Chapter 1

SISO Scheme and Linear Analysis

The mainstream methods of adaptive control for linear [7, 46, 57, 78, 89] and nonlinear [69] systems are applicable only for regulation to *known* set points or reference trajectories. In some applications, the reference-to-output map has an *extremum* (i.e., a maximum or a minimum) and the objective is to select the set point to keep the output at the extremum value. The uncertainty in the reference-to-output map makes it necessary to use some sort of adaptation to find the set point which extremizes (maximizes or minimizes) the output. This problem, called *extremum control* or *self-optimizing control*, was popular in the 1950s and 1960s [21, 36, 42, 59, 63, 81, 82, 83, 85, 90, 92], much before the theoretical breakthroughs in adaptive linear control of the 1980's. In fact, the emergence of extremum control dates as far back as the 1922 paper of Leblanc [73], whose scheme may very well have been the first "adaptive" controller reported in the literature. The method of sinusoidal perturbation used in this work has been the most popular of extremum-seeking schemes. In fact, it is the only method that permits fast adaptation, going beyond numerically based methods that need the plant dynamics to settle down before optimization. It is therefore on these extremum seeking schemes that this book is focused.

The purpose of this chapter is to lay a conceptual foundation for extremum seeking and develop familiarity with the methods used for analysis and thereby ease understanding of the more intricate analysis in subsequent chapters. Section 1.1 provides the stability test of the simplest possible extremum seeking scheme—a static map, to ease the reader into the problem. Section 1.2 provides the problem formulation, with Section 1.2.1 supplying linear time-varying (LTV) stability analysis, Section 1.2.2 a linear time-invariant (LTI) stability test, and Section 1.2.3 a design algorithm for a general single parameter extremum seeking scheme. To impress upon the reader the process of design, and the power of the method, we present a simulation study with a difficult

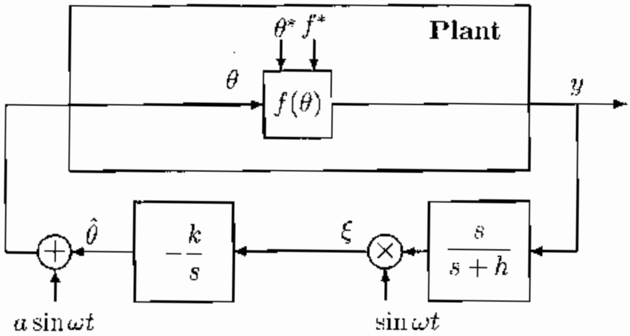

Figure 1.1: Basic extremum seeking scheme

toy model in Section 1.3.

1.1 Extremum Seeking for a Static Map

Figure 1.1 shows a basic extremum seeking loop for a static map. We posit $f(\theta)$ of the form:

$$f(\theta) = f^* + \frac{f''}{2}(\theta - \theta^*)^2 \tag{1.1}$$

where $f'' > 0$. Any C^2 function $f(\theta)$ can be approximated locally by Equ. (1.1). The assumption $f'' > 0$ is made without loss of generality. If $f'' < 0$, we just replace k $(k > 0)$ in Figure 1.1 with $-k$. The purpose of the algorithm is to make $\theta - \theta^*$ as small as possible, so that the output $f(\theta)$ is driven to its minimum f^*.

The perturbation signal $a \sin \omega t$ fed into the plant helps to get a measure of gradient information of the map $f(\theta)$. We give next an elementary intuitive explanation as to how the scheme "works" with a rigorous analysis to follow in the subsequent sections.

We start by noting that $\hat{\theta}$ in Figure 1.1 denotes the estimate of the unknown optimal input θ^*. Let

$$\tilde{\theta} = \theta^* - \hat{\theta}$$

denote the estimation error. Thus,

$$\theta - \theta^* = a \sin \omega t - \tilde{\theta},$$

which, when substituted into Equ. (1.1), gives

$$y = f^* + \frac{f''}{2}\left(\tilde{\theta} - a \sin \omega t\right)^2. \tag{1.2}$$

Expanding this expression further, and applying the basic trigonometric identity $2\sin^2\omega t = 1 - \cos 2\omega t$, one gets

$$
\begin{aligned}
y &= f^* + \frac{f''}{2}\tilde{\theta}^2 - af''\tilde{\theta}\sin\omega t + \frac{a^2 f''}{2}\sin^2\omega t && (1.3)\\
&= f^* + \frac{a^2 f''}{4} + \frac{f''}{2}\tilde{\theta}^2 - af''\tilde{\theta}\sin\omega t + \frac{a^2 f''}{4}\cos 2\omega t. && (1.4)
\end{aligned}
$$

The washout (high pass) filter

$$\frac{s}{s+h},$$

applied to the output, serves to remove f^*, namely,

$$\frac{s}{s+h}[y] \approx \frac{f''}{2}\tilde{\theta}^2 - af''\tilde{\theta}\sin\omega t + \frac{a^2 f''}{4}\cos 2\omega t. \tag{1.5}$$

This signal is then "demodulated" by multiplication with $\sin\omega t$, giving

$$\xi \approx \frac{f''}{2}\tilde{\theta}^2\sin\omega t - af''\tilde{\theta}\sin^2\omega t + \frac{a^2 f''}{4}\cos 2\omega t\sin\omega t. \tag{1.6}$$

As we shall see, it is the second term, and specifically the DC component (or constant) in $\sin^2\omega t$, that is crucial. Again applying $2\sin^2\omega t = 1 - \cos 2\omega t$, as well as the identity

$$2\cos 2\omega t\sin\omega t = \sin 3\omega t - \sin\omega t,$$

we arrive at

$$\xi \approx -\frac{af''}{2}\tilde{\theta} + \frac{af''}{2}\tilde{\theta}\cos 2\omega t + \frac{a^2 f''}{8}(\sin\omega t - \sin 3\omega t) + \frac{f''}{2}\tilde{\theta}^2\sin\omega t. \tag{1.7}$$

Noting that, because θ^* is constant,

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}},$$

we get

$$
\begin{aligned}
\tilde{\theta} \approx \frac{k}{s}\Bigg[&-\frac{af''}{2}\tilde{\theta} + \frac{af''}{2}\tilde{\theta}\cos 2\omega t\\
&+\frac{a^2 f''}{8}(\sin\omega t - \sin 3\omega t) + \frac{f''}{2}\tilde{\theta}^2\sin\omega t\Bigg]. && (1.8)
\end{aligned}
$$

First, we neglect the last term because it is quadratic in $\tilde{\theta}$ and we are interested only in local analysis:

$$
\begin{aligned}
\tilde{\theta} \approx \frac{k}{s}\Bigg[&-\frac{af''}{2}\tilde{\theta}\\
&+\frac{af''}{2}\tilde{\theta}\cos 2\omega t\\
&+\frac{a^2 f''}{8}(\sin\omega t - \sin 3\omega t)\Bigg]. && (1.9)
\end{aligned}
$$

The last two rows are high frequency signals. When passed through an integrator, they get greatly attenuated. Thus, we neglect them, getting

$$\tilde{\theta} \approx \frac{k}{s}\left[-\frac{af''}{2}\tilde{\theta}\right] \tag{1.10}$$

or

$$\dot{\tilde{\theta}} \approx -\frac{kaf''}{2}\tilde{\theta}. \tag{1.11}$$

Since $kf'' > 0$, this is a stable system. Thus, we conclude that $\tilde{\theta} \to 0$, or, in terms of the original problem, $\hat{\theta}(t)$ converges to within a small distance of θ^*.

Looking back at our "hand waving" analysis, it is important to note that our approximations hold only when ω is large (in a qualitative sense) relative to k, a, h, and f''. We shall explore this further in the coming sections. The following bare-bones result sums up the properties of the basic extremum seeking loop in Figure 1.1:

Theorem 1.1 (Extremum Seeking) *For the system in Figure 1.1, the output error $y - f^*$ achieves local exponential convergence to an $O(a^2 + 1/\omega^2)$ neighborhood of the origin provided the perturbation frequency ω is sufficiently large, and $\frac{1}{1+L(s)}$ is asymptotically stable, where*

$$L(s) = \frac{kaf''}{2s}. \tag{1.12}$$

We omit the rigorous proof as this result is subsumed in a more general result we prove in the following section. We draw the reader's attention to the fact that $\frac{1}{1+L(s)}$ is *always* asymptotically stable. We make the wording of this theorem tautological for symmetry with the more general results to come where $\frac{1}{1+L(s)}$ has to be made asymptotically stable by design. The convergence result in the theorem is second order, i.e., $O(a^2 + 1/\omega^2)$, because we are operating around a point of zero slope.

1.2 Single Parameter Extremum Seeking for Plants with Dynamics

Figure 1.2 shows the nonlinear plant with linear dynamics along with the extremum seeking loop. We let $f(\theta)$ be a function of the form:

$$f(\theta) = f^*(t) + \frac{f''}{2}\left(\theta - \theta^*(t)\right)^2, \tag{1.13}$$

where $f'' > 0$ is constant but unknown. The purpose of extremum seeking is to make $\theta - \theta^*(t)$ as small as possible, so that the output $F_o(s)[f(\theta)]$ is driven

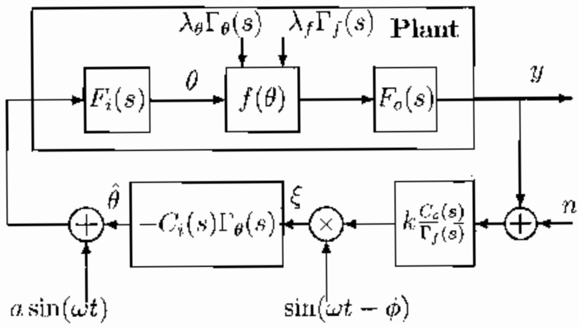

Figure 1.2: Extension of the extremum seeking algorithm to non-step changes in θ^* and f^*

to its extremum $F_o(s)[f^*(t)]$. The optimal input and output, θ^* and f^*, are allowed to be time varying. Let us denote their Laplace transforms by

$$\begin{aligned}\mathcal{L}\{\theta^*(t)\} &= \lambda_\theta\Gamma_\theta(s)\\ \mathcal{L}\{f^*(t)\} &= \lambda_f\Gamma_f(s).\end{aligned}$$

If θ^* and f^* happen to be constant (step functions),

$$\begin{aligned}\mathcal{L}\{\theta^*(t)\} &= \frac{\lambda_\theta}{s}\\ \mathcal{L}\{f^*(t)\} &= \frac{\lambda_f}{s}.\end{aligned}$$

While λ_θ and λ_f are unknown, the Laplace transform (qualitative) form of θ^* and f^* is known, and is included in the washout filter

$$\frac{C_o(s)}{\Gamma_f(s)} = \frac{s}{s+h}$$

(where in the static case we had chosen $C_o(s) = 1/(s+h)$) and in the estimation algorithm

$$C_i(s)\Gamma_\theta(s) = \frac{1}{s}$$

(where in the static case we had chosen $C_i(s) = 1$). Let us first shed more light on $\Gamma_\theta(s)$ and $\Gamma_f(s)$ and then return to discuss $C_i(s)$ and $C_o(s)$.

By allowing $\theta^*(t)$ and $f^*(t)$ to be time varying, we are allowing for the possibility of having to optimize a system whose commanded operation is non-constant. For example, if we have to ramp up the power of a gas turbine engine, we know the shape of, say, $f^*(t)$,

$$\mathcal{L}\{f^*(t)\} = \frac{\lambda_f}{s^2},$$

but we don't know λ_f (and λ_θ). We include $\Gamma_f(s) = 1/s^2$ into the extremum seeking scheme to compensate for the fact that f^* is not constant. The inclusion of $\Gamma_\theta(s)$ and $\Gamma_f(s)$ into the respective blocks in the feedback branch of Figure 1.2 follows the well known *internal model principle.* In its simplest form, this principle guides the use of an integrator in a proportional-integral (PI) controller to achieve a zero steady-state error. When applied, in a very generalized manner, to the extremum seeking problem, it allows us to track time-varying maxima or minima.

We return now to the compensators $C_o(s)$ and $C_i(s)$. Their presence is motivated by the dynamics $F_o(s)$ and $F_i(s)$, but also by the reference signals $\Gamma_\theta(s)$ and $\Gamma_f(s)$. For example, if we are tracking an input ramp,

$$\Gamma_\theta(s) = \frac{1}{s^2},$$

we get a double integrator in the feedback loop, which poses a threat to stability. Rather than choosing $C_i(s) = 1$, we would choose $C_i(s) = s + 1$ (or something similar) to reduce the relative degree of the loop. The compensators $C_i(s)$ and $C_o(s)$ are crucial design tools for satisfying stability conditions and achieving desired convergence rates.

We now make assumptions upon the system in Figure 1.2 that underlie the analysis to follow:

Assumption 1.2 *$F_i(s)$ and $F_o(s)$ are asymptotically stable and proper.*

Assumption 1.3 *$\Gamma_f(s)$ and $\Gamma_\theta(s)$ are strictly proper rational functions and poles of $\Gamma_\theta(s)$ that are not asymptotically stable are not zeros of $F_i(s)$.*

This assumption forbids delta function variations in the map parameters and also the situation where tracking of the extremum is not possible.

Assumption 1.4 *$\frac{C_o(s)}{\Gamma_f(s)}$ and $C_i(s)\Gamma_\theta(s)$ are proper.*

This assumption ensures that the filters $\frac{C_o(s)}{\Gamma_f(s)}$ and $C_i(s)\Gamma_\theta(s)$ in Figure 1.2 can be implemented. Since $C_i(s)$ and $C_o(s)$ are at our disposal to design, we can always satisfy this assumption. The analysis does not explicitly place conditions upon the dynamics of the parameters $\Gamma_\theta(s)$ and $\Gamma_f(s)$, however, for any design to yield a nontrivial region of attraction around the extremum, they cannot be faster than plant dynamics in $F_i(s)$ and $F_o(s)$. The signal n in Figure 1.2 denotes the measurement noise.

1.2.1 Single Parameter LTV Stability Test

We first provide background for the result on output extremization below. The equations describing the single parameter extremum seeking scheme in

Figure 1.2 are:

$$\begin{aligned} y &= F_o(s)\left[f^*(t) + \frac{f''}{2}(\theta - \theta^*(t))^2\right] & (1.14)\\ \theta &= F_i(s)\left[a\sin(\omega t) - C_i(s)\Gamma_\theta(s)[\xi]\right] & (1.15)\\ \xi &= k\sin(\omega t - \phi)\frac{C_o(s)}{\Gamma_f(s)}[y+n]\,. & (1.16) \end{aligned}$$

For the purpose of analysis, we define the tracking error $\tilde{\theta}$ and output error $\tilde{y}$:

$$\begin{aligned} \tilde{\theta} &= \theta^*(t) - \theta + \theta_0 & (1.17)\\ \theta_0 &= F_i(s)\left[a\sin(\omega t)\right] & (1.18)\\ \tilde{y} &= y - F_o(s)[f^*(t)]. & (1.19) \end{aligned}$$

In terms of these definitions, we can restate the goal of extremum seeking as driving output error $\tilde{y}$ to a small value by tracking $\theta^*(t)$ with θ. With the present method, we cannot drive $\tilde{y}$ to zero because of the sinusoidal perturbation θ_0. We now provide a minimally restrictive stability test for the system in Figure 1.2:

Proposition 1.5 (Single Parameter Extremum Seeking: LTV Test) *For the system in Figure 1.2, under Assumptions 1.2, 1.3, and 1.4, the output error $\tilde{y}$ achieves local exponential convergence to an $O(a^2)$ neighborhood of the origin provided $n = 0$ and:*

1. *$\pm j\omega$ is not a zero of $F_i(s)$.*

2. *Zeros of $\Gamma_f(s)$ that are not asymptotically stable are also zeros of $C_o(s)$.*

3. *Poles of $\Gamma_\theta(s)$ that are not asymptotically stable are not zeros of $C_i(s)$.*

4. *$C_o(s)$ is asymptotically stable and the eigenvalues of the matrix $\mathbf{\Phi}(T,0)$ lie inside the unit circle, where $T = 2\pi/\omega$ and $\mathbf{\Phi}(T,0)$ is the solution at time T of the matrix differential equation*

$$\dot{\mathbf{\Phi}} = \mathbf{A}(t)\mathbf{\Phi}(t,0),\ \ \mathbf{\Phi}(0,0) = \mathbf{I}, \tag{1.20}$$

and $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t,0)$, $\mathbf{x}(0) = \mathbf{x}_0$ is a state space representation of the LTV differential equation

$$\mathrm{den}\{H_i(s)\}[\tilde{\theta}] = -f''\mathrm{num}\{H_i(s)\}\left[\sin(\omega t - \phi)H_o(s)\left[\theta_0(t)\tilde{\theta}\right]\right], \tag{1.21}$$

where

$$\begin{aligned} H_i(s) &= C_i(s)\Gamma_\theta(s)F_i(s)\\ H_o(s) &= k\frac{C_o(s)}{\Gamma_f(s)}F_o(s). \end{aligned}$$

Proof of Proposition 1.5: Setting $n = 0$ and substituting Eqns. (1.15) and (1.18) in Eqn. (1.17) yields

$$\tilde{\theta} = \theta^* + H_i(s)[\xi] \tag{1.22}$$

Further, substitution for ξ from Eqn. (1.16) and for y from Eqn. (1.14) yields

$$\tilde{\theta} = \theta^* + H_i(s)\left[\sin(\omega t - \phi)H_o(s)\left[f^* + \frac{f''}{2}(\theta - \theta^*)^2\right]\right]. \tag{1.23}$$

Using $\theta - \theta^* = \theta_0 - \tilde{\theta}$ from Eqn. (1.17), we get

$$\begin{aligned}\tilde{\theta} &= \theta^* + H_i(s)\left[\sin(\omega t - \phi)H_o(s)\left[f^* + \frac{f''}{2}(\theta_0 - \tilde{\theta})^2\right]\right]\\ &= \theta^* + H_i(s)\left[\sin(\omega t - \phi)H_o(s)\left[f^* + \frac{f''}{2}(\theta_0^2 - 2\theta_0\tilde{\theta} + \tilde{\theta}^2)\right]\right]. \end{aligned} \tag{1.24}$$

We drop the higher order term $\tilde{\theta}^2$ (this is justified by Lyapunov's first method, as we have already written the system in terms of error variable $\tilde{\theta}$ thus transforming the problem to stability of the origin) and simplify the expression in Eqn. (1.24) using Lemmas A.1, A.2, Assns. 1.2, 1.3, and 1.4 and asymptotic stability of $\frac{C_o(s)}{\Gamma_f(s)}$ and $C_o(s)$:

$$\begin{aligned}&\sin(\omega t - \phi)H_o(s)[f^*(t)] - \lambda_f \sin(\omega t - \phi)\mathcal{L}^{-1}\left(H_o(s)\Gamma_f(s)\right)\\ &\quad - \sin(\omega t - \phi)(\epsilon^{-t}) = \epsilon^{-t} \end{aligned} \tag{1.25}$$

$$\sin(\omega t - \phi)H_o(s)[\theta_0^2] = C_1 a^2 \sin(\omega t + \mu_1) + C_2 a^2 \sin(3\omega t + \mu_2) + \epsilon^{-t} \tag{1.26}$$

where C_1, C_2, μ_1, μ_2 are constants (these can be determined from the frequency response of $H_o(s)$), and ϵ^{-t} denotes exponentially decaying terms. Now denote

$$u_{13}(t) = a^2\frac{f''}{2}\left[C_1 \sin(\omega t + \mu_1) + C_2 \sin(3\omega t + \mu_2)\right]. \tag{1.27}$$

The tracking error equation, Eqn. (1.24) after linearization (effected simply by dropping $\tilde{\theta}^2$ terms as we have expressed the system as an ODE in $\tilde{\theta}$) can be rewritten as

$$\tilde{\theta} = \theta^* + H_i(s)\left[u_{13}(t) + \epsilon^{-t} - f'' \sin(\omega t - \phi)H_o(s)\left[\theta_0\tilde{\theta}\right]\right]. \tag{1.28}$$

Multiplying both sides of Eqn. (1.28) with the denominator of $H_i(s)$, we get

$$\mathrm{den}\{H_i(s)\}[\tilde{\theta}] = \epsilon^{-t} + \mathrm{num}\{H_i(s)\}\left[u_{13}(t) + \epsilon^{-t} - f'' \sin(\omega t - \phi)H_o(s)\left[\theta_0\tilde{\theta}\right]\right]. \tag{1.29}$$

The θ^* term drops out or becomes an exponentially decaying term when operated upon by den$\{\Gamma_\theta(s)\}$ contained in den$\{H_i(s)\}$. We now write a state space representation of the LTV system in Eqn. (1.29):

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}u_{13}(t), \ \mathbf{A}(t+T) = \mathbf{A}(t), \ T = 2\pi/\omega. \tag{1.30}$$

The system has a state transition matrix $\mathbf{\Phi}(t,0)$ given by the solution of

$$\dot{\mathbf{\Phi}} = \mathbf{A}(t)\mathbf{\Phi}(t,0), \ \mathbf{\Phi}(0,0) = \mathbf{I}. \tag{1.31}$$

The system is exponentially stable if the eigenvalues of the matrix $\mathbf{\Phi}(T,0)$ (numerically calculated above) lie within the unit circle by Property 5.11 in reference [97]. As the persistent part of the non-homogenous forcing term in Eqn. (1.29) is $O(a^2)$, we have convergence of $\tilde{\theta}$ to $O(a^2)$[1] and therefore the convergence of $\tilde{y} = y - F_o(s)[f^*(t)] = F_o(s)\left[f''/2(\tilde{\theta}-\theta_0)^2\right]$ to $O(a^2)$. Q.E.D.

1.2.2 Single Parameter LTI Stability Test

While the result above permits determination of stability of extremum seeking loops in a wide variety of cases, it is not a convenient *design* tool as it requires calculation of the state transition matrix of an LTV system. For this purpose, we provide a result below that permits systematic design in a variety of situations. To this end, we introduce the following notation:

$$H_o(s) \ = \ k\frac{C_o(s)}{\Gamma_f(s)}F_o(s) \triangleq H_{osp}(s)H_{obp}(s) \triangleq H_{osp}(s)(1+H_{obp}^{sp}(s)) \tag{1.32}$$

where $H_{osp}(s)$ denotes the strictly proper part of $H_o(s)$ and $H_{obp}(s)$ its biproper part, and k is chosen to ensure

$$\lim_{s\to 0} H_{osp}(s) = 1. \tag{1.33}$$

Now we make an additional assumption upon the plant:

Assumption 1.6 *Let the smallest in absolute value among the real parts of all of the poles of $H_{osp}(s)$ be denoted by a. Let the largest among the moduli of all of the poles of $F_i(s)$ and $H_{obp}(s)$ be denoted by b. The ratio $M = a/b$ is sufficiently large.*

---

[1]Exponential stability of the homogenous part of the linear system in Eqn. 1.29 implies $\mathcal{L}$-stability of the system with forcing. Boundedness of the inhomogenous forcing yields boundedness of the solution of the forced system as in Example 6.3 in Khalil [64], and the bound on the forcing gives the order of $\tilde{\theta}$.

With this assumption, we separate the slow and fast dynamics in the LTV system in Eqn. (1.29) as

$$\begin{aligned}
\mathrm{den}\{H_i(s)\}[\tilde{\theta}] &= \epsilon^{\ t} + \mathrm{num}\{H_i(s)\}\left[u_{13}(t) + \epsilon^{\ t} - f'' \sin(\omega t - \phi))y'_{osp}\right] \\
y'_{osp} &= (1 + H^{sp}_{obp}(s))[y_{osp}] && (1.34) \\
y_{osp} &= H_{osp}(s)\left[\theta_0\tilde{\theta}\right], && (1.35)
\end{aligned}$$

and write the following state-space representation for the fast dynamics:

$$\begin{aligned}
\frac{1}{M}\dot{\mathbf{x}}_{osp} &= \mathbf{A}_{osp}\mathbf{x}_{osp} + \mathbf{B}_{osp}[\theta_0\tilde{\theta}] && (1.36) \\
y_{osp} &= \mathbf{C}_{osp}\mathbf{x}_{osp}, && (1.37)
\end{aligned}$$

where the eigenvalues of $M\mathbf{A}_{osp}$ are the poles of $H_{osp}(s)$. Reduction of the fast dynamics in Eqn. (1.36) by singular perturbation yields the substitution

$$y_{osp} = \mathbf{C}_{osp}\mathbf{A}^{-1}_{osp}\mathbf{B}_{osp}[\theta_0\tilde{\theta}] = \theta_0\tilde{\theta}, \tag{1.38}$$

using Eqn. (1.33) to form a reduced order model.

The purpose of this assumption is to use a singular perturbation reduction of the output dynamics and provide the LTI SISO stability test of the theorem stated below. If the assumption were made upon the output dynamics $F_o(s)$ alone, the design would be restricted to plants with fast output dynamics $F_o(s)$. Hence, for generality in the design procedure, the assumption of fast poles is made upon the strictly proper factor $H_{osp}(s)$ of $H_o(s)$. Its purpose is to deal with the strictly proper part of $F_o(s)$. If we have slow poles in a strictly proper $F_o(s)$, *we can introduce a biproper* $\frac{C_o(s)}{\Gamma_f(s)}$ *with an equal number of fast poles* to permit analysis based design. For example, if

$$F_i(s) = \frac{1}{s+1}, \text{ and } F_o(s) = \frac{1}{(s+1)(2s+3)},$$

with constant f^* and θ^* (giving $\Gamma_\theta(s) = \Gamma_f(s) = 1/s$) we may set

$$C_o(s) = \frac{(s+4)}{(s+5)(s+6)}$$

and $k = 60$ to give

$$H_o(s) = \frac{C_o(s)}{\Gamma_f(s)}F_o(s) = \frac{60s(s+4)}{(s+1)(2s+3)(s+5)(s+6)}.$$

We can factor the fast dynamics as

$$H_{osp}(s) = \frac{30}{(s+5)(s+6)}$$

and the slow biproper dynamics as

$$H_{obp}(s) = 1 + H_{obp}^{sp}(s) = 1 + \frac{1.5(s-1)}{(s+1)(s+1.5)}.$$

This gives, in the terms of Assumption 1.6, the smallest pole in absolute value in $H_{osp}(s)$, $a = 5$, the largest of the moduli of poles in $F_i(s)$ and $H_{obp}(s)$ as $b = 1.5$, giving their ratio $M = a/b = 3.33$. The singular perturbation reduction reduces the fast dynamics $H_{osp}(s) = \frac{30}{(s+5)(s+6)}$ to its unity gain, and we deal with stability of the reduced order model via the method of averaging to deduce stability conditions for the overall system in the theorem below. Hence, for all situations, we can first perform systematic design, and then, if necessary, reduce the order of the output filter and check for stability using the LTV test above.

To keep the proof brief, we make an additional assumption:

Assumption 1.7 *$H_i(s)$ is strictly proper.*

This assumption is very easy to satisfy. Either $F_i(s)$ is strictly proper or, if it is biproper, one would choose $C_i(s)\Gamma_\theta(s)$ strictly proper. For example, if $F_i(s)$ is biproper and $\Gamma_\theta(s) = 1/s$, $C_i(s) = 1$ satisfies this assumption. The assumption is made only for the purpose of keeping the proof of the following theorem brief. The formation of a state space representation of the reduced order system for averaging becomes more intricate when $H_i(s)$ is biproper, because of the need to account for a factor of ω when time varying terms are differentiated, and this distracts from the main theme of the proof.

Theorem 1.8 (Single Parameter Extremum Seeking) *For the system in Figure 1.2, under Assumptions 1.2–1.7, the output error $\tilde{y}$ achieves local exponential convergence to an $O(a^2+\delta^2)$ neighborhood of the origin, where $\delta = 1/\omega + 1/M$ provided $n = 0$ and:*

1. *Perturbation frequency ω is sufficiently large, and $\pm j\omega$ is not a zero of $F_i(s)$.*

2. *Zeros of $\Gamma_f(s)$ that are not asymptotically stable are also zeros of $C_o(s)$.*

3. *Poles of $\Gamma_\theta(s)$ that are not asymptotically stable are not zeros of $C_i(s)$.*

4. *$C_o(s)$ and $\frac{1}{1+L(s)}$ are asymptotically stable, where*

$$L(s) = \frac{af''}{4}\mathrm{Re}\{e^{j\phi}F_i(j\omega)\}H_i(s). \tag{1.39}$$

Proof. We rewrite the linearized model Eqn. (1.28)[2] after reduction of the fast dynamics $H_{osp}(s)$ to its unity static gain (from Assumption 1.6 and Eqn. (1.38)) and get

$$\tilde{\theta} = \theta^* + H_i(s)\left[u_{13}(t) + \epsilon^{-t} - f'' \sin(\omega t - \phi)(1 + H_{obp}^{sp}(s))\left[\theta_0 \tilde{\theta}\right]\right]. \quad (1.40)$$

Using Lemmas A.1, and A.2, we obtain[3]

$$\begin{aligned} \tilde{\theta} &= \theta^* + H_i(s)\left[u_{13}(t) + \epsilon^{-t} - af''/4\mathrm{Re}\{e^{j\phi} F_i(j\omega)\}\tilde{\theta} + v_1 + v_2\right] && (1.41) \\ v_1 &= af''/4\mathrm{Re}\{e^{j(2\omega t - \phi)} F_i(j\omega)\}\tilde{\theta} && (1.42) \\ v_2 &= \sin(\omega t - \phi) H_{obp}^{sp}(s)\left[v_3\right] && (1.43) \\ v_3 &= |F_i(j\omega)| \sin(\omega t + \angle F_i(j\omega))\tilde{\theta}. && (1.44) \end{aligned}$$

We next move the term $af''/4\mathrm{Re}\{e^{j\phi} F_i(j\omega)\} H_i(s)\tilde{\theta} = L(s)\tilde{\theta}$ to the left hand side, and divide both sides of Eqn. (1.41) with $1 + L(s)$:

$$\begin{aligned} \tilde{\theta} &= \frac{1}{1 + L(s)}[\theta^*] + Y_i(s)\left[u_{13}(t) + \epsilon^{-t} + v_1 + v_2\right] \\ &= \epsilon^{-t} + Y_i(s)\left[u_{13}(t) + v_1 + v_2\right], && (1.45) \end{aligned}$$

where $Y_i(s) = \frac{H_i(s)}{1+L(s)} = \frac{\mathrm{num}\{Y_i(s)\}}{\mathrm{num}\{1+L(s)\}}$ is asymptotically stable because the poles of $H_i(s)$ are cancelled by zeros of $\frac{1}{1+L(s)}$, and $\frac{1}{1+L(s)}$ is asymptotically stable; zeros in $\frac{1}{1+L(s)}$ cancel poles in $\theta^*(s) = \lambda_\theta \Gamma_\theta(s)$ resulting in exponentially decaying terms which are all consolidated (the asymptotically stable transfer function $Y_i(s)$ acting on exponentially decaying terms produces exponentially decaying terms). Now, $Y_i(s)$ is strictly proper from Assumption 1.7, and can therefore be written as $Y_i(s) = \frac{1}{s+p_0} Y_i'(s)$, where $Y_i'(s)$ is proper and p_0 is a pole of $Y_i(s)$ (and therefore asymptotically stable). In terms of their partial fraction expansions, we can write $Y_i'(s) = A_0 + \sum_{k=1}^{n} \frac{A_k}{s+p_k}$, and $H_{obp}^{sp}(s) = \sum_{j=1}^{m} \frac{B_j}{s+p_j}$. Multiplying both sides of Eqn. (1.45) with $s+p_0$ and using the partial fraction expansions, we get

$$\dot{\tilde{\theta}} + p_0 \tilde{\theta} = \epsilon^{-t} + A_0(u_{13}(t) + v_1 + v_2)$$

---

[2]The use of Lyapunov's first method in the proof of Proposition 1.5 and here is what makes the stability result of Theorem 1.8 local.

[3]Note that Eqn. (1.41) contains an additional term of the form $H_i(s)[\sin(\omega t - \phi) H_o(s)[\epsilon^{-t}\tilde{\theta}]]$ which comes from ϵ^{-t} in $\theta_0(t) = a\mathrm{Im}\{F_i(j\omega)e^{j\omega t}\} + \epsilon^{-t}$. We drop this term from subsequent analysis because it does not affect closed loop stability or asymptotic performance. It can be accounted for in three ways. One is to perform averaging over an infinite time interval in which all exponentially decaying terms disappear. The second way is to treat $\epsilon^{-t}\tilde{\theta}$ as a vanishing perturbation via Corollary 5.4 in Khalil [64], observing that ϵ^{-t} is integrable. The third way is to express ϵ^{-t} in state space format and let $\epsilon^{-t}\tilde{\theta}$ be dominated by other terms in a local Lyapunov analysis.

$$+\sum_{k=1}^{n}\left[u_{13,k}(t)+v_{1,k}+v_{2,k}\right] \tag{1.46}$$

$$v_2=\sin(\omega t-\phi)\sum_{j=1}^{m}(v_{3,j})$$

$$u_{13,k}=\frac{A_k}{s+p_k}u_{13};\ v_{1,k}=\frac{A_k}{s+p_k}v_1$$

$$v_{2,k}=\frac{A_k}{s+p_k}v_2,\ v_{3,j}=\frac{B_j}{s+p_j}v_3.$$

Now, we can write the system above in state-space form[4]:

$$\begin{aligned}\dot{\mathbf{x}} &= A(t)\mathbf{x}+A_{12}\mathbf{x}_e+Bu_{13}(t);\quad \tilde{\theta}=x_1 &(1.47)\\ \dot{\mathbf{x}}_e &= A_e\mathbf{x}_e. &(1.48)\end{aligned}$$

Eqn. (1.48) is a representation for the ϵ^{-t}. We get Eqns. (1.47), (1.48) into the standard form for averaging by using the transformation $\tau=\omega t$, and then averaging the right hand side of the equations w.r.t. time from 0 to $T=2\pi/\omega$, i.e., $\frac{1}{T}\int_0^T(\cdot)d\tau$ treating states $\mathbf{x}$, $\mathbf{x}_e$ as constant to get:

$$\begin{aligned}\frac{d\mathbf{x}_{av}}{d\tau} &= \frac{1}{\omega}\left(A_{av}\mathbf{x}_{av}+A_{12}\mathbf{x}_{eav}\right),\ \tilde{\theta}_{av}=x_{1av} &(1.49)\\ \frac{d\mathbf{x}_{eav}}{d\tau} &= \frac{1}{\omega}A_e\mathbf{x}_{eav}, &(1.50)\end{aligned}$$

which is a state-space representation of the system in the $\tau=\omega t$ time-scale, and $A_{av}=\frac{1}{T}\int_0^T(A(\tau))d\tau$. This gives

$$\begin{aligned}&\dot{\tilde{\theta}}_{av}+p_0\tilde{\theta}_{av}=\epsilon^{-t}\\ &+\sum_{k=1}^{n}\left[u_{13,k,av}+v_{1,k,av}+v_{2,k,av}\right] &(1.51)\\ &\dot{u}_{13,k,av}+p_ku_{13,k,av}=0,\ \dot{v}_{1,k,av}+p_kv_{1,k,av}=0\\ &\dot{v}_{2,k,av}+p_kv_{2,k,av}=0,\ \dot{v}_{3,j,av}+p_jv_{3,j,av}=0\end{aligned}$$

in the original time-scale. As all of the poles p_k for all k and p_j for all j are asymptotically stable (from asymptotic stability of $H_o(s)$ and $\frac{1}{1-L(s)}$), all of the terms on the right hand side of Eqn. (1.51) for $\tilde{\theta}_{av}$ are exponentially decaying, i.e., we have

$$\tilde{\theta}_{av}=\frac{1}{s+p_0}\left[\epsilon^{-t}\right], \tag{1.52}$$

which decays to zero because $\frac{1}{s+p_0}$ is asymptotically stable by asymptotic stability of $\frac{1}{1+L(s)}$. Hence, by a standard averaging theorem such as Theorem 8.3

[4]Note that $A(t)$ here is different from $\mathbf{A}(t)$ in the proof of Proposition 1.5.

in Khalil [64], we see that if ω, a, ϕ, $C_i(s)$ and $C_o(s)$ are such that $\frac{1}{1-L(s)}$ is asymptotically stable, and ω is sufficiently large relative to other parameters of the state-space representation, solutions starting from small initial conditions converge exponentially to a periodic solution in an $O(1/\omega)$ neighborhood of zero. Hence, the output $\tilde{\theta}(t)$ goes to a periodic solution $\tilde{\theta}_{per}(t) = O(1/\omega)$. We now proceed to put the system in the standard form for singular perturbation analysis through making the transformation $\delta\tilde{\theta} = \tilde{\theta}(t) - \tilde{\theta}_{per}(t)$ in Eqns. (1.34), (1.35) and get:

$$\begin{aligned}
&\mathrm{den}\{H_i(s)\}[\delta\tilde{\theta} + \tilde{\theta}_{per}] \\
&= \epsilon^{-t} + \mathrm{num}\{H_i(s)\}\left[u_{13}(t) + \epsilon^{-t} - f''\sin(\omega t - \phi))y'_{osp}\right] \\
&y'_{osp} = (1 + H^{sp}_{obp}(s))[y_{osp}] \qquad (1.53)\\
&y_{osp} = H_{osp}(s)\left[\theta_0(\delta\tilde{\theta} - \tilde{\theta}_{per})\right]. \qquad (1.54)
\end{aligned}$$

By linearity of the system described by the Eqns. (1.53), (1.54), we have that the reduced model in the new coordinates (replacing $H_{osp}(s)$ with its unity static gain) is given by

$$\begin{aligned}
&\mathrm{den}\{H_i(s)\}[\delta\tilde{\theta}] \\
&= -f''\mathrm{num}\{H_i(s)\}\left[\sin(\omega t - \phi))(1 + H^{sp}_{obp}(s))[\theta_0\delta\tilde{\theta}]\right], \qquad (1.55)
\end{aligned}$$

which has the state space representation

$$\dot{\mathbf{x}} = A(t)\mathbf{x}; \quad \delta\tilde{\theta} = x_1, \qquad (1.56)$$

where $A(t)$ is the same as in Eqn. (1.47). Hence $\delta\tilde{\theta}$ converges exponentially to the origin. This shows that the reduced model is exponentially stable. From exponential stability of $H_{osp}(s)$, we have exponential stability of the boundary layer model

$$\frac{d\mathbf{y}}{d\tau} = \mathbf{A}_{osp}\mathbf{y}, . \qquad (1.57)$$

Hence, by the Singular Perturbation Lemma A.3, we have that in the overall unreduced system in Eqns. (1.53), (1.54), the solution converges to an $O(1/M)$ neighborhood of the origin. Hence, $\delta\tilde{\theta}(t)$ converges to a $O(1/M)$ neighborhood of the origin. Therefore, $\tilde{\theta}$ converges exponentially to a $O(1/\omega) + O(1/M) = O(\delta)$ neighborhood of the origin. Further, the output error $\tilde{y}$ decays to $O(a^2 + \delta^2)$:

$$\tilde{y} = F_o(s)\left[\frac{f''}{2}(\theta - \theta^*)^2\right] = F_o(s)\left[\frac{f''}{2}(\tilde{\theta} - \theta_0)^2\right] = O(a^2 + \delta^2), \qquad (1.58)$$

which completes the proof. Q.E.D.

From Eqn. (1.39), we notice that $C_i(s)$ appears *linearly* in $L(s)$ (through $H_i(s) = C_i(s)\Gamma_\theta(s)F_i(s)$). This property allows systematic design using *linear control tools*. The conditions of Theorem 1.8 motivate the steps of a design algorithm below.

1.2.3 Single Parameter Compensator Design

In the design guidelines that follow, we set $\phi = 0$ which can be used separately for fine-tuning.

Algorithm 1.2.1 (Single Parameter Extremum Seeking)

1. *Select the perturbation frequency ω sufficiently large and not equal to any frequency in noise, and with $+j\omega$ not equal to any imaginary axis zero of $F_i(s)$.*

2. *Set perturbation amplitude a so as to obtain small steady state output error $\tilde{y}$.*

3. *Design $C_o(s)$ asymptotically stable, with zeros of $\Gamma_f(s)$ that are not asymptotically stable as its zeros, and such that $\frac{C_o(s)}{\Gamma_f(s)}$ is proper. In the case where dynamics in $F_o(s)$ are slow and strictly proper, use as many fast poles in $C_o(s)$ as the relative degree of $F_o(s)$, and as many zeros as needed to have zero relative degree of the slow part $H_{obp}(s)$ to satisfy Assumption 1.6.*

4. *Design $C_i(s)$ by any linear SISO design technique such that it does not include poles of $\Gamma_\theta(s)$ that are not asymptotically stable as its zeros, $C_i(s)\Gamma_\theta(s)$ is proper, and $\frac{1}{1+L(s)}$ is asymptotically stable.*

We examine these design steps in detail:

Step 1: Since the averaging assumption is only qualitative, we may be able to choose ω only slightly larger than the plant time constants, as we show in the examples in Sections 1.3 and 2.3. Choice of ω equal to a frequency component persistent in the noise n can lead to a large steady state tracking error $\tilde{\theta}$. In fact, our analysis can be adapted to include a bounded noise signal satisfying $\lim_{T\to\infty} \frac{1}{T}\int_0^T n \sin\omega t dt = 0$. Finally, if $\pm j\omega$ is a zero of $F_i(s)$, the sinusoidal forcing will have no effect on the plant.

Step 2: The perturbation amplitude a should be designed such that $a|F_i(j\omega)|$ is small; a itself may have to be large so as to produce a measurable variation in the plant output.

Step 3: In general, this design step will need designing a biproper $\frac{C_o(s)}{\Gamma_f(s)}$ when we have a slow and strictly proper $F_o(s)$ in order to satisfy Assumption 1.6. The use of fast poles in $\frac{C_o(s)}{\Gamma_f(s)}$ raises a possibility of noise deteriorating the feedback; however, the demodulation coupled with the integrating action of the input compensator prevents frequencies other than that of the forcing from entering into the feedback. While we have used the gain k in analysis to satisfy Assumption 1.6, this is not strictly necessary in design.

Step 4: We see from Algorithm 1.2.1 that $C_i(s)$ has to be designed such that $C_i(s)\Gamma_\theta(s)$ is proper; hence, for example, if $\Gamma_\theta(s) = \frac{1}{s^2}$, an improper $C_i(s) = 1+d_1s+d_2s^2$ is permissible. In the interest of robustness, it is desirable to design $C_i(s)$ and $C_o(s)$ to ensure minimum relative degree of $C_i(s)\Gamma_\theta(s)$ and $\frac{C_o(s)}{\Gamma_f(s)}$. This will help to provide lower loop phase and thus better phase margins. Simplification of the design for $C_i(s)$ is achieved by setting $\phi = -\angle(F_i(j\omega))$, and obtaining

$$L(s) = \frac{af''|F_i(j\omega)|}{4} H_i(s).$$

The attraction of extremum seeking is its ability to deal with uncertain plants. In our design, we can accommodate uncertainties in f'', $F_o(s)$, and $F_i(s)$, which appear as uncertainties in $L(s)$. Methods for treatment of these uncertainties are dealt with in texts such as [123]. Here we only show how it is possible to ensure robustness to variations in f''. Let $\widehat{f''}$ denote an a priori estimate of f''. Then we can represent $\frac{1}{1+L(s)}$ as $\frac{1}{1+L(s)} = \frac{1}{1+\left(1+\frac{\Delta f''}{\widehat{f''}}\right)P(s)}$, where $\Delta f'' = f'' - \widehat{f''}$, and $P(s) = \frac{\widehat{f''}}{f''}L(s)$, which is at our disposal because f'' in $P(s)$ gets cancelled by f'' in $L(s)$. We design $C_i(s)$ to minimize $\|\frac{P}{1+P}\|_{H_\infty}$ which maximizes the allowable $\Delta f'' < \widehat{f''}/\|\frac{P}{1+P}\|_{H_\infty}$ under which the system is still asymptotically stable.

1.3 Single Parameter Example

To illustrate the power of the method, we present a difficult toy benchmark problem. Bounded noise n is present in all simulations. Simulation results are plotted with $f^*(t), \theta^*(t)$ in dotted lines and y, θ in solid lines.

Benchmark Example. We use $F_i(s) = \frac{s-1}{s^2+3s+2}$, $F_o(s) = \frac{1}{s+1}$, $f(\theta) = f^*(t) + (\theta - \theta^*(t))^2$, where $f^*(t) = 0.01u(t-10)$, giving $\lambda_f\Gamma_f(s) = \frac{0.01e^{-10s}}{s}$, $\theta^*(t) = 0.01e^{0.01t}$, giving $\lambda_\theta\Gamma_\theta(s) = \frac{0.01}{s-.01}$, and $u(t-\tau)$ denotes the unit step at time τ. The benchmark example provides all the elements of difficulty that a design can face: a large relative degree of $F_i(s)F_o(s)$; unstable $\Gamma_\theta(s)$; non-minimum phase $F_i(s)$.

Using Algorithm 1.2.1 for the design, we set $\omega = 5$ rad/sec, $a = .05$, $\frac{C_o(s)}{\Gamma_f(s)} = \frac{s}{s+5}$, $\phi = -\angle(F_i(j5)) = .7955$, and $C_i(s)\Gamma_\theta(s) = 107.7\frac{s-4}{s-.01}$.

The design attains the desired goal of output minimization. The response in Figure 1.3 shows a slow transient and noise sensitivity in the parameter tracking. But, we note that because of the attenuation of the high frequency perturbation in the plant, the output y tracks the minimum $f^*(t)$ well. We note that we used $C_o(s) = 1/(s+5)$– a fast pole– as the pole in $F_o(s)$ is slow, and $F_o(s)$ is strictly proper.

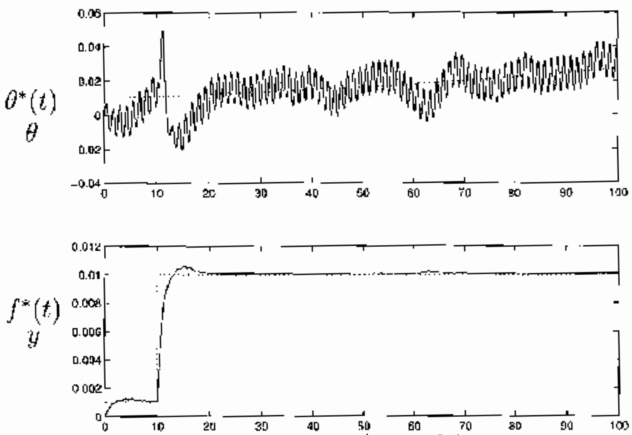

Figure 1.3: A simple $C_i(s) = s - 4$

Notes and References

Several books in the 1950-60s have been exclusively or partly dedicated to extremum seeking, including Tsien [109] (1954), Feldbaum [39] (1959), Krasovskii [66] (1963), Wilde [119] (1964), and Chinaev [29] (1969). Among the surveys on extremum seeking, we find the one by Sternby [105] particularly useful, as well as Section 13.3 in Astrom and Wittenmark [7] which puts extremum control among the most promising future areas for adaptive control. Among the many applications of extremum control overviewed in [105] and [7] are combustion processes (for IC engines, steam generating plants, and gas furnaces), grinding processes, solar cell and radio telescope antenna adjustment to maximize the received signal, and blade adjustment in water turbines and wind mills to maximize the generated power.

Extremum seeking witnessed a resurgence of interest after the publication of the stability studies in [70] and [67]. Several analyses of extremum seeking schemes [16, 37, 96, 113, 122] were presented at ACC 2000. The need for rigorous design guidelines guaranteeing performance was strongly felt in [67, 96]. Besides, numerical optimization based extremum seeking methods were used successfully. Extremum seeking via triangular search as in Zhang [122], was employed to attenuate combustor thermoacoustic oscillations and minimize diffuser losses at United Technologies Research Center (UTRC); extremum seeking via sliding modes, introduced by Korovin and Utkin [65], and analyzed and applied by Ozguner and co-workers [35, 51] and others [28, 106] on a variety of automotive problems. Even more recently, extremum seeking has been used for control of electromechanical valves [93], and for optimization of IC engine cam timing [94].

The first stability analysis of extremum seeking for a general nonlinear plant was developed in [70]. In [67], dynamic compensation was proposed for providing stability guarantees and fast tracking of changes in plant operating conditions for single parameter extremum seeking. The result in [5], on which this chapter is based, supplies the systematic design, and moreover, limits adaptation speed only by the speed of the plant dynamics.