_i}(V\cdot W)$ defined as in Section 4 of Chapter 0. This is of particular interest in the case of complementary dimension. For example, if C and D are two curves in $\mathbb{P}^2$ of degree d_1 and d_2 and having no component in common—that is, intersecting only in points—we see that they can have at most d_1d_2 points of intersection. This is a weak form of

Bezout's Theorem. *Two relatively prime polynomials* $f(x,y)$, $g(x,y) \in \mathbb{C}[x,y]$ *of degrees* d_1 *and* d_2 *can have at most* d_1d_2 *simultaneous solutions.*

The degree also behaves well with respect to the geometric operations of projection and coning. Clearly, if $V\subset\mathbb{P}^n$ is any variety, $p\in\mathbb{P}^n$ any point not on V, and $\pi_p: V\to\mathbb{P}^{n-1}$ the projection onto a hyperplane, then

$$\deg(V) = \deg(\pi_p(V)):$$

the number of points of intersection of $\pi(V)$ with a generic $(n-k-1)$-plane $\mathbb{P}^{n-k-1}$ in $\mathbb{P}^{n-1}$ is just the number of points of intersection of V with the $(n-k)$-plane $\mathbb{P}^{n-k}=\overline{\mathbb{P}^{n-k-1},p}$ in $\mathbb{P}^n$; since by Bertini the generic $\mathbb{P}^{n-k}$ through p meets V transversely, this is just the degree of V.

Coning is an operation we have not previously encountered. If $V\subset\mathbb{P}^n$ is any variety, $p\in\mathbb{P}^n$ at any point lying off V, we take the *cone* $\overline{p,V}$ *through* p *over* V to be the union of the lines through p meeting V. That $\overline{p,V}$ is a variety is easy to see: it is the image under projection on the first factor of the incidence correspondence $I \subset \mathbb{P}^n \times \mathbb{P}^n$ defined by

$$I = \{(q,r):\ r \in V, p \wedge q \wedge r = 0\},$$

itself an analytic subvariety of $\mathbb{P}^n \times \mathbb{P}^n$. (Alternatively, if in homogeneous coordinates $p = [0,\dots,0,1]$, let $\mathbb{P}^{n-1}$ be the hyperplane $X_n = 0$; if the image $\pi_p(V) \subset \mathbb{P}^{n-1}$ of V under projection from p is cut out in $\mathbb{P}^{n-1}$ by polynomials $\{F_\alpha(X_0,\dots,X_{n-1})\}$, then the cone $\overline{p,V}$ is cut out by the polynomials $\{\tilde{F}_\alpha(X_0,\dots,X_n) = F_\alpha(X_0,\dots,X_{n-1})\}$.) Now if $H \subset \mathbb{P}^n$ is a generic hyperplane, not containing p, then the intersection of H with the cone $\overline{p,V}$ will be simply the projection $\pi_p(V)$ of V from p into H; so

$$\begin{aligned}\deg(\overline{p,V}) &= \deg(H \cap \overline{p,V})\\ &= \deg(\pi_p(V)) = \deg(V).\end{aligned}$$

Another variety we may associate with a variety $V \subset \mathbb{P}^n$ is its *chordal variety* $C(V)$, defined to be the union of all lines meeting V twice or, in the limiting case, tangent to V. $C(V)$ is the image under projection on the third factor of the closure of the incidence correspondence $I \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ defined by

$$I = \{(p,q,r):\ p \neq q \in V, p \wedge q \wedge r = 0\}.$$

$\bar{I}$ is an analytic subvariety of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, and so $C(V)$ is an analytic variety in $\mathbb{P}^n$. Note that since projection on the first factor maps I onto V with $(\dim V + 1)$-dimensional fibers, I has dimension $2\cdot\dim V + 1$. $C(V)$ will thus have dimension at most $2\cdot\dim V + 1$; generally, this will be exact. In particular, since the projection π_p of a smooth variety into a hyperplane will be an embedding if and only if $p \notin C(V)$, we see that if $n > 2\cdot\dim V + 1$, then V may be smoothly projected into a hyperplane. Thus

Any smooth algebraic variety of dimension k *may be embedded in* $\mathbb{P}^{2k+1}$.

As we shall see, the degree of the chordal variety $C(V)$ of a variety does not depend on the degree of V alone.

A variety $V \subset \mathbb{P}^n$ is called *nondegenerate* if it does not lie in a hyperplane. We have the following condition on the degree of a nondegenerate variety:

If $V \subset \mathbb{P}^n$ *is an irreducible, nondegenerate,* k*-dimensional variety, then*

$$\deg(V) \geq n - k + 1.$$

We prove this first for V a curve in $\mathbb{P}^n$. Any n points of V lie in a hyperplane H, and if the degree of V were less than n, then H, having n

points in common with V, would have a curve in common with V; being irreducible, V would then lie in H.

Turning to the general case, we have to show that the generic hyperplane section $H \cap V$ of an irreducible nondegenerate variety V of dimension $\geqslant 2$ is again irreducible and nondegenerate in H. The latter part is clear: the condition that $H \cap V$ be degenerate is a closed one on $H \in \mathbb{P}^{n*}$, and since V itself is nondegenerate, we can find n points of V spanning a hyperplane, so not every hyperplane section of V can be degenerate.

The former half of our assertion—that the generic hyperplane section of an irreducible variety is irreducible—is somewhat harder. We note first that in case V is smooth, this follows easily from the Bertini theorem and the Lefschetz theorem on hyperplane sections: by Bertini, the generic hyperplane section $H \cap V$ is smooth, and so by Lefschetz,

$$H_0(H \cap V, \mathbb{C}) \cong H_0(V, \mathbb{C}) \cong \mathbb{C};$$

i.e., $H \cap V$ is connected. Thus, if $H \cap V$ were reducible, the components of $H \cap V$ would have to meet each other; but their points of intersection would be singular points of $H \cap V$, and so this cannot happen.

To prove the assertion in the general case requires a different approach. We argue as follows: let $p \in V$ be any smooth point, and let $\mathbb{P}^{n-2} \subset \mathbb{P}^n$ be an $(n-2)$-plane meeting V transversely at p; let Z be the irreducible component of $V \cap \mathbb{P}^{n-2}$ containing p. Now consider the pencil $\{H_\lambda\}$ of hyperplanes in $\mathbb{P}^n$ containing $\mathbb{P}^{n-2}$. Each hyperplane section $H_\lambda \cap V$ of V contains Z, but since each H_λ intersects V transversely at p, p—being a smooth point of $H_\lambda \cap V$—can lie on at most one of the irreducible components of $H_\lambda \cap V$ for each λ. Let V' be the union of the irreducible components of the sections $H_\lambda \cap V$ that contain Z. Then V' is an open k-dimensional analytic variety contained in V, and hence its closure $\overline{V'}$ must be all of V; thus $H_\lambda \cap V = H_\lambda \cap V'$ is irreducible for generic λ.

Now the original lemma follows readily from the curve case: if $V \subset \mathbb{P}^n$ is any irreducible nondegenerate k-dimensional variety of degree d, then the generic intersection of V with $k-1$ hyperplanes is an irreducible, nondegenerate curve of degree d in $\mathbb{P}^{n-k+1}$, and so

$$d \geqslant n - k + 1.$$

We can restate the lemma as follows: any irreducible k-dimensional variety $V \subset \mathbb{P}^n$ of degree d must lie in a linear space of dimension $d + k - 1$; as a corollary, then, we see again that any variety of degree one in $\mathbb{P}^n$ is a linear subspace.

We shall see that varieties that realize this lower bound on the degree—e.g., curves of degree n in $\mathbb{P}^n$, surfaces of degree $n-1$ in $\mathbb{P}^n$, etc.—are of a special character.

Tangent Spaces to Algebraic Varieties

To a variety $V \subset \mathbb{P}^n$ and a smooth point $p \in V$ is associated a linear subspace of $\mathbb{P}^n$, the *tangent space to V at p*. This may be defined in several ways; we mention two here.

1. The complement of a hyperplane $H \subset \mathbb{P}^n$ is isomorphic to $\mathbb{C}^n$ via Euclidean coordinates; we may take the tangent space to $V \subset \mathbb{P}^n$ at p to be the closure in $\mathbb{P}^n$ of the usual tangent subspace $T_p(V) \subset T_p(\mathbb{C}^n)$. Explicitly, if $x_1, \ldots, x_n$ are Euclidean coordinates on $\mathbb{P}^n$ in a neighborhood of $p = (\alpha_1, \ldots, \alpha_n)$ and V is cut out by functions $\{f_\alpha(x_1, \ldots, x_n)\}$, this is just the linear subspace of $\mathbb{P}^n$ defined by

$$\sum_{i=1} \frac{\partial f_\alpha}{\partial x_i}(p) \cdot (x_i - \alpha_i) = 0.$$

2. Alternatively, if V is given in terms of homogeneous coordinates $X_0, \ldots, X_n$ as the locus of polynomials $\{F_\alpha(X_1, \ldots, X_n)\}$, this is the linear subspace

$$\sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot X_i = 0,$$

where the differentiation is formal: if f_α is the inhomogeneous form of F_α, then $\partial f_\alpha / \partial x_i$ is the inhomogeneous form of $\partial F_\alpha / \partial X_i$, and by virtue of the relation

$$\frac{1}{d} \cdot \sum_{i=0}^{n} \frac{\partial F}{\partial X_i} = F,$$

where $d = \deg(F)$, we can write

$$\begin{aligned}
\sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot X_i &= X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot x_i \\
&= X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot (x_i - \alpha_i) \\
&= X_0 \cdot \sum_{i=1}^{n} \frac{\partial f_\alpha}{\partial x_i}(p) \cdot (x_i - \alpha_i),
\end{aligned}$$

so the homogeneous form describes the same subspace.

In a similar way we may define the *tangent cone* to a variety $V \subset \mathbb{P}^n$ at a (possibly singular) point $p \in V$. First, if V is a hypersurface cut out by the homogeneous polynomial F, and p a point of multiplicity k on V—so that all the partial derivatives of F of order $\leqslant k-1$ vanish—we take the tangent cone to V at p to be the locus

$$T_p(V) = \left(\sum \frac{\partial^k F}{\partial^{i_0} X_0 \cdots \partial^{i_n} X_n}(p) \cdot X_0^{i_0} \cdots X_n^{i_n} = 0 \right).$$

In general, we will take the tangent cone to a variety $V \subset \mathbb{P}^n$ at a point p to be the intersection of the tangent cones at p to all the hypersurfaces containing V near p.

This may be realized alternately as the union of the tangent lines at p to all curves lying on V and passing through p; or as the limiting position of chords $\lim_{\lambda \to 0} \overline{p, q(\lambda)}$ where $q(\lambda)$ is an arc in V with $q(0) = p$.

4. THE KODAIRA EMBEDDING THEOREM

Line Bundles and Maps to Projective Space

We will be concerned in this section with determining exactly when a compact complex manifold is an algebraic variety, i.e., when it can be embedded in projective space. We first establish a basic formalism for maps to $\mathbb{P}^N$.

Let M be a compact complex manifold, $L \to M$ a holomorphic line bundle. Recall that to any subspace E of the vector space $H^0(M, \mathcal{O}(L))$ is associated the linear system

$$|E| = \{(s)\}_{s \in E} \subset \operatorname{Div}(M)$$

of divisors on M. Since M is compact, $(s) = (s')$ only if $s = \lambda s'$ for some nonzero constant $\lambda \in \mathbb{C}$; thus $|E|$ is parametrized by the projective space $\mathbb{P}(E)$.

Suppose in addition that the linear system $|E|$ has no base points, i.e., that not all $s \in E$ vanish at any point $p \in M$. Then for each $p \in M$ the set of sections $s \in E$ vanishing at p forms a hyperplane $\tilde{H}_p \subset E$—or, equivalently, the set of divisors $D \in |E|$ containing p forms a hyperplane H_p in $\mathbb{P}(E)$—and so we can define a map

$$\iota_E : M \to \mathbb{P}(E)^*,$$

by sending $p \in M$ to $H_p \in \mathbb{P}(E)^*$.

We can describe the map ι_E more explicitly as follows. Choose a basis $s_0, \dots, s_N$ for E. If we let $s_{i,\alpha} = \varphi_\alpha^*(s_i) \in \mathcal{O}(U)$ for any trivialization φ_α of L over an open set $U \in M$, it is clear that the point $[s_{0,\alpha}(p), \dots, s_{N,\alpha}(p)] \in \mathbb{P}^N$ is independent of the trivialization φ_α chosen; we denote this point by $[s_0(p), \dots, s_N(p)]$. In terms of the identifications $\mathbb{P}(E)^* \cong \mathbb{P}^N$ corresponding to the choice of basis $s_0, \dots, s_N$, then, the map ι_E is given by

$$\iota_E(p) = [s_0(p), \dots, s_N(p)].$$

We see from this representation that ι_E is holomorphic.

Now let H be the hyperplane bundle on $\mathbb{P}^N$. The pullback bundle $\iota_E^*(H)$ on M is given by the divisor (s_i)—that is,

$$L = \iota_E^*(H).$$

Moreover, any section $s=\Sigma a_i s_i \in E$ is the pullback of the section $\Sigma a_i Z_i$ of H on $\mathbb{P}^N$; i.e.,

$$E = \iota_E^*\big(H^0(\mathbb{P}^N, \mathcal{O}(H))\big) \subset H^0(M, \mathcal{O}(L)).$$

Thus $\iota_E: M \to \mathbb{P}^N$ determines both the line bundle L and the subspace $E \subset H^0(M, \mathcal{O}(L))$, and we have a basic dictionary

$$\left\{\begin{array}{l}\text{nondegenerate maps}\\ f\colon\ M\to\mathbb{P}^N,\text{ modulo}\\ \text{projective}\\ \text{transformations}\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{line bundles } L\to M\\ \text{with } E\subset H^0(M,\mathcal{O}(L))\\ \text{such that } |E| \text{ has no base points}\end{array}\right\}$$

where the choice of homogeneous coordinates on $\mathbb{P}^N$ corresponds to the choice of basis $s_0,\ldots,s_N$ for E.

We will often write ι_L for $\iota_{H^0(M,\mathcal{O}(L))}$ and ι_D for $\iota_{[D]}$.

Note that the degree of the image of M under ι_E—that is, the intersection of M with n general hyperplanes in $\mathbb{P}^n$—is just the n-fold self-intersection of a representative divisor $D \in |E|$, that is,

$$\deg(\iota_E M) = c_1(L)^n.$$

A variety $V \subset \mathbb{P}^n$ is called *normal* if the linear system on V giving the embedding $\iota: V \hookrightarrow \mathbb{P}^n$ is complete, that is, if the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(H)) \to H^0(V, \mathcal{O}(H))$$

is surjective. Note that *any hypersurface* $V \subset \mathbb{P}^n$ *is normal*: from the exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(H-V) \to \mathcal{O}_{\mathbb{P}^n}(H) \xrightarrow{r} \mathcal{O}_V(H) \to 0$$

we have an exact sequence of cohomology groups

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H)) \xrightarrow{r} H^0(V, \mathcal{O}_V(H)) \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H-V)),$$

But

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H-V)) = H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((1-d)H)) = 0$$

so r must be surjective. Note that two normal varieties $V, V' \subset \mathbb{P}^n$ will be *projectively isomorphic*—that is, V may be carried into V' by an automorphism of $\mathbb{P}^n$—if V is biholomorphic to V' via a mapping carrying $H_{V'}$ to H_V. In particular, if V and V' are smooth hypersurfaces of dimension $\geqslant 3$

and degree $d \neq n+1$ in $\mathbb{P}^n$, then by the adjunction formula

$$K_V = \left(K_{\mathbb{P}^n} \otimes [V]\right)|_V = [(d-n-1)H]$$

and likewise for V'. But by the Lefschetz theorem on hyperplane sections

$$H^1(V,\mathcal{O}) \cong H^1(\mathbb{P}^n,\mathcal{O}) = 0, \qquad H^2(V,\mathcal{O}) \cong H^2(\mathbb{P}^n,\mathcal{O}) = 0$$

so from the long exact cohomology sequence associated to the exponential sheaf sequence and the Lefschetz theorem again

$$\operatorname{Pic}(V) = H^1(V,\mathcal{O}^*) \cong H^2(V,\mathbb{Z}) \cong H^2(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$$

and likewise for V'. Thus if $\varphi: V \to V'$ is biholomorphic,

$$\varphi^* K_{V'} = K_V \Rightarrow \varphi^*(H|_{V'}) = H|_V,$$

so V and V' are projectively isomorphic. In conclusion

Two smooth hypersurfaces of dimension $\geqslant 3$ *and degree* $\mathrm{d} \neq \mathrm{n}+1$ *in* $\mathbb{P}^{\mathrm{n}}$ *are isomorphic if and only if they are projectively isomorphic; or, equivalently,*

Any automorphism of a smooth hypersurface of dimension $\geqslant 3$ *and degree* $\mathrm{d} \neq \mathrm{n}+1$ *in* $\mathbb{P}^{\mathrm{n}}$ *is induced by an automorphism of* $\mathbb{P}^{\mathrm{n}}$.

This result in fact holds for surfaces V of degree $d \neq 4$ in $\mathbb{P}^3$ as well: to apply the previous argument, we need to know only that $H^2(V,\mathbb{Z})$ contains no torsion; this follows from the fact that V is simply connected (Lefschetz theorem once more), and the statement of Poincaré duality for the torsion part $H_{*,\mathrm{tor}}$ of homology:

$$H_{i,\mathrm{tor}}(M,\mathbb{Z}) \cong H^{n-i+1}_{\mathrm{tor}}(M,\mathbb{Z}).$$

We may illustrate the correspondence between maps to projective space and base-point-free linear systems with a classical example: the *Veronese map* associated to the line bundle dH on $\mathbb{P}^n$. We have seen that the global sections of dH correspond to homogeneous polynomials of degree d in $Z = [Z_0, \ldots, Z_n]$, so that if $\{Z^\alpha = Z_0^{\alpha_0} \cdots Z_n^{\alpha_n}\}$ denotes the set of monomials of degree d in Z, then the Veronese map is given by

$$[Z_0, \ldots, Z_n] \mapsto [\ldots, Z^\alpha, \ldots].$$

It is easily verified that the Veronese map is a smooth embedding, with the property that every hypersurface of degree d in $\mathbb{P}^n$ becomes a hyperplane section of $\iota_{dH}(\mathbb{P}^n) \subset \mathbb{P}^N$. Here are a few cases:

1. The Veronese map

$$\iota_{nH}: \mathbb{P}^1 \to \mathbb{P}^n$$

is given, in terms of the Euclidean coordinate $t = Z_1/Z_0$ on $\mathbb{P}^1$, by

$$t \mapsto [1, t, t^2, \ldots, t^n].$$

Its image is a nondegenerate curve of degree n, called the *rational normal curve*.

Conversely, if $C \subset \mathbb{P}^n$ is an irreducible, nondegenerate curve of degree n, let $p_1, \ldots, p_{n-1}$ be any $n-1$ independent points of C, $V = \overline{p_1, \ldots, p_{n-1}} \cong \mathbb{P}^{n-2}$ their linear span, and $\{H_\lambda\}_{\lambda \in \mathbb{P}^1}$ the pencil of hyperplanes in $\mathbb{P}^n$ containing V. Each hyperplane H_λ will then intersect C in n points: $p_1, \ldots, p_{n-1}$, and an additional point we will call $q(\lambda)$. (In case H_λ is the hyperplane containing V and tangent to C at p_i, the point $q(\lambda) = p_i$.) Every point of C will lie on a unique hyperplane H_λ, and so the map $q: \mathbb{P}^1 \to C$ is an isomorphism. Since moreover nH is the unique line bundle of degree n on $\mathbb{P}^1$, it follows that *every irreducible nondegenerate curve of degree* n *in* $\mathbb{P}^n$ *is projectively isomorphic to the rational normal curve.*

2. In terms of Euclidean coordinates $s = Z_1/Z_0$, $t = Z_2/Z_0$ on $\mathbb{P}^2$, the Veronese map $f = \iota_{2H}: \mathbb{P}^2 \to \mathbb{P}^5$ is given by

$$(s,t) \mapsto [1, s, t, s^2, st, t^2].$$

The image $S = f(\mathbb{P}^2)$ is a nondegenerate surface of degree $c_1(f^*H_{\mathbb{P}^5})^2 = c_1(2H_{\mathbb{P}^2})^2 = 4$; note that this degree is minimal in the sense of the last section.

We digress for a moment to discuss a curious feature of the Veronese surface $S \subset \mathbb{P}^5$: *it is the unique nondegenerate surface in* $\mathbb{P}^5$ *whose variety of chords* $C(S) = \cup_{p,q \in S} \overline{pq}$ *is a proper subvariety of* $\mathbb{P}^5$. To see this, note that for any point $p \in \mathbb{P}^5$ lying on the chord $\overline{f(u), f(u')}$ of S, the line $L = \overline{uu'} \subset \mathbb{P}^2$ is mapped into a curve of degree

$$\#(H_{\mathbb{P}^5} \cdot f(L)) = \#(2H_{\mathbb{P}^2} \cdot L) = 2$$

$\mathbb{P}^5$, hence by the result of p. 173 is a conic lying in a 2-plane $V_2 \subset \mathbb{P}^5$. Now $p \in \overline{f(u), f(u')} \subset V_2$, and any line through p in V_2 must intersect $f(L)$ twice, so that *any point of* $\mathbb{P}^5$ *lying on a chord of* S *lies on infinitely many chords of* S. In particular, if we let L_0 be the line $(s=0)$ in $\mathbb{P}^2$, and let $u_0 = L_0 \cap L$, then the line $\overline{f(u_0), p} \subset \mathbb{P}^5$ is a chord of S. Thus

$$C(S) = \bigcup_{\substack{p \in L_0 \\ q \in \mathbb{P}^2}} \overline{f(p), f(q)},$$

from which we see that $C(S)$ is of dimension at most four. Explicitly, we describe $C(S)$ as the locus

$$\{\alpha \cdot f(s,t) + (1-\alpha) \cdot f(0,t')\} = \{[1, \alpha s, \alpha t + (1-\alpha)t', \alpha s^2, \alpha st, \alpha t^2 + (1-\alpha)t'^2]\}.$$

Now we solve for α, s, t, and t': given $X = [X_0, \ldots, X_5] \in C(S)$, X must be the point $\alpha \cdot f(s,t) + (1-\alpha) \cdot f(0,t')$ for the values

$$s = X_3/X_1, \quad t = X_4/X_1, \quad \alpha = X_1^2/X_0X_3,$$
$$t' = (X_2X_3 - X_1X_4)/(X_0X_3 - X_1^2).$$

Consequently the coordinates of $X \in C(S)$ must satisfy

$$\begin{aligned} X_5/X_0 &= \alpha t^2 + (1-\alpha)t'^2 \\ &= X_4^2/X_0X_3 + (X_2X_3 - X_1X_4)^2/\big(X_0X_3(X_0X_3 - X_1^2)\big), \end{aligned}$$

i.e.,

$$\big(X_0X_3 - X_1^2\big)X_5 = X_0X_4^2 + X_2^2X_3 - 2X_1X_2X_4,$$

and we see that the variety of chords of the Veronese surface in $\mathbb{P}^5$ is a cubic hypersurface.

We may state the original question of this section as: Given $L \to M$ a holomorphic line bundle, when is $\iota_L : M \to \mathbb{P}^N$ an embedding? First, in order for ι_L to be well-defined the linear system $|L|$ cannot have any base points, i.e., for each $x \in M$ the restriction map

$$H^0(M, \mathcal{O}(L)) \xrightarrow{r_x} L_x$$

must be surjective. Granted this, ι_L will be an embedding if

1. ι_L *is one-to-one*. Clearly this is the case if and only if for all x and y in M, there exists a section $s \in H^0(M, \mathcal{O}(L))$ vanishing at x but not at y, i.e., if and only if the restriction map

$$(*) \qquad H^0(M, \mathcal{O}(L)) \xrightarrow{r_{x,y}} L_x \otimes L_y$$

is surjective for all $x \neq y \in M$. Note that if L satisfies this condition, then L must be base-point-free.

2. ι_L *has nonzero differential everywhere*. If φ_α is a trivialization of L near x, then this is the case if and only if for all $v^* \in T_x^*(M)$, there exists $s \in H^0(M, \mathcal{O}(L))$ with $s_\alpha(x) = 0$ and $ds_\alpha(x) = v^*$ where $s_\alpha = \varphi_\alpha^* s$. We can express this requirement more intrinsically as follows: let $\mathcal{I}_x \subset \mathcal{O}$ denote the sheaf of holomorphic functions on M vanishing at x, and let $\mathcal{I}_x(L)$ be the sheaf of sections of L vanishing at x. If s is any section of $\mathcal{I}_x(L)$ defined near x, and $\varphi_\alpha, \varphi_\beta$ are trivializations of L in a neighborhood U of x, then writing $s_\alpha = \varphi_\alpha^* s$, $s_\beta = \varphi_\beta^* s$, $s_\alpha = g_{\alpha\beta} s_\beta$, we have

$$\begin{aligned} d(s_\alpha) &= d(s_\beta)\cdot g_{\alpha\beta} + dg_{\alpha\beta}\cdot s_\beta \\ &= d(s_\beta)\cdot g_{\alpha\beta} \qquad \text{at } x. \end{aligned}$$

Thus we have a well-defined sheaf map

$$d_x : \mathcal{I}_x(L) \to T_x^{*\prime} \otimes L_x$$

and condition 2 can be stated as requiring that the map

$$(**) \qquad H^0(M, \mathcal{I}_x(L)) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x$$

be surjective for all $x \in M$. Note that $(**)$ is the limiting case of $(*)$ when $y \to x$.

The result we are aiming for is the

Kodaira Embedding Theorem. *Let* M *be a compact complex manifold and* $L \to M$ *a positive line bundle. Then there exists* k_0 *such that for* $k \geqslant k_0$, *the map*

$$\iota_{L^k}: M \to \mathbb{P}^N$$

is well-defined and is an embedding of M.

Let us consider how one might go about proving this. The first thing to do is to fit the maps (*) and (**) into exact sequences and try to use our vanishing theorems directly. To this end, let $\mathcal{I}_{x,y}(L)$ denote the sheaf of sections of L vanishing at x and y, and $\mathcal{I}_x^2(L)$ the sheaf of sections of L vanishing to order 2 at x, i.e., sections s of $\mathcal{I}_x(L)$ such that $d_x(s)=0$. We have exact sheaf sequences

$$0 \to \mathcal{I}_{x,y}(L) \to \mathcal{O}(L) \xrightarrow{r_{x,y}} L_x \oplus L_y \to 0$$

and

$$0 \to \mathcal{I}_x^2(L) \to \mathcal{I}_x(L) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x \to 0;$$

so that to show that the maps (*) and (**) are surjective, it would suffice to prove that

$$H^1\big(M, \mathcal{I}_x^2(L)\big) = H^1\big(M, \mathcal{I}_{x,y}(L)\big) = 0;$$

indeed, replacing L by L^k and using $H^1(M, \mathcal{O}(L^k))=0$ for $k \geqslant k_1$, the reader may check that our theorem is equivalent to this vanishing theorem for high powers of L. The problem is that unless M is of dimension 1 neither of the sheaves $\mathcal{I}_{x,y}(L)$ and $\mathcal{I}_x^2(L)$ is the sheaf of sections of a holomorphic vector bundle—for $E \to M$ a holomorphic vector bundle and $V \subset M$ a subvariety, the kernel of the restriction map $\mathcal{O}_M(E) \to \mathcal{O}_V(E)$ is the sheaf of sections of a vector bundle if and only if V is of codimension 1 in M—and so we cannot get a direct grip on them using our technique of harmonic theory. $\mathcal{I}_x^2$ and $\mathcal{I}_{x,y}$ are examples of *coherent sheaves*, a class of sheaves broader than, but closely related to, sheaves of sections of holomorphic vector bundles. The theory of coherent sheaves will be discussed in Chapter 5.

Another approach to the problem might be to emulate the proof of the proposition on p. 161 and do an induction on the dimension of M—for example, if we could find a smooth hypersurface $V \subset M$ containing x and y, then to show the map (*) surjective, we would only have to prove it for $L|_V$ on V and show that the restriction map

$$H^0(M, \mathcal{O}_M(L)) \to H^0(V, \mathcal{O}_V(L))$$

was surjective, i.e., that

$$H^1(M, \mathcal{O}_M(L - V)) = 0.$$

But this is very nearly presupposing the result to be proved: a priori, M need not have any divisors on it at all.

It is clear by now that our difficulty lies in the simple fact that, unless M is a Riemann surface, a point on M is not a divisor. We can overcome this problem by means of a beautiful classical construction called *blowing up*, which transforms points on a complex manifold into divisors.

Blowing Up

We will first describe the blow-up of the origin in a disc Δ in $\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ be Euclidean coordinates in Δ and $l = [l_1, \ldots, l_n]$ corresponding homogeneous coordinates on $\mathbb{P}^{n-1}$. Let $\tilde{\Delta} \subset \Delta \times \mathbb{P}^{n-1}$ be the submanifold of $\Delta \times \mathbb{P}^{n-1}$ given by the quadratic relations

$$\tilde{\Delta} = \{(z, l) : z_i l_j = z_j l_i \quad \text{for all } i, j\}.$$

If we consider points $l \in \mathbb{P}^{n-1}$ as lines in $\mathbb{C}^n$, then writing these equations as $z \wedge l = 0$ we see that this is just the *incidence correspondence* defined as $\{(z, l) : z \in l\}$.

Now $\tilde{\Delta}$ maps onto Δ via projection on the first factor $\pi : (z, l) \mapsto z$; from the geometric interpretation it follows that the map is an isomorphism away from the origin in Δ, and $\pi^{-1}(0)$ is just the projective space of lines in Δ. In effect, $\tilde{\Delta}$ *consists of all the lines through the origin in* Δ *made disjoint*. $\tilde{\Delta}$, together with its projection map π to Δ, is called the *blow-up* of Δ at 0. The real points of the blow-up of $\Delta \subset \mathbb{C}^2$ are pictured in Figure 1.

Note that we have encountered the manifold $\tilde{\Delta}$ before: together with the projection $\pi' : \tilde{\Delta} \to \mathbb{P}^{n-1}$ on the second factor it is the universal bundle J on $\mathbb{P}^{n-1}$.

Now let M be a complex manifold of dimension n, $x \in M$ any point, and $z : U \to \Delta$ a coordinate polydisc centered around $x \in M$. The restriction of the projection map

$$\pi : \tilde{\Delta} - E \longrightarrow U - \{x\} \subset M$$

gives an isomorphism between a neighborhood of $E = \pi^{-1}x$ in $\tilde{\Delta}$ and a neighborhood of x in M; we define the *blow-up* $\tilde{M}_x$ *of* M *at* x to be the complex manifold

$$\tilde{M}_x = M - \{x\} \cup_\pi \tilde{\Delta}$$

obtained by replacing $\Delta \subset M$ with $\tilde{\Delta}$, together with the natural projection map $\pi : \tilde{M}_x \to M$. Again, the projection $\pi : \tilde{M}_x - \{\pi^{-1}(x)\} \to M - \{x\}$ is an isomorphism; the inverse image $\pi^{-1}(x)$ in $\tilde{M}_x$ is called the *exceptional divisor* of the blow-up, and is usually denoted E or E_x.

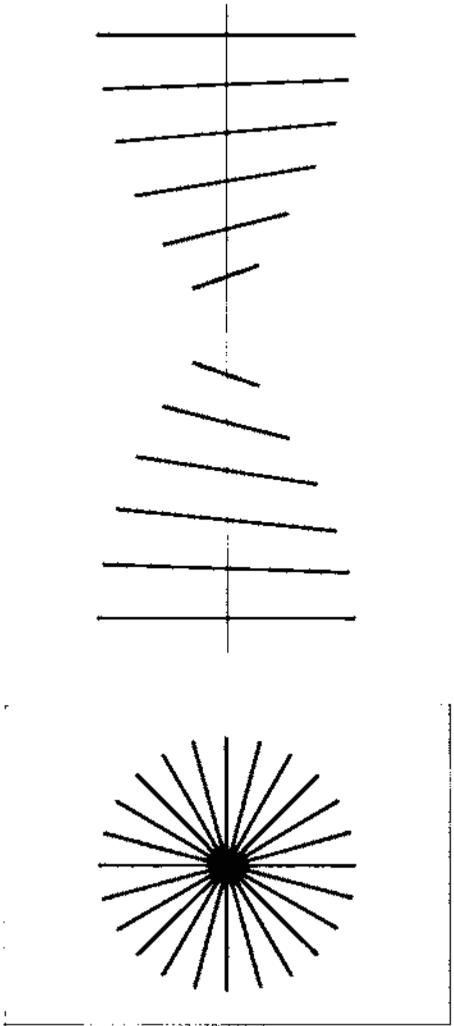

Figure 1

Note that the blow-up $\tilde{M} \to M$ is independent of the coordinates used in the disc Δ: if $\{z_i' = f_i(z)\}$ are other coordinates in Δ with $f_i(0) = 0$, $\tilde{\Delta}'$ the blow up of Δ in terms of these coordinates, then the isomorphism

$$f\colon \tilde{\Delta} - E \to \tilde{\Delta}' - E'$$

may be extended over E by setting $f(0, l) = (0, l')$, where

$$l_j' = \sum \frac{\partial f_j}{\partial z_i}(0) \cdot l_i.$$

Indeed, we see from this that the identification

$$E \longrightarrow \mathbb{P}(T_x(M))$$

given by

$$(0,l) \mapsto \left[\sum l_i \cdot \frac{\partial}{\partial z_i} \right]$$

is likewise independent of the coordinate system chosen.

Now we will describe the geometry of $\tilde{M}_x$ near E in more detail. First, we give local coordinates near E on $\tilde{M}_x$: let $z=(z_1,\dots,z_n)$ be local coordinates on $U \ni x$ with center x. Then

$$\tilde{U} = \pi^{-1}(U) = \left\{ (z,l) \in U \times \mathbb{P}^{n-1} : z_i l_j = z_j l_i \right\};$$

and we set

$$\tilde{U}_i = (l_i \neq 0) \subset \tilde{U}.$$

In this way we obtain an open cover of the neighborhood $\tilde{U}$ of E, and in each open set $\tilde{U}_i$ we have local coordinates $z(i)_j$:

$$z(i)_j = \frac{l_j}{l_i} = \frac{z_j}{z_i}, \qquad j \neq i,$$

and

$$z(i)_i = z_i.$$

The map $\pi : \tilde{M}_x \to M$ is given in $\tilde{U}_i$ by

$$(z(i)_1,\dots,z(i)_i,\dots,z(i)_n) \mapsto (z(i)_i \cdot z(i)_1,\dots,z(i)_i,\dots,z(i)_i \cdot z(i)_n)$$

and the divisor E is given by

$$E = (\tilde{z}(i)_i = 0)$$

in $\tilde{U}_i$. In $\tilde{U}_i \cap \tilde{U}_j$,

$$\begin{aligned} z(i)_k &= z(j)_i^{-1} \cdot z(j)_k, \\ z(i)_j &= z(j)_j^{-1}, \\ z_i &= z(j)_i \cdot z_j. \end{aligned}$$

Now, since $E = (z_i)$ in $\tilde{U}_i$, the line bundle $[E]$ is given in $\tilde{U}$ by transition functions

$$g_{ij} = z(j)_i = \frac{z_i}{z_j} = \frac{l_i}{l_j}, \text{ in } U_i \cap U_j$$

and so we can realize $[E]|_{\tilde{U}}$ by identifying the fiber

(*) $$[E]_{(z,l)} = \{\lambda(l_1,\dots,l_n), \lambda \in \mathbb{C}\}.$$

In particular, we see that *the line bundle* $[E]|_E$ *is just the universal bundle* $J = -H$ *on* $E \cong \mathbb{P}^{n-1}$.

Dually, the line bundle $[-E] = [E]^*$ has as fiber over any point $(z,l) \in \tilde{U}$ the space of linear functionals on the line $l \subset \mathbb{C}^n$; $[-E]|_E$ is the hyperplane bundle on E.

Now we have seen that E is naturally identified with $\mathbb{P}(T_x'(M))$, so that the global sections of $[-E]$ over E correspond exactly to the linear functionals on the tangent space, i.e.,

$$(**) \qquad H^0(E, \mathcal{O}_E(-E)) = T_x^{*\prime}(M).$$

On the other hand, given a function f on U vanishing at x, the function $\pi^* f \in \mathcal{O}(\tilde{U})$ vanishes along E and so can be considered as a section of $[-E]$ over $\tilde{U}$. By explicit computation we check that for any $f \in \mathcal{I}_x(U)$ *the restriction to* E *of the section* $\pi^* f \in \mathcal{O}(-E)(\tilde{U})$ *corresponds, via the identification* $(**)$, *to the differential* $df(x)$ *of* f *at* x, i.e., the diagram

$$\begin{array}{ccc} H^0(\tilde{U}, \mathcal{O}(-E)) & \xrightarrow{r_E} & H^0(E, \mathcal{O}(-E)) \\ \uparrow & & \| \\ H^0(U, \mathcal{I}_x) & \xrightarrow{d_x} & T_x^{*\prime}(U) \end{array}$$

commutes.

This correspondence reflects a basic aspect of the local analytic character of blow-ups: the infinitesimal behavior of functions, maps, or differential forms at the point x of M is transformed into global phenomena on $\tilde{M}$. Indeed, in classical terminology, a point in the exceptional divisor of the blow-up of M at x was called an "infinitely near point" of x; the exceptional divisor itself was called an "infinitesimal neighborhood" of x.

The next thing to do is to compute the curvature of the line bundles $[E]$ and $[-E]$ on $\tilde{M}$. We construct a metric on $[E]$ as follows: let h_1 be the metric on $[E]|_{\tilde{U}}$ given, in terms of the representation $(*)$ of E, by

$$|(l_1, \ldots, l_n)|^2 = \|l\|^2.$$

Let $\sigma \in H^0(\tilde{M}, \mathcal{O}([E]))$ be the above global section of $[E]$ on $\tilde{M}$ with $(\sigma) = E$, so that σ is nonzero on $\tilde{M} - E$; let h_2 be the metric on $[E]|_{\tilde{M}-E}$ given by

$$|\sigma(z)| \equiv 1.$$

For $\varepsilon > 0$, denote by U_ε the ball $(\|z\| < \varepsilon)$ around x in U and set $\tilde{U}_\varepsilon = \pi^{-1}(U_\varepsilon)$; let ρ_1, ρ_2 be a partition of unity for the cover $\{\tilde{U}_{2\varepsilon}, \tilde{M} - \tilde{U}_\varepsilon\}$ of $\tilde{M}$, and let h be the global metric given by

$$h = \rho_1 \cdot h_1 + \rho_2 \cdot h_2.$$

We will compute the curvature of $[E]$ with this metric. For notational convenience, let $\Omega_{[E]}$ denote $\sqrt{-1}/2$ times the curvature $\Theta_{[E]}$ of $[E]$. It is necessary to consider three cases:

1. On $\tilde{M}-\tilde{U}_{2\varepsilon}$, $\rho_2 \equiv 1$ so $|\sigma|^2 \equiv 1$; consequently

$$\Omega_{[E]} = dd^c \log \frac{1}{|\sigma|^2} \equiv 0.$$

2. On $\tilde{U}_\varepsilon - E \cong U_\varepsilon - \{x\}$, let σ be given in terms of the representation (*) by

$$\sigma(z,l) = z;$$

then

$$\Omega_{[E]} = dd^c \log \frac{1}{\|z\|^2} = -dd^c \log \|z\|^2,$$

i.e., $-\Omega_{[E]}$ is just the pullback $\pi'^*\omega$ of the associated (1,1)-form ω of the Fubini-Study metric on $\mathbb{P}^{n-1}$ under the map $\pi': \tilde{U}_\varepsilon \to \mathbb{P}^{n-1}$ given by $(z,l) \mapsto l$. Thus

$$-\Omega_{[E]} \geqslant 0 \qquad \text{on } \tilde{U}_\varepsilon - E.$$

3. We have seen that $-\Omega_{[E]} = \pi'^*\omega$ on $\tilde{U}_\varepsilon - E$; by continuity it follows that $-\Omega_{[E]} = \pi'^*\omega$ throughout $\tilde{U}_\varepsilon$, and in particular

$$-\Omega_{[E]}|_E = \omega > 0$$

on E.

Summing up, if we let $\Omega_{[-E]}$ be $\sqrt{-1}/2$ times the curvature form of the dual metric in $[E]^* = [-E]$, we have

$$\Omega_{[-E]} = -\Omega_{[E]} = \begin{cases} 0 & \text{on } \tilde{M}-\tilde{U}_{2\varepsilon}, \\ \geqslant 0 & \text{on } \tilde{U}_\varepsilon, \\ >0 & \text{on } T'_x(E) \subset T'_x(\tilde{M}) \quad \text{for all } x \in E. \end{cases}$$

The point of this computation is the following: let $L \to M$ be a positive line bundle with a metric h_L whose curvature form Θ_L is $2/\sqrt{-1}$ times a positive form Ω_L. Then if Ω_{π^*L} is $\sqrt{-1}/2$ times the curvature form of the induced metric on the bundle $\pi^*L \to \tilde{M}$,

$$\Omega_{\pi^*L} = \pi^*\Omega_L,$$

hence $\Omega_{\pi^*L} > 0$ on $\tilde{M} - E$. Moreover, for any $x \in E$ and tangent vector $v \in T_x(\tilde{M})$,

$$\langle \Omega_{\pi^*L}; v, \bar{v} \rangle = \langle \Omega_L; \pi_* v, \overline{\pi_* v} \rangle \geqslant 0$$

with equality holding if and only if $\pi_*(v) = 0$, i.e., if and only if v is tangent to E. Thus

$$\Omega_{\pi^*L} = \begin{cases} \geqslant 0 & \text{everywhere,} \\ >0 & \text{on } \tilde{M} - E, \\ >0 & \text{on } T'_x(\tilde{M})/T'_x(E) \quad \text{for all } x \in E, \end{cases}$$

and the form

$$\begin{aligned} \Omega_{\pi^*L^k \otimes [-E]} &= \Omega_{\pi^*L^k} + \Omega_{[-E]} \\ &= k\Omega_{\pi^*L} + \Omega_{[-E]} \end{aligned}$$

is positive everywhere in $\tilde{U}_\varepsilon$ and $\tilde{M} - \tilde{U}_{2\varepsilon}$. Moreover, since the form $\Omega_{[-E]}$ is bounded below in $\tilde{U}_{2\varepsilon} - \tilde{U}_\varepsilon$ and Ω_{π^*L} is strictly positive there, we see that $\Omega_{\pi^*L^k \otimes [-E]}$ is everywhere positive for k sufficiently large; i.e., *there exists* k_0 *such that* $\pi^*L^k - E$ *is a positive line bundle on* $\tilde{M}$ *for* $k \geqslant k_0$.

Note that by the same argument, for any positive integer n the bundle $\pi^*L^k - nE$ will be positive for $k \gg 0$.

We need to establish one more relation between $\tilde{M}_x$ and M:

Lemma. $K_{\tilde{M}} = \pi^* K_M + (n-1)E$.

Proof. This is easy in case M has a nontrivial meromorphic n-form ω. In terms of local coordinates $z_1, \ldots, z_n$ in a neighborhood U of x, write

$$\omega(z) = \frac{f(z)}{g(z)} \cdot dz_1 \wedge \cdots \wedge dz_n.$$

Now let $z(i)_j$ be local coordinates in as before. The map π is given in $\tilde{U}_i$ by

$$(z(i)_1, \ldots, z(i)_n) \to (z(i)_1 z_i, \ldots, z_i, \ldots, z(i)_n z_i),$$

and so

$$\begin{aligned} \pi^*\omega &= \pi^*(f/g) \cdot d(z(i)_1 z_i) \wedge \cdots \wedge dz_i \wedge \cdots \wedge d(z(i)_n z_i) \\ &= \pi^*(f/g) \cdot z_i^{(n-1)} dz(i)_1 \wedge \cdots \wedge dz(i)_n. \end{aligned}$$

Thus we see that in a neighborhood of $E = \pi^{-1}(x_0)$, the divisor $(\pi^*\omega)$ is given by $\pi^*(\omega) + (n-1)E$. Since clearly $(\pi^*\omega) = \pi^*(\omega)$ away from E,

$$K_{\tilde{M}} = [(\pi^*\omega)] = \pi^* K_M + (n-1)E$$

as desired. Thus the formula is proved under the assumption that M has a meromorphic n-form; this is the easiest way to see the result.

To prove the lemma in general, we let $\underline{U} = \{U_0, U_\alpha\}_\alpha$ be an open coordinate cover of M with $x \in U_0$, $x \notin U_\alpha$ and all sets U_α having nonempty intersection with U_0 lying in one coordinate patch with coordinates $z_1, \ldots, z_n$. Let

$$\underline{\tilde{U}} = \{ \tilde{U}_\alpha = \pi^{-1} U_\alpha, \quad \tilde{U}_i = \pi^{-1} U_0 \cap (l_i \neq 0) \}$$

be a corresponding cover for $\tilde{M}$; we compute the transition functions $\{g_{ij}, g_{i\alpha}, g_{\alpha\beta}\}$ for $K_{\tilde{M}}$ in terms of the coordinates $z(i)_j$ on $\tilde{U}_i$ and $w_{i,\alpha} = \pi^* w_{i,\alpha}$ on $\tilde{U}_\alpha$, where $\{w_{i,\alpha}\}_i$ are coordinates on U_α in M. First we have in $\tilde{U}_1 \cap \tilde{U}_2$

$$\begin{aligned} z(2)_1 &= z(1)_2^{-1}, \\ z_2 &= z(1)_2 \cdot z_1, \\ z(2)_i &= z(1)_i \cdot z(1)_2^{-1}, \qquad i \neq 1,2, \end{aligned}$$

and so the Jacobian matrix for the change of coordinates is

$$J_{12} = \begin{pmatrix} 0 & -z(1)_2^{-2} & 0 & \cdots & & & & & 0 \\ z(1)_2 & z_1 & 0 & \cdots & & & & & 0 \\ 0 & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & -z(1)_j \cdot z(1)_2^{-2} & 0 & \cdots & 0 & z(1)_2^{-1} & 0 & \cdots & 0 \\ \vdots & & & & & & & & \end{pmatrix};$$

in general

$$g_{ij} = \det J_{ij} = z(1)_j^{-n+1}.$$

Similarly, in $\tilde{U}_\alpha \cap \tilde{U}_1$

$$w_{1,\alpha} = z_1, \qquad w_{i,\alpha} = z_1 \cdot z(1)_i,$$

$$J_{1\alpha} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ z(1)_i & 0 & \cdots & z_1 & \cdots & 0 \\ \vdots & & & & & \end{pmatrix},$$

and in general

$$g_{i\alpha} = z_i^{(n-1)}.$$

Also

$$g_{\alpha\beta} = \pi^* g'_{\alpha\beta},$$

where $g'_{\alpha\beta}$ are the transition functions for K_M with respect to coordinates $w_{i,\alpha}$ in U_α, $w_{i,\beta}$ in U_β.

Now E is given in $\tilde{U}_i$ by (z_i), in $\tilde{U}_\alpha$ by (1); so the transition functions for $[E]$ over $\tilde{\underline{U}}$ are

$$\begin{aligned} h_{ij} &= \frac{z_i}{z_j} = z(i)_j^{-1}, \\ h_{i\alpha} &= z_i, \\ h_{\alpha\beta} &= 1. \end{aligned}$$

Thus the transition functions for the bundle $K_{\tilde{M}}\otimes[E]^{-n+1}$ are

$$f_{ij} = z(i)_j^{-n+1}\cdot z(i)_j^{n-1} = 1,$$
$$f_{i\alpha} = {}_iz_i^{n-1}\cdot {}_iz_i^{-n+1} = 1,$$
$$f_{\alpha\beta} = \pi^* g_{\alpha\beta},$$

and we see that $K_{\tilde{M}}-(n-1)E$ is just the pullback via π of the bundle on M given by transition functions

$$e_{0\alpha} = 1, \qquad e_{\alpha\beta} = g_{\alpha\beta};$$

i.e., $K_{\tilde{M}}-(n-1)E=\pi^* K_M$. Q.E.D.

We will develop a much more complete picture of the geometry of blow-ups later on in the chapter on surfaces; for the time being, we have enough information to proceed to the proof of the embedding theorem.

Proof of the Kodaira Theorem

Again, let $L \to M$ be a positive line bundle on the compact complex manifold M. We want to prove that there exists k_0 such that

1. The restriction map

$$H^0(M,\mathcal{O}(L^k)) \xrightarrow{r_{x,y}} L_x^k \oplus L_y^k$$

is surjective for all $x \neq y \in M$, $k \geqslant k_0$; and

2. The differential map

$$H^0(M,\mathcal{I}_x(L^k)) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x^k$$

is surjective for all $x \in M$, $k \geqslant k_0$.

To prove assertion 1, let $\tilde{M} \xrightarrow{\pi} M$ denote the blow-up of M at both x and y, $E_x=\pi^{-1}(x)$ and $E_y=\pi^{-1}(y)$ the exceptional divisors of the blow-up; for notational convenience, let E denote the divisor E_x+E_y and $\tilde{L}=\pi^* L$. (Here we are tacitly assuming that $n=\dim(M) \geqslant 2$; in case M is a Riemann surface, all the arguments that follow will be valid for $\tilde{M}=M$, $\pi = id$.)

Consider the pullback map on sections

$$\pi^*: H^0(M,\mathcal{O}_M(L^k)) \to H^0(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k)).$$

For any global section $\tilde{\sigma}$ of $\tilde{L}^k$, the section of L^k given by σ over $M-\{x,y\}$ extends by Hartogs' theorem to a global section $\sigma \in H^0(M,\mathcal{O}(L^k))$, and so we see that π^* is an isomorphism. Furthermore, by definition $\tilde{L}^k$ is trivial along E_x and E_y, i.e.,

$$(\tilde{L}^k)|_{E_x} = E_x \times L_x^k, \quad (\tilde{L}^k)|_{E_y} = E_y \times L_y^k,$$

so that

$$H^0(E,\mathcal{O}_E(\tilde{L}^k)) \cong L_x^k \oplus L_y^k,$$

and if r_E denotes the restriction map to E, the diagram

$$\begin{array}{ccc} H^0(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k)) & \xrightarrow{r_E} & H^0(E,\mathcal{O}_E(\tilde{L}^k)) \\ \uparrow & & \| \\ H^0(M,\mathcal{O}(L^k)) & \xrightarrow{r_{x,y}} & L_x^k \oplus L_y^k \end{array}$$

commutes. Thus to prove assertion 1 for x and y, we have to show the map r_E is surjective.

Now, on $\tilde{M}$ we have the exact sheaf sequence

$$0 \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k) \xrightarrow{r_E} \mathcal{O}_E(\tilde{L}^k) \to 0.$$

Choose k_1 such that $L^{k_1} + K_M^*$ is positive on M. By virtue of the computation on p. 186, we can choose k_2 such that $\tilde{L}^k - nE$ is positive on $\tilde{M}$ for $k \geqslant k_2$. By the previous lemma

$$K_{\tilde{M}} = \tilde{K}_M + (n-1)E,$$

where $\tilde{K}_M = \pi^* K_M$; and so for $k \geqslant k_0 = k_1 + k_2$,

$$\begin{aligned} \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E) &= \Omega^n_{\tilde{M}}\big(\tilde{L}^k - E + K^*_{\tilde{M}}\big) \\ &= \Omega^n_{\tilde{M}}\big((\tilde{L}^{k_1} + \tilde{K}^*_M) \otimes (\tilde{L}^{k'} - nE)\big) \end{aligned}$$

with $k' \geqslant k_2$. Now by hypothesis. $\tilde{L}^{k'} - nE$ has a positive definite curvature form on $\tilde{M}$; $L^{k_1} + K_M^*$ has a positive curvature form on M, and so $(\tilde{L}^{k_1} + \tilde{K}_M^*)$ has a positive semidefinite one on $\tilde{M}$. Thus the line bundle $(\tilde{L}^{k_1} + \tilde{K}_M^*) + \tilde{L}^{k'} - nE$ is positive on $\tilde{M}$, and by the Kodaira vanishing theorem,

$$\begin{aligned} H^1\big(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)\big) &= H^1\big(\tilde{M},\Omega^n_{\tilde{M}}\big((\tilde{L}^{k_1} + \tilde{K}^*_M) + (\tilde{L}^{k'} - nE)\big)\big) \\ &= 0 \qquad \text{for } k \geqslant k_0. \end{aligned}$$

Hence the map

$$r_E:\ H^0\big(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k)\big) \to H^0\big(E,\mathcal{O}_E(\pi^* L^k)\big)$$

is surjective for $k \geqslant k_0$, and so assertion 1 is proved for x and y.

Assertion 2 is proved similarly. Let $\tilde{M} \xrightarrow{\pi} M$ now denote the blow-up of M at x, $E = \pi^{-1}(x)$ the exceptional divisor. Again, the pullback map

$$\pi^*:\ H^0\big(M,\mathcal{O}_M(L^k)\big) \to H^0\big(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k)\big)$$

is an isomorphism. Further, if $\sigma \in H^0(M,\mathcal{O}_M(L^k))$, then $\sigma(x)=0$ if and only if $\tilde{\sigma} = \pi^*\sigma$ vanishes on E; thus π^* restricts to give an isomorphism

$$\pi^*:\ H^0\big(M,\mathcal{I}_x(L^k)\big) \xrightarrow{\sim} H^0\big(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)\big).$$

As before, we can identify

$$H^0\big(E,\mathcal{O}_E(\tilde{L}^k - E)\big) = L_x^k \otimes H^0\big(E,\mathcal{O}_E(-E)\big) \cong L_x^k \otimes T_x^{*\prime},$$

and the diagram

$$\begin{array}{ccc} H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)) & \xrightarrow{r_E} & H^0(E, \mathcal{O}_E(\tilde{L}^k - E)) \\ \wr\!\!\uparrow \pi^* & & \| \\ H^0(M, \mathcal{I}_x(L^k)) & \xrightarrow{d_x} & T_x^{*\prime} \otimes L_x^k \end{array}$$

commutes. Thus we must prove that r_E is surjective for $k \gg 0$.

On $\tilde{M}$, there is an exact sequence

$$O \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k - 2E) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E) \xrightarrow{r_E} \mathcal{O}_E(\tilde{L}^k - E) \to 0.$$

Again, choose k_1 such that $L^{k_1} + K_M^*$ is positive on M and k_2 such that $\tilde{L}^{k'} - (n+1)E$ is positive on $\tilde{M}$ for $k' \geqslant k_2$. For $k \geqslant k_0 = k_1 + k_2$

$$\mathcal{O}_{\tilde{M}}(\tilde{L}^k - 2E) = \Omega^n_{\tilde{M}}\big((\tilde{L}^{k_1} + \tilde{K}^*_M) \otimes (\tilde{L}^{k'} - (n+1)E)\big)$$

with $k' \geqslant k_2$. It follows by the Kodaira vanishing theorem that

$$H^1\big(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - 2E)\big) = 0$$

for $k \geqslant k_0$; hence r_E is surjective on global sections and assertion 2 is proved for arbitrary fixed x.

All that remains now to be proved is that we can find one value of k_0 such that assertions 1 and 2 hold for all choices of x and y and all $k \geqslant k_0$. But clearly if ι_{L^k} is defined at x and y and $\iota_{L^k}(x) \neq \iota_{L^k}(y)$, the same will be true for x' near x and y' near y, and likewise if ι_{L^k} is smooth at x it will be smooth at x' near x and separate points $x' \neq x''$ near x. Since M is compact, then, the result follows. Q.E.D.

Before proceeding to some examples and corollaries, we give a somewhat more intrinsic restatement of the theorem:

Kodaira Embedding Theorem. *A compact complex manifold* M *is an algebraic variety—i.e., is embeddable in projective space—if and only if it has a closed, positive* (1,1)*-form* ω *whose cohomology class* $[\omega]$ *is rational.*

Proof. If $[\omega] \in H^2(M, \mathbb{Q})$, then for some k, $[k\omega] \in H^2(M, \mathbb{Z})$; in the exact sequence

$$H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \xrightarrow{\iota_*} H^2(M, \mathcal{O})$$

$\iota_*([k\omega]) = 0$, and so there exists a holomorphic line bundle $L \to M$ with $c_1(L) = [k\omega]$. The line bundle L will then be positive. Q.E.D.

A metric whose (1,1)-form is rational is called a *Hodge metric*.

Corollary. *If* M, M′ *are algebraic varieties, then* M × M′ *is.*

Proof. If ω, ω' are closed, integral, positive $(1,1)$-forms on M, M', respectively, and $\pi: M \times M' \to M$, $\pi': M \times M' \to M'$ are the projection maps, then $\pi^*\omega + \pi'^*\omega'$ is again closed, integral, and positive of type $(1,1)$. Q.E.D.

A classical example of this is the *Segré map* $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ given by the complete linear system of the line bundle $\pi_1^* H \otimes \pi_2^* H$ on $\mathbb{P}^n \times \mathbb{P}^m$. For example, the Segré map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given, in terms of homogeneous coordinates $[z_0, z_1]$ and $[w_0, w_1]$ on $\mathbb{P}^1$, by

$$([z_0, z_1], [w_0, w_1]) \mapsto [z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1].$$

The image is just the quadric hypersurface $(X_0 X_3 = X_1 X_2)$ in $\mathbb{P}^3$.

Corollary. *If* M *is an algebraic variety,* $\tilde{M} \xrightarrow{\pi} M$ *the blow-up of* M *at a point* x, *then* $\tilde{M}$ *is algebraic.*

Proof. We have seen in the course of the proof of the embedding theorem that if $L \to M$ is positive and $E = \pi^{-1}(x)$, then $\pi^* L^k - E$ is positive for $k \gg 0$.

Corollary. *If* $\tilde{M} \xrightarrow{\pi} M$ *is a finite unbranched covering of compact complex manifolds, then* M *is algebraic if and only if* $\tilde{M}$ *is.*

Proof. Clearly, if $L \to M$ is positive, then $c_1(\pi^* L) = \pi^* c_1(L)$ implies that $\pi^* L$ is positive. Conversely, say ω is an integral, positive $(1,1)$-form on $\tilde{M}$. For any $p \in M$, we have isomorphisms of a neighborhood U of p in M with neighborhoods U_i of the points $q_i \in \pi^{-1}(p)$; we can define a $(1,1)$-form ω' on M by

$$\omega'(p) = \sum_{q \in \pi^{-1}(p)} \omega(q).$$

Then ω' is closed and of type $(1,1)$, and if $\eta \in H_{DR}^{2n-2}(M)$ is any integral cohomology class, then

$$\int_M \omega' \wedge \eta = \frac{1}{m} \int_{\tilde{M}} \omega \wedge \pi^* \eta \in \mathbb{Q},$$

where m is the number of sheets of the cover. Thus $[\omega']$ is rational.

DEFINITION. We say that a line bundle $L \to M$ over an algebraic variety is *very ample* if $H^0(M, \mathcal{O}(L))$ gives an embedding $M \to \mathbb{P}^N$, i.e., if there exists an embedding $f: M \hookrightarrow \mathbb{P}^N$ such that $L = f^* H$.

Now from the proof of the Kodaira embedding theorem, we see

Corollary. *If* $E \to M$ *is any line bundle and* $L \to M$ *a positive line bundle, then for* $k \gg 0$, *the bundle* $L^k + E$ *is very ample.*

5. GRASSMANNIANS

Definitions; The Cell Decomposition and Schubert Cycles

In this section, we will construct and describe the Grassmannians, a fundamental family of compact complex manifolds. Grassmannians may be thought of as a generalization of projective space; the analogy will be apparent throughout.

Let V be a complex vector space of dimension n. The *Grassmannian* $G(k,V)$ is defined to be the set of k-dimensional linear subspaces of V; we write $G(k,n)$ for $G(k,\mathbb{C}^n)$. Given a k-plane Λ in $\mathbb{C}^n$, we may represent Λ by a set of k row vectors in $\mathbb{C}^n$ spanning Λ, i.e., by a $k\times n$ matrix

$$\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{kn} \end{pmatrix}$$

of rank k. Clearly any such matrix represents an element of $G(k,n)$ and any two such matrices A,A' represent the same element of $G(k,n)$ if and only if $\Lambda = g\Lambda'$ for some $g\in \mathrm{GL}_k$.

For every multiindex $I=\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}$ of cardinality k, let $V_{I^\circ}\subset \mathbb{C}^n$ be the $(n-k)$-plane in $\mathbb{C}^n$ spanned by the vectors $\{e_j: j\notin I\}$, and let

$$U_I = \{\Lambda\in G(k,n):\ \Lambda\cap V_{I^\circ}=\{0\}\};$$

U_I is just the set of $\Lambda\in G(k,n)$ such that the Ith $k\times k$ minor of one, and hence for any, matrix representation for Λ is nonsingular. Any $\Lambda\in U_I$ has a unique matrix representation Λ^I whose Ith $k\times k$ minor is the identity matrix, e.g., any $\Lambda\in U_{\{1,\ldots,k\}}$ can be represented uniquely by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & 1 & & & \vdots & * & & \\ \vdots & & & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & & & \cdots & 1 & * & \cdots & * \end{pmatrix}$$

(Note that the row vectors of such a matrix representative for $\Lambda\in U_I$ are just the points of intersection of Λ with the affine $(n-k)$-planes $\{V_{I^\circ}+e_j: j\in I\}$.) Conversely, any $k\times n$ matrix of the form above represents a k-plane $\Lambda\in U_I$; thus the $k(n-k)$ entries of the I°th $k\times(n-k)$ minor $\Lambda^I_{I^\circ}$ of Λ^I give a bijection of sets

$$\varphi_I:\ U_I\to\mathbb{C}^{k(n-k)}$$

for each I. Note that $\varphi_I(U_I \cap U_{I'})$ is open in $\mathbb{C}^{k(n-k)}$ for all I, I'; we claim that in fact the map $\varphi_I \circ \varphi_{I'}^{-1}$ is holomorphic on this open set and hence that the maps φ_I give $G(k,n)$ the structure of a complex manifold. But this is clear: if, for $\Lambda \in U_I \cap U_{I'}$, we let $\Lambda^I_{I'}$ be the I'th $k \times k$ minor of Λ^I, then

$$\Lambda^{I'} = (\Lambda^I_{I'})^{-1} \cdot \Lambda^I,$$

and since the entries of $(\Lambda^I_{I'})^{-1}$ vary holomorphically with the entries of Λ^I, $\varphi_I \circ \varphi_{I'}^{-1}$ is holomorphic.

With this topology $G(k,n)$ is compact and connected, since the unitary group U_n maps surjectively and continuously onto $G(k,n)$ by the map

$$g \mapsto g(V_k),$$

where $V_k = \{e_1, \ldots, e_k\} \subset \mathbb{C}^n$. The full linear group GL_n likewise acts transitively on $G(k,n)$.

Note in particular that $G(1,n)$ is biholomorphic to $\mathbb{P}^{n-1}$ as a complex manifold: the "matrix representative" $(v_1, \ldots, v_n)$ for a line $\Lambda \in G(1,n)$ corresponds, via the natural set-theoretic identification of $G(1,n)$ with $\mathbb{P}^{n-1}$, to the homogeneous coordinates of $\Lambda \in \mathbb{P}^{n-1}$, and

$$\Lambda^{(i)} = \left(\frac{v_1}{v_i}, \ldots, 1, \ldots, \frac{v_n}{v_i} \right),$$

so

$$\varphi_{\{i\}} = \Lambda \mapsto \left(\frac{v_1}{v_i}, \ldots, \frac{v_n}{v_i} \right),$$

i.e., the coordinates on $G(1,n)$ given by $\varphi_{\{i\}}$ are just the Euclidean coordinates on $\mathbb{P}^{n-1}$. Dually, we have $G(n-1,n) \simeq \mathbb{P}^{n-1*}$, the projective space of hyperplanes in $\mathbb{P}^{n-1}$.

Finally, we note that $G(k,n)$ can be considered either as the set of linear k-planes Λ in $\mathbb{C}^n$, or equivalently as the set of $(k-1)$-planes $\overline{\Lambda}$ in $\mathbb{P}^{n-1}$. Our viewpoint in this section will for the most part be the former, as it is easier to keep track of dimension and codimension of cycles, but when Grassmannians arise in geometric questions we will generally want to think of them in the latter way.

The Cell Decomposition

Recall that the cell decomposition

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^1 \cup \mathbb{C}^0$$

of $\mathbb{P}^n = G(1, n+1)$ is obtained by choosing a *flag*

$$V = \left(V_1 \subsetneqq \cdots \subsetneqq V_{n-1} \subsetneqq V_n \subsetneqq \mathbb{C}^{n+1} \right)$$

of linear subspaces of $\mathbb{C}^{n+1}$ and taking $W_i \cong \mathbb{C}^{i-1} = \{l \subset \mathbb{C}^{n+1} : l \subset V_i, l \not\subset V_{i-1}\}$. The same technique works to give a cell decomposition of the Grassmannian: if we set $V_i = \{e_1, \ldots, e_i\} \subset \mathbb{C}^n$, then the set of $\Lambda \in G(k,n)$ whose intersection with each V_i is of a specified dimension turns out, as we shall see, to be a simple cell. The set-up is as follows: for every $\Lambda \in G(k,n)$ consider the increasing sequence of subspaces

$$(*) \qquad 0 \subset \Lambda \cap V_1 \subset \Lambda \cap V_2 \subset \cdots \subset \Lambda \cap V_{n-1} \subset \Lambda \cap V_n = \Lambda.$$

For generic Λ, $\Lambda \cap V_i$ will be zero for $i \leqslant n-k$, and $(i+k-n)$-dimensional thereafter—indeed, we have seen that the set of such Λ is just the open set $U_{I^0} \cong \mathbb{C}^{k(n-k)} \subset G(k,n)$. Now, for any sequence of integers $a_1, \ldots, a_k$, set

$$W_{a_1,\ldots,a_k} = \{\Lambda \in G(k,n) : \dim(\Lambda \cap V_{n-k+i-a_i}) = i\}.$$

We observe that $\dim(\Lambda + V_{n-k+i-a_i}) = n - a_i$, and consequently $W_{a_1,\ldots,a_k}$ will be empty unless $a_1, \ldots, a_k$ is a nonincreasing sequence of integers $\leqslant n-k$. Since $\dim(\Lambda \cap V_{n-k+i-a_i}) = i$ if and only if the rank of the *last* $k \times (k + a_i - i)$ minor of a matrix representative for Λ is exactly $k-i$ it follows that the closure

$$\overline{W_{a_1,\ldots,a_k}} = \{\Lambda : \dim(\Lambda \cap V_{n-k+i-a_i}) \geqslant i\}$$

is an analytic subvariety of $G(k,n)$.

We can choose a special basis for a k-plane $\Lambda \in W_{a_1,\ldots,a_k}$ as follows: let v_1 be a generator for the line $\Lambda \cap V_{n-k+1-a_1}$, normalized so that $\langle v_1, e_{n-k+1-a_1} \rangle = 1$; i.e.,

$$v_1 = (*, *, \ldots, *, 1, 0, \ldots, 0).$$

Now take v_2 so that v_1 and v_2 together span $\Lambda \cap V_{n-k+2-a_2}$, normalized so that

$$\langle v_2, e_{n-k+1-a_1} \rangle = 0, \qquad \langle v_2, e_{n-k+2-a_2} \rangle = 1.$$

Continue in this way, choosing v_i so that $v_1, \ldots, v_i$ span $\Lambda \cap V_{n-k+i-a_i}$ and such that

$$\langle v_i, e_{n-k+j-a_j} \rangle = \begin{cases} 0, & j < i, \\ 1, & j = i. \end{cases}$$

Clearly, the choice of v_i at each stage is completely specified by these conditions; thus the k-plane Λ has a unique matrix representative of the form

$$
\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} =
\left(\begin{array}{cccccccccccccccccc}
 & & & & & & & & & \overset{n-k}{|} & & & & & & & & \\
* & * & * & 1 & 0 & 0 & \cdots & & & & & & & & & & \cdots & 0 \\
* & * & * & 0 & * & 1 & 0 & 0 & \cdots & & & & & & & & \cdots & 0 \\
* & * & * & 0 & * & 0 & * & * & * & 1 & 0 & 0 & & & & & & 0 \\
 & & & 0 & & 0 & & & & 0 & & & & & & & & \\
\vdots & & & \vdots & & \vdots & & & & \vdots & & & & & & & & \vdots \\
* & * & * & 0 & * & 0 & * & * & * & 0 & * & * & * & 1 & 0 & 0 & & 0 \\
 & & & 0 & & 0 & & & & 0 & & & & 0 & & & & \\
 & & & \vdots & & \vdots & & & & \vdots & & & & \vdots & & & & \vdots \\
* & * & * & 0 & * & 0 & * & * & * & 0 & * & \cdots & & 0 & \cdots & * & * \; * \; * & *
\end{array}\right)
$$

$$|\longleftarrow a_1 \longrightarrow| \qquad \longleftarrow a_2 \longrightarrow| \qquad |\leftarrow a_3 \rightarrow|$$

Conversely, any matrix of this form describes a k-plane $\Lambda \in W_{a_1,\ldots,a_k}$. Since $(k^2+\Sigma a_i)$ entries are specified in the diagram and the rest are completely free to vary, we have homeomorphisms

$$W_{a_1,\ldots,a_k} \cong \mathbb{C}^{k(n-k)-\Sigma a_i};$$

consequently, the sets $W_{a_1,\ldots,a_k}$ give a cell decomposition of $G(k,n)$. Since we have cells only in even dimensions, all boundary maps are zero, and we deduce the

Proposition. *The integral homology of the Grassmannian* $G(k,n)$ *has no torsion and is freely generated by the cycles* $\sigma_{a_1,\ldots,a_k} = [\overline{W_{a_1,\ldots,a_k}}]$ *in real codimension* $2\Sigma a_i$, *where* $\{(a_1,\ldots,a_k)\}$ *ranges over all nonincreasing sequences of integers between* 0 *and* $n-k$. *In particular, all cohomology in* $G(k,n)$ *is analytic.*

In general, for any flag $V=(V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n)$ in $\mathbb{C}^n$ we let

$$\sigma_a(V) = \{\Lambda : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}.$$

Clearly the homology class of the subvariety $\sigma_a(V)$ is independent of the flag chosen, since we can find a continuous family of linear automorphisms of $\mathbb{C}^n$ taking any flag into any other. The subvarieties $\sigma_a(V)$ are called the *Schubert cycles* of the Grassmannian.

The simplest example of a Grassmannian different from projective space is the $G(2,4)$ of 2-planes in $\mathbb{C}^4$. The Schubert cycles on $G(2,4)$ are

$$
\begin{aligned}
\text{codim } 1: &\quad \sigma_{1,0}(V_2)=\{\Lambda:\ \dim(\Lambda\cap V_2)\geqslant 1\},\\
\text{codim } 2: &\quad \sigma_{1,1}(V_3)=\{\Lambda:\ \Lambda\subset V_3\},\\
&\quad \sigma_{2,0}(V_1)=\{\Lambda:\ \Lambda\supset V_1\},\\
\text{codim } 3: &\quad \sigma_{2,1}(V_1,V_3)=\{\Lambda:\ V_1\subset\Lambda\subset V_3\}.
\end{aligned}
$$

Alternatively, if we think of $G(2,4)$ as the set of lines l in $\mathbb{P}^3$ and fix the projective flag $p\in l_0\subset h$ consisting of a point, line, and hyperplane in $\mathbb{P}^3$, then

$$
\begin{aligned}
\sigma_{1,0}(l_0) &= \{l:\ l\cap l_0\neq\varnothing\},\\
\sigma_{2,0}(p) &= \{l:\ p\in l\},\\
\sigma_{1,1}(h) &= \{l:\ l\in h\},\\
\sigma_{2,1}(p,h) &= \{l:\ p\in l\subset h\}.
\end{aligned}
$$

The Schubert Calculus

Now that we have determined the additive cohomology of $G(k,n)$, we would like to describe its multiplicative structure—that is, to express the intersection of general Schubert cycles σ_a,σ_b as a linear combination of other Schubert cycles in homology.

The first task is to write down the intersection pairing in complementary dimensions. To do this, let

$$\sigma_a(V)=\{\Lambda:\ \dim(\Lambda\cap V_{n-k+i-a_i})\geqslant i\}$$

and

$$\sigma_b(V')=\{\Lambda:\ \dim(\Lambda\cap V'_{n-k+i-b_i})\geqslant i\}$$

be general Schubert cycles. Then for each i and any $\Lambda\in\sigma_a(V)\cap\sigma_b(V')$,

$$
\begin{aligned}
&\dim(\Lambda\cap V_{n-k+i-a_i})\geqslant i,\\
&\dim(\Lambda\cap V'_{n-k+(k-i+1)-b_{k-i+1}})\geqslant k-i+1\\
&\Rightarrow \Lambda\cap V_{n-k+i-a_i}\cap V'_{n-i+1-b_{k-i+1}}\neq(0).
\end{aligned}
$$

But now if $a_i+b_{k-i+1}>n-k$, we have

$$
\begin{aligned}
(n-k+1-a_i)+(n-i+1-b_{k-i+1}) &= 2n-k+1-(a_i+b_{k-i+1})\\
&\leqslant n,
\end{aligned}
$$

and so we can choose our flags V and V' such that $V_{n-k+i-a_i}$ and $V'_{n-i+1-b_{k-i+1}}$ intersect only at the origin. Consequently the cycles $\sigma_a(V)$

and $\sigma_b(V')$ can be made disjoint, i.e.,

$$^{\#}(\sigma_a \cdot \sigma_b) = 0 \qquad \text{unless } a_i + b_{k-i+1} \leqslant n-k, \quad \text{for all } i.$$

Now suppose σ_a and σ_b are cycles of complementary dimension, so that

$$\sum a_i + \sum b_i = k(n-k);$$

then

$$a_i + b_{k-i+1} \leqslant n-k \text{ for all } i \to b_{k-i+1} = n-k-a_i,$$

i.e., the cycle σ_a has intersection number zero with all Schubert cycles in complementary dimension except $\sigma_{n-k-a_n,\ldots,n-k-a_1}$. Since the Schubert cycles form an integral basis for $H_*(G(k,n),\mathbb{Z})$, it follows either by Poincaré duality and the fact that analytic cycles intersect positively or by direct examination that

$$^{\#}(\sigma_a, \sigma_{n-k-a_n,\ldots,n-k-a_1}) = 1.$$

Summing up, then, we have the formula

$$^{\#}(\sigma_a \cdot \sigma_b) = \delta_{(a_1,\ldots,a_k)}^{(n-k-b_k,\ldots,n-k-b_1)}.$$

This enables us to express an arbitrary cycle γ on $G(k,n)$ as a linear combination of Schubert cycles, by computing intersections, i.e.,

$$\gamma = \sum {}^{\#}(\gamma \cdot \sigma_{n-k-a_k,\ldots,n-k-a_1}) \cdot \sigma_a,$$

and in particular reduces the problem of computing the intersection of pairs of Schubert cycles in arbitrary dimension to the problem of computing triple intersections in complementary dimension:

$$(\sigma_a \cdot \sigma_b) = \sum {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{n-k-c_k,\ldots,n-k-c_1}) \cdot \sigma_c.$$

As an example, for any hypersurface $W \subset \mathbb{P}^n$ of degree 2, let $\tau(W) \subset G(2,n+1)$ denote the set of lines in $\mathbb{P}^n$ lying on W. $\tau(W)$ is clearly an analytic cycle in $G(2,n+1)$, and since a line $l \subset \mathbb{P}^n$ lies on W if and only if three points of l lie on W, $\tau(W)$ has complex codimension 3. $G(2,n+1)$ has only two Schubert cycles of codimension 3—$\sigma_{3,0}$ and $\sigma_{2,1}$—and so we can write

$$\tau(W) = {}^{\#}(\tau(W) \cdot \sigma_{n-1,n-4}) \cdot \sigma_{3,0} + {}^{\#}(\tau(W) \cdot \sigma_{n-2,n-3}) \cdot \sigma_{2,1}.$$

Now, $\sigma_{n-1,n-4}$ is the set of lines in $\mathbb{P}^n$ containing a point p and contained in a 4-plane $V_4 \subset \mathbb{P}^n$; if we choose our point p to lie off W, clearly $\tau(W)$ will be disjoint from $\sigma_{n-1,n-4}$. On the other hand, $\sigma_{n-2,n-3}$ is the cycle of lines meeting a line $l_0 \subset \mathbb{P}^n$ and contained in a 3-plane $S \subset \mathbb{P}^n$ containing l_0. Generically, $W' = W \cap S$ will be a smooth quadric surface in $S \simeq \mathbb{P}^3$, with l_0 meeting it at two points p_1 and p_2; clearly any line $l \subset \tau(W) \cap \sigma_{n-2,n-3}$

will pass through either p_1 or p_2. But any line on W' through p_i must lie in the tangent plane $T_{p_i}(W')$; and $T_{p_i}(W')\cap W$ is a singular curve of degree 2, hence consists of two lines. Thus $\tau(W)$ meets $\sigma_{n-2,n-3}$ in four points generically, and so

$$\tau(W)\sim 4\cdot\sigma_{2,1}.$$

In particular, if W and W' are two generic quadric hypersurfaces in $\mathbf{P}^4$, meeting transversally in a smooth surface S, then by the above S will have

$$^{\#}(\tau(W)\cdot\tau(W'))_{G(2,5)} = {}^{\#}(4\sigma_{2,1}\cdot 4\sigma_{2,1})_{G(2,5)} = 16$$

lines in $\mathbb{P}^4$ lying on it. We will verify this in Section 4 of Chapter 4.

Similarly, we will be able to compute the homology class of $\tau(W)\subset G(2,n+1)$ for other hypersurfaces of low degree, once we know a few more things about special cases.

Before we go on to consider general intersections, we want to offer two general observations.

First, we will alter our formalism slightly, as follows: for *any* sequence $a=a_1,a_2,\dots$ of nonnegative integers, we let $\sigma_a(V)$ denote the cycle

$$\sigma_a(V)=\{\Lambda:\ \dim(\Lambda\cap V_{n-k+i-a_i})\geqslant i\}\subset G(k,n)$$

so that the symbol σ_a can be used to refer to a Schubert cycle in any Grassmannian. Of course, σ_a will be null in $G(k,n)$ unless $a_i\leqslant n-k$ for all i, $a_i=0$ for all $i>k$, and a is nonincreasing.

Now, the inclusion $\mathbb{C}^n\to\mathbb{C}^{n+1}$ induces inclusions

$$\iota_1:\ G(k,n)\to G(k,n+1)$$

and

$$\iota_2:\ G(k,n)\to G(k+1,n+1)$$

obtained by sending $\Lambda\subset\mathbb{C}^n$ to $\Lambda\subset\mathbb{C}^{n+1}$ and $\Lambda\oplus\{e_{n+1}\}\subset\mathbb{C}^{n+1}$, respectively. Under these inclusions, it is not hard to see that for appropriate choices of flags V in $\mathbb{C}^n$ and V' in $\mathbb{C}^{n+1}$,

$$\sigma_a(V)=\iota_1^{-1}(\sigma_a(V'))=\iota_2^{-1}(\sigma_a(V')),$$

i.e., if we denote the Poincaré dual of σ_a by $\tilde{\sigma}_a$,

$$\iota_1^*\tilde{\sigma}_a=\iota_2^*\tilde{\sigma}_a=\tilde{\sigma}_a.$$

Thus any formula

$$(\sigma_a\cdot\sigma_b)=\sum n_c\cdot\sigma_c$$

for the intersection of Schubert cycles in $G(k,n+1)$ or $G(k+1,n+1)$ holds as well in $G(k,n)$, and we can define the *universal Schubert coefficients*

$\delta(a,b;c)$ to be such that the formula

$$(\sigma_a \cdot \sigma_b) = \sum \delta(a,b;c) \cdot \sigma_c$$

holds in all $G(k,n)$.

Note that by our first computation, we have

$$\delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{n-k-c_k,\ldots,n-k-c_1})_{G(k,n)}$$

for any k,n such that σ_c is nonnull in $G(k,n)$, i.e., such that $c_i \leqslant n-k$ for all i and $c_i = 0$ for all $i > k$. In particular, if we let $l(c)$ denote the length of the sequence c, that is, the number of nonzero entries, we may take $k = l(c)$, $n-k = c_1$ in the above to obtain

$$(*) \qquad \delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{c_1-c_k,\ldots,c_1-c_2}) \qquad \text{in } G(l(c), l(c)+c_1).$$

As an immediate consequence, we see that $\delta(a,b;c)=0$ if σ_a or σ_b is null in $G(l(c), l(c)+c_1)$, i.e., $\delta(a,b;c)=0$ if either

1. $c_1 < a_1$ or $c_1 < b_1$, or
2. $l(c) < l(a)$ or $l(c) < l(b)$.

Next, note that for any vector space W of dimension n, we have a natural isomorphism

$$*: \; G(k,W) \longrightarrow G(n-k, W^*)$$

defined by

$$*\Lambda = \mathrm{Ann}(\Lambda) = \{ l \subset V^* : \; l(\Lambda) = 0 \}.$$

Let $V = \{ V_1 \subset V_2 \subset \cdots \subset V_n = W \}$ be a flag in W, and let $V^* = \{ V_1^* \subset V_2^* \subset \cdots \subset V_n^* = W^* \}$ be the dual flag in W^* given by

$$V_i^* = \mathrm{Ann}(V_{n-i}).$$

By linear algebra, for Λ any k-plane in W,

$$\dim(\Lambda \cap V_{n-k+i-a_i}) \geqslant i \Longleftrightarrow \dim(*\Lambda \cap V^*_{k-i+a_i}) \geqslant a_i,$$

Thus, *for any* a, *the image* $*\sigma_a \subset G(n-k,n)$ *of the Schubert cycle* $\sigma_a \subset G(k,n)$ *is the Schubert cycle* a^*, *where* a^* *is defined to be the smallest nonincreasing sequence such that*

$$a^*_{a_i} \geqslant i \qquad \text{for all } i.$$

For example,

$$*(\sigma_2) = \sigma_{1,1}, \qquad *(\sigma_{2,1,1}) = \sigma_{3,1}.$$

In general, we will have

$$\delta(a,b;c) = \delta(a^*, b^*; c^*),$$

and so we may expect that any formula for the intersection of Schubert cycles σ_a, σ_b gives a dual formula, when applied to $\sigma_{a^*}, \sigma_{b^*}$.

Note that

$$l(a^*) = a_1 \quad \text{and} \quad a_1^* = l(a)$$

so that the formulas 1 and 2 above are, as expected, equivalent under the $*$ map.

We turn now to the original problem of computing $\delta(a,b;c)$ for general a, b, and c. We will first give a reduction that allows us to compute effectively in many cases.

Our basic technique is simply a linear algebra reduction to smaller Grassmannians. For example, consider a triple of indices α, β, γ such that $\alpha+\beta+\gamma = 2k+1$. Then for any k-plane $\Lambda \in \sigma_a(V) \cap \sigma_b(V') \cap \sigma_c(V'')$,

$$\begin{aligned} &\dim\left(\Lambda \cap V_{n-k+\alpha-a_\alpha}\right) \geqslant \alpha, \\ &\dim\left(\Lambda \cap V'_{n-k+\beta-b_\beta}\right) \geqslant \beta, \\ &\dim\left(\Lambda \cap V''_{n-k+\gamma-c_\gamma}\right) \geqslant \gamma \\ \Rightarrow &\dim\left(\Lambda \cap V_{n-k+\alpha-a_\alpha} \cap V'_{n-k+\beta-b_\beta} \cap V''_{n-k+\gamma-c_\gamma}\right) \geqslant 1. \end{aligned}$$

Thus $^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_c) = 0$ in $G(k,n)$ if

$$(k-\alpha+a_\alpha)+(k-\beta+b_\beta)+(k-\gamma+c_\gamma) > n-1,$$

i.e., if

$$a_\alpha + b_\beta + c_\gamma > n-k.$$

Suppose on the other hand that $a_\alpha + b_\beta + c_\gamma = n-k$, i.e., that generically chosen subspaces $V_{n-k+\alpha-a_\alpha}$, $V'_{n-k+\beta-b_\beta}$, and $V''_{n-k+\gamma-c_\gamma}$ will intersect in a line $L \subset \mathbb{C}^n$. Then any $\Lambda \in \sigma_a(V) \cap \sigma_b(V') \cap \sigma_c(V'')$ must contain L. Let L^0 denote a subspace complementary to L in $\mathbb{C}^n$ and let π denote the projection of $\mathbb{C}^n$ onto L^0 with kernel L. Let

$$\begin{aligned} \bar{V}_1 &= \pi(V_1), \\ &\vdots \\ \bar{V}_{n-k+\alpha-a_\alpha-1} &= \pi(V_{n-k+\alpha-a_\alpha-1}) = \pi(V_{n-k+\alpha-a_\alpha}), \\ &\vdots \\ \bar{V}_{n-2} &= \pi(V_{n-1}), \\ \bar{V}_{n-1} &= \pi(V_n) = L^0, \end{aligned}$$

and define $\bar{V}_i'$ and $\bar{V}_i''$ similarly. Then $\bar{V} = \{\bar{V}_i\}$, $\bar{V}' = \{\bar{V}_i'\}$, and $\bar{V}'' = \{\bar{V}_i''\}$

are transverse flags in L^0, and for any $(k-1)$-plane $\overline{\Lambda} \subset L^0$, we see that

$$\begin{aligned}\Lambda = \overline{L,\overline{\Lambda}} &\in \sigma_a(V) \cap \sigma_b(V') \cap \sigma_c(V'') \\ &\Leftrightarrow \overline{\Lambda} \in \sigma_{a_1,\ldots,\hat{a}_\alpha,\ldots,a_k}(\overline{V}) \cap \sigma_{b_1,\ldots,\hat{b}_\beta,\ldots,b_k}(\overline{V}') \cap \sigma_{c_1,\ldots,\hat{c}_\gamma,\ldots,c_k}(\overline{V}'').\end{aligned}$$

Thus we have the

Reduction Formula I. *For any three indices* $0 \leqslant \alpha,\beta,\gamma \leqslant \mathrm{k}$ *with* $\alpha+\beta+\gamma = 2\mathrm{k}+1$,

$$^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_c)_{G(k,n)} = \begin{cases} 0 & \textit{if } a_\alpha + b_\beta + c_\gamma > n-k, \\ ^{\#}(\sigma_{a-a_\alpha} \cdot \sigma_{b-b_\beta} \cdot \sigma_{c-c_\gamma})_{G(k-1,n-1)} & \textit{if } a_\alpha + b_\beta + c_\gamma = n-k. \end{cases}$$

Note that in case we take $\beta = \gamma = k$, this reduction applies if $a_1 = n-k$; in case we take $\gamma = k$, it applies if $a_i + b_{k+1-i} = n-k$ for any i.

As suggested, we can apply this first reduction to the intersection of cycles $^{\#}(\sigma_{a^*} \cdot \sigma_{b^*} \cdot \sigma_{c^*})$ in $G(n-k,n)$; we obtain

Reduction Formula II. *For any three coefficients* $\mathrm{a}_\alpha, \mathrm{b}_\beta, \mathrm{c}_\gamma$ *with* $\mathrm{a}_\alpha + \mathrm{b}_\beta + \mathrm{c}_\gamma \geqslant 2(\mathrm{n}-\mathrm{k})+1$,

$$\begin{aligned} &^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_c)_{G(k,n)} \\ &= \begin{cases} 0 \qquad \textit{if } \alpha+\beta+\gamma > k \\ ^{\#}(\sigma_{a_1-1,\ldots,a_\alpha-1,a_{\alpha+1},\ldots,a_k} \cdot \sigma_{b_1-1,\ldots,b_\beta-1,b_{\beta+1},\ldots,b_k} \cdot \sigma_{c_1-1,\ldots,c_\gamma-1,c_{\gamma+1},\ldots,c_k})_{G(k,n-1)} \\ \qquad \textit{if } \alpha+\beta+\gamma = k. \end{cases}\end{aligned}$$

For the purposes of this formula, we may set $a_0 = b_0 = c_0 = n-k$ formally; thus in case we take $\gamma = \beta = 0$, this reduction applies if $a_k \neq 0$, and if we take $\gamma = 0$, it applies in case $a_i + b_{k-i} \geqslant n-k+1$ for some i.

Note also that if the sequence $b_1 - 1, \ldots, b_\beta - 1, b_{\beta+1}, \ldots$ appearing in the formula is no longer nonincreasing—i.e., if $b_\beta = b_{\beta+1}$—then the intersection number is zero: just apply the formula to $\alpha, \beta+1, \gamma$. Thus we may use the formula in all circumstances, if we adopt the convention that σ_b is null for b not a nonincreasing sequence.

As a sample calculation, we compute the coefficient $\delta(311,21;521)$ of σ_{521} in the expression for $(\sigma_{311} \cdot \sigma_{21})$ as a linear combination of Schubert cycles. By $(*)$ and the reductions we have

$$\begin{aligned} \delta(311,21;521) &= {}^{\#}(\sigma_{311} \cdot \sigma_{21} \cdot \sigma_{43}) && \text{in } G(3,8) \\ &= {}^{\#}(\sigma_2 \cdot \sigma_{21} \cdot \sigma_{43}) && \text{in } G(3,7) \\ &= {}^{\#}(\sigma_2 \cdot \sigma_{21} \cdot \sigma_3) && \text{in } G(2,6) \\ &= {}^{\#}(\sigma_2 \cdot \sigma_1 \cdot \sigma_3) && \text{in } G(2,5) \\ &= {}^{\#}(\sigma_2 \cdot \sigma_1) && \text{in } G(1,4) = \mathbb{P}^3 \\ &= 1. \end{aligned}$$

The two formulas given here will not apply every time, but in low codimension will yield the answer more often than not. They work especially well in case one of the factors σ_a is a *special Schubert cycle*, defined to be one of the form $\sigma_{a,0,0,\ldots}$. In this case, we can use the reductions to obtain the general

Pieri's Formula. *If* a = a, 0, 0, …, *then for any* b,

$$(\sigma_a \cdot \sigma_b) = \sum_{\substack{b_i \leqslant c_i \leqslant b_{i-1} \\ \Sigma c_i = a + \Sigma b_i}} \sigma_c.$$

Proof. We want to show that, for σ_c of appropriate codimension,

$$\delta(a,b;c) = \begin{cases} 1, & \text{if } b_i \leqslant c_i \leqslant b_{i-1}, \\ 0, & \text{otherwise.} \end{cases}$$

We have, setting $k = l(c)$,

$$\delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{c_1 - c_k, \ldots, c_1 - c_2, 0}) \qquad \text{in } G(k, k+c_1).$$

To start, suppose that $c_i < b_{i-1}$ for some i. Then we have

$$c_i + b_{i-1} + (c_1 - c_i) \geqslant 2c_1 + 1,$$

and applying the second reduction formula with $\sigma = 0$, $\beta = i-1$, and $\gamma = k - i + 1$, we obtain

$$\delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_{b_1 - 1, \ldots, b_{i-1} - 1, b_i, \ldots} \cdot \sigma_{c_1 - c_k - 1, \ldots, c_1 - c_i - 1, c_1 - c_{i+1}, \ldots}) \qquad \text{in } G(k, k+c_1-1)$$

$$= \delta(a, b'; c')$$

where

$$b' = b_1 - 1, \ldots, b_{i-1} - 1, b_i, \ldots$$

and

$$c' = c_1 - 1, \ldots, c_{i-1} - 1, c_i, \ldots.$$

Now

$$(b_i \leqslant c_i \leqslant b_{i-1} \text{ for all } i) \Leftrightarrow (b'_i \leqslant c'_i \leqslant b'_{i-1} \text{ for all } i)$$

and of course

$$b'_{i-1} - c'_i = b_{i-1} - c_i - 1 \geqslant 0.$$

Thus we may assume from the start that $c_i \geqslant b_{i-1}$ for all i. Since $\Sigma c_i = a + \Sigma b_i$, it follows that $a \geqslant c_1$; and so there are three cases:

1. If $c_i > b_{i-1}$ for some i, then $a > c_1$ and so $\delta(a,b;c) = 0$.
2. If $c_i < b_i$ for any i, then $c_i \geqslant b_{i-1}$ implies that $b_i > b_{i-1}$, i.e., the sequence b is not nonincreasing and σ_b is taken to be null; so $\delta(a,b;c) = 0$.
3. If $b_i \leqslant c_i \leqslant b_{i-1}$ for all i, it follows that $c_i = b_{i-1}$ for all i, hence $a = c_1, b_k = 0$, and applying the first reduction formula with $\alpha = 1$ and

$\beta = \gamma = k$ we have

$$\begin{aligned}\delta(a,b;c) &= {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{c_1-c_k,\ldots,c_1-c_2,0}) &&\text{in } G(k,k+c_1)\\ &= {}^{\#}(\sigma_b \cdot \sigma_{c_1-c_k,\ldots,c_1-c_2}) &&\text{in } G(k-1,k+c_1-1)\\ &= {}^{\#}(\sigma_b \cdot \sigma_{c_1-b_{k-1},\ldots,c_1-b_1})\\ &= 1.\end{aligned}$$

Q.E.D.

Our final result on Schubert cycles is a formula that expresses the general Schubert cycle as a polynomial in the special Schubert cycles $\sigma_{b,0,\ldots}$.

We proceed as follows: for $\sigma_{a_1,\ldots,a_d}$ any Schubert cycle, we consider the cycle

$$(*) \qquad \tilde{\sigma}_a = \sum_{j=1}^{d} (-1)^j \sigma_{a_1,\ldots,a_{j-1},a_{j+1}-1,\ldots,a_d-1} \cdot \sigma_{a_j+d-j}.$$

Note that $\tilde{\sigma}_a$ has the same dimension as σ_a. Now, we can by Pieri's formula write out each of the intersections in the sum $(*)$ as a sum of Schubert cycles. Let $\sigma_{c_1,\ldots,c_d}$ be any Schubert cycle; if σ_c appears in this expression, consider the sequence of integers

$$c_1 - 1, c_2 - 2, \ldots, c_d - d.$$

By Pieri, at most one of these numbers will lie in each of the $(d+1)$ closed intervals

$$\begin{gathered}[a_1 - 1, n-k],\\ [a_2 - 2, a_1 - 2],\\ \vdots\\ [a_d - d, a_{d-1} - d],\\ [-d-1, a_d - d - 1],\end{gathered}$$

and so exactly one of these intervals will fail to contain an integer $c_i - i$. By cases, then:

1. If no integer $c_i - i$ lies in the interval $[-d-1, a_d - d - 1]$, then
$$c_i - i \in [a_i - i, a_{i-1} - i],$$

and σ_c can appear only in the last term of the sum $(*)$. But since

$$c_i \geq a_i \quad \text{and} \quad \sum c_i = \sum a_i,$$

it follows that $c = a$. The Schubert cycle $\sigma_{a_1,\ldots,a_d}$ thus appears once in $(*)$, with coefficient $(-1)^d$.

2. If no integer $c_i - i$ appears in the interval $[a_k - k, a_{k-1} - k]$, then we have

$$\begin{aligned} c_1 - 1 &\in [a_1 - 1, n - k], \\ &\vdots \\ c_{k-1} - k + 1 &\in [a_{k-1} - k + 1, a_k - k + 1], \\ c_k - k &\in [a_{k+1} - k - 1, a_k - k - 1], \\ &\vdots \\ c_d - d &\in [-d - 1, a_d - d - 1], \end{aligned}$$

i.e.,

$$a_i \leqslant c_i \leqslant a_{i-1}, \qquad i = 1, \dots, k-1,$$

and

$$a_{i+1} - 1 \leqslant c_i \leqslant a_i - 1, \qquad i = k, \dots, d.$$

In this case, the Schubert cycle σ_c will appear twice in the expression for (*): once in the kth term, and once in the $(k-1)$st term. Since these two have opposite sign, σ_c will not appear in the final expression for $\tilde{\sigma}_a$.

3. If the interval $[a_1 - 1, n - k]$ is unoccupied, we have

$$c_i - i \in [a_{i+1} - i - 1, a_i - i - 1]$$

for each i—but then $c_i \leqslant a_i - 1$, and hence $\sum c_i < \sum a_i$, so σ_c cannot appear in (*).

We have, then, the formula

$$(**) \qquad (-1)^d \sigma_{a_1, \dots, a_d} = \sum_{j=1}^{d} (-1)^j \sigma_{a_1, \dots, a_{j-1}, a_{j+1}-1, \dots, a_d - 1} \cdot \sigma_{a_j + d - j}.$$

Note that since each factor on the right has length $< d$, this already implies that σ_a is expressible as a polynomial in the special Schubert cycles $\sigma_{b,0,\dots}$, i.e., that

The cohomology ring of the Grassmannian G(k, n) *is generated by the classes of the special Schubert cycles.*

Now, we will use the relation (**) to prove *Giambelli's formula*

$$\sigma_{a_1, \dots, a_d} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+d-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+d-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & & \\ \vdots & & & & \vdots \\ \sigma_{a_d-d+1} & & \cdots & & \sigma_{a_d} \end{vmatrix}.$$

We will prove this by induction; clearly it is true for $d=1$. Assume that it holds for $d-1$; expanding by cofactors along the left-hand row, the determinant is given by

$$\sum(-1)^j \sigma_{a_j+d-j}\cdot\begin{vmatrix} \sigma_{a_1} & \cdots & \sigma_{a_1+d-2} \\ \vdots & & \vdots \\ \sigma_{a_{j-1}-j} & \cdots & \sigma_{a_{j-1}+d-j} \\ \sigma_{a_{j+1}-j-2} & \cdots & \sigma_{a_{j+1}+d-j-2} \\ \vdots & & \vdots \\ \sigma_{a_d-d+1} & \cdots & \sigma_{a_d-1} \end{vmatrix}$$

$$\begin{aligned} &= \sum(-1)^j \sigma_{a_j+d-j}\cdot\sigma_{a_1,\ldots,a_{j-1},a_{j+1}-1,\ldots,a_d-1} \\ &= \sigma_{a_1,\ldots,a_d}, \end{aligned}$$

and the formula is proved. Q.E.D.

Note that Pieri's formula together with the formula (**) give an algorithm for evaluating an arbitrary intersection of Schubert cycles.

The Schubert calculus will appear frequently in the remainder of the book, in a variety of contexts; for the time being we give some applications of our formulas to elementary problems in enumerative geometry. Perhaps the simplest nontrivial such problem is the question: given four lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$ in general position, how many lines meet all four? The answer is easily obtained: since the set of lines meeting L_i is just the Schubert cycle $\sigma_1(L_i)$, the answer is just the fourfold self-intersection number of σ_1 in $G(2,4)$; this is

$$\begin{aligned} \sigma_1^4 &= \sigma_1^2\cdot(\sigma_{1,1}+\sigma_2) \\ &= \sigma_1\cdot(2\sigma_{2,1}) \\ &= 2. \end{aligned}$$

In general, the number of lines meeting four $(n+1)$-planes in general position in $\mathbb{P}^{2n+1}$ is given by the fourfold self-intersection of σ_n in $G(2,2n+2)$; this is

$$\begin{aligned} (\sigma_n)^4 &= \left(\sigma_n^2\right)^2 \\ &= \left(\sum_{i=0}^{n}\sigma_{2n-i,i}\right)^2 \\ &= n+1. \end{aligned}$$

In a similar vein, the number of lines in $\mathbb{P}^4$ meeting six 2-planes in general position is given by σ_1^6 in $G(2,5)$; we have

$$\sigma_1^3 = \sigma_1(\sigma_{1,1}+\sigma_2) = 2\sigma_{2,1}+\sigma_3,$$

so

$$\sigma_1^6 = (2\sigma_{2,1}+\sigma_3)^2 = 4+1 = 5.$$

Universal Bundles

Let $\mathbb{C}^n \times G(k,n)$ denote the trivial vector bundle of rank n over $G(k,n)$. We define the *universal subbundle* $S \to G(k,n)$ to be the subbundle of $\mathbb{C}^n \times G(k,n)$ whose fiber at each point $\Lambda \in G(k,n)$ is just the subspace $\Lambda \subset \mathbb{C}^n$. S is clearly a holomorphic subbundle of $\mathbb{C}^n \times G(k,n)$—explicitly, in each open $U_I \subset G(k,n)$ the row vectors of the normalized matrix representatives for $\Lambda \in U_I$ give a frame for S over U_I; transition functions relative to these frames are given in $U_I \cap U_{I'}$ by $g_{U_I U_{I'}} = \Lambda_I \cdot \Lambda_{I'}^{-1}$. The quotient bundle $Q = \mathbb{C}^n / S$ is called the *universal quotient bundle* on $G(k,n)$. Note that under the identification $*: G(k,n) \to G(n-k,n)$, the universal subbundle on $G(n-k,n)$ corresponds to the *dual* of the universal quotient bundle in $G(k,n)$, and likewise $Q \to G(n-k,n)$ pulls back to the dual $S^* \to G(k,n)$. Note in particular that the universal subbundle $S \to G(1,n) \cong \mathbb{P}^{n-1}$ is just the universal line bundle mentioned earlier.

Now let $E \to M$ be any holomorphic vector bundle of rank k on a complex manifold M, $V \subset H^0(M, \mathcal{O}(E))$ an n-dimensional vector space of global holomorphic sections, and suppose that the values $\{\sigma(x)\}_{\sigma \in V}$ of the sections σ in V span E_x for all $x \in M$. Then for each $x \in M$, the subspace $\Lambda_x \subset V$ of sections $\sigma \in V$ vanishing at x is an $(n-k)$-dimensional subspace; accordingly, we obtain a map

$$\iota_V: \ M \to G(n-k, V) = G(k, V^*)$$

with

$$E = \iota_V^* S^* \quad \text{and} \quad V = \iota_V^*\big(H^0(G(k,n), \mathcal{O}(S^*))\big)$$

just as for line bundles. Explicitly, if we choose a basis $\sigma_1, \dots, \sigma_n$ for V and a frame $e_1, \dots, e_k$ for E locally and write

$$\sigma_i = \sum a_{i\alpha} e_\alpha,$$

then in terms of the corresponding identification $G(n-k, V) \cong G(k, V^*)$ the map ι_V is given by

$$x \mapsto \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix},$$

so that ι_V is clearly holomorphic.

As in the case of line bundles, we have an embedding theorem:

Theorem. *For* M *any compact complex manifold,* L→M *a positive line bundle and* E→M *any holomorphic vector bundle, then for* m *sufficiently large, the map* $\iota_{E \otimes L^m}$ *is an embedding.*

Proof. Most of the work has been done for us already by the Kodaira embedding theorem: since M has a positive line bundle, we may take

$M \subset \mathbb{P}^N$ an algebraic variety and $L \to M$ the hyperplane bundle.

Now $\iota_{E\otimes L^m}$ will be 1-1 if for all $x, y \in M$, the restriction map

$$(*) \qquad H^0(M, \mathcal{O}(E\otimes L^m)) \to (E\otimes L^m)_x \oplus (E\otimes L^m)_y$$

is surjective. Similarly, we have a differential map

$$(**) \qquad H^0(M, \mathcal{I}_x(E\otimes L^m)) \to T_x^{*\prime} \otimes (E\otimes L^m)_x$$

defined as for line bundles; $\iota_{E\otimes L^m}$ will be smooth at x if this map is surjective. The compactness argument used in the proof of the Kodaira embedding theorem again assures us that to prove the result, it is sufficient to show that for any particular choice of x and y, the above two maps are surjective for m sufficiently large.

We proceed by induction on the dimension of M. For any $x,y \in M$, consider the linear system of hyperplane sections of $M \subset \mathbb{P}^N$ containing x and y: by Bertini's theorem, the generic element of this system is smooth outside the base locus $\{x,y\}$ of the system, and it is easy to see that, unless M is a curve with $T_x(M) = T_y(M) \subset \mathbb{P}^N$ (which circumstance we can always avoid by embedding M differently), the generic element of the system will be smooth at x and y as well. Thus we can find a smooth hyperplane section $V = H \cap M$ of M containing x and y. Consider the sequence

$$0 \to \mathcal{O}_M(E\otimes L^{m-1}) \to \mathcal{O}_M(E\otimes L^m) \to \mathcal{O}_V(E\otimes L^m) \to 0.$$

By Theorem B, there exists m_1 such that for $m > m_1$, $H^1(M, \mathcal{O}(E\otimes L^{m-1})) = 0$, so that the restriction map

$$H^0(M, \mathcal{O}(E\otimes L^m)) \to H^0(V, \mathcal{O}(E\otimes L^m))$$

will be surjective. On the other hand, by induction there exists m_2 such that for $m > m_2$,

$$H^0(V, \mathcal{O}_V(E\otimes L^m)) \to (E\otimes L^m)_x \oplus (E\otimes L^m)_y$$

is surjective. For $m > m_0 = \max(m_1, m_2)$, then, the map $(*)$ will be surjective.

Similarly, for each of a generating set of cotangent vectors $\{\omega_\alpha\}$ for $T_x^{*\prime}$ we can find a smooth hyperplane section V_α of M through x, such that ω_α is not in the kernel of the natural projection map $T_x^{*\prime}(M) \to T_x^{*\prime}(V_\alpha)$. Then by induction we can find m_α such that for $m > m_\alpha$, the differential map

$$H^0(V_\alpha, \mathcal{I}_x(E\otimes L^m)) \to T_x^{*\prime}(V_\alpha) \otimes (E\otimes L^m)_x$$

is surjective. Likewise, from the sequence

$$0 \to \mathcal{O}_M(E\otimes L^{m-1}) \to \mathcal{I}_{x,M}(E\otimes L^m) \to \mathcal{I}_{x,V_\alpha}(E\otimes L^m) \to 0$$

we see that for $m > m_1$ as before,

$$H^0(M, \mathcal{I}_x(E\otimes L^m)) \to H^0(V_\alpha, \mathcal{I}_x(E\otimes L^m))$$

is surjective. Thus for $m > m_0' = \max(m_1, m_\alpha)$, we have

$$\begin{array}{ccc} H^0(M, \mathcal{I}_x(E\otimes L^m)) & \xrightarrow{d_x} & T_x^{*\prime}(M)\otimes(E\otimes L^m)_x \\ \downarrow & & \downarrow \\ H^0(V_\alpha, \mathcal{I}_x(E\otimes L^m)) & \xrightarrow{d_x} & T_x^{*\prime}(V_\alpha)\otimes(E\otimes L^m)_x \end{array}$$

for all α, i.e., the map (**) is surjective. Q.E.D.

The Plücker Embedding

We close this section by describing the classical Plücker embedding of the Grassmannian $G(k,n)$ in projective space; this will illustrate both the Kodaira embedding theorem and Chow's theorem. The embedding line bundle over $G(k,n)$ will be $L = \det S^* = \det Q$. L may be seen to be positive by introducing a suitable metric with a positive curvature form in a similar manner to the Fubini-Study metric on projective space; rather than do this, however, we shall give the Plücker embedding directly. The *Plücker map*

$$p: \ G(k,n) \to \mathbb{P}(\wedge^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$$

simply sends a k-plane $\Lambda = \mathbb{C}\{v_1, \ldots, v_k\} \subset \mathbb{C}^n$ to the multivector $v_1 \wedge \cdots \wedge v_k$. Explicitly, in terms of the basis $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{\#I=k}$ for $\wedge^k \mathbb{C}^n$, this map is given by

$$\Lambda \mapsto [\ldots, |\Lambda_I|, \ldots],$$

i.e., the homogeneous coordinates of the map are just the determinants $|\Lambda_I|$ of all the $k\times k$ minors Λ_I of a matrix representative of Λ. It follows that (1) p is holomorphic, (2) p takes every Schubert cycle of the form

$$\sigma_1(V) = \{\Lambda \in G(k,n): \ \dim(\Lambda \cap V_{n-k}) \geqslant 1\}$$

into a hyperplane section of $p(G(k,n)) \subset \mathbb{P}^{\binom{n}{k}-1}$. We can always find, for $\Lambda \neq \Lambda' \in G(k,n)$, an $(n-k)$-plane V_{n-k} such that $\Lambda \cap V_{n-k} \neq (0)$, $\Lambda' \cap V_{n-k} = (0)$, so p is 1-1; and since, in each open set $U_I = \{\Lambda : |\Lambda_I| \neq 0\}$ the Euclidean coordinates on $G(k,n)$ described above appear as

$$a_{jk} = \frac{|\Lambda_{I-j+k}|}{|\Lambda_I|},$$

the map p has nonzero differential. Thus the Plücker mapping is an embedding.

Now we shall determine equations which define the Plücker image of $G(k, V)$ in $\mathbb{P}(\wedge^k V)$. What we are asking for are the conditions that a multivector $\Lambda \in \bigwedge^k V$ be *decomposable*, i.e., of the form

$$\Lambda = v_1 \wedge \cdots \wedge v_k.$$

For this we pose the more general problem of determining the minimal linear subspace $W \subset V$ such that Λ is in the image of

$$\bigwedge^k W \to \bigwedge^k V.$$

If $\dim W = l$, then $l \geq k$ with equality holding if and only if Λ is decomposable.

Recall the contraction operator

$$i(v^*): \bigwedge^k V \to \bigwedge^{k-1} V$$

defined for $v^* \in V^*$ by

$$\langle i(v^*)\Lambda, \Xi \rangle = \langle \Lambda, v^* \wedge \Xi \rangle$$

for all $\Xi \in (\bigwedge^{k-1} V)^* \cong \bigwedge^{k-1} V^*$. We associate to Λ the linear spaces

$$\Lambda^{\perp} = \{ v^* \in V^* : i(v^*)\Lambda = 0 \} \subset V^*$$

and

$$W = \operatorname{Ann}(\Lambda^{\perp}) \subset V.$$

Lemma. *W is the minimal subspace of V such that Λ is in the image of $\bigwedge^k W \to \bigwedge^k V$.*

Proof. Let $w_1, \ldots, w_l$ be a basis for W, and complete it by $u_{l+1}, \ldots, u_n$ to a basis for V. Denote the dual basis of V^* by $\{w_i^*, u_\alpha^*\}$. Setting $U = \mathbb{C}\{u_{l+1}, \ldots, u_n\}$, the direct sum decomposition $V = W \oplus U$ induces

$$\bigwedge^k V \simeq \bigwedge^k W \oplus (\bigwedge^{k-1} W \otimes U) \oplus (\bigwedge^{k-2} W \otimes \bigwedge^2 U) \oplus \cdots.$$

We want to show that Λ lies in the first factor. Write the component of Λ in the second factor as $\Sigma_{\alpha=l+1}^n \Lambda_\alpha \otimes u_\alpha$, where $\Lambda_\alpha \in \bigwedge^{k-1} W$. Since

$$i(u_\alpha^*): \bigwedge^{k-m} W \otimes \bigwedge^m U \to \bigwedge^{k-m} W \otimes \bigwedge^{m-1} U$$

and $i(u_\alpha^*)\Lambda = 0$, we deduce that all $\Lambda_\alpha = 0$. Similarly, the other factors of Λ in $\bigwedge^{k-m} W \otimes \bigwedge^m U$ ($m \geq 2$) are zero, and consequently $\Lambda \in \bigwedge^k W$.

It is easy to see that W is the minimal such subspace. Q.E.D.

We now define

$$W' = \{ w \in W : w \wedge \Lambda = 0 \}.$$

If Λ is decomposable, then clearly $W' = W$. Conversely, if Λ is not decomposable so that $\dim W = l > k$, then since the pairing $\bigwedge^k W \otimes \bigwedge^{l-k} W \to \bigwedge^l W$ is nondegenerate we deduce that $W' \neq W$. So Λ is decomposable if and only if $W' = W$.

We now express this condition by duality, in two ways. For the first we use the operator

$$i(\Xi): \bigwedge^k V \to V^*$$

defined for $\Xi\in\bigwedge^{k+1}V^*$ by

$$\langle i(\Xi)\Lambda, v\rangle = \langle \Xi, \Lambda\wedge v\rangle$$

for all $v\in V$. We observe that, by the definition of Λ^-, for $v\in W$ the left-hand side depends only on the image of Ξ under the natural projection

$$\bigwedge^{k+1}V^*\to\bigwedge^{k+1}\left(\frac{V^*}{\bigwedge^{\perp}}\right)\cong\Lambda^{k+1}W^*.$$

Consequently, the condition $\Lambda\wedge w=0$ for all $w\in W$ is equivalent to $i(\Xi)\Lambda\in\Lambda^{\perp}$ for all Ξ, which is in turn equivalent to

$$(*)\qquad i(i(\Xi)\Lambda)\Lambda = 0 \qquad \text{for all } \Xi\in\bigwedge^{k+1}V^*.$$

The left-hand side of $(*)$ gives $\binom{n}{k+1}$ quadratic forms in the homogeneous coordinates Λ_I of $p(G(k,V))$; setting them equal to zero gives the classical *Plücker relations*. In sum,

the image of the Grassmannian under the Plücker embedding $p: G(k,V)\to\mathbb{P}(\Lambda^k V)$ *is cut out by the linear system of quadrics given by* $(*)$.

Alternatively, we may characterize W as being the image of

$$\bigwedge^{k-1}V^*\to V$$

under the map

$$\Xi\to i(\Xi)\Lambda,\qquad \Xi\in\bigwedge^{k-1}V^*.$$

Then the condition $W'=W$ is equivalent to

$$(**)\qquad (i(\Xi)\Lambda)\wedge\Lambda = 0\qquad\text{for all }\Xi\in\bigwedge^{k-1}V^*.$$

For example, suppose that

$$\Lambda=\tfrac{1}{2}\sum_{i,j}\lambda_{ij}e_i\wedge e_j,\qquad \lambda_{ij}+\lambda_{ji}=0,$$

is a bivector. Since for $v^*\in V^*$

$$(i(v^*)\Lambda)\wedge\Lambda=\tfrac{1}{2}i(v^*)(\Lambda\wedge\Lambda),$$

we may rewrite the conditions $(**)$ as

$$\Lambda\wedge\Lambda=0.$$

When $n=4$ we find the single equation

$$\lambda_{12}\lambda_{34}-\lambda_{13}\lambda_{24}+\lambda_{14}\lambda_{23}=0$$

expressing the condition that $\Lambda\in\mathbb{P}(\bigwedge^2\mathbb{C}^4)\cong\mathbb{P}^5$ be decomposable. In other words, $G(2,4)$ *is naturally realized as a nonsingular quadric hypersurface in* $\mathbb{P}^5$. We will see more of this in the final chapter.