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Permutations and Combinations

1.1. DEFINITIONS

A permutation is an ordered selection of objects from a set S.
A combination is an unordered selection of objects from a set S.

We may or may not permit repetition in our permutations and combinations. Thus, selecting two letters from the three letters a, b, c, we have nine permutations with repetitions permitted:

$$aa, ab, ac, ba, bb, bc, ca, cb, cc.$$

We have six permutations without repetitions:

$$ab, ac, ba, bc, ca, cb.$$

We have six combinations with repetitions permitted:

$$aa, bb, cc, ab, ac, bc,$$

and three combinations without repetitions:

$$ab, ac, bc.$$

The number of permutations of n things taken r at a time, without repetition, written ${}_nP_r$, is easily evaluated. For in a permutation $a_1 a_2 \cdots a_r$ we may choose a_1 as any of the n objects, a_2 as any one of the remaining $(n-1)$ objects, and having chosen $a_1 a_2 \cdots a_i$, we may take a_{i+1} as any one of the $(n-i)$ remaining objects. Hence,

$$ {}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!} = (n)_r. \tag{1.1.1}$$

A combination of n things taken r at a time without repetition, say, $a_1a_2\cdots a_r$, will lead to $r!$ different permutations, namely, all $r!$ permutations of $a_1, \ldots, a_r$. Hence the number of combinations of n things taken r at a time, written ${}_nC_r$, is given by

$$
{}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}, \tag{1.1.2}
$$

which is the familiar binomial coefficient. Indeed, in the product $(x+y)^n = (x+y)\cdots(x+y)$, the coefficient of the term x^ry^{n-r} is the number of ways of choosing r of the factors $x+y$, from which we take an x, and then y from the remaining $n-r$ factors $x+y$. We note that

$$
{}_nC_r = {}_nC_{n-r}. \tag{1.1.3}
$$

The number of permutations of n things taken r at a time, repeats permitted, is n^r, since in $a_1a_2\cdots a_r$ there are n choices for each of $a_1, a_2, \ldots, a_r$ in turn.

To find the number of combinations of n things taken r at a time with repeats permitted, we cannot simply divide n^r by an appropriate factor, since different combinations may yield a different number of permutations. Thus, taking combinations of a, b, c, d, e three at a time, the combination abc gives six permutations, the combination aab gives three permutations, and the combination aaa gives only one permutation. Here we use the device of counting a different set, which is in one-to-one correspondence with our given set. To a given combination (say, bbd), let us adjoin the entire set $abcde$ and write the whole set in order $abbbcdde$ and then insert marks separating the different letters thus: $a\,|\,bbb\,|\,c\,|\,dd\,|\,e$. In general, to a combination of r letters, repeats permitted, from a set of n letters adjoin all n letters and write in order the set of $(n+r)$ letters and then insert $(n-1)$ marks between the different letters. Thus, with $(n+r)$ positions to be filled and $(n+r-1)$ spaces between these positions, we are to insert $(n-1)$ marks. The number of ways of doing this is

$$
\binom{n+r-1}{n-1} = \binom{n+r-1}{r}. \tag{1.1.4}
$$

There is a one-to-one correspondence between the ways of inserting the $(n-1)$ marks in the $(n+r-1)$ spaces and the combinations with repeats of n things taken r at a time. Hence, this number is $\binom{n+r-1}{r}$. The expression for the number of combinations, n things taken r at a time, without repeats and with repeats, are similar in form. Thus, for five things taken three at a time, the numbers are, respectively,

$$
\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} \quad \text{and} \quad \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}.
$$

where the factors in the numerators decrease in one case and increase in the other.

The number of combinations with repeats permitted of n things taken r at a time is the number of solutions $(x_1, x_2, \ldots, x_n)$ in nonnegative integers x_i of

$$r = x_1 + x_2 + \cdots + x_n, \tag{1.1.5}$$

where x_i is the number of times the ith object is included in the combination. This suggests another, but similar, evaluation of the number of combinations. Put $y_i = x_i + 1$, $i = 1, \ldots, n$. Then (1.1.5) becomes

$$n + r = y_1 + y_2 + \cdots + y_n, \tag{1.1.6}$$

and, the number of solutions $(y_1, \ldots, y_n)$ of (1.1.6) in positive integers y_i is clearly the same as the number of solutions of (1.1.5) in nonnegative integers. If we take $(n + r)$ dots and place $(n - 1)$ marks in the $(n + r - 1)$ spaces between the dots, we may take y_1 as the number in the first set of dots, y_2 as the number in the second set, and so on. Thus, again we see that the number of solutions of (1.1.6) is $\binom{n+r-1}{n-1}$, and this is in turn the number of nonnegative solutions of (1.1.5), and so it is the number of combinations of n things r at a time with repeats permitted.

As a further application of this method we may find the number of combinations of $1, 2, \ldots, n$ taken r at a time without including repeats or consecutive numbers. Let us list $1, 2, \ldots, n$ in order and put a mark after each number selected. If there are x_1 numbers before the first mark, x_2 between the first and second mark, and finally x_{r+1} after the last mark, then these determine the choice and

$$n = x_1 + x_2 + \cdots + x_{r+1}, \tag{1.1.7}$$

where $x_1 \geqslant 1$, $x_2 \geqslant 2, \ldots, x_r \geqslant 2$, and $x_{r+1} \geqslant 0$. We now write

$$n - r + 2 = x_1 + (x_2 - 1) + \cdots + (x_r - 1) + (x_{r+1} + 1), \tag{1.1.8}$$

giving a representation of $n - r + 2$ as a sum of $(r + 1)$ positive integers, and this number is $\binom{n-r+1}{r}$, the number of ways of putting r marks in $(n - r + 1)$ spaces.

There are an enormous number of identities involving binomial coefficients, a few of which follow:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n; \tag{1.1.9a}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0, & n > 0 \\ 1, & n = 0; \end{cases} \tag{1.1.9b}$$

$$\sum_{k=1}^{n} k(-1)^k \binom{n}{k} = \begin{cases} 0, & n > 1 \\ k, & n = 1; \end{cases} \tag{1.1.9c}$$

$$\sum_{k=r}^{n} (-1)^k \binom{k}{r}\binom{n}{k} = \begin{cases} 0, & n > r \\ (-1), & n = r. \end{cases} \tag{1.1.9d}$$

These may be derived from the relation

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

To obtain (1.1.9a) and (1.1.9b), put $x = 1$ and $x = -1$, respectively. For (1.1.9c), differentiate with respect to x and then put $x = -1$. For (1.1.9d), differentiate r times with respect to x, divide by $r!$, and put $x = -1$.

If we have permutations $a_1 a_2 \cdots a_n$ of n objects, of which b_1 are of one kind, b_2 of a second kind, and b_i of an ith kind for i running to r, where naturally $b_1 + b_2 + \cdots + b_r = n$, we may first replace the b_i objects of the ith kind by distinct objects in every case, and then we have $n!$ permutations. But by identifying the like objects, we have counted each permutation $b_1! b_2! \cdots b_r!$ times. Hence, the number of permutations is

$$\frac{n!}{b_1! b_2! \cdots b_r!}, \qquad b_1 + b_2 + \cdots + b_r = n, \tag{1.1.10}$$

the familiar multinomial coefficient.

1.2. APPLICATIONS TO PROBABILITY

In a given situation let us suppose that there are n possible outcomes, which we label $x_1, x_2, \ldots, x_n$, and which are mutually exclusive. We assign to the outcome x_i a number $p_i = p(x_i)$, where p_i is a real number, $p_i \geqslant 0$, and $p_1 + p_2 + \cdots + p_n = 1$. If an event E occurs along with the possibilities $x_{i_1}, \ldots, x_{i_m}$ and not otherwise, we define the probability of E as $p(E) = p_{i_1} + \cdots + p_{i_m}$. The assignment of the initial probabilities $p_1, p_2, \ldots, p_n$ is not a mathematical problem, but is an estimate of the relative likelihoods of the different outcomes, and in any actual case the validity of the mathematical calculation of $p(E)$ depends on the correctness of this assignment.

There are many practical situations in which it seems reasonable to consider the n outcomes as equally likely, and so we take $p_1 = p_2 \cdots = p_n = 1/n$. In this case the probability of the event E occurring along with m possible outcomes, but no others, is $p(E) = m/n$. In such a situation the calculation of $p(E)$ becomes the purely combinatorial problem of calculating m, the number of possible outcomes yielding the event E. In throwing at random a die whose six faces are numbered from 1 to 6, it seems reasonable to assume that any face is as likely to come on top as any other, if the die is of uniform density. In this case, we take $p_1 = p_2 = \cdots = p_6 = 1/6$, where p_i is the probability that the face numbered i will come up. If we are merely interested in whether or not a 6 comes up, we consider only two possible outcomes, putting $p = 1/6$ as the probability for a 6 and a $p' = 5/6$ as the probability of not getting a 6.

Let us suppose we have N urns numbered from 1 to N. We are to place at random n balls in the urns, where $n < N$. We ask the probability that each of the urns numbered 1 to n will contain exactly one ball. This probability depends on two things: (1) whether the balls are distinguishable or indistinguishable, and (2) whether there is an exclusion principle that does not allow a second ball to be placed in an urn that already contains one ball. If the n balls are distinguishable and there is no exclusion principle, there will be N^n ways of placing the n balls in the N urns. There will be $n!$ ways of placing them in the urns numbered $1, \ldots, n$, placing one in each of these urns. With these conventions, the probability is

$$p(E) = \frac{n!}{N^n}. \tag{1.2.1}$$

If the balls are distinguishable and there is an exclusion principle, the first ball may be placed in any one of N urns, the next in any one of $(N - 1)$ urns, and the ith in any one of $(N - i + 1)$ urns, whence the number of ways of placing the n balls in the N urns is ${}_NP_n = N(N - 1)\cdots(N - n + 1)$. They may be placed in urns $1, \ldots, n$ in $n!$ ways, and under these conventions, the probability is

$$p(E) = \frac{n!}{{}_NP_n} = \frac{1}{\binom{N}{n}}. \tag{1.2.2}$$

If the balls are indistinguishable and there is no exclusion principle, we are asking for the solutions of $x_1 + x_2 + \cdots + x_N = n$ in nonnegative integers x_i, where x_i is the number of balls placed in the ith urn. This, as we noted in the preceding section, is the number of combinations of N things taken n at a time with repeats permitted, and is $\binom{N+n-1}{n}$. Exactly one of these is the solution $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = x_N = 0$; in this case our probability is

$$p(E) = \frac{1}{\binom{N+n-1}{n}}. \tag{1.2.3}$$

From the physical standpoint, "indistinguishable" means that one combination is as likely as another.

If the balls are indistinguishable and there is an exclusion principle, the number of ways of placing the balls is merely the number of combinations of N things taken n at a time without repeats, and this is ${}_NC_n = \binom{N}{n}$. The choice of the first n urns is a single one of these combinations, and here the probability is

$$p(E) = \frac{1}{\dbinom{N}{n}}. \tag{1.2.4}$$

Note that this is the same as (1.2.2), so that with an exclusion principle, the probability is the same whether the balls are distinguishable or not.

In statistical physics we consider a collection of n particles, which may be protons, electrons, mesons, neutrons, neutrinos, or photons, each of which may be in any of N "states," which may be energy levels. The macroscopic state of the system of the n particles is a vector $x = (x_1, x_2, \ldots, x_N)$, where x_i is the number of particles in the ith state. The probability of any single macroscopic state depends on whether or not the particles are distinguishable and whether or not the particles obey the Pauli exclusion principle, which says that no two (indistinguishable) particles may be in the same state. If the particles are considered distinguishable and do not obey the exclusion principle, the probability of any single macroscopic state is given by (1.2.1) and the particles are said to obey the Maxwell–Boltzmann statistics. If the particles are indistinguishable and do not obey the exclusion principle, the probability is given by (1.2.3), and they are said to obey the Bose–Einstein statistics. If they are indistinguishable and do obey the exclusion principle, the probability is given by (1.2.4), and the particles are said to obey the Fermi–Dirac statistics. Electrons, protons, and neutrons obey Fermi–Dirac statistics. Photons and pi-mesons obey Bose–Einstein statistics. The case (1.2.2) of distinguishable particles with an exclusion principle does not arise in physics.

At high temperatures, when the number N is large and the different microscopic states are approximately equally likely, the Fermi–Dirac and Bose–Einstein statistics are essentially the same as the classical Maxwell–Boltzmann statistics. At low temperatures, the low-energy levels are more likely than the high-energy ones, and then the preceding models must be modified accordingly.

PROBLEMS

1. Prove

$$\sum_{i=0}^{m} \binom{r}{i}\binom{s}{m-i} = \binom{r+s}{m}.$$

Hint: $(1 + x)^r(1 + x)^s = (1 + x)^{r+s}$. Give an alternate proof of this identity by considering the number of ways of choosing a committee of m people out of a group of r men and s women.

2. A flag is to be designed with 13 horizontal stripes colored red, white, or blue, subject to the condition that no stripe be of the same color as the one above it. In how many ways may this be done?

3. How many positive integers less than 10^n (in the decimal scale) have their digits in nondecreasing order?

4. In how many ways may n identical gifts be given to r children (a) under no restriction and (b) if each child must receive at least one gift?

5. A hand of five cards is selected from a deck of $4n$ cards that contains four different suits, each with n cards, $n \geqslant 5$, numbered $1, \ldots, n$. Rank in order of increasing frequency, depending on the value of n, the following hands: a straight flush (five consecutively numbered cards of the same suit), four of a kind (four cards having the same number), full house (three cards of one number, the other two of another number), flush (five cards of the same suit), straight (five consecutively numbered cards), three of a kind (three cards of the same number), two pair (two cards of one number, two others of a second number), and a pair (two cards of a number).