1

FUNDAMENTALS OF MACHINE VIBRATION AND CLASSICAL SOLUTIONS

This chapter is focused on practical applications of mechanical vibrations theory. The reader may want to supplement the chapter with one of the vibration textbooks in the reference list at the end of the chapter if he has no background in the theory.

THE MAIN SOURCES OF VIBRATION IN MACHINERY

The most common sources of vibration in machinery are related to the inertia of moving parts in the machine. Some parts have a reciprocating motion, accelerating back and forth. In such a case Newton's laws require a force to accelerate the mass and also require that the force be reacted to the frame of the machine. The forces are usually periodic and therefore produce periodic displacements observed as vibration. For example, the piston motion in the slider-crank mechanism of Fig. 1-1 has a fundamental frequency equal to the crankshaft speed but also has higher frequencies (harmonics). The dominant harmonic is twice crankshaft speed (2nd harmonic). Figure 1-2a shows the displacement of the piston. It looks almost like a sine wave but it is slightly distorted by higher-order harmonics due to the nonlinear kinematics of the mechanism. Fig. 1-2b shows the acceleration of the piston, where the 2nd harmonic is amplified since the acceleration amplitude is frequency-squared times the displacement amplitude.

Even without reciprocating parts, most machines have rotating shafts and wheels that cannot be perfectly balanced, so according to Newton's laws, there must be a rotating force vector at the bearing supports of each

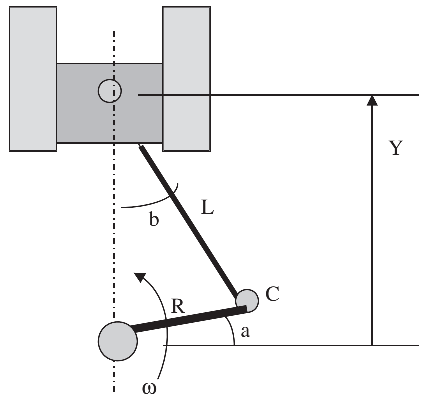

Figure 1-1 Slider-crank mechanism.

rotor to produce the centripetal acceleration of the mass center. Most of these force vectors are rotating and therefore produce a rotating displacement vector (all real machine parts are elastic) that can be observed as an orbit if two orthogonal vibration transducers are employed. Each of the transducers will produce a time trace similar to Fig. 1-2a or 1-2b. Harmonics and resulting distortion similar to Fig. 1-2a and 1-2b can be produced by shaft misalignment or by nonlinearity of the bearing stiffness. The fundamental frequency of the X and Y (orthogonal) vibration vectors is shaft speed ω, so the fundamental vibration is $x(t) = X\cos(\omega t)$ and $y(t) = Y\sin(\omega t)$. This type of vibration is referred to as *forced response* or *synchronous response to unbalance*. The vibration amplitude can become very large if the excitation frequency (rotor speed for example) becomes close to one of the natural frequencies of the machine structure. This is called a *resonance* or a *critical speed*, but it is not an unstable motion since the amplitude does not grow with time (unless there is no damping).

Another type of machine vibration problem, less common but more difficult to deal with, can come from the characteristic natural vibration frequencies (eigenvalues) of the machine structure and its supports, even if no imbalance or excitation is present. Natural frequencies die out in static structures due to the energy dissipated by damping, but in rotating machines they can grow larger with time. This is known as *self-excited instability* or *rotordynamic instability*. It is an innate potential characteristic of some rotating machines, especially when fluid pressures are present (e.g., bearings, impellers, turbine wheels, or seals).

Every real structure has an infinite number of natural frequencies, but many machinery vibration problems involve just one of these frequencies. That is why the simple single degree of freedom (SDOF) model (with just one natural frequency) presented in vibration textbooks [1–3] can be

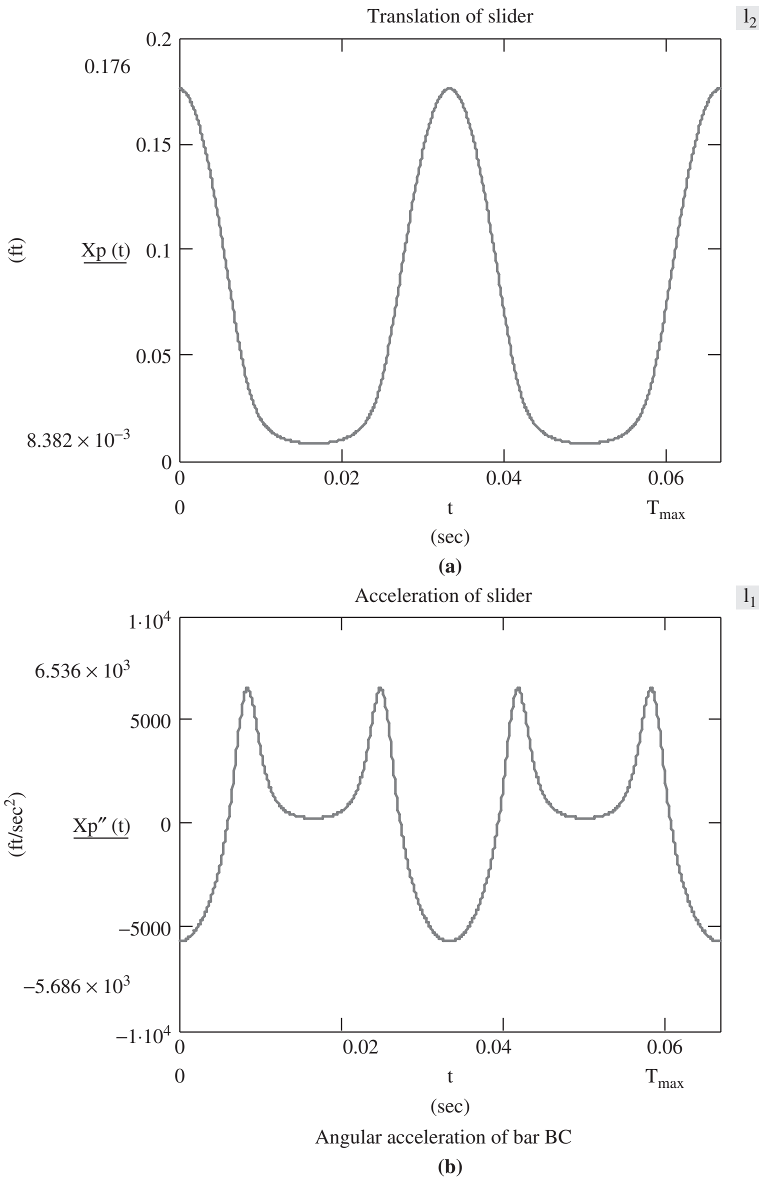

Figure 1-2 (a) Displacement of the piston, and (b) acceleration of the piston.

useful for analyzing vibration in machines. In fact, a SDOF model, consisting of one rigid mass, one spring, and one damper can be constructed to represent the vibration characteristics of any real machine in the neighborhood of a particular natural frequency of interest. This is called a *modal model*. To make physical sense out of complex machinery vibration data, or from realistic computer simulations of machinery vibration, the details of the SDOF mathematical model, its variations, and its solutions must be burned indelibly into the mind of the vibration engineer.

THE SINGLE DEGREE OF FREEDOM (SDOF) MODEL

The SDOF model as seen in most vibration textbooks is shown in Fig. 1-3. Here it will be referred to as system A. The stiffness, damping, and mass are k, c, and m, respectively. The undamped natural frequency is given by

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{rad/sec} \tag{1-1}$$

The circular frequency ω_n can be converted to hertz (Hz) (cycles/sec) as $f_n = \omega_n/2\pi$, or to revolutions per minute (rpm) as $N = 60 f_n$.

With a sinusoidal force applied to the mass, the differential equation of motion

$$m\ddot{x} + c\dot{x} + kx = F\sin(\omega t) \tag{1-2}$$

has a solution made up of two parts: (1) the particular solution for x that gives $F\sin(\omega t)$ on the right-hand side, and (2) the homogeneous solution for x that gives zero on the right-hand side. The sum of the two solutions, of course, gives $F\sin(\omega t)$, which satisfies the equality sign. The two solutions represent the two types of machine vibration described in the previous section, that is, forced response and characteristic (free) vibration. The particular solution for forced response is

$$x_p(t) = F\sin(\omega t + \phi)\Big/\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2} \tag{1-3}$$

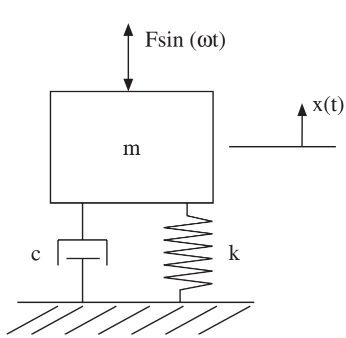

Figure 1-3 Single degree of freedom vibration model (system A).

Notice that the frequency ω of the forced vibration response is the same as the frequency of the excitation. The angle ϕ gives the time ϕ/ω by which the response x lags the excitation force F. For analyzing a vibration problem it is important to understand how k, c, and m influence the response amplitude. They have different effects depending on the frequency ratio ω/ω_n, as we shall see in the section to follow. Looking at Eq. 1-3 we can see that the amplitude X of the forced vibration response is

$$X = F\Big/\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2} \tag{1-4}$$

which depends on k, c, m, ω, and F. Notice that the denominator gets small when the exciting frequency ω is ω_n (Eq. 1-1) unless the damping coefficient c is large. A plot of Eq. 1-4 is shown in Fig. 1-7. It is called the *Bode amplitude plot* or the *frequency response plot* for system A.

The homogeneous part of the solution (for free vibration) with $F = 0$ is given by

$$x_h(t) = Ae^{st} \tag{1-5}$$

where s is a complex number, $s = \lambda + i\omega_d$. s is called the *eigenvalue*. Using the law of exponents, Eq. 1-5 can be rewritten as

$$x_h(t) = Ae^{\lambda t}e^{i\omega_d t} \tag{1-6}$$

where

$$e^{i\omega_d t} = \cos(\omega_d t) + i\,\sin(\omega_d t) \tag{1-7}$$

Equation 1-5 or 1-6 satisfies the differential Eq. 1-2 with $F = 0$ provided that the real part of the eigenvalue is $\lambda = -c/2m$ and the imaginary part is the square root of $\omega_d^2 = k/m - (c/2m)^2$. The amplitude A in Eq. 1-5 is of little interest here since it is determined only by the initial condition that instigates the free vibration. In rotating machinery, the differential equations are more complicated but still are of the same class as (1-2) and have the same form of homogeneous solution as (1-5). The imaginary part of s, ω_d, is the damped natural frequency. Notice that it becomes equal to ω_n, Eq. 1-1, when the damping coefficient $c = 0$.

The real part λ of the eigenvalue s determines how fast the free vibration dies out. It is often converted into a *damping ratio* $\zeta = c/c_{\text{cr}}$, where the critical damping $c_{\text{cr}} = 2m\omega_n$. Critical damping is the amount required to prevent free vibration (and no more). The conversion equation is $\zeta = -\lambda/\omega_n$. Figure 1-4a shows free vibration with $\zeta = 0.05$ (5% of critical damping); Fig. 1-4b shows the same system with $\zeta = 0.25$ (25% of critical damping). If a free vibration is graphed like Fig. 1-4, the damping can be expressed as the natural logarithm of

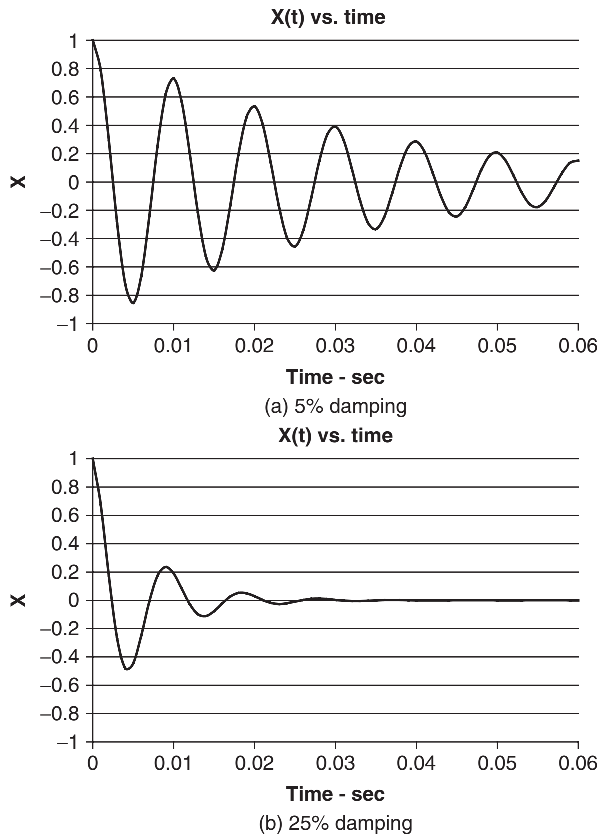

Figure 1-4 (a) Free vibration with 0.05 damping ratio; (b) free vibration with 0.25 damping ratio.

the ratio of successive amplitudes X_n/X_{n+1}. The logarithmic decrement $\delta = \ln(X_n/X_{n+1}) = 2\pi\zeta/\left(1+\zeta^2\right)^{1/2}$. The inverse expression is often useful: $\zeta = \delta/\left[(2\pi)^2 + \delta^2\right]^{1/2}$.

The algebraic sign of the real part of the eigenvalue λ is the mathematical test for vibration stability, i.e., whether the free vibration of frequency ω_d will die out or, in the unstable case, will grow with time. For example, in the simple system of Fig. 1-3, λ becomes positive if the damping c is negative. Negative damping is possible in mechanical systems, especially when fluid pressures are acting.

USING SIMPLE MODELS FOR ANALYSIS AND DIAGNOSTICS

Techniques and methods for solving vibration problems can often be developed by using the simple one degree of freedom model even though the real system is more complicated. The main purpose of the model is to provide an understanding of the type of problem being encountered so that the most effective type of "fix" can be identified. Sometimes a simple model can even yield useful approximations for the optimum parametric values, such as stiffness and damping to be employed. In contrast to the large and detailed finite element models being promoted by some for all diagnostic vibration analysis, this approach suggests that the engineer should first use the simplest possible model that contains the relevant physical characteristics and resort to the more detailed models only when the simple models do not yield sufficient guidance for modifications to the design or when improved accuracy is desired.

In addition to system A of Fig. 1-3, two more single degree of freedom models are shown in Figs. 1-5 and 1-6. All three of these systems have a single natural frequency determined by their modal mass and stiffness, but there are subtle differences between the three models that are related to the type of excitation.

The constant amplitude exciting force F in system A is generally unrealistic. Inertia forces in rotating machinery are proportional to speed squared. Model C in Fig. 1-6 has an unbalanced rotor so that the exciting force $F = m\omega^2 u$, where u is the offset of the center of rotor mass m from the axis of rotation. Note that the mass m is the rotating mass, not the total mass, so m on the left side of differential equation (1-2) must be replaced by the total mass M unless the nonrotating mass is negligible.

In some cases the excitation is a vibration displacement at the base, rather than a force. This is represented by system B in Fig. 1-5.

These small differences in the models produce different frequency response curves. The differences are useful in diagnosing problems and determining solutions. Obviously, to use these differences, the engineer must have a complete and thorough knowledge of the three models and their responses. The three systems illustrated in Figs. 1-3, 1-5, and 1-6 and their mathematical analyses are described in most vibration textbooks [1–3]. In some cases the damping should be included in the most realistic way possible, i.e., as viscous, Coulomb, hysteretic, or aerodynamic damping. However, if the damping is other than viscous, it may usually be

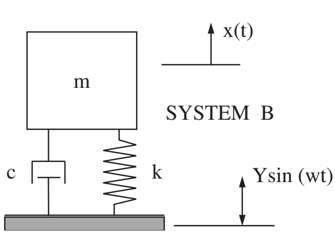

Figure 1-5 SDOF model with base excitation.

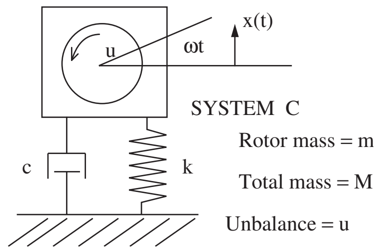

Figure 1-6 SDOF model with rotating unbalance.

represented by an equivalent viscous damping coefficient that varies with frequency [1, page 73]. For purely steel structures, it is usually less than 5% of the critical value. System B may have its predominant damping either (1) between the vibrating base and the modal mass, or (2) from the mass to ground. It is important to recognize the difference and set up the model correctly.

The frequency response curves for systems A, B, and C are plots of the amplitude of forced vibration versus the frequency. The response amplitude for system A is computed from Eq. 1-4 at each frequency, using appropriate values for k, c, m, and F. Figure 1-7 shows the response curve for system A with parameter values from Table 1-1. For plotting the curve, frequency ω (rad/sec) has been converted to rpm (cpm). X_static in the table is F/k, the displacement at zero frequency, which is the deflection of the spring under a static force F. Resonance is the undamped natural

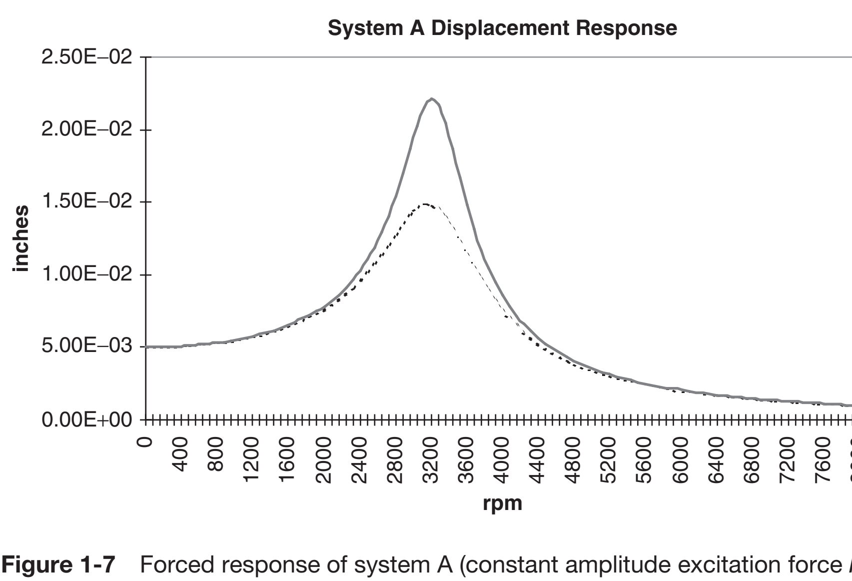

Figure 1-7 Forced response of system A (constant amplitude excitation force F).

Table 1-1 System A values for Fig. 1-7

	Data	Units
Input		
Mass	100	lb
Kstiff	30,000	lb/in
Cdamp	20	lb-sec/in
Force	150	lb
Freqstart	0	rpm
Freqstop	8000	rpm
Npoints	101	use 101
Output		
Resonance	3251.252	rpm
Zeta	0.11349	none
X_static	5.00E-03	in

frequency ω_n converted to cpm. Zeta is the critical damping ratio, i.e., the percentage of critical damping divided by 100. The solid curve in Fig. 1-7 has all the parametric values of Table 1-1.

The dashed curve in Fig. 1-7 has all the values of Table 1-1 except that the damping coefficient c has been increased from 20 lb-sec/in. (in the solid curve) to 30 lb-sec/in. The main effect of the increased damping is to reduce the vibration amplitude at the critical speed. It has very little effect at frequencies away from the critical speed. The critical speed (where the peak vibration occurs) is 3200 rpm for the solid curve and about 3150 rpm for the dashed curve. These are both slightly below the undamped natural frequency of 3251 cpm. Thus, damping tends to lower the critical speed. (This effect is reversed in system C (below) when the constant shaking force F is replaced with a rotating unbalance force $m\omega^2 u$). In Fig. 1-7, notice that the response amplitude X (= 5 mils at zero frequency) becomes large near the natural frequency, and approaches zero at very high frequencies. Figure 1-8 shows how the vibration X (the dashed curve) lags the force F with a phase angle ϕ (see Eq. 1-3). Figure 1-9 shows how the phase angle varies with frequency. More damping (the dashed curve) makes the phase angle change more gradually as the excitation frequency passes through ω_n. The phase angle is 90 degrees at the undamped natural frequency ω_n, regardless of the amount of damping. This fact is useful in determining the value of ω_n, since the phase angle can be measured but ω_n cannot be measured.

Graphs like Figs. 1-7 and 1-9 are often referred to as the frequency response curves, or Bode plots. If the parameter values (k, c, m) are changed, then the response curves will look similar but will have different

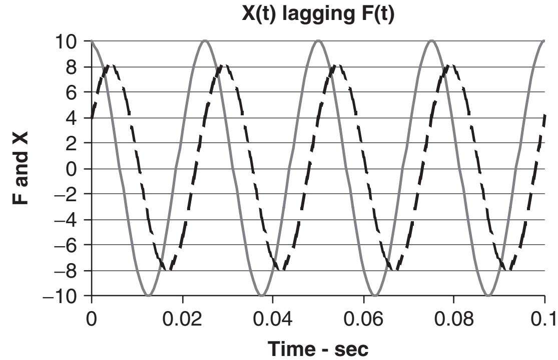

Figure 1-8 *X* (dashed) lagging force (solid).

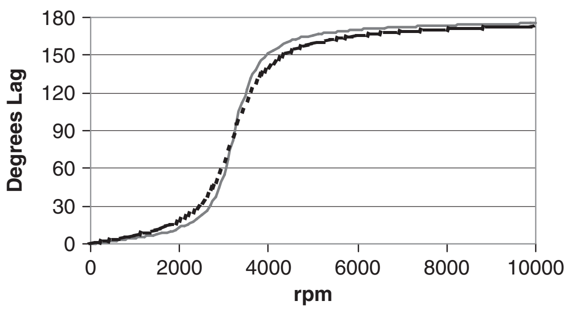

Figure 1-9 Phase lag response of system A.

values of response amplitude and phase. Increasing the damping generally brings the peak amplitude down but has a negligible effect at frequencies away from the natural frequency.

The necessity to plot many different curves for different values of F, k, and m is avoided by plotting the curve with dimensionless ratios as shown in Fig. 1-10. The abscissa in Fig. 1-10 is frequency ratio ω/ω_n; the ordinate Xk/F is X/X_static (the ratio of vibration amplitude to static displacement under the force F).

The frequency response of system B (Fig. 1-5, base vibration excitation) is given by

$$X = Y\sqrt{\frac{k^2 + (\omega c)^2}{\left(k - m\omega^2\right)^2 + (c\omega)^2}} \tag{1-8}$$

Figure 1-11 shows the response amplitude X calculated with the parametric values of Table 1-2. In the table, X_Base is the displacement amplitude Y of the vibrating support. Notice that damping in system B (the dashed

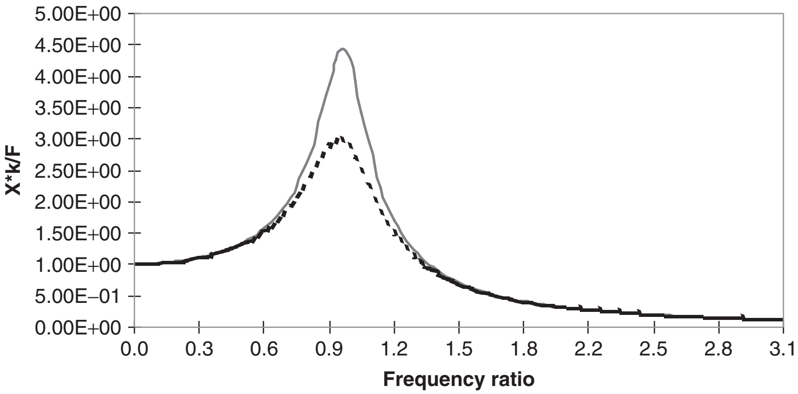

Figure 1-10 Dimensionless response of system A.

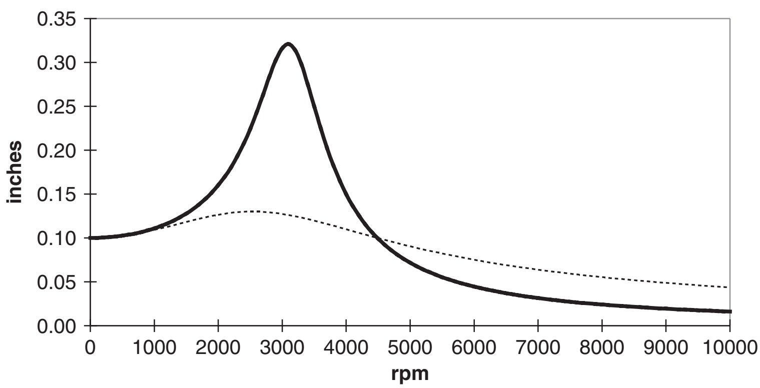

Figure 1-11 Response to base excitation of system B.

curve) actually increases the vibration response at high frequencies. Solving the differential equation for system B [1, page 66] shows that the crossover frequency is 1.4 times the undamped natural frequency. All the curves with different damping values cross at this frequency, and the amplitude there is the same as X_Base. The frequency range above this is called the *isolation range*, since the response there is reduced below what would be obtained with a hard support. A vibrating system with a fixed excitation frequency can be put into the isolation range by softening the spring Kstiff between the vibrating base and the mass.

The frequency response of system C (Fig. 1-6) is given by Eq. 1-9, where u is the unbalance (C.G. offset of the rotor), m is the rotor mass, and M is the total mass:

$$X = m\omega^2 u \Big/ \sqrt{\left(k - M\omega^2\right)^2 + (c\omega)^2} \tag{1-9}$$

Table 1-2 System B parameters for Fig. 1-11

	Data	Units
Input		
Mass	0.35	lb
Kstiff	100	lb/in
Cdamp	0.1	lb-sec/in
Cdamp2	0.4	lb-sec/in
X_Base	0.1	in
Freqstart	0	rpm
Freqstop	10,000	rpm
Npoints	101	use 101
Output		
Resonance	3172.897	rpm
Zeta	0.166132	none
Zeta2	0.66453	none

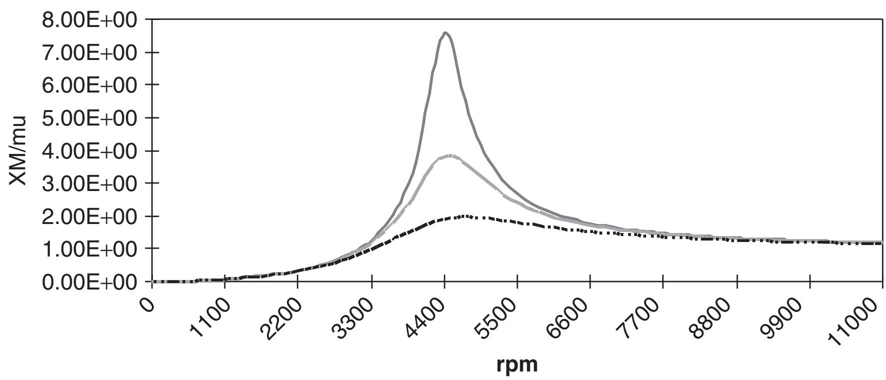

Figure 1-12 Response with an unbalanced rotor for three damping values.

The ratio X/u is often used and is sometimes called the *magnification factor*. The response calculated with the parametric values of Table 1-3 is shown in Fig. 1-12 with dimensionless amplitude XM/mu. In the table, m = Rotrmass and M = Rotrmass + Housmass. Note that XM/mu is approximately X/u in this case, since $m/M = 0.98$ (the housing mass is negligible). Figure 1-12 shows that system C response starts out at zero and damping in system C reduces the peak amplitude of vibration response and *raises* the critical speed. At very high frequencies the vibration amplitude approaches a limiting value determined by the amount of unbalance. Increasing the housing mass will reduce this value.

Table 1-3 System C parameters for Fig. 1-12

	Data	Units
Input		
Rotrmass	50	lb
Housmass	1	lb
Kstiff	28,000	lb/in
Cdamp	8	lb-sec/in
Cdamp2	16	lb-sec/in
Cdamp3	32	lb-sec/in
Unbalance	0.0015	in
Freqstart	0	rpm
Freqstop	11000	rpm
Npoint	101	use 101
Output		
Resonance	4398.29	rpm
Zeta	0.065798	none
Zeta2	0.131597	none
Zeta3	0.263193	none
Totalmass	51	lb
Massratio	0.980392	none

SIX TECHNIQUES FOR SOLVING VIBRATION PROBLEMS WITH FORCED EXCITATION

When vibration measurements from the real system are compared and identified with the theoretical response from the appropriate model (A, B, or C) one of the following techniques for reducing the vibration will often become apparent.

1. *Identify and reduce the excitation source.* This most obvious solution is also the one least likely to be possible in systems of type A or type B, but it should be investigated first. In rotating machinery (system C), this technique is implemented by balancing the rotating parts. Balancing will be effective only when the vibration frequency is equal to the speed of a rotating part or its integer harmonics, and this fact is the corollary of a diagnostic rule: *Frequency components in a measured spectrum that are synchronous with a rotating speed or one of its harmonics are often caused by rotating imbalance.* In a reciprocating machine (Fig. 1-1), balancing the 2nd harmonic often requires a separate unbalanced *balance*

shaft rotating at twice crankshaft speed to cancel out the inertia forces.

2. *Tune the natural frequency to a value further away from the frequency of excitation to avoid resonance.* A study of the frequency response curves for any of the systems A, B, or C reveals that the vibratory excitation is highly magnified at frequencies near the natural frequency. This magnification factor R, or Q factor as it is sometimes called, can typically range from 5 to 50 or more depending on the amount of damping. The excitation frequency can seldom be changed, but the natural frequency can sometimes be easily changed by changing the modal stiffness. This is one place where intelligent construction of the analytical model becomes important, since the modal stiffness may be made up of several real stiffnesses in parallel or in series. In parallel combinations the very low stiffnesses have little effect in determining the modal stiffness, while in series combinations the very high stiffnesses have little effect. The tuning method is effective only when the excitation frequency is constant or when it only varies over a narrow range.
3. *Isolate the modal mass from the vibratory excitation by making the modal stiffness very low.* Notice that all the response curves show a very low response to the vibratory excitation at frequencies much higher than the natural frequency (far to the right on the response curves). Once again, the excitation frequency usually cannot be changed but the natural frequency can be brought far down by a very soft modal stiffness, thus placing the system response far to the right of resonance on the response curve. This method is particularly effective in systems of type B. A typical application is isolating an electronics box from a vibrating vehicle frame.
4. *Add damping to the system.* Damping is added by incorporating mechanisms that dissipate vibratory energy into heat. When they work, damping mechanisms produce forces that act in opposition to the vibratory velocity. Contrary to popular belief, however, adding damping indiscriminately does not always reduce vibration. Damping does work well whenever operation is near resonance (and this is the operating condition most likely to cause a problem). At frequencies away from resonance damping has very little effect, except to increase the forces transmitted to ground at high frequencies far above resonance. In a system B application where isolation is used, damping added between the modal mass and the vibrating support will actually increase the vibration of the mass at high frequencies. In a system C (rotating machinery) application with rolling element bearings, adding damping to the bearing supports will increase the

dynamic bearing loads and shorten bearing life for operation at high supercritical speeds [4, page 14].

5. *Add a vibration absorber.* A vibration absorber is a separate spring–mass assembly, which is added to the original system to "absorb" the vibration. This method works well only under a strict set of conditions: (a) the excitation frequency must be constant and resonant (i.e., equal to a natural frequency of the system), (b) the absorber spring–mass assembly must be tuned to a natural frequency equal to the resonant frequency of the original system, (c) the absorber mass should be at least 20 percent of the modal mass of the original system, and (d) the absorber spring–mass assembly should not have much damping. Under all of these conditions the modal mass of the original system will stand still while the absorber mass vibrates with a large amplitude. Since the absorber adds a degree of freedom to the analytical model, it follows that mathematical analysis of absorber performance requires at least a two degree of freedom model with two differential equations to solve simultaneously [2, page 293].
6. *Stiffen the system.* This method is listed last because it is valid only for systems of type A, but often is mistakenly suggested for all type systems. On the dimensionless response curves for system A (Fig. 1-10), notice on the vertical amplitude axis that the vibration amplitude X is determined by multiplying the graph value by F/k. Thus, the vibration amplitude can be made smaller at any frequency by raising the stiffness k. Once again, this applies only to systems of type A in which there is a constant amplitude force excitation that does not vary with frequency.

SOME EXAMPLES WITH FORCED EXCITATION

Illustrative Example 1

Problem: Figure 1-13 shows a car towing a trailer. This car/trailer system has a vibration problem in the direction of travel, which occurs only during braking. The car has a warped front brake disk, which produces a vibratory braking torque and braking force P. The trailer hitch is flexible in the direction of travel such as might be produced by installing the hitch ball directly onto a lightly constructed rear bumper. During braking the vibration frequency decreases as the front wheel speed decreases. At some particular speed the excitation frequency becomes equal to the natural frequency of the car/trailer system and the amplitude becomes very large. The car and trailer move longitudinally as rigid bodies (out of phase) in the vibratory motion.

Analysis: Let the car displacement be X_1 and the trailer be X_2 (relative to the displacement produced by travel speed). There are two degrees of freedom (dof), but the system can be reduced to 1 dof because there is no spring to ground. The two differential equations (one for each dof) can be combined by subtracting one from the other (because the first mode has zero frequency). $X = X_1 - X_2$ is a *modal coordinate*. This produces the system A differential equation (1-2), where m_e is the modal or equivalent mass and P_e is the modal or equivalent force as follows:

$$m_e\ddot{X} + KX = P_e \tag{1-10}$$

where

$$m_e = \frac{m_1 m_2}{m_1 + m_2} \tag{1-11}$$

$$P_e = \frac{m_2 P}{m_1 + m_2} \tag{1-12}$$

Figure 1-14 is the dimensionless response curve with the modal parameters and with a small amount of damping added to keep the amplitudes positive.

See Problem 5-6 in [1] for the torsional analogy to this problem. The coordinates X_1 and X_2 describe the displacement of the car and trailer, respectively, as rigid bodies. This model has two degrees of freedom, but since neither the car nor the trailer has spring connections to ground, only one degree of freedom is relevant. Any movement of the system in which the car and trailer move in unison is irrelevant since they cannot vibrate together in phase; hence, the vibration coordinate of interest is the *relative* displacement $X = X_1 - X_2$. Mathematically this is the modal coordinate of the second mode, as the first mode has zero natural frequency. In the model of Fig. 1-13 the vibratory braking torque has been translated into a vibratory braking force with amplitude P and frequency ω equal to the rotational speed of the front wheel. The two differential equations in X_1 and X_2 have been subtracted one from another to produce the single differential equation in x (this is possible only when there are no springs to ground). Inspection of the resulting differential equation in X shows that

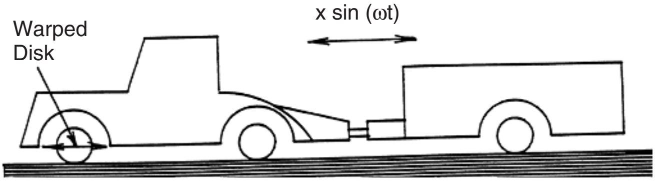

Figure 1-13 Car and trailer with flexible bumper/hitch.

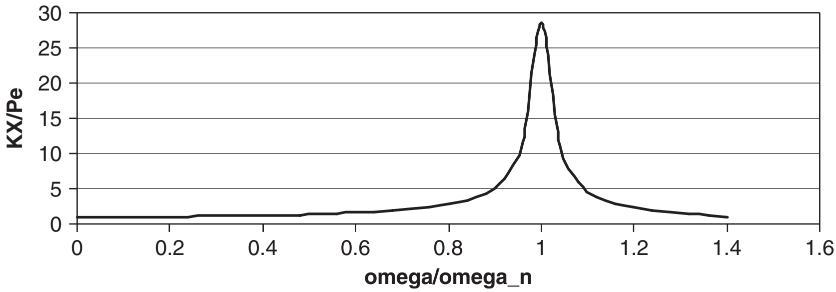

Figure 1-14 Dimensionless response of the car and trailer to the warped brake disk.

the modal mass m_e is $m_1 m_2 (m_1 + m_2)$ and the equivalent excitation force P_e is $Pm_2/(m_1 + m_2)$. The modal stiffness is simply the hitch connection stiffness K. The system is of Type A since *the differential equation has exactly the same form as the system A equation 1–2*. This is true because the equivalent excitation is a *constant force* $F = P_e$. Notice that the modal mass can be easily calculated from the weights of the car and trailer, and the modal stiffness K can be measured directly by applying static forces to the hitch or by measuring the resonant frequency and calculating $K = \omega^2 m_e$. The numerical magnitude of P need not be known to arrive at useful solutions as will be seen below.

Solution: Consider the six different methods described above for reducing vibration. The first method, reducing the source, could be implemented by replacing the warped brake disk and would be the ideal solution. If for some reason this cannot be done, consider the remaining methods. Tuning or absorption will not work because the excitation frequency is variable. Isolation will not work because the excitation frequency goes all the way down to zero. Damping would help at frequencies near resonance, but requires the addition of an expensive damping element to the flexible hitch connection at the rear bumper. Method 6, stiffening the system, can be implemented by stiffening the rear bumper or hitch connection and would be the best approach if the warped brake disk cannot be corrected or replaced. On the dimensionless response curve, note the effect of K on the value of X at every frequency. Stiffening the system in this case will reduce the response at all frequencies.

Illustrative Example 2

Problem: It is desired to mount an electronics package onto a vibrating surface with assurance that the electronics will survive. This generally requires that testing be done to define the limits of the vibratory environment that could damage the electronics in the package and also that the vibration amplitude of the mounting surface be known as a function of frequency (preferably from testing). In this example an electronic box weighing 0.35 lb is to be supported on a bracket that is welded to a vibrating bulkhead (Fig. 1-15). The excitation is rotating unbalance. A rubber mounting pad is to be designed as a vibration *isolator*. The vibration limits specified by the electronics manufacturer are shown in Fig. 1-16. The bulkhead vibration measured with an accelerometer is shown in Fig. 1-17.

Analysis: This type of problem is almost always addressed with method 3 (isolation) and is modeled by system B. It is helpful to plot the system B response curve in terms of dimensionless parameters as shown in Fig. 1-18. The frequency ratio is the ratio of the exciting frequency ω to the undamped natural frequency ω_n.

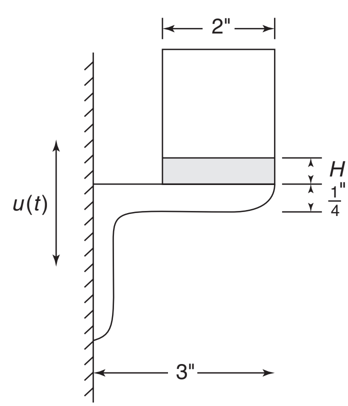

Figure 1-15 Electronic box installation.

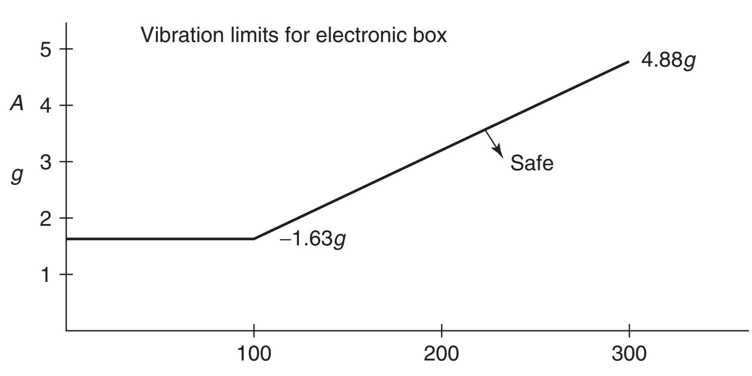

Figure 1-16 Vibration limits for the electronics.

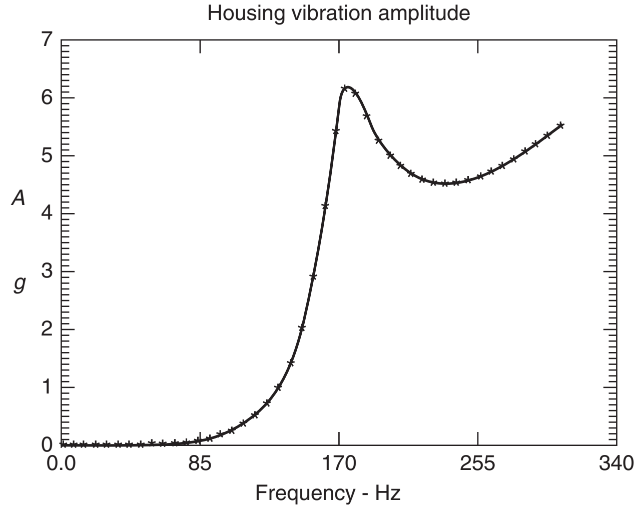

Figure 1-17 Measured bulkhead vibration, *g*'s.

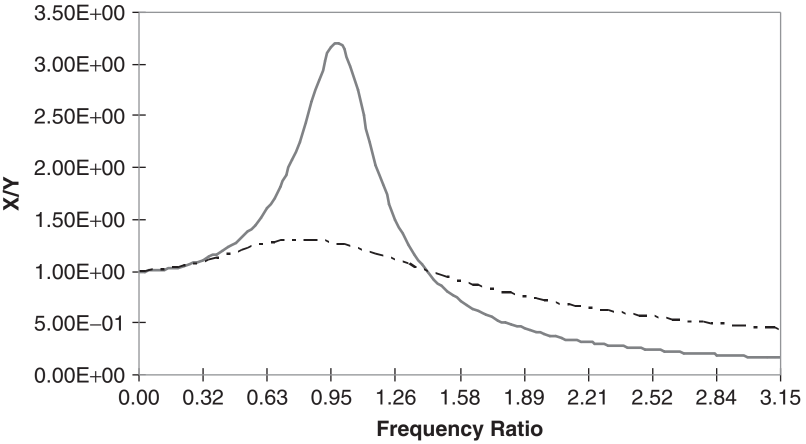

Figure 1-18 Dimensionless system B response (transmissibility).

The dimensionless *transmissibility* (ordinate) in Fig. 1-18 is the ratio of the vibratory amplitude of the electronics box to the vibratory amplitude of the mounting surface. There are two different curves for two different damping factors. The dashed curve line is for the higher damping. Inspection of the response curve shows that the best place to be on the response curve is far to the right at very high-frequency ratios, and with less damping. This can be accomplished if the mounting stiffness can be made soft enough and with relatively small damping. It is helpful in quantifying a solution to fit a curve to the measured vibration of the mounting surface, which is shown in Fig. 1-17. The problem may now be stated mathematically as follows: At every frequency in the operating range the product of the transmissibility and the amplitude in g's at the mounting surface must be less than the vibration limit shown in Fig. 1-16. The function $\tau = X/Y$ that generates the transmissibility curve in Fig. 1-18 is the dimensionless form of Eq. 1-8. The function is

$$\tau = \sqrt{\frac{1+\eta^2}{\left(1-r^2\right)^2+\eta^2}} \tag{1-13}$$

where η is a damping factor and r is the frequency ratio ω/ω_n. For viscous damping the damping factor is

$$\eta = 2\xi r \tag{1-14}$$

where ξ is the ratio of the viscous damping coefficient to the critical value and r is the frequency ratio. For hysteretic damping, which elastomeric materials exhibit, the damping factor η is simply the loss factor of the elastomeric material (generally published by the manufacturer with other material properties).

Solution: The transmissibility function can be multiplied by the curve-fitted function for A_b (Fig. 1-17) and compared with the vibration limits in Fig. 1-16. This process is well suited for computer coding and it is found that the modal stiffness k should be 320 lb/in or less to keep the vibration amplitude of the electronic box below the specified limits at all frequencies. The solution is found to be insensitive to variations in the loss factor for typical rubber materials. The analysis shows that the damping should be small. If the electronic box is mounted on a bracket as shown in Fig. 1-15, then the bracket becomes part of the modal stiffness for the system. The metal bracket, however, is much stiffer than 320 lb/in, so a rubber pad will probably be needed to get the support stiffness down low enough. A single pad, or several pads, made of butadiene compound can be sized so that $AE/t = 320$ lb/in, where A is the total contact area of the pads, E is the elastic modulus, and t is the pad thickness. Then the bracket and the rubber in series will have $k < 320$ lb/in and a composite loss factor less than the rubber alone.

Illustrative Example 3

Problem: A typical beach house structure is shown in Fig. 1-19. The house is built on tall piers (without the cross braces shown with question marks).

At the beach, the piers are set into soft damp sand and this gives the structure a significant amount of damping to attenuate lateral vibration. When such a house is built at locations with a hard rock foundation, the damping is much less, about 1 percent of the critical value (Q factor $= 50$). There are typically two modes of lateral vibration in which the house vibrates as a rigid body with the piers acting as cantilever beams to produce lateral stiffness. A rectangular house plan produces orthotropic stiffness (with the higher stiffness in the longer direction) and consequently the two modes. The two natural frequencies are about 3 and 5 Hz.

Consider the response of the structure to running a washing machine with a vertical rotating axis (the tub). The unbalanced tub spins up from zero to a speed much higher than the natural frequency of the house on piers and then coasts back down to zero, thus producing resonance twice in each of the two modes during each spin cycle. The amplitude at resonance is about 1/4″, which is enough to rattle dishes.

Analysis: It is tempting to simply rely on experience and recall that stiffening the system in the car/trailer problem reduced the response at all frequencies. It would thus seem that cross braces should be added to the pier support structure as shown in Fig. 1-19. However, this problem is of type C (rotating excitation) instead of type A. Look at the dimensionless group of variables on the vertical axis of the system C response curve (Fig. 1-12) and notice that the stiffness k does not directly appear.

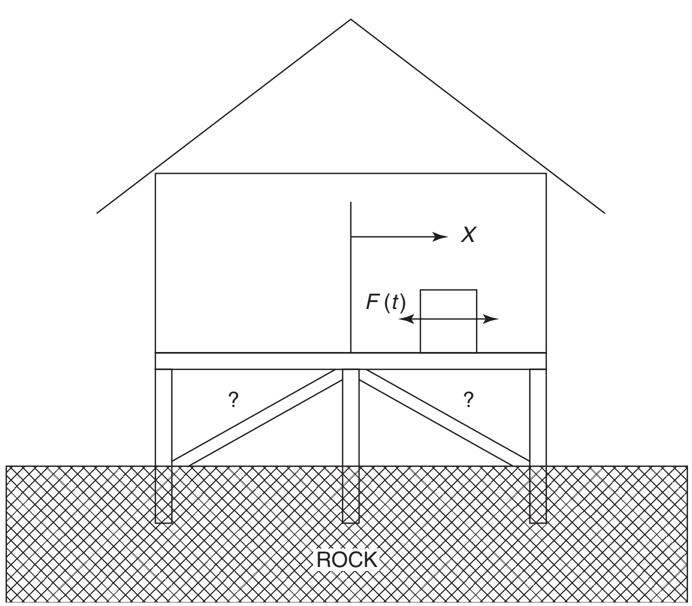

Figure 1-19 Beach house on piers.

To get the vibration amplitude from the curve, one multiplies the graphed value by mu/M, that is, the rotor mass times the unbalance divided by the total mass. In system C the excitation is *centrifugal force*, which increases as the square of rotating frequency. Increasing the stiffness of such a system raises resonance to a higher frequency where the excitation force is higher. Figure 1-20 shows the effect of adding stiffness to the supports. The assumed parameters are shown in Table 1-4. Notice that the housing mass in this case is much larger than the rotor mass. Although X/u is plotted here, the basic dimensionless group is XM/mu, so the large housing mass reduces the peak vibration to be less than the unbalance. But it is still unacceptable and adding stiffness makes it worse.

Now consider the other methods described above to reduce the vibration. Method 1, reducing the source, might be implemented by replacing the washing machine (which already has an automatic balancer). If method 1 is not practical, we move to method 2 or 3, tuning or isolation, neither of which work because the frequency of excitation is variable and starts at zero. Method 5, absorption, would require an absorber mass 1/5 the mass of the house and would introduce additional natural frequencies in the operating range. This leaves only method 4, damping, which is the parameter we lost by moving away from the beach and cementing the piers into hard rock.

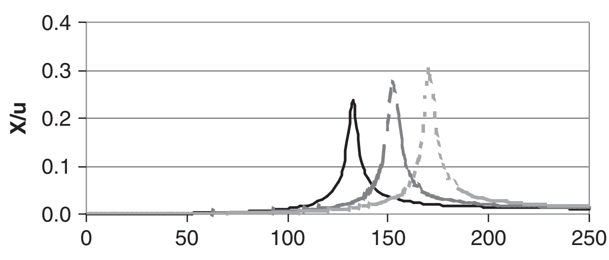

Figure 1-20 Stiffness raises the vibration amplitude in system C.

Table 1-4 Parameter values for the beach house

Rotrmass	100	lb
Housmass	12,000	lb
Kstiff	6,000	lb/in
Kstiff2	8,000	lb/in
Kstiff3	10,000	lb/in
Cdamp	15	lb-sec/in
Unbalance	1	in

Solution: Figure 1-21 illustrates the tremendous reduction in resonant vibration that can be obtained by increasing the damping. The two lower curves have 30 and 60 lb-sec/in of viscous damping, repectively.

To determine the existing damping, the fraction ξ of critical damping (*damping ratio*) can be obtained by measuring the logarithmic decay δ of free vibration and calculating

$$\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{1-15}$$

The effective viscous modal damping coefficient is then given by

$$C = 2m\omega_n\xi \tag{1-16}$$

where m is the mass of the house (lb-sec^2/in, not lb) and ω_n is the natural frequency in radians/sec ($= 2\pi$ times 2.25 Hz or about 14 rad/sec).

Since the damping must act on motion of the house relative to the ground (i.e., absolute motion), there are practical problems associated with installing it. A single damping element at one point would probably flex the house structure and possibly fail at the attachment point. A number of steel cables from the tops of piers out to ground anchors, with viscous shock absorbers (*dashpots*), could likely be made to work but might be a visual detraction from the house. In this case the sum of all the damping coefficients of the added dashpots acting in the same direction should be 2 or 3 times the value of the existing C calculated from formula 1-16.

In actual practice, this problem was solved by replacing the washing machine with one that has a much better automatic balancing mechanism, with the rotating tub on much softer supports.

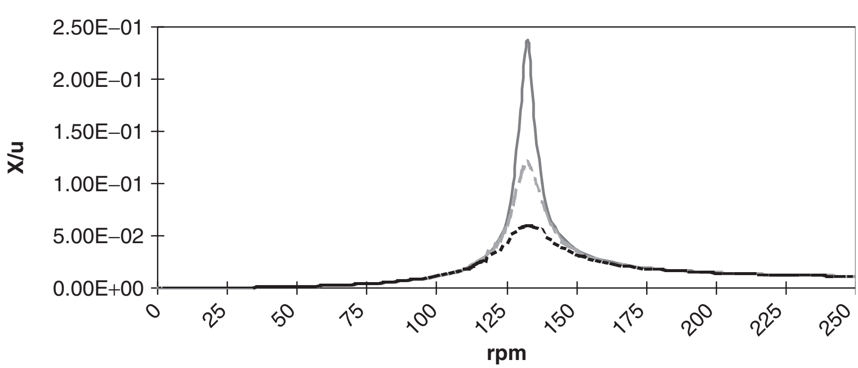

Figure 1-21 Effect of damping on the beach house vibration response.

Illustrative Example 4

Problem: A power turbine rotor in a turboprop aircraft engine has large vibration on start-up when the lube oil is hot. The rotor is mounted on squirrel cage bearing supports with stiffness much lower than the ball bearings themselves. (This is common practice in aircraft turbine engines). Figure 1-22 shows the rotor–bearing assembly mounted on pedestals in the Turbomachinery Laboratory at Texas A&M University. Figure 1-23 shows the measured vibration response with oil at three temperatures ranging from 94 to 204°F (operating temperature). The squeeze film damper becomes less effective due to the loss of viscosity at higher temperatures, which almost doubles the peak vibration response.

It is desired to minimize the amplitude of response at the critical speed, independent of temperature. The rotor speed of aircraft engines is highly variable and it is impossible to avoid passing through some of the lower critical speeds on start-up.

Analysis: It is often tempting to do the analysis with intuition, which suggests stiffening the bearing supports—perhaps even mounting the ball bearings solidly in the engine housing. Recall once again, however, that this approach moves the resonance to higher speeds where the force of the unbalanced rotor mass is higher by the square of rotor speed.

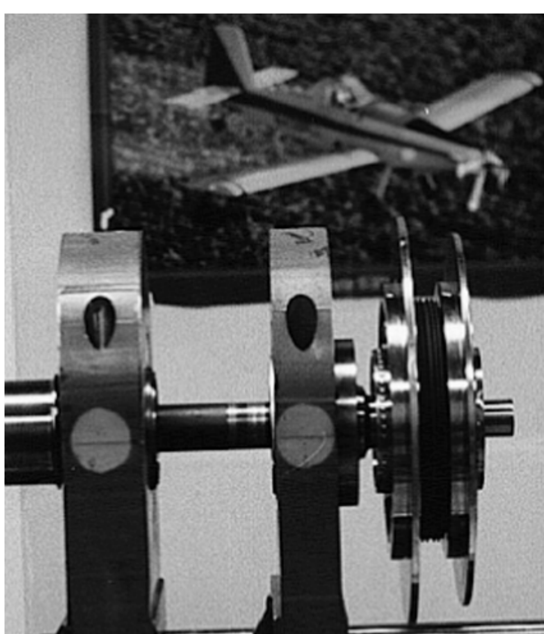

Figure 1-22 Power turbine rotor.

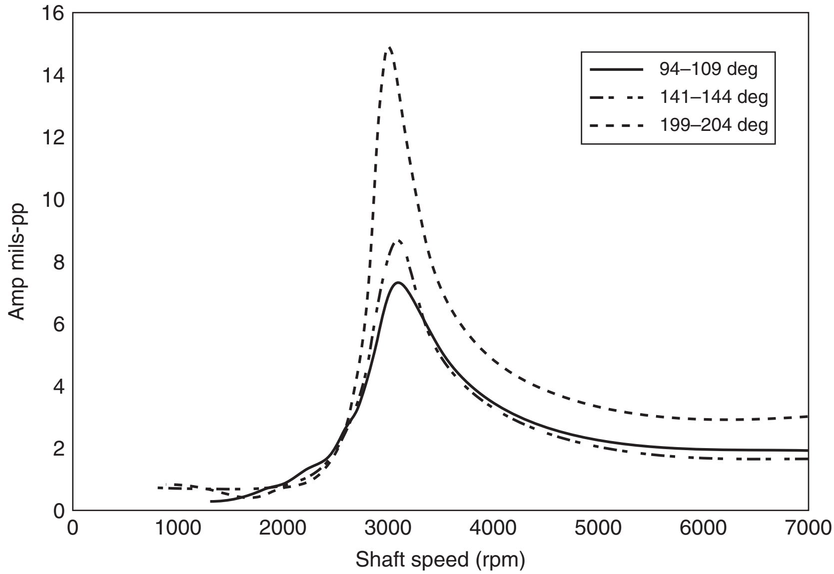

Figure 1-23 Response to unbalance at three oil temperatures.

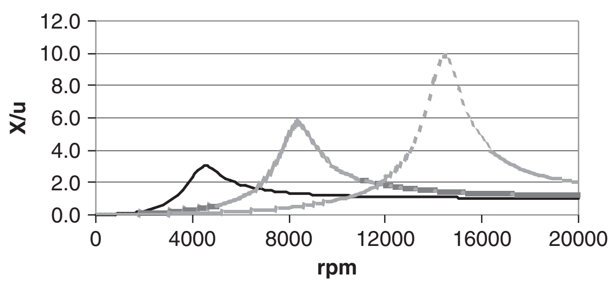

Figure 1-24 Effect of increased support stiffness on rotor response.

It would also move the critical speed up into the operating speed range. Figure 1-24 shows graphically the effect of stiffening the supports and shows that the amplitude of vibration actually *increases* with support stiffness. Other undesirable things happen as well to high-speed rotors when support stiffness is raised, as will be shown in following chapters on rotordynamics. Method 1, balancing the rotor, is often used in cases like this. It works, but tedious precision balancing is required, and the state of balance usually tends to degrade with operation time. Since we want to reduce the amplitude at the critical speed (near resonance), Fig. 1-12 shows that damping is the preferred approach. Like most aircraft turbine engines, this system already has squeeze film bearing dampers for this purpose. But the damping coefficient should be independent of temperature, and squeeze film dampers depend on the viscosity of the lube oil. Figure 1-25 shows the computed effect of damping on the unbalance response of this rotor at its first critical speed.

Solution: Experiments in the Turbomachinery Laboratory at Texas A&M University have shown that bearing supports made of woven wire mesh can provide the same effective damping as a squeeze film damper at operating temperature, since their damping is independent of temperature. Figure 1-26 shows the response of this power turbine on metal mesh supports, measured at three temperatures up to 210°F. More about this new type of bearing damper is presented in Chapter 5.

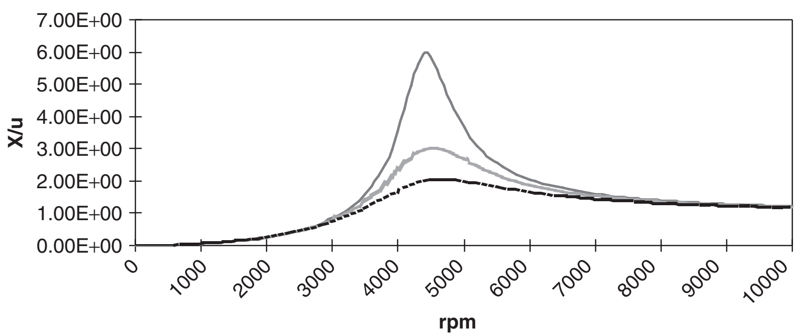

Figure 1-25 Effect of bearing support damping on the power turbine.

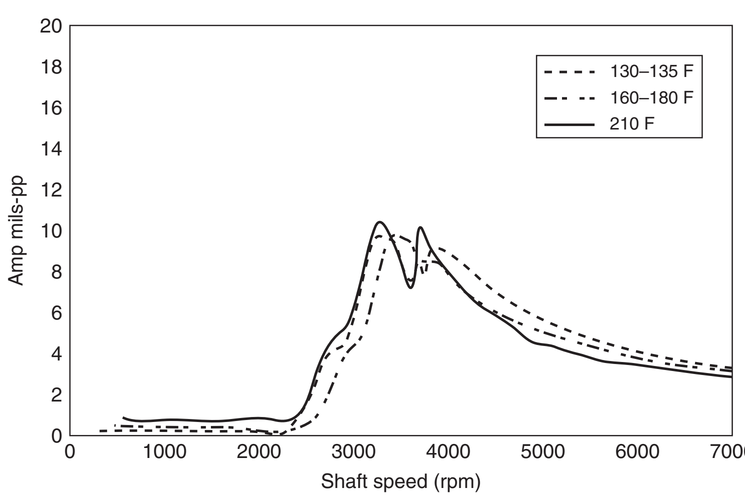

Figure 1-26 Power turbine response on metal mesh at three temperatures.

SOME OBSERVATIONS ABOUT MODELING

The last example above raises a question about the adequacy of a single degree of freedom model to represent the power turbine rotor-bearing system. In this case it was found that the CG of the rotor was directly above the outboard bearing support. This suggested that the first critical speed would be a mode with very little pitch (i.e., a cylindrical whirl mode, not conical) and therefore with little gyroscopic effect. The rotor was also modeled with XLROTOR using 17 stations with 68 degrees of freedom as shown in Fig. 1-27. The computed response to unbalance in Fig. 1-28 is identical to the response computed from the one degree of freedom (dof) model. The judgments required in constructing an appropriate one-dof model for the power turbine must be based on some knowledge about rotordynamics. This material is presented in following chapters.

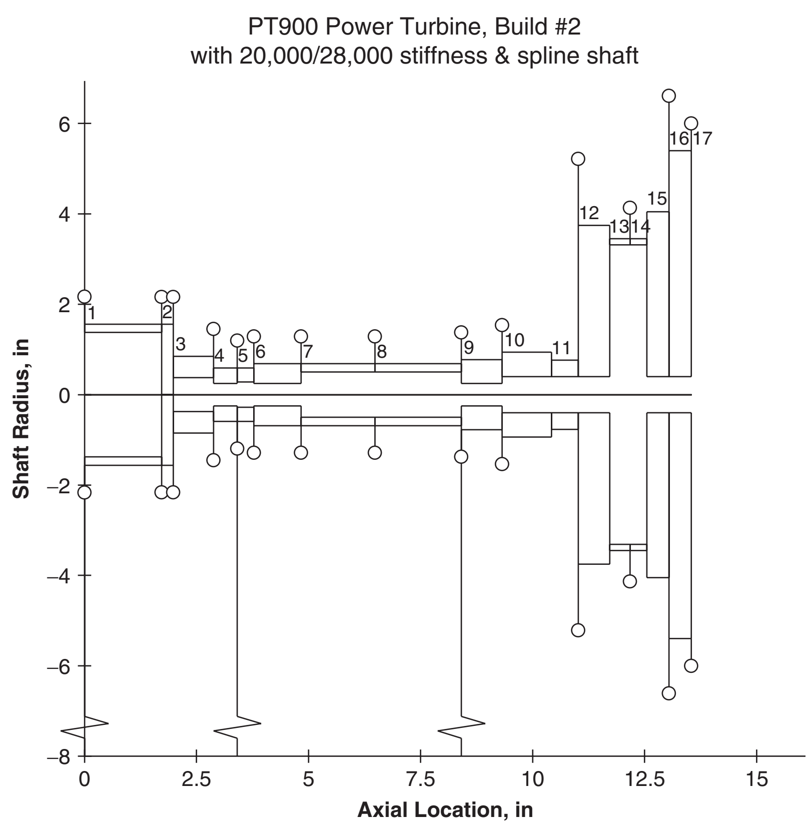

Figure 1-27 Computer model of the power turbine with 68 degrees of freedom.

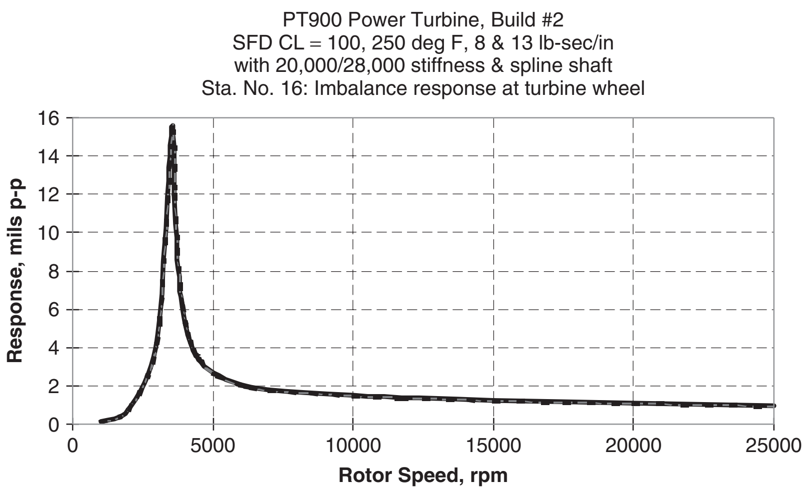

Figure 1-28 Response curve computed from the Fig. 1-27 model.

UNSTABLE VIBRATION

None of the examples presented above involve unstable vibration as represented by the homogeneous solutions (1-5) and (1-6) to the differential equation. Furthermore, all six of the presented techniques for solving vibration problems are based on the particular solution, i.e., response to excitation. Unstable vibration in nonrotating structures is rare. Those few cases usually involve fluid flow across a transverse member that sheds Von Karman vortices. The fluid pressure on the transverse member acts in phase with the vibratory velocity of the member, thus producing a negative damping coefficient. Since the real part of the eigenvalue is $\lambda = -c/2m$ in Eq. 1-6, negative c produces a growing exponential function and the vibration amplitude grows without limit. The frequency of unstable vibration is always the natural frequency ω_d of the system, independent of any external exciting frequency.

Figure 1-29 shows a simple apparatus (from Den Hartog [5]) in which negative damping can be generated. The flow of air around the beam of semicircular cross section produces a pressure distribution, which pushes the beam in the same direction as its instantaneous velocity (for motion in a vertical plane), as shown in Fig. 1-30. The differential equation for the vertical translational displacement Y of free vibration is

$$m\ddot{Y} - c\dot{Y} + kY = 0 \tag{1-17}$$

where m is the mass of the beam, c is the (negative) damping coefficient, and k is the total effective stiffness of all the springs. The solution of Eq. 1-17 is

$$Y(T) = Ae^{st} \tag{1-18}$$

where A is a constant, and the values of s that satisfy Eq. 1-17 are the complex conjugate eigenvalues. They are

$$s = \frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{1-19}$$

The positive real part of the eigenvalue indicates that the natural frequency of the beam will be unstable, with an amplitude that grows exponentially with time. In practice, the motion will become bounded at some finite amplitude large enough to render the linear equation (1-17) no longer valid. Note that every term in Eq. 1-17 contains the coordinate Y or one of its derivatives. This makes the equation *homogeneous*, which is a general property of the type of equations used to predict instabilities.

Practical examples of this phenomenon do exist. The oscillating eddies of the air are called *Von Karman vortices*. The beam could be a long pipe in a heat exchanger, a vertical smokestack, an electrical transmission

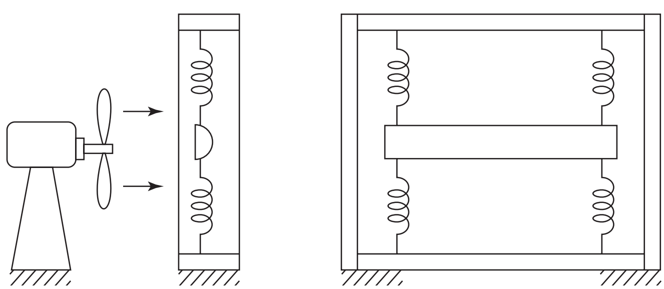

Figure 1-29 Apparatus to demonstrate unstable vibration. From Den Hartog [5].

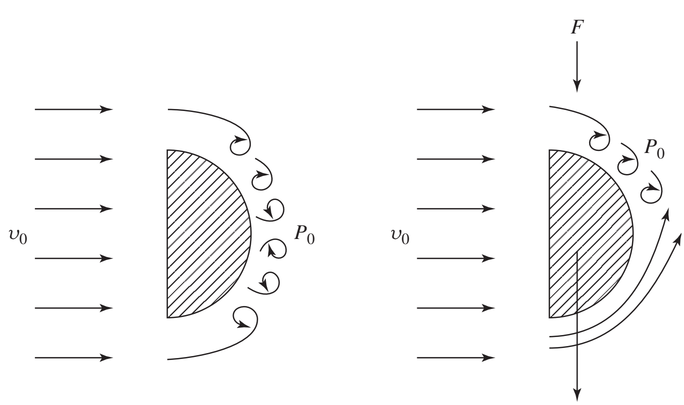

Figure 1-30 Negative damping produced by aerodynamic flow separation. From Den Martog [5].

wire, or a guy wire. However, unstable vibration is much more common in rotating machinery than in structures and can be very destructive. In rotating machinery it is called *rotordynamic instability*. It is generally caused by cross-coupled stiffness instead of negative direct damping. It will be analyzed and discussed in chapters to follow.

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[3] Den Hartog, J. P. *Mechanical Vibrations*, 4th ed. New York: McGraw-Hill, 1956.

[4] Vance, J. M. *Rotordynamics of Turbomachinery*. New York: Wiley, 1988.

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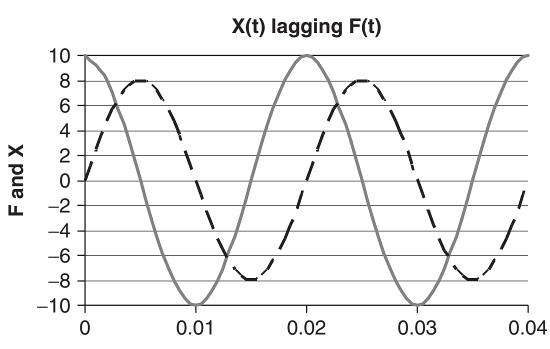
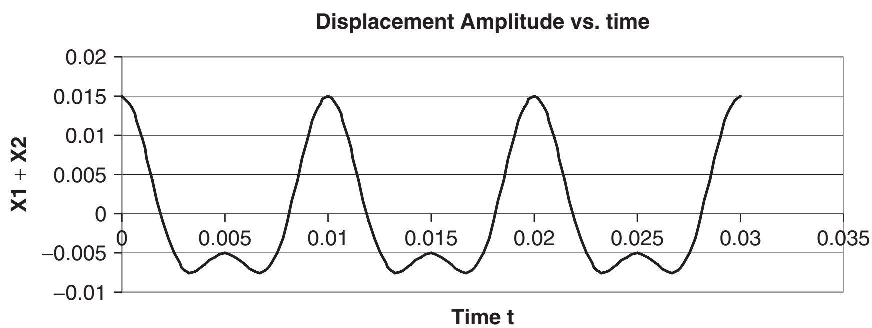
EXERCISES

1-1. Use Excel or some other application to plot the force F and displacement X versus time t for a synchronous vibration at machine speed 3000 rpm with no higher harmonics. The peak force and amplitude values are 10 lb and 8 mils (0.008″), respectively. Let the displacement lag the force by 90°. Show that the acceleration amplitude is 790 times larger than the displacement and is equivalent to 2.04g.

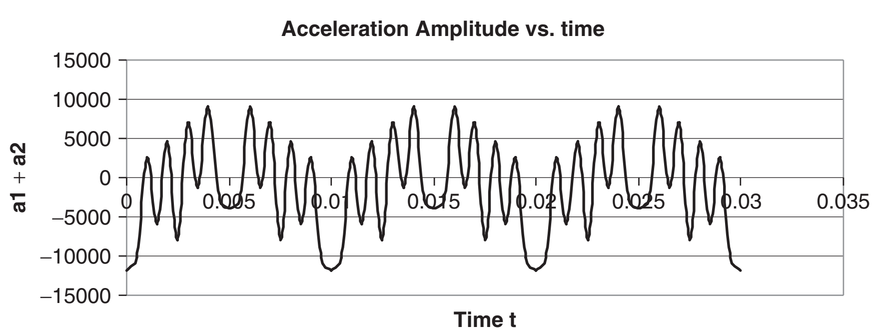


Ex. Figure 1-1

1-2. Use Excel or some other application to plot the total displacement amplitude ($x_1 + x_2$) and the total acceleration amplitude ($a_1 + a_2$) versus time for a machine vibration containing synchronous component $x_1(t)$ and a second harmonic $x_2(t)$ with half the amplitude. The fundamental amplitude is 10 mils (0.010″). The machine speed is 6000 rpm. For measurements, note that a displacement transducer is preferable to an accelerometer here if the primary interest is synchronous vibration.



Ex. Figure 1-2a



Ex. Figure 1-2b

1-3. Substitute Eq. 1-3 into Eq. 1-2 to show that the particular solution satisfies the differential equation.

1-4. Substitute Eq. 1-5 into Eq. 1-2 to show that the homogeneous solution satisfies the differential equation with $F = 0$. Show that the real part of the eigenvalue $\lambda = -c/2m$ and the imaginary part is the square root of $\omega_d^2 = k/m - (c/2m)^2$.

1-5. Referring to Eq. 1-6, the solution to the homogeneous differential equation for free vibration with no damping is $Ae^{i\omega t}$, since $\lambda = 0$.

Show that the solution can also can be written as $A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t)$, where A_1 and A_2 are real numbers and $\omega_n = (k/m)^{1/2}$, provided that A is an arbitrary complex number.

1-6. Show that eq. 1-6 with nonzero damping can be expressed as $x_h(t) = e^{-\frac{c}{2m}t}[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]$. Assuming initial conditions to give $A_2 = 0$, take the ratio of successive amplitudes X_n/X_{n+1} to show that the logarithmic decrement $\delta = \ln(X_n/X_{n+1}) = 2\pi\zeta/\left(1+\zeta^2\right)^{1/2}$, where $\zeta = c/2m\omega_n$. *Hint:* Note that the period of the damped vibration is $2\pi/\omega_d$.

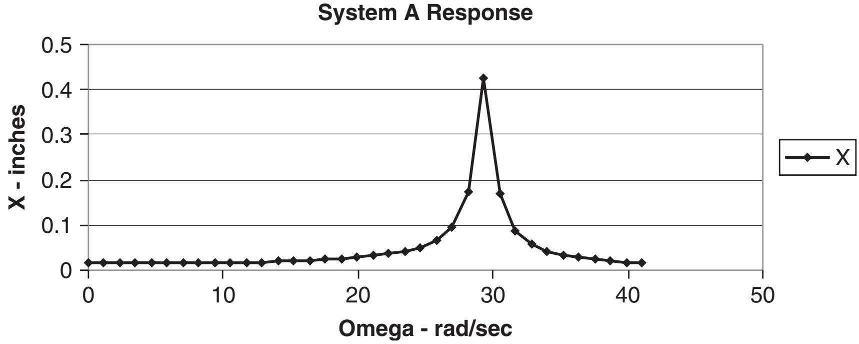
1-7. The tuning method on page 14 states that intelligent construction of the analytical model is important, since the modal stiffness may be made up of several real stiffnesses in parallel or in series. In parallel combinations the very low stiffnesses have little effect in determining the modal stiffness, while in series combinations the very high stiffnesses have little effect.

a. Show that the effective stiffness of k_1 and k_2 in parallel is practically k_1 if $k_1 = 100k_2$.
b. Show that the effective stiffness of k_1 and k_2 in series is practically k_2 if $k_1 = 100k_2$.

1-8. Derive the dimensionless form of Eq. 1-3 for the purpose of plotting Fig. 1-14.

1-9. Referring to Illustrative Example 1 and Fig. 1-13:

a. Derive the differential equation in X_1 for the car, with P_e as the excitation force.
b. Derive the differential equation in X_2 for the trailer.
c. Divide each equation by the mass and subtract the X_2 equation from the X_1 equation.
d. Do the math to obtain Eq. 1-10.
e. Use Excel or some other application to plot Fig. 1-14 in dimensional variables (X versus ω). Assume a speed range 70 mph down to zero with a tire diameter $D = 30''$ and brake excitation force $P = 100$ lb. Assume resonance occurs at 50 mph. Assume the car weighs 3000 lb and the trailer weighs 1000 lb. Include small damping $C = 2$ lb-sec/in using Eq. 1-4 so that the amplitude curve is always positive. Vary the stiffness K and note how it changes the response curve.

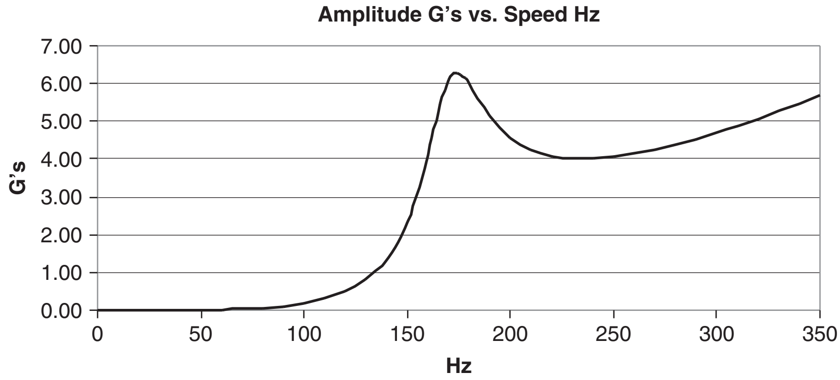


Ex. Figure 1-9

1-10. See Illustrative Example 2, where it is suggested that "it is helpful in quantifying a solution to fit a curve to the measured vibration of the mounting surface, which is shown in Fig. 1-17." Develop a mathematical function that will approximate the data in Fig. 1-17. *Solution:* This can be done with existing curve-fit software, but a more instructive approach is to realize that the excitation is likely due to some rotating unbalance since the data begin at the origin, which is unique to system C. The reason that the data do not look like Fig. 1-12 for system C is that the data are acceleration, not displacement. Multiplication by ω^2 and division by acceleration of gravity g converts Eq. 1-9 for system C to acceleration in g's. Assuming a small housing mass and dividing numerator and denominator by $M = m$ yields

$$\frac{\omega^2 X}{g} = \frac{\omega^4 u/g}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}} \tag{1-20}$$

The broad-banded peak response suggests a damping ratio ζ about equal to 0.085. The peak acceleration in the data is $6.2g$ or $2396\,\text{in/sec}^2$. This allows a calculation of the unbalance $u = 0.00035''$ to match the peak acceleration. The critical speed is seen to be about 170 Hz, so $\omega_n = 1068\,\text{rad/sec}$. The angular velocity ω must be converted to hertz $= \omega/(2\pi)$ for the graph. With these values, Eq. 1-20 produces the graph shown here.



Ex. Figure 1-10