CHAPTER 1

Introduction to Electrostatics

We begin our discussion of electrodynamics with the subject of *electrostatics*: phenomena involving time-independent distributions of charge and fields. For most readers this material is in the nature of a review. In this chapter especially we do not elaborate significantly. We introduce concepts and definitions that are important for later discussion and present some essential mathematical apparatus. In subsequent chapters the mathematical techniques are developed and applied.

One point of physics should be mentioned. Historically, electrostatics developed as a science of *macroscopic* phenomena. As indicated at the end of the Introduction, such idealizations as point charges or electric fields at a point must be viewed as mathematical constructs that permit a description of the phenomena at the macroscopic level, but that may fail to have meaning microscopically.

1.1 Coulomb's Law

All of electrostatics stems from the quantitative statement of Coulomb's law concerning the force acting between charged bodies at rest with respect to each other. Coulomb, in an impressive series of experiments, showed experimentally that the force between two small charged bodies separated in air a distance large compared to their dimensions

varies directly as the magnitude of each charge,
varies inversely as the square of the distance between them,
is directed along the line joining the charges, and
is attractive if the bodies are oppositely charged and repulsive if the bodies have the same type of charge.

Furthermore it was shown experimentally that the total force produced on one small charged body by a number of the other small charged bodies placed around it is the *vector* sum of the individual two-body forces of Coulomb. Strictly speaking, Coulomb's conclusions apply to charges in vacuum or in media of negligible susceptibility. We defer consideration of charges in dielectrics to Chapter 4.

1.2 Electric Field

Although the thing that eventually gets measured is a force, it is useful to introduce a concept one step removed from the forces, the concept of an electric field due to some array of charged bodies. At the moment, the electric field can be

defined as the force per unit charge acting at a given point. It is a vector function of position, denoted by **E**. One must be careful in its definition, however. It is not necessarily the force that one would observe by placing one unit of charge on a pith ball and placing it in position. The reason is that one unit of charge may be so large that its presence alters appreciably the field configuration of the array. Consequently one must use a limiting process whereby the ratio of the force on the small test body to the charge on it is measured for smaller and smaller amounts of charge.* Experimentally, this ratio and the direction of the force will become constant as the amount of test charge is made smaller and smaller. These limiting values of magnitude and direction define the magnitude and direction of the electric field **E** at the point in question. In symbols we may write

$$\mathbf{F} = q\mathbf{E} \tag{1.1}$$

where **F** is the force, **E** the electric field, and q the charge. In this equation it is assumed that the charge q is located at a point, and the force and the electric field are evaluated at that point.

Coulomb's law can be written down similarly. If **F** is the force on a point charge q_1, located at $\mathbf{x}_1$, due to another point charge q_2, located at $\mathbf{x}_2$, then Coulomb's law is

$$\mathbf{F} = kq_1q_2\frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \tag{1.2}$$

Note that q_1 and q_2 are algebraic quantities, which can be positive or negative. The constant of proportionality k depends on the system of units used.

The electric field at the point $\mathbf{x}$ due to a point charge q_1 at the point $\mathbf{x}_1$ can be obtained directly:

$$\mathbf{E}(\mathbf{x}) = kq_1\frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3} \tag{1.3}$$

as indicated in Fig. 1.1. The constant k differs in different systems of units.† In electrostatic units (esu), $k = 1$ and unit charge is chosen as that charge that exerts a force of one dyne on an equal point charge located one centimeter away. The esu unit of charge is called the *statcoulomb*, and the electric field is measured in *statvolts per centimeter*. In the SI system, which we employ here, $k = (4\pi\epsilon_0)^{-1} = 10^{-7}c^2$, where $\epsilon_0 \approx 8.854 \times 10^{-12}$ farad per meter (F/m) is called the permittivity of free space. The SI unit of charge is the *coulomb* (C), and the electric field is measured in *volts per meter* (V/m). One coulomb (1 C) produces an electric field

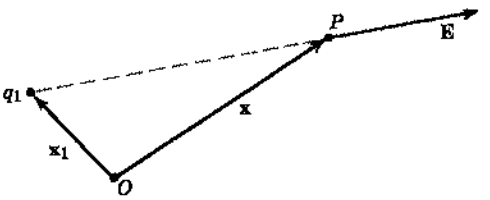


Figure 1.1

*The discreteness of electric charge (see Section 1.1) means that this mathematical limit is impossible to realize physically. This is an example of a mathematical idealization in macroscopic electrostatics.

†The question of units is discussed in detail in the Appendix.

of approximately 8.9874×10^9 V/m (8.9874 GV/m) at a distance of 1 meter. One electron ($q \approx 1.602 \times 10^{-19}$ C) produces a field of approximately 1.44×10^{-9} V/m (1.44 nV/m) at 1 meter.

The experimentally observed linear superposition of forces due to many charges means that we may write the electric field at $\mathbf{x}$ due to a system of point charges q_i, located at $\mathbf{x}_i$, $i = 1, 2, \ldots, n$, as the vector sum:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|^3} \tag{1.4}$$

If the charges are so small and so numerous that they can be described by a charge density $\rho(\mathbf{x}')$ [if Δq is the charge in a small volume $\Delta x\, \Delta y\, \Delta z$ at the point $\mathbf{x}'$, then $\Delta q = \rho(\mathbf{x}')\, \Delta x\, \Delta y\, \Delta z$], the sum is replaced by an integral:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x' \tag{1.5}$$

where $d^3x' = dx'\, dy'\, dz'$ is a three-dimensional volume element at $\mathbf{x}'$.

At this point it is worthwhile to introduce the *Dirac delta function*. In one dimension, the delta function, written $\delta(x - a)$, is a mathematically improper function having the properties:

1. $\delta(x - a) = 0$ for $x \neq a$, and
2. $\int \delta(x - a)\, dx = 1$ if the region of integration includes $x = a$, and is zero otherwise.

The delta function can be given an intuitive, but nonrigorous, meaning as the limit of a peaked curve such as a Gaussian that becomes narrower and narrower, but higher and higher, in such a way that the area under the curve is always constant. L. Schwartz's theory of distributions is a comprehensive rigorous mathematical approach to delta functions and their manipulations.*

From the definitions above it is evident that, for an arbitrary function $f(x)$,

3. $\int f(x)\, \delta(x - a)\, dx = f(a)$.

The integral of $f(x)$ times the derivative of a delta function is simply understood if the delta function is thought of as a well-behaved, but sharply peaked, function. Thus the definition is

4. $\int f(x)\, \delta'(x - a)\, dx = f'(a)$

where a prime denotes differentiation with respect to the argument.

If the delta function has as argument a function $f(x)$ of the independent variable x, it can be transformed according to the rule,

5. $\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|} \delta(x - x_i)$

where $f(x)$ is assumed to have only simple zeros, located at $x = x_i$.

In more than one dimension, we merely take products of delta functions in each dimension. In three dimensions, for example, with Cartesian coordinates,

6. $\delta(\mathbf{x} - \mathbf{X}) = \delta(x_1 - X_1)\, \delta(x_2 - X_2)\, \delta(x_3 - X_3)$

*A useful, rigorous account of the Dirac delta function is given by *Lighthill*. See also *Dennery and Krzywicki* (Section III.13). (Full references for items cited in the text or footnotes by italicized author only will be found in the Bibliography.)

is a function that vanishes everywhere except at $\mathbf{x} = \mathbf{X}$, and is such that

$$7.\quad \int_{\Delta V} \delta(\mathbf{x} - \mathbf{X})\, d^3x = \begin{cases} 1 & \text{if } \Delta V \text{ contains } \mathbf{x} = \mathbf{X} \\ 0 & \text{if } \Delta V \text{ does not contain } \mathbf{x} = \mathbf{X} \end{cases}$$

Note that a delta function has the dimensions of an inverse volume in whatever number of dimensions the space has.

A discrete set of point charges can be described with a charge density by means of delta functions. For example,

$$\rho(\mathbf{x}) = \sum_{i=1}^{n} q_i\, \delta(\mathbf{x} - \mathbf{x}_i) \tag{1.6}$$

represents a distribution of n point charges q_i, located at the points $\mathbf{x}_i$. Substitution of this charge density (1.6) into (1.5) and integration, using the properties of the delta function, yields the discrete sum (1.4).

1.3 *Gauss's Law*

The integral (1.5) is not always the most suitable form for the evaluation of electric fields. There is another integral result, called *Gauss's law*, which is sometimes more useful and furthermore leads to a differential equation for $\mathbf{E}(\mathbf{x})$. To obtain Gauss's law we first consider a point charge q and a *closed* surface S, as shown in Fig. 1.2. Let r be the distance from the charge to a point on the surface, $\mathbf{n}$ be the outwardly directed unit normal to the surface at that point, da be an

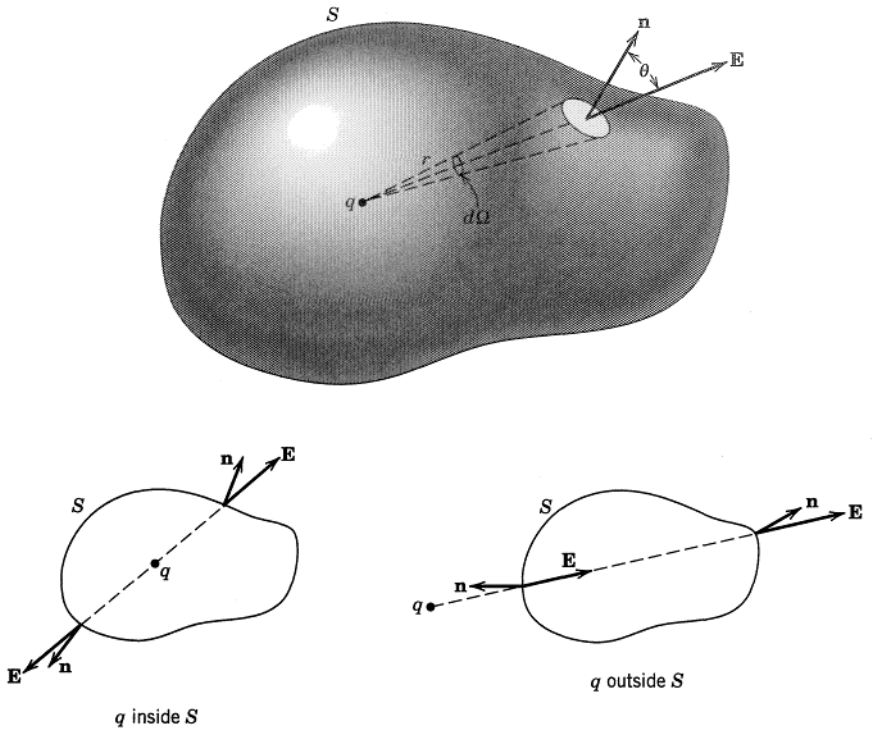


Figure 1.2 Gauss's law. The normal component of electric field is integrated over the closed surface S. If the charge is inside (outside) S, the total solid angle subtended at the charge by the inner side of the surface is 4π (zero).

element of surface area. If the electric field **E** at the point on the surface due to the charge q makes an angle θ with the unit normal, then the normal component of **E** times the area element is:

$$\mathbf{E} \cdot \mathbf{n}\, da = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}\, da \tag{1.7}$$

Since **E** is directed along the line from the surface element to the charge q, $\cos\theta\, da = r^2\, d\Omega$, where $d\Omega$ is the element of solid angle subtended by da at the position of the charge. Therefore

$$\mathbf{E} \cdot \mathbf{n}\, da = \frac{q}{4\pi\epsilon_0}\, d\Omega \tag{1.8}$$

If we now integrate the normal component of **E** over the whole surface, it is easy to see that

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \begin{cases} q/\epsilon_0 & \text{if } q \text{ lies } \textit{inside } S \\ 0 & \text{if } q \text{ lies } \textit{outside } S \end{cases} \tag{1.9}$$

This result is Gauss's law for a single point charge. For a discrete set of charges, it is immediately apparent that

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \sum_i q_i \tag{1.10}$$

where the sum is over only those charges *inside* the surface S. For a continuous charge density $\rho(\mathbf{x})$, Gauss's law becomes:

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x \tag{1.11}$$

where V is the volume enclosed by S.

Equation (1.11) is one of the basic equations of electrostatics. Note that it depends upon

the inverse square law for the force between charges,
the central nature of the force, and
the linear superposition of the effects of different charges.

Clearly, then, Gauss's law holds for Newtonian gravitational force fields, with matter density replacing charge density.

It is interesting to note that, even before the experiments of Cavendish and Coulomb, Priestley, taking up an observation of Franklin that charge seemed to reside on the outside, but not the inside, of a metal cup, reasoned by analogy with Newton's law of universal gravitation that the electrostatic force must obey an inverse square law with distance. The present status of the inverse square law is discussed in Section I.2.

1.4 *Differential Form of Gauss's Law*

Gauss's law can be thought of as being an integral formulation of the law of electrostatics. We can obtain a differential form (i.e., a differential equation) by

using the divergence theorem. The *divergence theorem* states that for any well-behaved vector field $\mathbf{A}(\mathbf{x})$ defined within a volume V surrounded by the closed surface S the relation

$$\oint_S \mathbf{A} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{A}\, d^3x$$

holds between the volume integral of the divergence of $\mathbf{A}$ and the surface integral of the outwardly directed normal component of $\mathbf{A}$. The equation in fact can be used as the definition of the divergence (see *Stratton*, p. 4).

To apply the divergence theorem we consider the integral relation expressed in Gauss's theorem:

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x$$

Now the divergence theorem allows us to write this as

$$\int_V (\nabla \cdot \mathbf{E} - \rho/\epsilon_0)\, d^3x = 0 \tag{1.12}$$

for an arbitrary volume V. We can, in the usual way, put the integrand equal to zero to obtain

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{1.13}$$

which is the differential form of Gauss's law of electrostatics. This equation can itself be used to solve problems in electrostatics. However, it is often simpler to deal with scalar rather then vector functions of position, and then to derive the vector quantities at the end if necessary (see below).

1.5 Another Equation of Electrostatics and the Scalar Potential

The single equation (1.13) is not enough to specify completely the three components of the electric field $\mathbf{E}(\mathbf{x})$. Perhaps some readers know that a vector field can be specified almost* completely if its divergence and curl are given everywhere in space. Thus we look for an equation specifying curl $\mathbf{E}$ as a function of position. Such an equation, namely,

$$\nabla \times \mathbf{E} = 0 \tag{1.14}$$

follows directly from our generalized Coulomb's law (1.5):

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x'$$

The vector factor in the integrand, viewed as a function of $\mathbf{x}$, is the negative gradient of the scalar $1/|\mathbf{x} - \mathbf{x}'|$:

$$\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)$$

*Up to the gradient of a scalar function that satisfies the Laplace equation. See Section 1.9 on uniqueness.

Since the gradient operation involves $\mathbf{x}$, but not the integration variable $\mathbf{x}'$, it can be taken outside the integral sign. Then the field can be written

$$\mathbf{E}(\mathbf{x}) = \frac{-1}{4\pi\epsilon_0}\nabla\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{1.15}$$

Since the curl of the gradient of any well-behaved scalar function of position vanishes ($\nabla\times\nabla\psi = 0$, for all ψ), (1.14) follows immediately from (1.15).

Note that $\nabla\times\mathbf{E} = 0$ depends on the central nature of the force between charges, and on the fact that the force is a function of relative distances only, but does not depend on the inverse square nature.

In (1.15) the electric field (a vector) is derived from a scalar by the gradient operation. Since one function of position is easier to deal with than three, it is worthwhile concentrating on the scalar function and giving it a name. Consequently we define the *scalar potential* $\Phi(\mathbf{x})$ by the equation:

$$\mathbf{E} = -\nabla\Phi \tag{1.16}$$

Then (1.15) shows that the scalar potential is given in terms of the charge density by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{1.17}$$

where the integration is over all charges in the universe, and Φ is arbitrary only to the extent that a constant can be added to the right-hand side of (1.17).

The scalar potential has a physical interpretation when we consider the work done on a test charge q in transporting it from one point (A) to another point (B) in the presence of an electric field $\mathbf{E}(\mathbf{x})$, as shown in Fig. 1.3. The force acting on the charge at any point is

$$\mathbf{F} = q\mathbf{E}$$

so that the work done in moving the charge from A to B is

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{l} = -q\int_A^B \mathbf{E}\cdot d\mathbf{l} \tag{1.18}$$

The minus sign appears because we are calculating the work done *on* the charge against the action of the field. With definition (1.16) the work can be written

$$W = q\int_A^B \nabla\Phi\cdot d\mathbf{l} = q\int_A^B d\Phi = q(\Phi_B - \Phi_A) \tag{1.19}$$

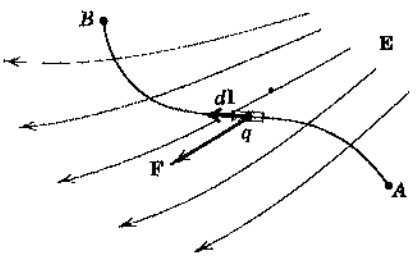


Figure 1.3

which shows that $q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field.

From (1.18) and (1.19) it can be seen that the line integral of the electric field between two points is independent of the path and is the negative of the potential difference between the points:

$$\int_A^B \mathbf{E} \cdot d\mathbf{l} = -(\Phi_B - \Phi_A) \tag{1.20}$$

This follows directly, of course, from definition (1.16). If the path is closed, the line integral is zero,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.21}$$

a result that can also be obtained directly from Coulomb's law. Then application of *Stokes's theorem* [if $\mathbf{A}(\mathbf{x})$ is a well-behaved vector field, S is an arbitrary open surface, and C is the closed curve bounding S,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da$$

where $d\mathbf{l}$ is a line element of C, $\mathbf{n}$ is the normal to S, and the path C is traversed in a right-hand screw sense relative to $\mathbf{n}$] leads immediately back to $\nabla \times \mathbf{E} = 0$.

1.6 Surface Distributions of Charges and Dipoles and Discontinuities in the Electric Field and Potential

One of the common problems in electrostatics is the determination of electric field or potential due to a given surface distribution of charges. Gauss's law (1.11) allows us to write down a partial result directly. If a surface S, with a unit normal $\mathbf{n}$ directed from side 1 to side 2 of the surface, has a surface-charge density of $\sigma(\mathbf{x})$ (measured in coulombs per square meter) and electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ on either side of the surface, as shown in Fig. 1.4, then Gauss's law tells us immediately that

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \sigma/\epsilon_0 \tag{1.22}$$

This does not determine $\mathbf{E}_1$ and $\mathbf{E}_2$ unless there are no other sources of field and the geometry and form of σ are especially simple. All that (1.22) says is that there

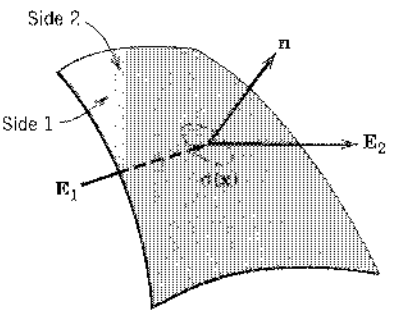


Figure 1.4 Discontinuity in the normal component of electric field across a surface layer of charge.

is a discontinuity of σ/ϵ_0 in the normal component of electric field in crossing a surface with a surface-charge density σ, the crossing being made in the direction of $\mathbf{n}$.

The tangential component of electric field can be shown to be continuous across a boundary surface by using (1.21) for the line integral of $\mathbf{E}$ around a closed path. It is only necessary to take a rectangular path with negligible ends and one side on either side of the boundary.

An expression for the potential (hence the field, by differentiation) at any point in space (not just at the surface) can be obtained from (1.17) by replacing $\rho\, d^3x$ by $\sigma\, da$:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, da' \tag{1.23}$$

For volume or surface distributions of charge, the potential is everywhere continuous, even within the charge distribution. This can be shown from (1.23) or from the fact that $\mathbf{E}$ is bounded, even though discontinuous across a surface distribution of charge. With point or line charges, or dipole layers, the potential is no longer continuous, as will be seen immediately.

Another problem of interest is the potential due to a dipole-layer distribution on a surface S. A dipole layer can be imagined as being formed by letting the surface S have a surface-charge density $\sigma(\mathbf{x})$ on it, and another surface S', lying close to S, have an equal and opposite surface-charge density on it at neighboring points, as shown in Fig. 1.5. The dipole-layer distribution of strength $D(\mathbf{x})$ is formed by letting S' approach infinitesimally close to S while the surface-charge density $\sigma(\mathbf{x})$ becomes infinite in such a manner that the product of $\sigma(\mathbf{x})$ and the local separation $d(\mathbf{x})$ of S and S' approaches the limit $D(\mathbf{x})$:

$$\lim_{d(\mathbf{x})\to 0} \sigma(\mathbf{x})\, d(\mathbf{x}) = D(\mathbf{x})$$

The direction of the dipole moment of the layer is normal to the surface S and in the direction going from negative to positive charge.

To find the potential due to a dipole layer we can consider a single dipole and then superpose a surface density of them, or we can obtain the same result by performing mathematically the limiting process described in words above on the surface-density expression (1.23). The first way is perhaps simpler, but the second gives useful practice in vector calculus. Consequently we proceed with

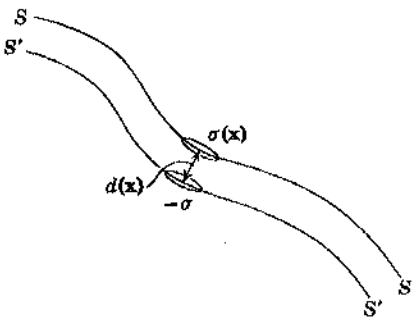


Figure 1.5 Limiting process involved in creating a dipole layer.

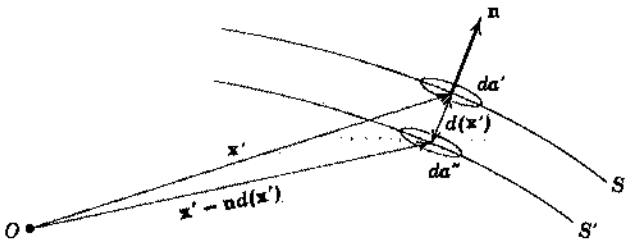


Figure 1.6 Dipole-layer geometry.

the limiting process. With $\mathbf{n}$, the unit normal to the surface S, directed away from S', as shown in Fig. 1.6, the potential due to the two close surfaces is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,da' - \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'+\mathbf{n}d|}\,da''$$

For small d we can expand $|\mathbf{x}-\mathbf{x}'+\mathbf{n}d|^{-1}$. Consider the general expression $|\mathbf{x}+\mathbf{a}|^{-1}$, where $|\mathbf{a}| \ll |\mathbf{x}|$. We write a Taylor series expansion in three dimensions:

$$\frac{1}{|\mathbf{x}+\mathbf{a}|} = \frac{1}{x} + \mathbf{a}\cdot\nabla\left(\frac{1}{x}\right) + \cdots$$

In this way we find that as $d \to 0$ the potential becomes

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{x}')\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)da' \tag{1.24}$$

In passing we note that the integrand in (1.24) is the potential of a point dipole with dipole moment $\mathbf{p} = \mathbf{n}\,D\,da'$. The potential at $\mathbf{x}$ caused by a dipole $\mathbf{p}$ at $\mathbf{x}'$ is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1.25}$$

Equation (1.24) has a simple geometrical interpretation. We note that

$$\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)da' = -\frac{\cos\theta\,da'}{|\mathbf{x}-\mathbf{x}'|^2} = -d\Omega$$

where $d\Omega$ is the element of solid angle subtended at the observation point by the area element da', as indicated in Fig. 1.7. Note that $d\Omega$ has a positive sign if θ is

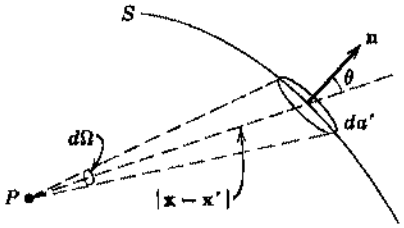


Figure 1.7 The potential at P due to the dipole layer D on the area element da' is just the negative product of D and the solid angle element $d\Omega$ subtended by da' at P.

an acute angle (i.e., when the observation point views the "inner" side of the dipole layer). The potential can be written:

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{x}')\, d\Omega \tag{1.26}$$

For a constant surface-dipole-moment density D, the potential is just the product of the moment divided by $4\pi\epsilon_0$ and the solid angle subtended at the observation point by the surface, regardless of its shape.

There is a discontinuity in potential in crossing a double layer. This can be seen by letting the observation point come infinitesimally close to the double layer. The double layer is now imagined to consist of two parts, one being a small disc directly under the observation point. The disc is sufficiently small that it is sensibly flat and has constant surface-dipole-moment density D. Evidently the total potential can be obtained by linear superposition of the potential of the disc and that of the remainder. From (1.26) it is clear that the potential of the disc alone has a discontinuity of D/ϵ_0 in crossing from the inner to the outer side, being $-D/2\epsilon_0$ on the inner side and $+D/2\epsilon_0$ on the outer. The potential of the remainder alone, with its hole where the disc fits in, is continuous across the plane of the hole. Consequently the total potential jump in crossing the surface is:

$$\Phi_2 - \Phi_1 = D/\epsilon_0 \tag{1.27}$$

This result is analogous to (1.22) for the discontinuity of electric field in crossing a surface-charge density. Equation (1.27) can be interpreted "physically" as a potential drop occurring "inside" the dipole layer; it can be calculated as the product of the field between the two layers of surface charge times the separation before the limit is taken.

1.7 *Poisson and Laplace Equations*

In Sections 1.4 and 1.5 it was shown that the behavior of an electrostatic field can be described by the two differential equations:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{1.13}$$

and

$$\nabla \times \mathbf{E} = 0 \tag{1.14}$$

the latter equation being equivalent to the statement that $\mathbf{E}$ is the gradient of a scalar function, the scalar potential Φ:

$$\mathbf{E} = -\nabla\Phi \tag{1.16}$$

Equations (1.13) and (1.16) can be combined into one partial differential equation for the single function $\Phi(\mathbf{x})$:

$$\nabla^2\Phi = -\rho/\epsilon_0 \tag{1.28}$$

This equation is called the *Poisson equation*. In regions of space that lack a charge density, the scalar potential satisfies the *Laplace equation*:

$$\nabla^2\Phi = 0 \tag{1.29}$$

We already have a solution for the scalar potential in expression (1.17):

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' \tag{1.17}$$

To verify directly that this does indeed satisfy the Poisson equation (1.28), we operate with the Laplacian on both sides. Because it turns out that the resulting integrand is singular, we invoke a limiting procedure. Define the "a-potential" $\Phi_a(\mathbf{x})$ by

$$\Phi_a(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{\sqrt{(\mathbf{x}-\mathbf{x}')^2+a^2}}\,d^3x'$$

The actual potential (1.17) is then the limit of the "a-potential" as $a \to 0$. Taking the Laplacian of the "a-potential" gives

$$\begin{aligned}\nabla^2\Phi_a(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right)d^3x' \\ &= -\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\left[\frac{3a^2}{(r^2+a^2)^{5/2}}\right]d^3x'\end{aligned} \tag{1.30}$$

where $r = |\mathbf{x}-\mathbf{x}'|$. The square-bracketed expression is the negative Laplacian of $1/\sqrt{r^2+a^2}$. It is well-behaved everywhere for nonvanishing a, but as a tends to zero it becomes infinite at $r = 0$ and vanishes for $r \neq 0$. It has a volume integral equal to 4π for arbitrary a. For the purposes of integration, divide space into two regions by a sphere of fixed radius R centered on $\mathbf{x}$. Choose R such that $\rho(\mathbf{x}')$ changes little over the interior of the sphere, and imagine a much smaller than R and tending toward zero. If $\rho(\mathbf{x}')$ is such that (1.17) exists, the contribution to the integral (1.30) from the exterior of the sphere will vanish like a^2 as $a \to 0$. We thus need consider only the contribution from inside the sphere. With a Taylor series expansion of the well-behaved $\rho(\mathbf{x}')$ around $\mathbf{x}' = \mathbf{x}$, one finds

$$\nabla^2\Phi_a(\mathbf{x}) = -\frac{1}{\epsilon_0}\int_0^R \frac{3a^2}{(r^2+a^2)^{5/2}}\left[\rho(\mathbf{x}) + \frac{r^2}{6}\nabla^2\rho + \cdots\right]r^2dr + O(a^2)$$

Direct integration yields

$$\nabla^2\Phi_a(\mathbf{x}) = -\frac{1}{\epsilon_0}\rho(\mathbf{x})\left(1 + O(a^2/R^2)\right) + O(a^2, a^2\log a)\,\nabla^2\rho + \cdots$$

In the limit $a \to 0$, we obtain the Poisson equation (1.28).

The singular nature of the Laplacian of $1/r$ can be exhibited formally in terms of a Dirac delta function. Since $\nabla^2(1/r) = 0$ for $r \neq 0$ and its volume integral is -4π, we can write the formal equation, $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$ or, more generally,

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{1.31}$$

1.8 Green's Theorem

If electrostatic problems always involved localized discrete or continuous distributions of charge with no boundary surfaces, the general solution (1.17) would

be the most convenient and straightforward solution to any problem. There would be no need of the Poisson or Laplace equation. In actual fact, of course, many, if not most, of the problems of electrostatics involve finite regions of space, with or without charge inside, and with prescribed boundary conditions on the bounding surfaces. These boundary conditions may be simulated by an appropriate distribution of charges outside the region of interest (perhaps at infinity), but (1.17) becomes inconvenient as a means of calculating the potential, except in simple cases (e.g., method of images).

To handle the boundary conditions it is necessary to develop some new mathematical tools, namely, the identities or theorems due to George Green (1824). These follow as simple applications of the divergence theorem. The divergence theorem:

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \oint_S \mathbf{A} \cdot \mathbf{n}\, da$$

applies to any well-behaved vector field $\mathbf{A}$ defined in the volume V bounded by the closed surface S. Let $\mathbf{A} = \phi\, \nabla\psi$, where ϕ and ψ are arbitrary scalar fields. Now

$$\nabla \cdot (\phi\, \nabla\psi) = \phi\, \nabla^2\psi + \nabla\phi \cdot \nabla\psi \tag{1.32}$$

and

$$\phi\, \nabla\psi \cdot \mathbf{n} = \phi \frac{\partial\psi}{\partial n} \tag{1.33}$$

where $\partial/\partial n$ is the normal derivative at the surface S (directed outward from inside the volume V). When (1.32) and (1.33) are substituted into the divergence theorem, there results *Green's first identity*:

$$\int_V (\phi\, \nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \oint_S \phi \frac{\partial\psi}{\partial n}\, da \tag{1.34}$$

If we write down (1.34) again with ϕ and ψ interchanged, and then subtract it from (1.34), the $\nabla\phi \cdot \nabla\psi$ terms cancel, and we obtain Green's second identity or *Green's theorem*:

$$\int_V (\phi\, \nabla^2\psi - \psi\, \nabla^2\phi)\, d^3x = \oint_S \left[\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right] da \tag{1.35}$$

The Poisson differential equation for the potential can be converted into an integral equation if we choose a particular ψ, namely $1/R \equiv 1/|\mathbf{x} - \mathbf{x}'|$, where $\mathbf{x}$ is the observation point and $\mathbf{x}'$ is the integration variable. Further, we put $\phi = \Phi$, the scalar potential, and make use of $\nabla^2\Phi = -\rho/\epsilon_0$. From (1.31) we know that $\nabla^2(1/R) = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, so that (1.35) becomes

$$\int_V \left[-4\pi\Phi(\mathbf{x}')\, \delta(\mathbf{x} - \mathbf{x}') + \frac{1}{\epsilon_0 R}\, \rho(\mathbf{x}')\right] d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right] da'$$

If the point $\mathbf{x}$ lies within the volume V, we obtain:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')}{R}\, d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da' \tag{1.36}$$

If $\mathbf{x}$ lies outside the surface S, the left-hand side of (1.36) is zero.* [Note that this is consistent with the interpretation of the surface integral as being the potential due to a surface-charge density $\sigma = \epsilon_0 \partial\Phi/\partial n'$ and a dipole layer $D = -\epsilon_0\Phi$. The discontinuities in electric field and potential (1.22) and (1.27) across the surface then lead to zero field and zero potential outside the volume V.]

Two remarks are in order about result (1.36). First, if the surface S goes to infinity and the electric field on S falls off faster than R^{-1}, then the surface integral vanishes and (1.36) reduces to the familiar result (1.17). Second, for a charge-free volume, the potential anywhere inside the volume (a solution of the Laplace equation) is expressed in (1.36) in terms of the potential and its normal derivative only on the surface of the volume. This rather surprising result is not a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both Φ and $\partial\Phi/\partial n$ (*Cauchy boundary conditions*) is an overspecification of the problem. This is discussed in detail in the next sections, where techniques yielding solutions for appropriate boundary conditions are developed using Green's theorem (1.35).

1.9 Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions

What boundary conditions are appropriate for the Poisson (or Laplace) equation to ensure that a unique and well-behaved (i.e., physically reasonable) solution will exist inside the bounded region? Physical experience leads us to believe that specification of the potential on a closed surface (e.g., a system of conductors held at different potentials) defines a unique potential problem. This is called a *Dirichlet problem*, or *Dirichlet boundary conditions*. Similarly it is plausible that specification of the electric field (normal derivative of the potential) everywhere on the surface (corresponding to a given surface-charge density) also defines a unique problem. Specification of the normal derivative is known as the *Neumann boundary condition*. We now proceed to prove these expectations by means of Green's first identity (1.34).

We want to show the uniqueness of the solution of the Poisson equation, $\nabla^2\Phi = -\rho/\epsilon_0$, inside a volume V subject to either Dirichlet or Neumann boundary conditions on the closed bounding surface S. We suppose, to the contrary, that there exist two solutions Φ_1 and Φ_2 satisfying the same boundary conditions. Let

$$U = \Phi_2 - \Phi_1 \tag{1.37}$$

Then $\nabla^2 U = 0$ inside V, and $U = 0$ or $\partial U/\partial n = 0$ on S for Dirichlet and Neumann boundary conditions, respectively. From Green's first identity (1.34), with $\phi = \psi = U$, we find

$$\int_V (U\,\nabla^2 U + \nabla U \cdot \nabla U)\, d^3x = \oint_S U \frac{\partial U}{\partial n}\, da \tag{1.38}$$

*The reader may complain that (1.36) has been obtained in an illegal fashion since $1/|\mathbf{x} - \mathbf{x}'|$ is not well-behaved inside the volume V. Rigor can be restored by using a limiting process, as in the preceding section, or by excluding a small sphere around the offending point, $\mathbf{x} = \mathbf{x}'$. The result is still (1.36).

With the specified properties of U, this reduces (for both types of boundary condition) to:

$$\int_V |\nabla U|^2 \, d^3x = 0$$

which implies $\nabla U = 0$. Consequently, inside V, U is constant. For Dirichlet boundary conditions, $U = 0$ on S so that, inside V, $\Phi_1 = \Phi_2$ and the solution is unique. Similarly, for Neumann boundary conditions, the solution is unique, apart from an unimportant arbitrary additive constant.

From the right-hand side of (1.38) it is evident that there is also a unique solution to a problem with mixed boundary conditions (i.e., Dirichlet over part of the surface S, and Neumann over the remaining part).

It should be clear that a solution to the Poisson equation with both Φ and $\partial\Phi/\partial n$ specified arbitrarily on a closed boundary (Cauchy boundary conditions) does not exist, since there are unique solutions for Dirichlet and Neumann conditions separately and these will in general not be consistent. This can be verified with (1.36). With arbitrary values of Φ and $\partial\Phi/\partial n$ inserted on the right-hand side, it can be shown that the values of $\Phi(\mathbf{x})$ and $\nabla\Phi(\mathbf{x})$ as $\mathbf{x}$ approaches the surface are in general inconsistent with the assumed boundary values. The question of whether Cauchy boundary conditions on an *open* surface define a unique electrostatic problem requires more discussion than is warranted here. The reader may refer to *Morse and Feshbach* (Section 6.2, pp. 692–706) or to *Sommerfeld* (*Partial Differential Equations in Physics*, Chapter II) for a detailed discussion of these questions. The conclusion is that electrostatic problems are specified only by Dirichlet *or* Neumann boundary conditions on a closed surface (part or all of which may be at infinity, of course).

1.10 Formal Solution of Electrostatic Boundary-Value Problem with Green Function

The solution of the Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S can be obtained by means of Green's theorem (1.35) and so-called Green functions.

In obtaining result (1.36)—not a solution—we chose the function ψ to be $1/|\mathbf{x} - \mathbf{x}'|$, it being the potential of a unit point source, satisfying the equation:

$$\nabla'^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.31}$$

The function $1/|\mathbf{x} - \mathbf{x}'|$ is only one of a class of functions depending on the variables $\mathbf{x}$ and $\mathbf{x}'$, and called *Green functions*, which satisfy (1.31). In general,

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.39}$$

where

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \tag{1.40}$$

with the function F satisfying the Laplace equation inside the volume V:

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0 \tag{1.41}$$

In facing the problem of satisfying the prescribed boundary conditions on Φ *or* $\partial\Phi/\partial n$, we can find the key by considering result (1.36). As has been pointed out already, this is not a solution satisfying the correct type of boundary conditions because both Φ *and* $\partial\Phi/\partial n$ appear in the surface integral. It is at best an integral relation for Φ. With the generalized concept of a Green function and its additional freedom [via the function $F(\mathbf{x}, \mathbf{x}')$], there arises the possibility that we can use Green's theorem with $\psi = G(\mathbf{x}, \mathbf{x}')$ and choose $F(\mathbf{x}, \mathbf{x}')$ to eliminate one or the other of the two surface integrals, obtaining a result that involves only Dirichlet or Neumann boundary conditions. Of course, if the necessary $G(\mathbf{x}, \mathbf{x}')$ depended in detail on the exact form of the boundary conditions, the method would have little generality. As will be seen immediately, this is not required, and $G(\mathbf{x}, \mathbf{x}')$ satisfies rather simple boundary conditions on S.

With Green's theorem (1.35), $\phi = \Phi$, $\psi = G(\mathbf{x}, \mathbf{x}')$, and the specified properties of G (1.39), it is simple to obtain the generalization of (1.36):

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{x}, \mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] da' \tag{1.42}$$

The freedom available in the definition of G (1.40) means that we can make the surface integral depend only on the chosen type of boundary conditions. Thus, for *Dirichlet boundary conditions* we demand:

$$G_D(\mathbf{x}, \mathbf{x}') = 0 \qquad \text{for } \mathbf{x}' \text{ on } S \tag{1.43}$$

Then the first term in the surface integral in (1.42) vanishes and the solution is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x}, \mathbf{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}\, da' \tag{1.44}$$

For *Neumann boundary conditions* we must be more careful. The obvious choice of boundary condition on $G(\mathbf{x}, \mathbf{x}')$ seems to be

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = 0 \qquad \text{for } \mathbf{x}' \text{ on } S$$

since that makes the second term in the surface integral in (1.42) vanish, as desired. But an application of Gauss's theorem to (1.39) shows that

$$\oint_S \frac{\partial G}{\partial n'}\, da' = -4\pi$$

Consequently the simplest allowable boundary condition on G_N is

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{S} \qquad \text{for } \mathbf{x}' \text{ on } S \tag{1.45}$$

where S is the total area of the boundary surface. Then the solution is

$$\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_N(\mathbf{x}, \mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}\, G_N\, da' \tag{1.46}$$

where $\langle\Phi\rangle_S$ is the average value of the potential over the whole surface. The customary Neumann problem is the so-called exterior problem in which the vol-

ume V is bounded by two surfaces, one closed and finite, the other at infinity. Then the surface area S is infinite; the boundary condition (1.45) becomes homogeneous; the average value $\langle\Phi\rangle_S$ vanishes.

We note that the Green functions satisfy simple boundary conditions (1.43) or (1.45) which do not depend on the detailed form of the Dirichlet (or Neumann) boundary values. Even so, it is often rather involved (if not impossible) to determine $G(\mathbf{x}, \mathbf{x}')$ because of its dependence on the shape of the surface S. We will encounter such problems in Chapters 2 and 3.

The mathematical symmetry property $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$ can be proved for the Green functions satisfying the Dirichlet boundary condition (1.43) by means of Green's theorem with $\phi = G(\mathbf{x}, \mathbf{y})$ and $\psi = G(\mathbf{x}', \mathbf{y})$, where $\mathbf{y}$ is the integration variable. Since the Green function, as a function of one of its variables, is a potential due to a unit point source, the symmetry merely represents the physical interchangeability of the source and the observation points. For Neumann boundary conditions the symmetry is not automatic, but can be imposed as a separate requirement.*

As a final, important remark we note the physical meaning of $F(\mathbf{x}, \mathbf{x}')/4\pi\epsilon_0$. It is a solution of the Laplace equation inside V and so represents the potential of a system of charges *external to the volume V*. It can be thought of as the potential due to an external distribution of charges chosen to satisfy the homogeneous boundary conditions of zero potential (or zero normal derivative) on the surface S when combined with the potential of a point charge at the source point $\mathbf{x}'$. Since the potential at a point $\mathbf{x}$ on the surface due to the point charge depends on the position of the source point, the external distribution of charge $F(\mathbf{x}, \mathbf{x}')$ must also depend on the "parameter" $\mathbf{x}'$. From this point of view, we see that the method of images (to be discussed in Chapter 2) is a physical equivalent of the determination of the appropriate $F(\mathbf{x}, \mathbf{x}')$ to satisfy the boundary conditions (1.43) or (1.45). For the Dirichlet problem with conductors, $F(\mathbf{x}, \mathbf{x}')/4\pi\epsilon_0$ can also be interpreted as the potential due to the surface-charge distribution induced on the conductors by the presence of a point charge at the source point $\mathbf{x}'$.

1.11 Electrostatic Potential Energy and Energy Density; Capacitance

In Section 1.5 it was shown that the product of the scalar potential and the charge of a point object could be interpreted as potential energy. More precisely, if a point charge q_i is brought from infinity to a point $\mathbf{x}_i$ in a region of localized electric fields described by the scalar potential Φ (which vanishes at infinity), the work done on the charge (and hence its potential energy) is given by

$$W_i = q_i\Phi(\mathbf{x}_i) \tag{1.47}$$

The potential Φ can be viewed as produced by an array of $(n - 1)$ charges $q_j (j = 1, 2, \ldots, n - 1)$ at positions $\mathbf{x}_j$. Then

$$\Phi(\mathbf{x}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{1.48}$$

*See K.-J. Kim and J. D. Jackson, *Am. J. Phys.* **61**, (12) 1144–1146 (1993).

so that the potential energy of the charge q_i is

$$W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{1.49}$$

The *total* potential energy of all the charges due to all the forces acting between them is:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{1.50}$$

as can be seen most easily by adding each charge in succession. A more symmetric form can be written by summing over i and j unrestricted, and then dividing by 2:

$$W = \frac{1}{8\pi\epsilon_0} \sum_i \sum_j \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{1.51}$$

It is understood that $i = j$ terms (infinite "self-energy" terms) are omitted in the double sum.

For a continuous charge distribution [or, in general, using the Dirac delta functions (1.6)] the potential energy takes the form:

$$W = \frac{1}{8\pi\epsilon_0} \int \int \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x \, d^3x' \tag{1.52}$$

Another expression, equivalent to (1.52), can be obtained by noting that one of the integrals in (1.52) is just the scalar potential (1.17). Therefore

$$W = \frac{1}{2} \int \rho(\mathbf{x})\Phi(\mathbf{x}) \, d^3x \tag{1.53}$$

Equations (1.51), (1.52), and (1.53) express the electrostatic potential energy in terms of the positions of the charges and so emphasize the interactions between charges via Coulomb forces. An alternative, and very fruitful, approach is to emphasize the electric field and to interpret the energy as being stored in the electric field surrounding the charges. To obtain this latter form, we make use of the Poisson equation to eliminate the charge density from (1.53):

$$W = \frac{\epsilon_0}{2} \int \Phi \, \nabla^2\Phi \, d^3x$$

Integration by parts leads to the result:

$$W = \frac{\epsilon_0}{2} \int |\nabla\Phi|^2 \, d^3x = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2 \, d^3x \tag{1.54}$$

where the integration is over all space. In (1.54) all explicit reference to charges has gone, and the energy is expressed as an integral of the square of the electric field over all space. This leads naturally to the identification of the integrand as an energy density w:

$$w = \frac{\epsilon_0}{2} |\mathbf{E}|^2 \tag{1.55}$$

This expression for energy density is intuitively reasonable, since regions of high fields "must" contain considerable energy.

There is perhaps one puzzling thing about (1.55). The energy density is pos-

itive definite. Consequently its volume integral is necessarily nonnegative. This seems to contradict our impression from (1.51) that the potential energy of two charges of opposite sign is negative. The reason for this apparent contradiction is that (1.54) and (1.55) contain "self-energy" contributions to the energy density, whereas the double sum in (1.51) does not. To illustrate this, consider two point charges q_1 and q_2 located at $\mathbf{x}_1$ and $\mathbf{x}_2$, as in Fig. 1.8. The electric field at the point P with coordinate $\mathbf{x}$ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3} + \frac{1}{4\pi\epsilon_0}\frac{q_2(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_2|^3}$$

so that the energy density (1.55) is

$$32\pi^2\epsilon_0 w = \frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_2^2}{|\mathbf{x}-\mathbf{x}_2|^4} + 2\,\frac{q_1q_2(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3\,|\mathbf{x}-\mathbf{x}_2|^3} \tag{1.56}$$

Clearly the first two terms are "self-energy" contributions. To show that the third term gives the proper result for the interaction potential energy we integrate over all space:

$$W_{\text{int}} = \frac{q_1q_2}{16\pi^2\epsilon_0}\int \frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3\,|\mathbf{x}-\mathbf{x}_2|^3}\,d^3x \tag{1.57}$$

A change of integration variable to $\boldsymbol{\rho} = (\mathbf{x}-\mathbf{x}_1)/|\mathbf{x}_1-\mathbf{x}_2|$ yields

$$W_{\text{int}} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\mathbf{x}_1-\mathbf{x}_2|}\times\frac{1}{4\pi}\int\frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\mathbf{n})}{\rho^3\,|\boldsymbol{\rho}+\mathbf{n}|^3}\,d^3\rho \tag{1.58}$$

where $\mathbf{n}$ is a unit vector in the direction $(\mathbf{x}_1-\mathbf{x}_2)$. Using the fact that $(\boldsymbol{\rho}+\mathbf{n})/|\boldsymbol{\rho}+\mathbf{n}|^3 = -\nabla_\rho(1/|\boldsymbol{\rho}+\mathbf{n}|)$, the dimensionless integral can easily be shown to have the value 4π, so that the interaction energy reduces to the expected value.

Forces acting between charged bodies can be obtained by calculating the change in the total electrostatic energy of the system under small virtual displacements. Examples of this are discussed in the problems. Care must be taken to exhibit the energy in a form showing clearly the factors that vary with a change in configuration and those that are kept constant.

As a simple illustration we calculate the force per unit area on the surface of a conductor with a surface-charge density $\sigma(\mathbf{x})$. In the immediate neighborhood of the surface the energy density is

$$w = \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \sigma^2/2\epsilon_0 \tag{1.59}$$

If we now imagine a small outward displacement Δx of an elemental area Δa of the conducting surface, the electrostatic energy decreases by an amount that is the product of energy density w and the excluded volume $\Delta x\,\Delta a$:

$$\Delta W = -\sigma^2\,\Delta a\,\Delta x/2\epsilon_0 \tag{1.60}$$

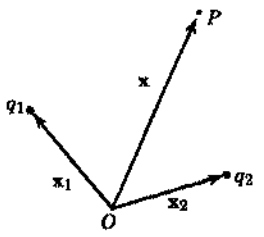


Figure 1.8

This means that there is an outward force per unit area equal to $\sigma^2/2\epsilon_0 = w$ at the surface of the conductor. This result is normally derived by taking the product of the surface-charge density and the electric field, with care taken to eliminate the electric field due to the element of surface-charge density itself.

For a system of n conductors, each with potential V_i and total charge Q_i $(i = 1, 2, \ldots, n)$ in otherwise empty space, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called coefficients of capacity. For a given configuration of the conductors, the linear functional dependence of the potential on the charge density implies that the potential of the ith conductor can be written as

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \qquad (i = 1, 2, \ldots, n)$$

where the p_{ij} depend on the geometry of the conductors. These n equations can be inverted to yield the charge on the ith conductor in terms of all the potentials:

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \qquad (i = 1, 2, \ldots, n) \tag{1.61}$$

The coefficients C_{ii} are called capacities or capacitances while the C_{ij}, $i \neq j$, are called coefficients of induction. The *capacitance of a conductor is therefore the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential.* Sometimes the capacitance of a system of conductors is also defined. For example, the capacitance of two conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor to the potential difference between them. The equations (1.61) can be used to express this capacitance in terms of the coefficients C_{ij}.

The potential energy (1.53) for the system of conductors is

$$W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j \tag{1.62}$$

The expression of the energy in terms of the potentials V_i and the C_{ij}, or in terms of the charges Q_i and the coefficients p_{ij}, permits the application of variational methods to obtain approximate values of capacitances. It can be shown, based on the technique of the next section (see Problems 1.17 and 1.18), that there are variational principles giving upper and lower bounds on C_{ii}. The principles permit estimation with known error of the capacitances of relatively involved configurations of conductors. High-speed computational techniques permit the use of elaborate trial functions involving several parameters. It must be remarked, however, that the need for a Green function satisfying Dirichlet boundary conditions in the lower bound makes the error estimate nontrivial. Further consideration of this technique for calculating capacitances is left to the problems at the end of this and subsequent chapters.

1.12 Variational Approach to the Solution of the Laplace and Poisson Equations

Variational methods play prominent roles in many areas of classical and quantum physics. They provide formal techniques for the derivation of "equations of mo-

tion" and also practical methods for obtaining approximate, but often accurate, solutions to problems not amenable to other approaches. Estimates of resonant frequencies of acoustic resonators and energy eigenvalues of atomic systems come readily to mind.

The far-reaching concept that physical systems in equilibrium have minimal energy content is generalized to the consideration of energy-like functionals. As an example, consider the functional

$$I[\psi] = \frac{1}{2}\int_V \nabla\psi \cdot \nabla\psi \, d^3x - \int_V g\psi \, d^3x \tag{1.63}$$

where the function $\psi(\mathbf{x})$ is well-behaved inside the volume V and on its surface S (which may consist of several separate surfaces), and $g(\mathbf{x})$ is a specified "source" function without singularities within V. We now examine the first-order change in the functional when we change $\psi \to \psi + \delta\psi$, where the modification $\delta\psi(\mathbf{x})$ is infinitesimal within V. The difference $\delta I = I[\psi + \delta\psi] - I[\psi]$ is

$$\delta I = \int_V \nabla\psi \cdot \nabla(\delta\psi) \, d^3x + \int_V g\delta\psi \, d^3x + \cdots \tag{1.64}$$

The neglected term is semipositive definite and is second order in $\delta\psi$. Use of Green's first identity with $\phi = \delta\psi$ and $\psi = \psi$ yields

$$\delta I = \int_V [-\nabla^2\psi - g] \, \delta\psi \, d^3x + \oint_S \delta\psi \frac{\partial\psi}{\partial n} \, da \tag{1.65}$$

Provided $\delta\psi = 0$ on the boundary surface S (so that the surface integral vanishes), the first-order change in $I[\psi]$ vanishes if $\psi(\mathbf{x})$ satisfies

$$\nabla^2\psi = -g \tag{1.66}$$

Recalling that the neglected term in (1.64) is semipositive definite, we see that $I[\psi]$ is a stationary minimum if ψ satisfies a Poisson-like equation within the volume V and the departures $\delta\psi$ vanish on the boundary. With $\psi \to \Phi$ and $g \to \rho/\epsilon_0$, the minimization of the functional yields the "equation of motion" of the electrostatic potential in the presence of a charge density and Dirichlet boundary conditions (Φ given on S and so $\delta\Phi = 0$ there).

The derivation of the Poisson equation from the variational functional is the formal aspect. Equally important, the stationary nature of the extremum of $I[\psi]$ permits a practical approach to an approximate solution for $\psi(\mathbf{x})$. We choose a flexible "trial" function $\psi(\mathbf{x}) = A\Psi(\mathbf{x}, \alpha, \beta, \ldots)$ that depends on a normalization constant A and some number of other parameters, $\alpha, \beta, \ldots$, and is constructed to satisfy the given boundary conditions on the surface S. The function Ψ may be a sum of terms with the parameters as coefficients, or a single function of several parameters; it should be chosen with some eye toward the expected form of the solution. (Intuition plays a role here!) Calculation of $I[\psi]$ gives the function, $I(A, \alpha, \beta, \ldots)$. We now vary the parameters to locate the extremum (actually a minimum) of $I(A, \alpha, \beta, \ldots)$. With the optimum parameters, the trial solution is the best possible approximation to the true solution with the particular functional form chosen. For the Laplace equation, the normalization constant is determined by the Dirichlet boundary values of ψ. For the Poisson equation, it is determined by the source strength $g(\mathbf{x})$, as well as the boundary values on S.

A different functional is necessary for Neumann boundary conditions. Sup-

pose that the boundary conditions on ψ are specified by $\partial\psi/\partial n|_S = f(\mathbf{s})$, where $\mathbf{s}$ locates a point on the surface S. The appropriate functional is

$$I[\psi] = \frac{1}{2}\int_V \nabla\psi \cdot \nabla\psi \, d^3x - \int_V g\psi d^3x - \oint_S f\psi \, da \tag{1.67}$$

The same steps as before with $\psi \to \psi + \delta\psi$ lead to the first-order difference in functionals,

$$\delta I = \int_V [-\nabla^2\psi - g] \, \delta\psi \, d^3x + \oint_S \left(\frac{\partial\psi}{\partial n} - f(\mathbf{s})\right) \delta\psi \, da \tag{1.68}$$

The requirement that δI vanish independent of $\delta\psi$ implies

$$\nabla^2\psi = -g \text{ within } V \qquad \text{and} \qquad \frac{\partial\psi}{\partial n} = f(\mathbf{s}) \text{ on } S \tag{1.69}$$

Again the functional is a stationary minimum for ψ satisfying (1.69). Approximate solutions can be found by the use of trial functions that satisfy the Neumann boundary conditions, just as described above for Dirichlet boundary conditions.

As a simple application to the Poisson equation, consider the two-dimensional problem of a hollow circular cylinder of unit radius centered on the z-axis, with an interior source density $g(\mathbf{x}) = g(\rho)$, azimuthally symmetric and independent of z. The potential vanishes at $\rho = 1$. The "equation of motion" for ψ (a function of ρ alone) in polar coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) = -g(\rho) \tag{1.70}$$

For trial functions we consider finite polynomials in powers of $(1 - \rho)$ and ρ. A three-parameter function of the first type is

$$\Psi_1 = \alpha_1(1 - \rho) + \beta_1(1 - \rho)^2 + \gamma_1(1 - \rho)^3 \tag{1.71}$$

This choice might seem natural because it automatically builds in the boundary condition at $\rho = 1$, but it contains a flaw that makes it a less accurate representation of ψ than the power series in ρ. The reason is that, if the source density g is well behaved and finite at the origin, Gauss's law shows that ψ has a maximum or minimum there with vanishing slope. The requirements at both the origin and $\rho = 1$ are met by a three-parameter trial function in powers of ρ:

$$\Psi_2 = \alpha\rho^2 + \beta\rho^3 + \gamma\rho^4 - (\alpha + \beta + \gamma) \tag{1.72}$$

We expect this trial function in general to be a better approximation to ψ than Ψ_1 for the same number of variational parameters. [We could, of course, impose the constraint, $\alpha_1 + 2\beta_1 + 3\gamma_1 = 0$ on (1.71) to get the proper behavior at the origin, but that would reduce the number of parameters from three to two.]

The functional integral (1.63) for Ψ_2 is easily shown to be

$$\frac{1}{2\pi} I[\Psi_2] = \left[\frac{1}{2}\alpha^2 + \frac{6}{5}\alpha\beta + \frac{4}{3}\alpha\gamma + \frac{3}{4}\beta^2 + \frac{12}{7}\beta\gamma + \gamma^2\right] - [e_2\alpha + e_3\beta + e_4\gamma] \tag{1.73}$$

where $e_n = \int_0^1 g(\rho)(\rho^n - 1)\,\rho\,d\rho$.

The integral for Ψ_1 has the same form as (1.73), but different coefficients. As described above, we seek an extremum of (1.73) by setting the partial derivatives with respect to the parameters α, β, and γ equal to zero. The three coupled algebraic linear equations yield the "best" values,

$$\begin{aligned} \alpha &= 225e_2 - 420e_3 + 210e_4 \\ \beta &= -420e_2 + \frac{2450}{3} e_3 - 420e_4 \\ \gamma &= 210e_2 - 420e_3 + \frac{441}{2} e_4 \end{aligned} \tag{1.74}$$

These values can be inserted into (1.73) to give $I[\Psi_2]_{min}$ as a not very illuminating function of the e_n. One would then find that the "kinetic" (first) bracket was equal to half the "potential" (second) bracket and opposite in sign, a characteristic of the extremum.

To go further we must specify $g(\rho)$. The results for the best trial functions Ψ_1 and Ψ_2 are shown in Fig. 1.9 for the source density,

$$g(\rho) = -5(1 - \rho) + 10^4\rho^5(1 - \rho)^5 \tag{1.75}$$

The choice of source is arbitrary and is chosen to give a potential that is not quite featureless. The "best" parameters for Ψ_2 are $\alpha = 2.915$, $\beta = -7.031$, and $\gamma = 3.642$. The variational integral has the value, $I[\Psi_2]_{\min} = -1.5817$, compared to $I[\psi]_{\text{exact}} = -1.6017$. The fractional error is 1.3%.

Note that the trial function Ψ_1 fails rather badly for $\rho < 0.3$ because it does

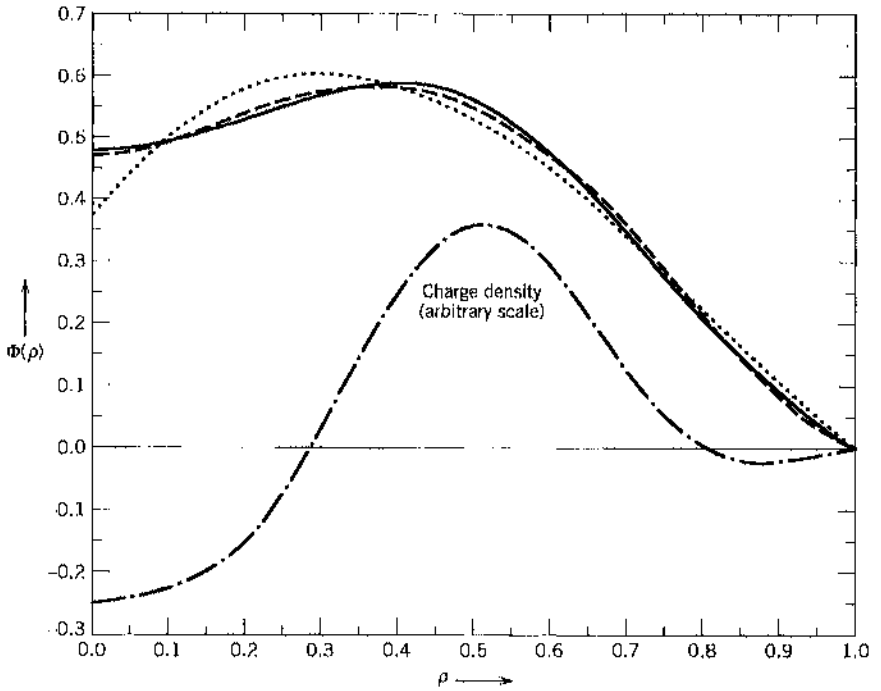


Figure 1.9 Comparison of the exact solution $\psi(\rho)$ (solid curve) with two variational approximations for the potential, Ψ_1 (dotted curve) and Ψ_2 (dashed curve). The charge density (1.75) is indicated by the dash-dot curve (arbitrary scale).

not respect the vanishing slope at $\rho = 0$. Nonetheless, it gives $I[\Psi_1]_{\min} = -1.5136$, which is somewhat, but not greatly, worse than Ψ_2 (5.5% error). The insensitivity of $I[\Psi]$ to errors in the trial function illustrates both a strength and a weakness of the variational method. If the principle is used to estimate eigenvalues (related to the value of $I[\Psi]$), it does well. Used as a method of estimating a solution $\psi \approx \Psi$, it can fail badly, at least in parts of the configuration space.

The reader will recognize from (1.70) that a polynomial source density leads to an exact polynomial solution for ψ, but the idea here is to illustrate the variational method, not to demonstrate a class of explicit solutions. Further illustration is left to the problems at the end of this and later chapters.

1.13 Relaxation Method for Two-Dimensional Electrostatic Problems

The relaxation method is an iterative numerical scheme (sometimes called iterative finite difference method) for the solution of the Laplace or Poisson equation in two dimensions. Here we present only its basic ideas and its connection with the variational method. First we consider the Laplace equation with Dirichlet boundary conditions within a two-dimensional region S with a boundary contour C. We imagine the region S spanned by a square lattice with lattice spacing h (and the boundary contour C approximated by a step-like boundary linking lattice sites along C). The independent variables are the integers (i, j) specifying the sites; the dependent variables are the trial values of the potential $\psi(i, j)$ at each site. The potential values on the boundary sites are assumed given.

To establish the variational nature of the method and to specify the iterative scheme, we imagine the functional integral $I[\psi]$ over S as a sum over small domains of area h^2, as shown in Fig. 1.10*a*. We consider the neighboring trial values of the potential as fixed, while the value at the center of the subarea is a variational quantity to be optimized. The spacing is small enough to permit us to approximate the derivatives in, say, the northeast quarter of the subarea by

$$\left(\frac{\partial\psi}{\partial x}\right)_{\text{NE}} = \frac{1}{h}(\psi_{\text{E}} - \psi_0); \qquad \left(\frac{\partial\psi}{\partial y}\right)_{\text{NE}} = \frac{1}{h}(\psi_{\text{N}} - \psi_0)$$

and similarly for the other three quarters. The functional integral over the northeast quarter is

$$\begin{aligned} I_{\text{NE}} &= \frac{1}{2}\int_0^{h/2} dx \int_0^{h/2} dy \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \\ &= \frac{1}{8}\left[(\psi_0 - \psi_{\text{N}})^2 + (\psi_0 - \psi_{\text{E}})^2\right] \end{aligned} \tag{1.76}$$

The complete integral over the whole (shaded) subarea is evidently

$$I = \frac{1}{4}\left[(\psi_0 - \psi_{\text{N}})^2 + (\psi_0 - \psi_{\text{E}})^2 + (\psi_0 - \psi_{\text{S}})^2 + (\psi_0 - \psi_{\text{W}})^2\right] \tag{1.77}$$

Minimizing this integral with respect to ψ_0 gives the optimum value,

$$(\psi_0)_{\text{optimum}} = \frac{1}{4}(\psi_{\text{N}} + \psi_{\text{E}} + \psi_{\text{S}} + \psi_{\text{W}}) \tag{1.78}$$

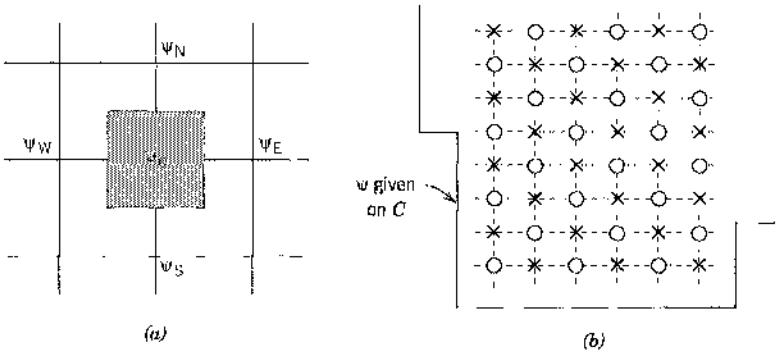


Figure 1.10 (*a*) Enlargement of one of the subareas in the functional integral (shaded). The trial values of the potential at the neighboring sites are labeled ψ_N, ψ_S, ψ_E, and ψ_W, while the value at the center of the subarea is ψ_0. (*b*) One possible iteration is to replace the trial values at the lattice sites (○) with the average of the values at the surrounding sites (×).

The integral is minimized if ψ_0 is equal to the average of the values at the "cross" points.

Now consider the whole functional integral, that is, the sum of the integrals over all the subareas. We guess a set of $\psi(i, j)$ initially and approximate the functional integral $I[\psi]$ by the sum of terms of the form of (1.77). Then we go over the lattice and replace half the values, indicated by the circles in Fig. 1.10*b*, by the average of the points (crosses) around them. The new set of trial values $\psi(i, j)$ will evidently minimize $I[\psi]$ more than the original set of values; the new set will be closer to the correct solution. Actually, there is no need to do the averaging for only half the points—that was just a replication for half of the subareas of the process for Fig. 1.10*a*.

There are many improvements that can be made. One significant one concerns the type of averaging. We could have taken the average of the values at the corners of the large square in Fig. 1.10*a* instead of the "cross" values. Or we could take some linear combination of the two. It can be shown (see Problem 1.22) by Taylor series expansion of any well-behaved function $F(x, y)$ that a particular weighted average,

$$\langle\langle F(x, y)\rangle\rangle = \frac{4}{5}\langle F\rangle_c + \frac{1}{5}\langle F\rangle_s \tag{1.79}$$

where the "cross" and "square" averages are

$$\langle F(x, y)\rangle_c = \frac{1}{4}[F(x+h, y) + F(x, y+h) + F(x-h, y) + F(x, y-h)] \tag{1.80a}$$

$$\langle F(x, y)\rangle_s = \frac{1}{4}[F(x+h, y+h) + F(x+h, y-h) + F(x-h, y+h) + F(x-h, y-h)] \tag{1.80b}$$

yields

$$\langle\langle F(x, y)\rangle\rangle = F(x, y) + \frac{3}{10}h^2\nabla^2 F + \frac{1}{40}h^4\nabla^2(\nabla^2 F) + O(h^6) \tag{1.81}$$

In (1.81) the Laplacians of F are evaluated at (x, y). If $F(x, y)$ is a solution of the Laplace equation, the weighted averaging over the eight adjacent lattice sites in (1.79) gives F at the center with corrections only of order h^6. Instead of (1.78), which is the same as (1.80a), a better iteration scheme uses $\psi_{\text{new}}(i, j) = \langle\langle\psi(i, j)\rangle\rangle + O(h^6)$. With either the "cross" or "square" averaging separately, the error is $O(h^4)$. The increase in accuracy with $\langle\langle\psi\rangle\rangle$ is at the expense of twice as much computation for each lattice site, but for the same accuracy, far fewer lattice sites are needed: $\langle\langle N\rangle\rangle = O(\langle N\rangle^{2/3})$, where $\langle\langle N\rangle\rangle$ is the number of sites needed with $\langle\langle\psi\rangle\rangle$ and $\langle N\rangle$ is the corresponding number with the "cross" or "square" average.

Equation (1.81) has an added advantage in application to the Poisson equation, $\nabla^2\psi = -g$. The terms of order h^2 and h^4 can be expressed directly in terms of the specified charge density and the simplest approximation for its Laplacian. It is easy to show that the new value of the trial function at (i, j) is generated by

$$\psi_{\text{new}}(i, j) = \langle\langle\psi(i, j)\rangle\rangle + \frac{h^2}{5} g(i, j) + \frac{h^2}{10}\langle g(i, j)\rangle_c + O(h^6) \qquad (1.82)$$

where $\langle g\rangle_c$ is the "cross" average of g, according to (1.80a).

A basic procedure for the iterative numerical solution of the Laplace or Poisson equation in two dimensions with Dirichlet boundary conditions is as follows:

1. A square lattice spacing h is chosen and the lattice sites, including the sites on the boundary, are labeled in some manner [which we denote here as (i, j)].
2. The values of the potential at the boundary sites are entered in a table of the potential at all sites.
3. A guess is made for the values, called $\Phi_{\text{old}}(i, j)$, at all interior sites. A constant value everywhere is easiest. These are added to the table or array of "starting" values.
4. The first iteration cycle begins by systematically going over the lattice sites, one by one, and computing $\langle\langle\Phi(i, j)\rangle\rangle$ with (1.79) or one of the averages in (1.80). This quantity (or (1.82) for the Poisson equation) is entered as $\Phi_{\text{new}}(i, j)$ in a table of "new" values of the potential at each site. Note that the sites next to the boundary benefit from the known boundary values, and so their $\langle\langle\Phi\rangle\rangle$ values are likely initially to be closer to the ultimate values of the potential than those for sites deep in the interior. With each iteration, the accuracy works its way from the boundaries into the interior.
5. Once all interior sites have been processed, the set of $\Phi_{\text{old}}(i, j)$ is replaced by the set of $\Phi_{\text{new}}(i, j)$, and the iteration cycle begins again.
6. Iterations continue until some desired level of accuracy is achieved. For example, one might continue iterations until the absolute value of the difference of old and new values is less than some preassigned value at every interior site.

The scheme just outlined is called Jacobian iteration. It requires two arrays of values of the potential at the lattice sites during each iteration. A better scheme, called Gauss–Seidel iteration, employs a trivial change: one replaces $\Phi_{\text{old}}(i, j)$ with $\Phi_{\text{new}}(i, j)$ as soon as the latter is determined. This means that during an iteration one benefits immediately from the improved values. Typically, at any given site, $\langle\langle\Phi\rangle\rangle$ is made up half of old values and half of new ones, depending

on the path over the lattice. There are many other improvements possible – consult *Press et al.*, *Numerical Recipes*, or some of the references cited at the end of the chapter. The relaxation method is also applicable to magnetic field problems, as described briefly in Section 5.14.

References and Suggested Reading

On the mathematical side, the subject of delta functions is treated simply but rigorously by

Lighthill
Dennery and Kryzwicki

For a discussion of different types of partial differential equations and the appropriate boundary conditions for each type, see

Morse and Feshbach, Chapter 6
Sommerfeld, *Partial Differential Equations in Physics*, Chapter II
Courant and Hilbert, Vol. II, Chapters III–VI

The general theory of Green functions is treated in detail by

Friedman, Chapter 3
Morse and Feshbach, Chapter 7

The general theory of electrostatics is discussed extensively in many of the older books. Notable, in spite of some old-fashioned notation, are

Maxwell, Vol. 1, Chapters II and IV
Jeans, Chapters II, VI, VII
Kellogg

Of more recent books, mention may be made of the treatment of the general theory by Stratton, Chapter III, and parts of Chapter II.

Readers interested in variational methods applied to electromagnetic problems can consult

Cairo and Kahan
Collin, Chapter 4
Sadiku, Chapter 4

and

Pólya and Szegö

for elegant and powerful mathematical techniques.

The classic references to relaxation methods are the two books by R. V. Southwell:

Relaxation Methods in Engineering Science, Oxford University Press, Oxford (1940).
Relaxation Methods in Theoretical Physics, Oxford University Press, Oxford (1946).

Physicists will be more comfortable with the second volume, but much basic material is in the first. More modern references on relaxation and other numerical methods are

Sadiku
Zhou

Problems

1.1 Use Gauss's theorem [and (1.21) if necessary] to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density per unit area on the surface.

1.2 The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function

$$D(\alpha; x, y, z) = (2\pi)^{-3/2}\alpha^{-3} \exp\left[-\frac{1}{2\alpha^2}(x^2 + y^2 + z^2)\right]$$

Consider a general orthogonal coordinate system specified by the surfaces $u =$ constant, $v =$ constant, $w =$ constant, with length elements du/U, dv/V, dw/W in the three perpendicular directions. Show that

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(u - u')\,\delta(v - v')\,\delta(w - w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

1.3 Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

(d) The same as part (c), but using spherical coordinates.

1.4 Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n \geq -3$), has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

1.5 The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

1.6 A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of

(a) two large, flat, conducting sheets of area A, separated by a small distance d;

(b) two concentric conducting spheres with radii a, b ($b > a$);

(c) two concentric conducting cylinders of length L, large compared to their radii a, b ($b > a$).

(d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

1.7 Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d, which is large compared with either radius. Show that the capacitance per unit length is given approximately by

$$C \simeq \pi\epsilon_0 \left(\ln \frac{d}{a} \right)^{-1}$$

where a is the geometrical mean of the two radii.

Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of 1.2×10^{-11} F/m if the separation of the wires was 0.5 cm? 1.5 cm? 5.0 cm?

1.8 **(a)** For the three capacitor geometries in Problem 1.6 calculate the total electrostatic energy and express it alternatively in terms of the equal and opposite charges Q and $-Q$ placed on the conductors *and* the potential difference between them.

(b) Sketch the energy density of the electrostatic field in each case as a function of the appropriate linear coordinate.

1.9 Calculate the attractive force between conductors in the parallel plate capacitor (Problem 1.6a) and the parallel cylinder capacitor (Problem 1.7) for

(a) fixed charges on each conductor;

(b) fixed potential difference between conductors.

1.10 Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

1.11 Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 and R_2 are the principal radii of curvature of the surface.

1.12 Prove *Green's reciprocation theorem*: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi' \, d^3x + \int_S \sigma\Phi' \, da = \int_V \rho'\Phi \, d^3x + \int_S \sigma'\Phi \, da$$

1.13 Two infinite grounded parallel conducting planes are separated by a distance d. A point charge q is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. (*Hint*: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple.)

1.14 Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\mathbf{y}$ and $\phi = G(\mathbf{x}, \mathbf{y})$, $\psi = G(\mathbf{x}', \mathbf{y})$, with $\nabla^2_{\mathbf{y}} G(\mathbf{z}, \mathbf{y}) = -4\pi\delta(\mathbf{y} - \mathbf{z})$. Find an expression for the difference $[G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}', \mathbf{x})]$ in terms of an integral over the boundary surface S.

(a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\mathbf{x}, \mathbf{x}')$ must be symmetric in $\mathbf{x}$ and $\mathbf{x}'$.

(b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\mathbf{x}, \mathbf{x}')$ to show that $G_N(\mathbf{x}, \mathbf{x}')$ is not symmetric in general, but that $G_N(\mathbf{x}, \mathbf{x}') + F(\mathbf{x})$ is symmetric in $\mathbf{x}$ and $\mathbf{x}'$, where

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x}, \mathbf{y})\, da_y$$

(c) Show that the addition of $F(\mathbf{x})$ to the Green function does not affect the potential $\Phi(\mathbf{x})$. See problem 3.26 for an example of the Neumann Green function.

1.15 Prove *Thomson's theorem*: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.

1.16 Prove the following theorem: If a number of conducting surfaces are fixed in position with a given total charge on each, the introduction of an uncharged, insulated conductor into the region bounded by the surfaces lowers the electrostatic energy.

1.17 A volume V in vacuum is bounded by a surface S consisting of several separate conducting surfaces S_i. One conductor is held at *unit* potential and all the other conductors at zero potential.

(a) Show that the capacitance of the one conductor is given by

$$C = \epsilon_0 \int_V |\nabla\Phi|^2\, d^3x$$

where $\Phi(\mathbf{x})$ is the solution for the potential.

(b) Show that the true capacitance C is always less than or equal to the quantity

$$C[\Psi] = \epsilon_0 \int_V |\nabla\Psi|^2\, d^3x$$

where Ψ is any trial function satisfying the boundary conditions on the conductors. This is a variational principle for the capacitance that yields an *upper bound*.

1.18 Consider the configuration of conductors of Problem 1.17, with all conductors except S_1 held at zero potential.

(a) Show that the potential $\Phi(\mathbf{x})$ anywhere in the volume V *and* on any of the surfaces S_i can be written

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')\, da'$$

where $\sigma_1(\mathbf{x}')$ is the surface charge density on S_1 and $G(\mathbf{x}, \mathbf{x}')$ is the Green function potential for a point charge in the presence of all the surfaces that are held at zero potential (but with S_1 absent). Show also that the electrostatic energy is

$$W = \frac{1}{8\pi\epsilon_0}\oint_{S_2} da \oint_{S_1} da'\ \sigma_1(\mathbf{x})G(\mathbf{x}, \mathbf{x}')\sigma_1(\mathbf{x}')$$

where the integrals are only over the surface S_1.

(b) Show that the variational expression

$$C^{-1}[\sigma] = \frac{\displaystyle\oint_{S_1} da \oint_{S_1} da'\sigma(\mathbf{x})G(\mathbf{x}, \mathbf{x}')\sigma(\mathbf{x}')}{4\pi\epsilon_0\left|\displaystyle\oint_{S_1} \sigma(\mathbf{x})\, da\right|^2}$$

with an arbitrary integrable function $\sigma(\mathbf{x})$ defined on S_1, is stationary for small variations of σ away from σ_1. Use Thomson's theorem to prove that the

reciprocal of $C^{-1}[\sigma]$ gives a *lower bound* to the true capacitance of the conductor S_1.

1.19 For the cylindrical capacitor of Problem 1.6c, evaluate the variational upper bound of Problem 1.17b with the naive trial function, $\Psi_1(\rho) = (b - \rho)/(b - a)$. Compare the variational result with the exact result for $b/a = 1.5, 2, 3$. Explain the trend of your results in terms of the functional form of Ψ_1. An improved trial function is treated by *Collin* (pp. 275-277).

1.20 In estimating the capacitance of a given configuration of conductors, comparison with known capacitances is often helpful. Consider two configurations of n conductors in which the $(n - 1)$ conductors held at zero potential are the same, but the one conductor whose capacitance we wish to know is different. In particular, let the conductor in one configuration have a closed surface S_1 and in the other configuration have surface S_1', with S_1' totally inside S_1.

(a) Use the extremum principle of Section 1.12 and the variational principle of Problem 1.17 to prove that the capacitance C' of the conductor with surface S_1' is less than or equal to the capacitance C of the conductor with surface S_1 that encloses S_1'.

(b) Set upper and lower limits for the capacitance of a conducting cube of side a. Compare your limits and also their average with the numerical value, $C = 0.655(4\pi\epsilon_0 a)$.

(c) By how much do you estimate the capacitance per unit length of the two-wire system of Problem 1.7 will change (larger? smaller?) if *one* of the wires is replaced by a wire of square cross section whose side is equal to its diameter?

1.21 A two-dimensional potential problem consists of a unit square area ($0 \le x \le 1$, $0 \le y \le 1$) bounded by "surfaces" held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z).

(a) Apply the variational principle (1.63) for the Poisson equation with the "variational" trial function $\Psi(x, y) = A \cdot x(1 - x) \cdot y(1 - y)$ to determine the best value of the constant A. [I use quotation marks around variational because there are no parameters to vary except the overall scale.]

(b) The exact (albeit series) solution for this problem is [see Problems 2.15 and 2.16]

$$4\pi\epsilon_0\Phi(x, y) = \frac{16}{\pi^3}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)\pi x]}{(2m+1)^3}\left\{1 - \frac{\cosh[(2m+1)\pi(y-\frac{1}{2})]}{\cosh[(2m+1)\pi/2]}\right\}$$

For $y = 0.25$ and $y = 0.5$, plot and compare the simple variational solution of part a with the exact solution as functions of x.

1.22 Two-dimensional relaxation calculations commonly use sites on a square lattice with spacing $\Delta x = \Delta y = h$, and label the sites by (i, j), where i, j are integers and $x_i = ih + x_0$, $y_j = jh + y_0$. The value of the potential at (i, j) can be approximated by the average of the values at neighboring sites. [Recall the relevant theorem about harmonic functions.] But what average?

(a) If $F(x, y)$ is a well-behaved function in the neighborhood of the origin, but not necessarily harmonic, by explicit Taylor series expansions, show that the "cross" sum

$$S_c = F(h, 0) + F(0, h) + F(-h, 0) + F(0, -h)$$

can be expressed as

$$S_c = 4F(0, 0) + h^2\nabla^2 F + \frac{h^4}{12}(F_{xxxx} + F_{yyyy}) + O(h^6)$$

(b) Similarly, show that the "square" sum,

$$S_S = F(h, h) + F(-h, h) + F(-h, -h) + F(h, -h)$$

can be expressed as

$$S_S = 4F(0, 0) + 2h^2\nabla^2 F + \frac{h^4}{3}(F_{xxxx} + F_{yyyy}) + \frac{h^4}{2}\nabla^2(\nabla^2 F) + O(h^6)$$

Here F_{xxxx} is the fourth partial derivative of F with respect to x, evaluated at $x = 0$, $y = 0$, etc. If $\nabla^2 F = 0$, the averages $S_c/4$ and $S_s/4$ each give the value of $F(0, 0)$, correct to order h^3 inclusive. Note that an improvement can be obtained by forming the "improved" average,

$$\langle\langle F(0, 0)\rangle\rangle = \frac{1}{5}\left[S_c + \frac{1}{4}S_s\right]$$

where

$$\langle\langle F(0, 0)\rangle\rangle = F(0, 0) + \frac{3}{10}h^2\nabla^2 F + \frac{h^4}{40}\nabla^2(\nabla^2 F) + O(h^6)$$

If $\nabla^2 F = 0$, then $\tilde{S}$ gives $F(0, 0)$, correct to order h^5 inclusive. For Poisson's equation, the charge density and its lowest order Laplacian can be inserted for the same accuracy.

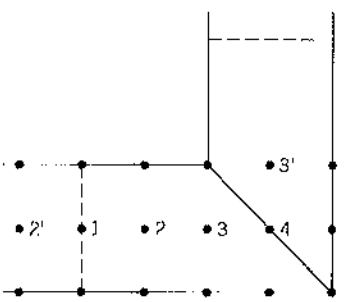
1.23 A transmission line consists of a long straight conductor with a hollow square region in its interior, with a square conductor of one-quarter the area of the hollow region centered in the empty space, with edges parallel to the inner sides of outer conductor. If the conductors are raised to different potentials, the potential and electric field in the space between them exhibit an eightfold symmetry; the basic unit is sketched in the accompanying figure. The efficacy of the relaxation method in determining the properties of the transmission line can be illustrated by a simple calculation.

(a) Using only the four interior points indicated in the figure, write down the relaxation equation for each point for the "cross" and the "improved" averaging schemes (defined in Problem 1.22) if the inner conductor has $\Phi = 100$ V and the outer has $\Phi = 0$. By performing either the relaxation iteration process or solving the set of algebraic equations for each scheme, find estimates for the potential at each of the four points for the two schemes.

(b) From the results of part a make the best estimate (or estimates) you can for the capacitance per unit length of the transmission line.

(c) (Optional) Using your favorite computational tools, repeat the relaxation calculation with half the lattice spacing (21 interior points) and compare.

Answer: $\Phi_1 = 48.87$ V, $\Phi_2 = 47.18$ V, $\Phi_3 = 38.34$ V, $\Phi_4 = 19.81$ V and $C = 10.23\ \epsilon_0$ F/m [from an accurate numerical calculation].



Problem 1.23

1.24 Consider solution of the two-dimensional Poisson equation problem of Problem 1.21, a unit square with zero potential on the boundary and a constant unit charge density in the interior, by the technique of relaxation. Choose $h = 0.25$ so that there are nine interior sites. Use symmetry to reduce the number of needed sites to three, at (0.25, 0.25), (0.5, 0.25), and (0.5, 0.5). With so few sites, it is easy to do the iterations with a block of paper and a pocket calculator, but suit yourself.

(a) Use the "improved grid" averaging of Problem 1.22 and the simple (Jacobian) iteration scheme, starting with $4\pi\epsilon_0\Phi = 1.0$ at all three interior sites. Do at least six iterations, preferably eight or ten.

(b) Repeat the iteration procedure with the same starting values, but using Gauss–Seidel iteration.

(c) Graph the two sets of results of each iteration versus iteration number and compare with the exact values, $4\pi\epsilon_0\Phi(0.25, 0.25) = 0.5691$, $4\pi\epsilon_0\Phi(0.5, 0.25) = 0.7205$, $4\pi\epsilon_0\Phi(0.5, 0.5) = 0.9258$. Comment on rate of convergence and final accuracy.