# 1 Puzzles, Patterns, and Mathematical Language



**Ask five mathematicians what mathematics is, and you might very well get five different answers, but there will no doubt be some underlying themes. Mathematics is about solving problems. Mathematics explains patterns. Mathematics is a set of statements deduced logically from axioms and definitions. Mathematics uses abstraction to model the real world. This introductory chapter lays a foundation for our study of discrete mathematics by addressing all these themes. In the second section, we will examine patterns in number sequences, and in particular, we will discuss the idea of a "recursive model" that will be important throughout the book. The subsequent sections address basic logic particularly as it applies to clarity of language. We are accustomed to people not saying exactly what they mean in conversation, and as a result, we are good at interpreting statements from context, but there is no room for ambiguity in mathematics. We end the chapter with a discussion of validity of the kinds of arguments that one might encounter in life.**

## 1.1 First Examples

In this section we will take a first look at some of the puzzles and games that will occupy us through the weeks ahead. We will not yet worry about the mathematics that lies behind these puzzles and games. The mathematical ideas will confront us soon enough. At this point, you should just immerse yourself in these examples to get a feeling for the type of problems we will be studying in this book. Make some time to perform the magic trick on a roommate, discuss the Josephus problem in class, solve the puzzles, and play the games. This will make the mathematics more meaningful to you as we get to each of the relevant topics behind these examples.

## A Magic Trick

At a recent magic show, the following trick was performed. Four playing cards* were stacked face up, with a heart at the bottom, then a club, then a diamond, and then a spade. A spectator held this packet while the blindfolded magician gave these instructions:

1. Turn the spade (the uppermost card) face down.
2. Move any number of cards, one at a time, from the top of the packet to the bottom.
3. Turn over the top two cards as one.
4. Move any number of cards, one at a time, from the top of the packet to the bottom.
5. Turn over the top two cards as one.
6. Move any number of cards, one at a time, from the top of the packet to the bottom.
7. Either turn over the entire stack or do not—your choice.
8. Turn over the topmost card.
9. Turn over the top two cards as one.
10. Turn over the top three cards as one.

At this point, the prestidigitator, while still blindfolded, correctly divined the state of the stack. The club was the only card facing the opposite way from the others!

This trick takes advantage of the fact that the actions that seem like shuffling to the spectator are actually preserving all the properties the magician cares about. We will see how to verify this "invariance of properties" in Section 2.4.

**Example 1** *Write down the initial state of the deck (from bottom to top) as H C D S. Using uppercase letters to indicate the card is face up and lowercase for face down, we see that after step 1 the deck would be H C D s. Starting from here, trace the state of the deck through step 10 if in steps 2, 4, and 6, we move exactly two cards from the top to the bottom, and in step 7 we turn over the packet.*

SOLUTION Using the notation described, we can show the state of the packet after each step.

| | | | |
|---|---|---|---|
| Step 1. | H C D s | Step 6. | S d c h |
| Step 2. | D s H C | Step 7. | H C D s |
| Step 3. | D s c h | Step 8. | H C D S |
| Step 4. | c h D s | Step 9. | H C s d |
| Step 5. | c h S d | Step 10. | H D S c |

□

* The reader who is unfamiliar with the anatomy of a deck of playing cards should consult Appendix A.

**Practice Problem 1** (**Note.** Practice problems allow the reader to stop and try an idea before moving on. The answers to all practice problems are given at the end of the section.) *The following questions refer to the magic trick described in Example 1.*

*(a)* *Repeat Example 1 but this time in step 7, do not turn over the deck.*

*(b)* *Repeat Example 1, but this time in steps 2, 4, and 6, move one, two, and three cards (respectively) from the top to the bottom, and in step 7 do not turn over the deck.*

## A Matter of Life and Death

In [32], the following legend about the first-century historian Flavius Josephus is recounted:

> *In the Jewish revolt against Rome, Josephus and 39 of his comrades were holding out against the Romans in a cave. With defeat imminent, they resolved that, like the rebels at Masada, they would rather die than be slaves to the Romans. They decided to arrange themselves in a circle. One man was designated as number one, and they proceeded clockwise killing every seventh man. . . . Josephus (according to the story) was among other things an accomplished mathematician; so he instantly figured out where he ought to sit in order to be the last to go. But when the time came, instead of killing himself he joined the Roman side.*

The solution is for Josephus to stand in the 24$^{\text{th}}$ position in the circle. It is yet another historical example of how those with a distaste for mathematics quickly become the chaff of evolution. That point aside, the problem rightfully raises the question of how Josephus could have quickly computed this position.

We will refer to this scenario as the "Josephus game" even though it does not sound like fun. We call it a game because it is very similar to methods that children use to decide things on the playground. Perhaps we should imagine the soldiers saying (in Latin, of course):

> *One potato, Two potato, Three potato, Four; Five potato, Six potato,*
> *Seven potato, More*

On the "More" of course, someone would be killed, which would be a shame, but it would certainly eliminate that person. This is much like the Josephus game where one skips by eight people between those who are eliminated instead of seven. Another playground example is "eeny-meeny-miney-moe," where the skip number is 16.

Here are some samples of the many questions one can ask about this type of game. We will begin an examination of the first question in the practice problems and exercises of this section. The others are more difficult. In fact, no one knows the answer to the last one.

1. If every second person is killed (instead of every seventh) when there are initially $n$ people in the circle, where should Josephus stand?
2. In some versions of the story, Josephus saves a friend by having the friend stand in a position so that the two of them are the last two people left alive. If every second

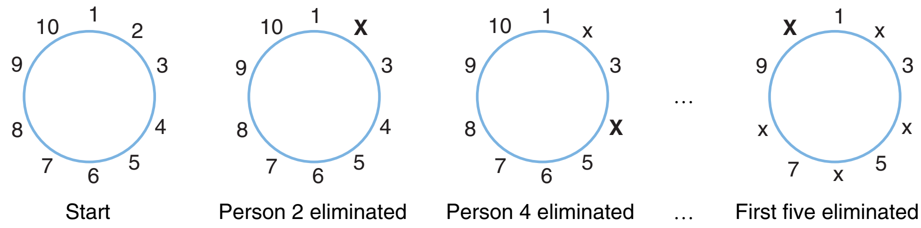


**Figure 1-1** The beginning of the Josephus "game."

person is killed (instead of every seventh) when there are initially $n$ people in the circle, where should *Josephus's friend* stand?

3. If Josephus is not allowed to move positions but is asked what the "skip number" should be (say Monty Hall is a Roman soldier, for instance), can he always respond in a way so that he will live? What about his friend?
4. Characterize all sets of people in the original circle who can be made to be the last people living by naming an appropriate skip number. In other words, can Josephus save any number of friends in any positions by naming an appropriate skip number?*

The Josephus problem is the basis for introducing recursive modeling in [29] as well as in [18]. In [32] the concept of *Josephus permutation* is introduced so that the entire killing process is captured instead of just the grisly end result. Later in this book, we will explore some aspects of Josephus permutations using recursion and induction.

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**Example 2** *If there are 10 people numbered 1 to 10 in a circle, and every other person is eliminated starting with person number 2, which person is the last standing?*

SOLUTION In Figure 1-1, we use the letter x to represent dead people and a capital **X** to highlight the most recent person eliminated.

In the interest of space, we will write the state of the game in a line rather than draw the circle each time. For example, after the first five people have been eliminated, we can write 1 x 3 x 5 x 7 x 9 **X** instead of drawing the last picture above, and we can still tell from the placement of the **X** that the next person to be eliminated is number 3. If we use this system, the remainder of the game proceeds like this:

- 1 x **X** x 5 x 7 x 9 x
- 1 x x x 5 x **X** x 9 x
- **X** x x x 5 x x x 9 x
- x x x x 5 x x x **X** x

Hence, person number 5 is the last one standing in this game. □

* This is given as a "Research problem" in [29].

### Practice Problem 2

*(a)* *If there are nine people numbered 1 to 9 in a circle, and every other person is eliminated starting with person number 2, which person is left?*

*(b)* *If there are eight people numbered 1 to 8 in a circle, and every other person is eliminated starting with person number 2, which person is left?*

*(c)* *If there are sixteen people numbered 1 to 16 in a circle, and every other person is eliminated starting with person number 2, which person is left?*

*(d)* *Explain why the answer to #3 is the same as for #2.*

## It's Just a Game

Sporting events present an interesting challenge for mathematical modeling since in sports lies a blend of strategy, skill, and luck that is hard to separate. We will simplify the analysis by using probabilities to simulate all three aspects. For example, if a baseball player bats 0.300 for a season, we will use $\frac{3}{10}$ as the probability of that player getting a hit in any given at-bat. This is certainly an oversimplification—in a given at-bat, the probability of a player getting a hit could be higher or lower depending on who is pitching, how many outs there are, or what he had for breakfast. However, using $\frac{3}{10}$ as if it is the real probability makes our calculations feasible, and it is a reasonable simplification.

We will look at two kinds of sports events in this course, each of which is characterized by the following examples from tennis:

1. In a certain tennis league, the first player to win two sets wins the match. If Player A has a 55% chance of winning a set against Player B, what is the probability that Player A wins the match? If these two players played many matches, what would you expect to be the average number of sets that determine a match?
2. In tennis, each player's score progresses from 0 to 15 to 30 to 40 to *game* with the wrinkle that a final score of *game* to 40 is impossible. A score of 40–40 is called "deuce," and a game can only be won from deuce if one player wins two consecutive points. (This is equivalent to the rule of "winning by two" in other sports.) If Player A has a 60% chance of winning a **point** against Player B, what is the probability that Player A wins the game? If these two players played many games, what would you expect to be the average number of points that determine a game?

The fundamental difference between these two examples is that the first one can be analyzed by exhaustively cataloging all possible matches, but the second one cannot because there are infinitely many possible ways that a game could progress. In Chapter 6, we will see how to handle each type of problem.

**Practice Problem 3** *Suppose that Player A and Player B play a tennis match, where Player A has a 55% chance of winning any given set and the first player to win two sets wins the match. (This is called a "best-of-three" match.) One example of how a match could go is, "A wins the first set, B wins the second, and A wins the third." This sequence of events can be represented more briefly by the simple string ABA. Every outcome of a match has a similar representation.*

1. *Suppose that Player A and Player B play a best-of-three match. List in an organized manner (using the representation described above) all the different ways the match could go.*
2. *Is every way you listed in Part 1 equally likely to occur? In particular, which do you think is more likely, that A wins in two sets, or that B wins in two sets?*
3. *Suppose that Player A and Player B play a single game that lasts for eight points. One way this could happen is for the points to be won in this order: ABABABAA. Give two other ways this could happen.*
4. *Explain why a single game cannot last for exactly seven points.*
5. *In an organized manner, list all the different ways Player A can win in six or fewer points.*

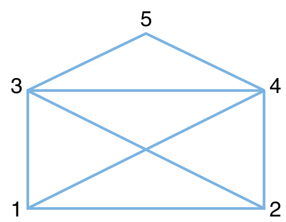

**Figure 1-2** A simple envelope.

## An Elementary Puzzle

It is not too difficult to draw the picture in Figure 1-2 without lifting your pencil from the paper and without retracing any lines. It is not as easy to decide whether the picture in Figure 1-3 can be drawn in the same way, but it can.

We will see how to make this decision for any picture without resorting to hours of trial and error. Surprisingly, this is related to an eighteenth-century puzzle that was solved by Leonhard Euler. Perhaps even more unexpected is that there are present-day applications of the same idea in many hard problems which have to do with scheduling tasks and constructing networks.

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**Example 3** *Starting at the lower left corner of the envelope (position 1 in Figure 1-2), describe a way to draw the envelope of Figure 1-2 without lifting your pencil and without retracing any lines.*

SOLUTION One possibility is to go up, then across the top, do the "flap," go diagonally to the lower right corner, then up, diagonally to the lower left corner, and finally go across the bottom. Using the numbering in Figure 1-2, we can represent this path by the list of corners we pass through:

$$1, 3, 4, 5, 3, 2, 4, 1, 2$$

□

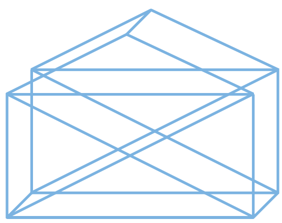
**Figure 1-3** A more complicated picture.

### Practice Problem 4

(a) *If you have not already done so, find a different solution to Example 3 than the one given.*

(b) *Do you think it is possible to start at the upper left corner (position 3)?*

## A Game of Skill?

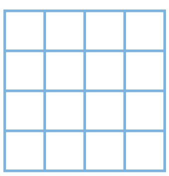
**Figure 1-4** The grid game.

Play this game with a friend. Start with a $4 \times 4$ grid as shown in Figure 1-4. Players alternate turns where on each turn a player

1. Chooses a row or column that has an empty space, and
2. Places new X's anywhere in that row or column.

| | | | |
|---|---|---|---|
| X | | X | X |
| X | X | X | |
| | | | X |
| X | X | X | X |

| | | | |
|---|---|---|---|
| | X | X | X |
| X | X | X | |
| X | | | X |
| X | | X | X |

**Figure 1-5** Grids for Practice Problem 5.

At least one new X must be placed on each turn, and the person to X the last available space wins. Does either player have a foolproof strategy for winning this game?

In Chapter 7, we will see why one player *does* always have a winning strategy in a broad category of similar games. We will also see how this fact does not help in most real-world examples, since the strategy is often too complicated to implement effectively.

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## Practice Problem 5

*(a)* *It's your turn to move, and the board looks like the grid on the left in Figure 1-5. Describe a winning strategy.*

*(b)* *It's your turn to move, and the board looks like the grid on the right in Figure 1-5. Can you win if your opponent plays intelligently?*

# Solutions to Practice Problems

1 We use the same notation as in Example 1, showing the state of the deck after each step.

(a) If in steps 2, 4, and 6, we move exactly two cards from the top to the bottom, and in step 7 we do not turn over the packet, the trick would proceed as follows:

| | | | |
|---|---|---|---|
| Step 1. | H C D s | Step 6. | S d c h |
| Step 2. | D s H C | Step 7. | S d c h |
| Step 3. | D s c h | Step 8. | S d c H |
| Step 4. | c h D s | Step 9. | S d h C |
| Step 5. | c h S d | Step 10. | S c H D |

(b) If in steps 2, 4, and 6, we move one, two, and three cards (respectively) from the top to the bottom, and in step 7 we do not turn over the deck, the trick would proceed as follows:

| | | | |
|---|---|---|---|
| Step 1. | H C D s | Step 6. | c h S d |
| Step 2. | s H C D | Step 7. | c h S d |
| Step 3. | s H d c | Step 8. | c h S D |
| Step 4. | d c s H | Step 9. | c h d s |
| Step 5. | d c h S | Step 10. | c S D H |

2 We will use the same notation as in Example 2.

(a) After the first four eliminations we have 1 x 3 x 5 x 7 X 9 and next in line is 1. The next four eliminations will go as follows:

— X x 3 x 5 x 7 x 9

— x x 3 x X x 7 x 9

— x x 3 x x x 7 x X

— x x 3 x x x X x x

(b) After the first four eliminations, we have 1 x 3 x 5 x 7 X, with 3 next in line. Then 3, 7, and 5 go, leaving 1.

(c) After the first eight eliminations, we have 1 x 3 x 5 x 7 x 9 x 11 x 13 x 15 X, with 3 next in line. After the next four eliminations, we have

1 x x x 5 x x x 9 x x x 13 x X x

with 5 next in line. Then 5, 13, and 9 go, leaving 1.

(d) After the first eight eliminations, there are eight people left (1, 3, 5, 7, 9, 11, 13, 15) with the second one next in line for extinction. This is really the same problem as starting with eight people, with the second one the first to go.

3 (a) You can write them out as AA, ABA, ABB, BAA, BAB, BB. Alternatively, the "game tree" in Figure 1-6 provides a handy visualization of the solutions.

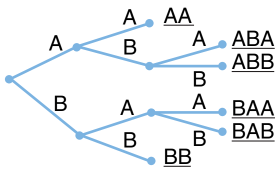


**Figure 1-6** A game tree for Practice Problem 3.

| First Three Points | Conclusion |
|---|---|
| AAA | A<br>BA<br>BBA |
| AAB | AA<br>ABA<br>BAA |
| ABA | AA<br>ABA<br>BAA |
| ABB | AAA |
| BAA | AA<br>ABA<br>BAA |
| BAB | AAA |
| BBA | AAA |
| BBB | AAAA |

**Table 1-1** Table for Practice Problem 3

(b) Because A is more likely to win each set, it seems reasonable that A is more likely to win two sets in a row than is B.

| | | | |
|---|---|---|---|
| | X | X | X |
| X | X | X | |
| X | X | | X |
| X | | X | X |

**Figure 1-7** Grid game in Practice Problem 5.

(c) ABBABAAA and BBBAAAAA are two of many possibilities.

(d) Here are the scores that total to seven points with the winner ahead by at least two points: 7–0, 6–1, 5–2. None of these are possible, since in each case the game would have ended when the winner reached four points.

(e) One way to organize is to think about the first three points, which can be either AAA, AAB, ABA, ABB, BAA, BAB, BBA, BBB. For each of these, we list in Table 1-1 all the possible conclusions of the game that result in a win by A.

4 (a) If we use the same notation as in Example 3, a different way to draw the envelope can be represented as

$$1, 2, 4, 5, 3, 1, 4, 3, 2$$

(b) In the graph theory chapter, we will see that this is not possible.

5 (a) By completely filling row 3, you leave two spaces, and your opponent cannot X out both at once.

(b) If you completely fill in any column or any row, your opponent can win. Likewise, if you fill in row 3, column 3 or row 4, column 2, then your opponent has a winning strategy. However, if you fill in row 3, column 2, you have the board in Figure 1-7 and you are guaranteed a win.

## Exercises for Section 1.1

Exercises numbered in bold face blue have answers or partial answers in the back of the book.

**1.** Suppose we use $C(a, b, c, d)$ to mean the result of doing the card trick with $a$, $b$, $c$, and $d$ describing what to do in steps 2, 4, 6, and 7, respectively. For example, $C(2, 2, 2, no)$ would mean to move two cards to the bottom at each of steps 2, 4, and 6, and then to not turn over the deck in step 7. Find the final configuration of the deck for each of the following games:

(a) $C(1, 1, 1, no)$

**(b)** $C(1, 1, 1, yes)$

(c) $C(3, 3, 3, no)$

**(d)** $C(3, 2, 1, yes)$

**2.** Explain why in the card trick, after each of steps 2–7, the packet of cards contains exactly one card reversed from the others.

**3.** Suppose we use $J(p, s)$ to mean the Josephus game with $p$ people and a skip amount of $s$. For example, $J(10, 2)$ means the game with 10 people in which every other person is eliminated, starting with person 2. Play the following Josephus games to determine the winner. Also decide where Josephus's friend should stand—that is,

(a)

| | | | |
|---|---|---|---|
| X | | X | X |
| X | X | X | |
| X | | X | X |
| X | X | X | X |

(b)

| | | | |
|---|---|---|---|
| X | | X | X |
| X | X | X | |
| X | | X | X |
| X | X | X | |

(c)

| | | | |
|---|---|---|---|
| | | X | X |
| X | X | X | |
| X | | X | X |
| X | X | X | |

**Figure 1-8** Grids for Exercise 12.

who is the last person who would be eliminated before the final survivor?

(a) $J(15, 3)$

**(b)** $J(32, 2)$

(c) $J(15, 2)$

**(d)** $J(5, 4)$

4. Suppose a tennis match is a "best of five" rather than a "best of three." Draw the complete game tree. If each of the outcomes is equally likely, what percentage of the time would you expect the match to last five sets?

**5.** We showed that a tennis game cannot consist of exactly seven points. What other numbers of total points are impossible?

**6.** Draw a game tree to show the fifteen ways in which Player A can win a tennis game in six or fewer points. (See Practice Problem 3.)

7. Suppose you toss three coins, a nickel, dime, and quarter, and record the results in that order. For example, HTH would mean head on the nickel, tails on the dime, and heads on the quarter.

**(a)** In a systematic way, list all the different results you could record.

**(b)** Draw a game tree for the recording of the results.

**(c)** On the game tree, label each possible result either 0, 1, 2, or 3, indicating how many heads it has. Do you think a person who tosses three coins is more likely to get all three heads, or to get exactly two heads?

**8.** Suppose you take a true-false test with three questions, and you answer all the questions.

**(a)** In a systematic way, write all the different answers you could give. Use the representation where, for example, TFF means answering true for Question 1, false for Question 2, and false for Question 3.

**(b)** Draw a game tree for the different answers.

**(c)** Suppose the correct answers on the test are TFT. On your game tree, label each possible result either 0, 1, 2, or 3, indicating how many correct answers it has. Do you think a person who guesses is more likely to get exactly one right, or to get exactly two right?

9. Explain the similarity between your answers in Exercises 7 and 8.

**10.** Explain the similarity between the answers in Exercises 7 and 8 and the list of how the first three points could go in a tennis game between Player A and Player B.

11. In solutions to the envelope puzzle, are there any other places on the envelope where it would be possible to start? After you've investigated this, make a complete list of where you were able to start, and for each of these write down where you finished. What do you notice?

12. Consider the grid game from Practice Problem 5. For each of the grids in Figure 1-8, if it's your turn, can you make a play that guarantees you a win no matter what your opponent does, or can your opponent win no matter what you do? Explain.

# 1.2 Number Puzzles and Sequences

## Guess the Next Number

One reason we focus on puzzles and games in this course is that much of mathematics seems like a puzzle or a game when viewed the right way. Most of you are familiar with the following type of mathematical puzzle.

**What is the next number in each of the following lists?**

1. 5, 7, 9, 11, 13, __________
2. 1, 9, 17, 25, 33, 41, __________

3. 1, 4, 9, 16, 25, 36, ________
4. 2, 4, 8, 16, 32, 64, ________
5. 1, 2, 6, 24, 120, 720, ________

The key to this type of puzzle is to find a pattern in the sequence of numbers. By “pattern,” we mean one of three things:

(i) Each term is related (by arithmetical operations) to previous terms.
(ii) Each term can be described relative to its position in the sequence.
(iii) The sequence merely enumerates a set of integers that the reader must recognize from the scant information given.

Occasionally, a sequence can be described in all three ways, as we see in our first example.

**Example 1** *How can the sequence* 1, 3, 5, 7, 9, . . . *be described in each of these ways?*

SOLUTION

(i) Each term is 2 more than the previous term and the first term is 1. When we describe a sequence in this way, we have to describe not only how each term is related to the previous term(s), but also how the sequence begins. For example, if we simply say, “Each term is 2 more than the previous term,” that could also describe the sequence 202, 204, 206, 208, . . . . The stipulation that the first term is 1 guarantees that we have a complete description.
(ii) The $n^{\text{th}}$ term is given by the formula $(2n - 1)$. So the first term is $2 \cdot 1 - 1 = 1$, the second term is $2 \cdot 2 - 1 = 3$, and so on.
(iii) The sequence merely enumerates the positive odd integers.

□

In this course, we will be concerned primarily with characterizations of the first two types. One reason we are interested in both types of characterizations is that each has its own particular strengths. The first type of description is often easier to spot and describe, such as in the sequence 1, 1, 2, 3, 5, 8, . . . , where each term is the sum of the two previous terms.

On the other hand, the second type of description means having a formula for calculating each term relative to its position in the sequence, and this makes it easy to calculate a particular term. For example, if the $n^{\text{th}}$ term of some sequence is given by the formula $3n - 7$, we can quickly determine that the $100^{\text{th}}$ term in the list is $3(100) - 7 = 293$.

**Practice Problem 1** *Consider the sequence* 4, 6, 8, 10, 12, . . . .

*(a) Describe the sequence in each of the three ways we have mentioned.*
*(b) If you know a certain term of this sequence is* 898, *what would the next three terms be? Which type of description is most useful for answering this question?*

(c) *What is the $1{,}000^{\text{th}}$ term of the sequence? Which type of description is most useful for answering this question?*

## Sequences and Sequence Notation

Before we examine some more complicated examples, we will introduce some notation and terminology.

**Definition** A **recursive formula** for a sequence is a formula where each term is described in relation to a previous term (or terms) of the sequence. This type of description must include enough information on how the list begins for the recursive relationship to determine every subsequent term in the list. This is sometimes called a **recurrence relation.**

**Definition** A **closed formula** for a sequence is a formula where each term is described only in relation to its position in the list.

**Definition** [**Sequence notation**] We usually use lowercase letters ($a, b$, etc.) to name sequences, and we use subscripting to indicate position in a sequence. The notation $a_n$ indicates the $n^{\text{th}}$ term of the sequence we are writing as $a$. We read $a_n$ as "$a$ subscript $n$," or more usually just "$a$ sub $n$."

**Example 2** *In the sequence of numbers* $1, 3, 5, 7, 9, \ldots,$

$a_1 = 1$ *means "the first term in the sequence is 1"*

$a_2 = 3$ *means "the second term in the sequence is 3"*

$a_3 = 5$ *means "the third term in the sequence is 5"*

$a_4 = 7$ *means "the fourth term in the sequence is 7"*

*and so on. A **closed formula** for this sequence is*

$$a_n = 2n - 1$$

*In words, this says, "the $n^{\text{th}}$ term of the sequence is given by the formula $2n - 1$."*

*A **recursive formula** for this sequence is*

$$a_1 = 1 \qquad \text{and} \qquad a_n = a_{n-1} + 2$$

*In words, this says, "the first term in the sequence is 1," and "each term in the sequence can be found by adding 2 to the previous term."*

Sequence notation is quite similar to the function notation you are familiar with from algebra courses. When we write $a_n = 2n - 1$, it is similar to writing $f(x) = 2x - 1$. In each case, the formula gives a rule for calculating the result for a particular given number. Just as $f(15)$ would be calculated by substituting 15 for $x$ in the formula $2x - 1$, so $a_{15}$ is found by substituting 15 for $n$ in the formula $2n - 1$. So $a_{15} = 2 \cdot 15 - 1 = 30 - 1 = 29$.

In this notation, we think of the subscripts as *ordinal numbers*. The ordinal numbers are *first, second, third, fourth, . . . , one-hundred-twenty-first*, and so on. They indicate ordering within a list—hence the term ordinal. So when we see a subscript, we can interpret it as a position within the sequence. For example, $a_{21}$ indicates the $21^{\text{st}}$ term in the sequence. Likewise, $a_n$ is the $n^{\text{th}}$ term in the sequence, and $a_{n-1}$ would be the $(n-1)^{\text{th}}$ term, that is, the term just before the $n^{\text{th}}$ term. So when we write $a_n = a_{n-1} + 2$, we can read it as "the $n^{\text{th}}$ term in the sequence is equal to the $(n-1)^{\text{th}}$ term plus 2." Since the $(n-1)^{\text{th}}$ term is the one immediately before the $n^{\text{th}}$ term, we sometimes just describe it as "the previous term."

In the preceding discussion, it is implicit that the subscripts start at $n = 1$. We will use this convention throughout our discussion of numerical sequences. For example, when we write the closed formula

$$a_n = 2n - 1$$

it will mean the same as writing

$$a_n = 2n - 1, \quad \text{for integers } n \geq 1$$

If for some reason we decide to begin with a subscript other than 1, we will carefully indicate that fact.

On the other hand, for a recursive formula we explicitly give some initial terms in the sequence, so the formula typically starts with the first term not explicitly given. So when we write the recursive formula

$$a_1 = 1 \qquad \text{and} \qquad a_n = a_{n-1} + 2$$

we mean that the recursive relationship holds for all values of $n$ beyond the explicitly given term for $n = 1$. That is, this is shorthand for

$$a_1 = 1 \qquad \text{and} \qquad a_n = a_{n-1} + 2, \quad \text{for integers } n \geq 2$$

The following examples illustrate more about the relationship between closed formulas and recursive formulas for sequences of numbers.

**Example 3** *For the sequence whose closed formula is $a_n = 2^n - 1$, do the following:*

1. *List the first five terms of the sequence.*
2. *Calculate the value of the $20^{\text{th}}$ term.*
3. *Give a formula for the $(k+1)^{\text{th}}$ term.*
4. *Give a formula for $a_{2i-3}$.*

SOLUTION

1. $a_1 = 2^1 - 1 = 1,\ a_2 = 2^2 - 1 = 3,\ a_3 = 2^3 - 1 = 7,\ a_4 = 2^4 - 1 = 15,\ a_5 = 2^5 - 1 = 31$.
2. $a_{20} = 2^{20} - 1 = 1{,}048{,}575$.
3. $a_{k+1} = 2^{k+1} - 1$. Just as we find $a_{20}$ by replacing $n$ with 20 in the formula, we find $a_{k+1}$ by replacing $n$ with $k + 1$ in the formula.
4. We solve this by replacing $n$ with $2i - 3$ in the formula, so $a_{2i-3} = 2^{2i-3} - 1$. □

**Example 4** *Consider the sequence whose recursive formula is* $a_1 = 11$ *and* $a_n = a_{n-1} + 5$.

1. *Write the first five terms of the sequence.*
2. *Write a recursive formula for the* $80^{\text{th}}$ *term.*
3. *Write a recursive formula for the* $(k+1)^{\text{th}}$ *term.*
4. *Write a recursive formula for* $a_{3j-2}$.

SOLUTION

1. We can solve the problem by plugging into the recursive formula as follows:

$$\begin{aligned} a_1 &= 11 \\ a_2 &= a_{2-1} + 5 = a_1 + 5 = 11 + 5 = 16 \\ a_3 &= a_{3-1} + 5 = a_2 + 5 = 16 + 5 = 21 \\ a_4 &= a_{4-1} + 5 = a_3 + 5 = 21 + 5 = 26 \\ a_5 &= a_{5-1} + 5 = a_4 + 5 = 26 + 5 = 31 \end{aligned}$$

   Alternatively, once we realize that the recursive formula says, "Each term is equal to the previous term plus 5," we can simply write down the sequence as 11, 16, 21, 26, 31.
2. In the formula $a_n = a_{n-1} + 5$, we replace $n$ with 80, giving $a_{80} = a_{80-1} + 5$, which simplifies to $a_{80} = a_{79} + 5$. In other words, the $80^{\text{th}}$ term is 5 more than the $79^{\text{th}}$ term.
3. We replace $n$ with $k+1$, giving $a_{k+1} = a_{(k+1)-1} + 5$, which simplifies to $a_{k+1} = a_k + 5$. Notice that $a_{k+1}$ is the next term in the sequence after $a_k$. We can express this formula in words as "the next term is equal to this term plus 5," which is all the original description says.
4. We replace $n$ with $3j - 2$, giving $a_{3j-2} = a_{(3j-2)-1} + 5$, which simplifies to $a_{3j-2} = a_{3j-3} + 5$.

□

## Discovering Patterns in Sequences

We will now return to the original problems in this section. We will try to do two things with these puzzles. First, we will determine the next number in each sequence. To do this, we will have to discover a pattern for the sequence. We will then try to use the discovered pattern to describe the sequence with either a closed formula or a recursive formula (or perhaps both). Here are the puzzles again:

1. 5, 7, 9, 11, 13, __________
2. 1, 9, 17, 25, 33, 41, __________
3. 1, 4, 9, 16, 25, 36, __________
4. 2, 4, 8, 16, 32, 64, __________
5. 1, 2, 6, 24, 120, 720, __________

Before reading on, try to discover the pattern and determine the next number in each sequence. Then we will discuss the answers.

*Explore more on the Web.*

**Example 5** *Give both closed and recursive formulas for the sequence that begins*

$$5, 7, 9, 11, 13$$

SOLUTION It appears that the sequence lists odd numbers in order, so the next number would be 15. We observe that each term is 2 more than the previous term. It is not too hard to use this fact to develop the recursive formula. We just have to remember that the "previous" number in the sequence can be expressed as $a_{n-1}$. The recursive description is

$$a_1 = 5 \qquad \text{and} \qquad a_n = a_{n-1} + 2$$

The closed formula requires a bit more thought. One approach that works well is to think of an easier sequence (i.e., one you know a closed formula for) with the same recursive pattern, and compare this sequence to the original. In this case, the sequence 2, 4, 6, 8, ... has the easy closed formula $b_n = 2n$ and the same recursive pattern, "each term is 2 more than the previous term." In Table 1-2 we line up the sequence whose closed formula we know with the sequence whose closed formula we wish to find. We notice that each term in the bottom row is 3 more than the corresponding term above it. That is, $a_n = b_n + 3$. Therefore, the closed formula for $a_n$ is

| $n$ | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| $b_n$ | 2 | 4 | 6 | 8 | ... |
| $a_n$ | 5 | 7 | 9 | 11 | ... |

**Table 1-2** Table of Values for Example 5

$$a_n = 2n + 3$$

$\square$

**Example 6** *Give both closed and recursive formulas for these sequences.*

1. *The sequence that begins* 1, 9, 17, 25, 33, 41.
2. *The sequence that begins* 1, 4, 9, 16, 25, 36.
3. *The sequence that begins* 2, 4, 8, 16, 32, 64.
4. *The sequence that begins* 1, 2, 6, 24, 120, 720.

SOLUTION

1. The obvious pattern here is that each term is 8 more than the previous term, so the recursive formula is easy to write

$$a_1 = 1 \qquad \text{and} \qquad a_n = a_{n-1} + 8$$

An easy sequence with this same pattern is $b_n = 8n$, and we can compare this with the original as shown in Table 1-3. Each term in the bottom row is 7 less than the term above it, so the closed formula is $a_n = 8n - 7$.

| $n$ | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| $b_n$ | 8 | 16 | 24 | 32 | ... |
| $a_n$ | 1 | 9 | 17 | 25 | ... |

**Table 1-3** Table of Values for Example 6

2. By now we may be in the habit of looking at differences between successive terms. For the sequence 1, 4, 9, 16, 25, 36, however, those differences change. But there is a pattern in the differences:

$$a_2 = a_1 + \boxed{3}$$
$$a_3 = a_2 + \boxed{5}$$
$$a_4 = a_3 + \boxed{7}$$

$$a_5 = a_4 + \boxed{9}$$
$$a_6 = a_5 + \boxed{11}$$
$$a_7 = a_6 + \boxed{??}$$

This might lead us to conclude that $a_7$ will be 13 more than $a_6$. Can we express the difference in terms of $n$? For $n = 2$ (i.e., when the subscript on the left-hand side of the equation is 2), the number in the box on the right-hand side of the equation is 3. When $n = 3$, the boxed number is 5, and so on. Following this pattern, we conclude that when the subscript on the left-hand side is $n$, the number in the box has formula $2n - 1$. This leads to our recursive formula for the sequence,

$$a_1 = 1 \qquad \text{and} \qquad a_n = a_{n-1} + (2n - 1)$$

For the closed formula, we need to find a pattern in the numbers themselves. In this case it helps to recognize that the sequence can be described verbally as "those positive integers that are perfect squares" ($1^2 = 1, 2^2 = 4, 3^2 = 9, \ldots$). This allows us to write the closed formula

$$a_n = n^2$$

3. For the sequence that begins 2, 4, 8, 16, 32, 64, each successive term is twice as large as the previous, so the next term will be 128. The recursive formula is therefore

$$a_1 = 2 \qquad \text{and} \qquad a_n = 2a_{n-1}$$

For the closed formula, we recognize that the sequence consists of "integers that are a power of 2," beginning at $2 = 2^1, 4 = 2^2, 8 = 2^3$, and we write

$$a_n = 2^n$$

4. When we analyze the sequence that begins 1, 2, 6, 24, 120, 720, we may get stuck for a while. The differences between the terms are 1, 4, 18, 96, 600, and there is no obvious pattern here. Having just worked a problem where each term is the previous term **times** some factor, we look at this to try to detect a pattern:

$$a_2 = a_1 \cdot \boxed{2}$$
$$a_3 = a_2 \cdot \boxed{3}$$
$$a_4 = a_3 \cdot \boxed{4}$$
$$a_5 = a_4 \cdot \boxed{5}$$
$$a_6 = a_5 \cdot \boxed{6}$$
$$a_7 = a_6 \cdot \boxed{??}$$

It appears that $a_7$ will be $a_6$ times 7, or 5,040. The recursive formula is

$$a_1 = 1 \qquad \text{and} \qquad a_n = a_{n-1} \cdot n$$

For the closed formula, notice that $a_1 = 1$, $a_2 = 1 \cdot 2$, $a_3 = 1 \cdot 2 \cdot 3$, $a_4 = 1 \cdot 2 \cdot 3 \cdot 4$, and so on. We write

$$a_n = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

**Note:** This "product of the integers from 1 to $n$" occurs in a variety of mathematical contexts. In fact, mathematicians have created a special notation for it, namely $n!$, which is read as "$n$ factorial." □

**Practice Problem 2** *For each sequence, identify the next number in the sequence. Then give either a closed formula, a recursive formula, or a verbal description. Give more than one description when you can.*

*(a)* 5, 10, 15, 20, 25, ________

*(b)* 5, 10, 20, 40, 80, ________

*(c)* 4, 9, 16, 25, 36, ________

Sometimes each term in a sequence is calculated from more than one of the previous terms. A simple example occurs in the famous *Fibonacci numbers*. This sequence was developed by Leonardo Pisano Fibonacci (1170–1250) as a recreational problem about the growth of rabbit populations, but it turns out to have wide-ranging applications. The sequence begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The sequence follows the rule that, starting with the third term, each term is the sum of the two terms that precede it. For example, $3 = 2 + 1, 5 = 3 + 2, 8 = 5 + 3$. If we use $F_n$ to represent the $n^{\text{th}}$ term of this sequence, this relationship can be stated succinctly as

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 3$$

where the initial conditions $F_1 = F_2 = 1$ must also be given to exactly describe the sequence. We should note that there *is* a formula that produces the $n^{\text{th}}$ Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

but it is not at all obvious from a casual examination of the sequence.

**Example 7** *Discover a closed formula for the sequence whose recursive formula is $a_1 = 2$, and $a_n = 3a_{n-1}$.*

SOLUTION To discover the formula, we first write down the first few terms of the sequence:

$$a_1 = 2, \quad a_2 = 6, \quad a_3 = 18, \quad a_4 = 54$$

In Table 1-4, we compare this to the simpler sequence $b_n = 3^n$, which also has the recursive pattern, "Each term is 3 times the previous term." Observe that each term in the bottom row is $2/3$ times the corresponding term in the $b_n$ row. Therefore,

| $n$ | 1 | 2 | 3 | 4 | … |
|---|---|---|---|---|---|
| $b_n$ | 3 | 9 | 27 | 81 | … |
| $a_n$ | 2 | 6 | 18 | 54 | … |

**Table 1-4** Table of Values for Example 7

$$a_n = \frac{2}{3} \cdot 3^n = 2 \cdot 3^{n-1}$$

Perhaps a more obvious connection between the two rows is that each term in the $a_n$ row is twice the *previous* term in the $b_n$ row. This relationship can be literally transcribed, $a_n = 2 \cdot b_{n-1}$, which results in the same closed formula as above. □

Discovering closed formulas is much more difficult than finding recursive formulas, so in this course, we will concentrate on how to *verify* a proposed closed formula for a sequence whose recursive description we already know. This will be a focus in the next chapter. Before leaving the topic altogether however, we will look at a classic example of a closed formula that comes from an important type of recursive description.

**Example 8** *Discover a closed formula for the sequence whose recursive formula is $s_1 = 1$, and $s_n = s_{n-1} + n$.*

SOLUTION This recursive description is of the type, "Each term is the previous term plus (something)." If we do not evaluate this sum at each stage, we can see an alternate way to describe $s_n$:

- $s_1 = 1$
- $s_2 = 1 + 2$
- $s_3 = (1 + 2) + 3$
- $s_4 = (1 + 2 + 3) + 4$

Hence, $s_n$ is the sum of the first $n$ positive integers, and so there is a nice way to look at $s_n$ in order to discover a closed formula for it. We will start by writing $s_n$ as a sum twice, once with terms in increasing order and once with terms in decreasing order:

$$s_n = 1 + 2 + \cdots + (n-1) + n$$
$$s_n = n + (n-1) \cdots + 2 + 1$$

Now by adding the terms aligned in "columns" above, we have

$$2s_n = (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

Since the right-hand side above consists of $(n+1)$ added to itself $n$ times, we can conclude that

$$s_n = \frac{n \cdot (n+1)}{2}$$

□

Although discovering a closed formula can be difficult, even when we already know a recursive formula, the converse is rarely true. If we know a closed formula, then finding an alternate recursive description is usually straightforward.

**Example 9** *Discover a recursive formula for the sequence whose closed formula is $a_n = 3n + 5$.*

SOLUTION To discover the formula, we first write down the first few terms of the sequence. Using the formula, we get $a_1 = 3 \cdot 1 + 5 = 8$, $a_2 = 3 \cdot 2 + 5 = 11$, $a_3 = 3 \cdot 3 + 5 = 14$, and so on. We summarize the results by just writing the answers: 8, 11, 14, 17, 20, 23, . . . . Then we look for a pattern that relates each term to the previous term, and we see that each one is 3 more than the previous. This leads to the recursive formula

$$a_1 = 8 \qquad \text{and} \qquad \text{for } n \geq 2, \quad a_n = a_{n-1} + 3$$

We use the subscript $n-1$ to indicate "the previous term," that is, "the term just before the $n^{\text{th}}$ term." □

Actually finding a recursive formula from a closed formula is often just a matter of doing some algebra. In the previous exercise, since we are *given* the closed formula $a_n = 3n + 5$, we know that $a_{n-1} = 3(n-1) + 5$, and so verifying

$$a_n = a_{n-1} + 3$$

is a simple matter of verifying algebraically that

$$3n + 5 = (3(n-1) + 5) + 3$$

Let's look at an example with a more complicated formula to see this idea in action.

**Example 10** *Verify that the sequence given by the closed formula $a_n = 3^n - 2$ satisfies the recursive formula*

$$a_n = 3 \cdot a_{n-1} + 4$$

SOLUTION Since $a_n = 3^n - 2$, we know that $a_{n-1} = 3^{n-1} - 2$, and so

$$\begin{aligned} 3 \cdot a_{n-1} + 4 &= 3 \cdot (3^{n-1} - 2) + 4 \\ &= 3^n - 6 + 4 \\ &= 3^n - 2 \\ &= a_n \end{aligned}$$

as desired. □

**Practice Problem 3** Verify that the sequence given by the closed formula $a_n = n^2 + n$ satisfies the recursive formula

$$a_n = a_{n-1} + 2n$$

Our final examples illustrate some ways we can use what we know about a sequence to learn more about the sequence without "starting from scratch."

*Explore more on the Web.*

**Example 11** *For the sequence whose recursive formula is $a_1 = 11$, and $a_n = a_{n-1} + 5$, suppose someone tells you they have already calculated the $213^{\text{th}}$ term of the sequence, and the answer is 1,071. What is $a_{214}$?*

SOLUTION We do not need to start over from $a_1$ to solve the problem. We simply calculate the $214^{\text{th}}$ term from the $213^{\text{th}}$. Replacing $n$ with 214 in the recursive formula gives $a_{214} = a_{213} + 5$. Since we know $a_{213}$ is 1,071, we get

$$\begin{aligned} a_{214} &= a_{213} + 5 \\ &= 1{,}071 + 5 \\ &= 1{,}076 \end{aligned}$$

□

*Explore more on the Web.*

**Example 12** *For the sequence whose closed formula is $a_n = 2^n - 1$, do the following:*

1. *Calculate the value of the $20^{\text{th}}$ term.*
2. *Someone tells you they have calculated the sum of the first 19 terms, and the answer was 1, 048, 555. What is the sum of the first 20 terms?*

SOLUTION

1. $a_{20} = 2^{20} - 1 = 1{,}048{,}575$.
2. This is similar to the previous example, except that it involves summing up the terms. Again we don't have to start over from the first term; we can reason that the sum of the first 20 terms can be calculated by adding the $20^{\text{th}}$ term to the sum of the first 19 terms. So our answer is $1{,}048{,}555 + 1{,}048{,}575 = 2{,}097{,}130$. We can write this in symbols as

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{20} &= (a_1 + a_2 + a_3 + \cdots + a_{19}) + a_{20} \\ &= 1{,}048{,}555 + 1{,}048{,}575 \\ &= 2{,}097{,}130 \end{aligned}$$

□

Does it strike you as odd that in this sequence the $20^{\text{th}}$ term is exactly 20 more than the sum of the first 19 terms? Do you think the $21^{\text{st}}$ term will be exactly 21 more than the sum of the first 20 terms? Look at Exercises 19 and 20 below to explore this some more.

## Notation for Sums

Example 12 demonstrates an important source of number patterns in mathematics. Summing sequences of numbers is essential in mathematical applications ranging from the analysis of algorithms to integral calculus. We will study sums in more detail in Chapter 2, but for now we will just get acquainted with the traditional notation for sums.

**Definition** For a sequence of numbers $a_k$ with $k \geq 1$, we use the notation

$$\sum_{k=1}^{n} a_k$$

to denote the sum of the first $n$ terms of the sequence. This is called *sigma notation* for the sum. In informal situations, we will sometimes write "$a_1 + a_2 + \cdots + a_n$" instead of using sigma notation.

A simple variation on this notation is to write $\sum_{k=m}^{n} a_k$ or $a_m + a_{m+1} + \cdots + a_n$ (where $m \leq n$) whenever we want to sum the numbers in the sequence starting with the $m^{\text{th}}$ term and ending with the $n^{\text{th}}$ term. This notation is easier to read than it is to write, since writing it requires that we know a closed form for the sequence $a_k$ at the outset, and this is not always the case.

**Example 13** *Write each of the following sums using sigma notation:*

1. *The sum of the first 10 numbers in the sequence* $a_k = \frac{1}{k}$ *with* $k \geq 1$.
2. $2 + 4 + 8 + 16 + 32 + 64$
3. $2 + 6 + 18 + 54 + 162$
4. $(-4) + (-1) + 2 + 5 + 8 + 11 + 14$

SOLUTION In each case, we must first determine a closed form for the terms $a_k$ to be summed, and then determine the correct indices of the first term and the last term in the sum.

1. In this case, the form of $a_k$ is given, so we simply have $\sum_{k=1}^{10} \frac{1}{k}$.
2. The terms being summed have closed formula $a_k = 2^k$, the first term in the sum is $a_1 = 2$, and the last term in the sum is $a_6 = 64$. Hence, this sum is represented by $\sum_{k=1}^{6} 2^k$.
3. The terms being summed have closed formula $a_k = 2 \cdot 3^{k-1}$, the first term in the sum is $a_1 = 2$, and the last term in the sum is $a_5 = 162$. Hence, this sum is represented by $\sum_{k=1}^{5} 2 \cdot 3^{k-1}$.
4. The terms being summed have closed formula $a_k = 3k - 7$, the first term in the sum is $a_1 = -4$, and the last term in the sum is $a_7 = 14$. Hence, this sum is represented by $\sum_{k=1}^{7} 3k - 7$.

□

**Practice Problem 4** *Evaluate each of the following sums:*

*(a)* $\sum_{k=1}^{6} (2k-1)$

*(b)* $\sum_{k=0}^{4} 3^k$

*(c)* $\sum_{k=3}^{3} k^2$

*(d)* $\sum_{k=1}^{5} \frac{1}{k(k+1)}$

Sigma notation will be used periodically throughout this book, so it is very important that you become comfortable using it.

## Solutions to Practice Problems

1 (a) **Method 1**: The first term is 4, and each term is 2 more than the previous. **Method 2**: The $n^{\text{th}}$ term is given by $2(n+1)$. **Method 3**: These are the even integers starting at 4.

(b) From Method 1 above, the next three terms are 900,902,904.

(c) From Method 2 above, the $1{,}000^{\text{th}}$ term is $2(1{,}000 + 1) = 2{,}002$.

2 (a) The next number is 30. A closed formula is $a_n = 5n$, a recursive formula is $a_1 = 5$, $a_n = a_{n-1} + 5$, and a description is "multiples of 5."

(b) Next is 160. A closed formula is $a_n = 5 \cdot 2^n$, and a recursive formula is $a_1 = 5$, $a_n = 2a_{n-1}$.

(c) Next is 49. A closed formula is $a_n = (n+1)^2$, a recursive formula is $a_1 = 4$, $a_n = a_{n-1} + (2n+1)$, and a description is "perfect squares starting at 4."

3 Since $a_n = n^2 + n$, we know that $a_{n-1} = (n-1)^2 + (n-1)$, and so

$$\begin{aligned} a_{n-1} + 2n &= \left((n-1)^2 + (n-1)\right) + 2n \\ &= (n^2 - 2n + 1) + (n-1) + 2n \\ &= n^2 + n \\ &= a_n \end{aligned}$$

as desired.

4 (a) $1 + 3 + 5 + 7 + 9 + 11 = 36$

(b) $1 + 3 + 9 + 27 + 81 = 121$

(c) 9

(d) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$

## Exercises for Section 1.2

**1.** For each of these sequences, determine the next number in the sequence. Be able to explain how you got it. Then give either a closed formula or a recursive formula. If you can give both, do so.

(a) 2, 4, 6, 8, 10, 12, __________

**(b)** 4, 9, 16, 25, 36, __________

(c) 2, 5, 10, 17, 26, 37, __________ (HINT. How does this relate to the sequence 1, 4, 9, 16, 25, 36?)

**(d)** 2, 4, 8, 16, 32, 64, __________

(e) 1, 2, 4, 8, 16, 32, __________

(f) 1, 3, 7, 15, 31, 63, ___________
(g) 2, 5, 8, 11, 14, 17, ___________
(h) 4, 16, 64, 256, 1,024, ___________
(i) 5, 10, 20, 40, 80, 160, ___________
(j) 5, 9, 17, 33, 65, 129, ___________
(k) 2, 5, 10, 50, 500, ___________
(l) 1, 5, 9, 13, 17, ___________
(m) 3, 6, 9, 12, 15, ___________
(n) 3, 5, 9, 17, 33, ___________

**2.** For each of the following sequences, calculate $a_{k-1}$ and $a_{k+1}$, and algebraically simplify the expressions:
(a) $a_n = 5n - 2$
(b) $a_n = 3n^2 + n$
(c) $a_n = 2n + 7$
(d) $a_n = 3^n + 4$
(e) $a_n = 2^{3n+1} - 1$
(f) $a_n = \frac{n(2n-1)(n+2)}{6}$

**3.** If you form the decreasing list $n, n-1, n-2, \ldots$, what will be the $k^{\text{th}}$ number in the list? Your answer should involve variables $n$ and $k$.

**4.** Here are verbal descriptions of sequences. Use these descriptions to write recursive formulas for the sequences.
(a) The sequence starts with 2, and each entry is 2 more than the previous entry.
(b) The sequence starts with 1, and each entry is 6 more than the previous entry.
(c) The sequence is obtained by starting with 2, and each subsequent entry is 1 more than twice the previous entry.
(d) The sequence is obtained by starting with 2, and each subsequent entry is the square of the previous entry.

**5.** For each of these sequences described in English, write out the first five (or more if necessary) terms of the sequence. Use your answer to discover a closed formula for the sequence.
(a) The sequence starts with 2, and each entry is 2 more than the previous entry.
(b) The sequence starts with 1, and each entry is 6 more than the previous entry.
(c) The sequence obtained starting with 2, and each subsequent entry is 1 more than twice the previous entry.
(d) The sequence obtained starting with 2, and each subsequent entry is the square of the previous entry.

**6.** For each of the given sequences, use the given closed formula and algebra to check whether the given recursive formula is true:
(a) Given that $a_n = 2^n - 1$ for all $n \geq 1$, is it true that $a_n = 2 \cdot a_{n-1} + 1$?
(b) Given that $a_n = n^2$ for all $n \geq 1$, is it true that $a_n = a_{n-1} + 2n$?
(c) Given that $a_n = 5n + 3$ for all $n \geq 1$, is it true that $a_n = 5 \cdot a_{n-1} - 3$?
(d) Given that $a_n = 1 - \frac{1}{2^n}$ for all $n \geq 1$, is it true that $a_n = a_{n-1} + \frac{1}{2^n}$?

**7.** For each of these sequences given in closed form, write out the first five (or more if necessary) terms of the sequence. Use your answer to discover a recursive formula for the sequence.
(a) $a_n = 3n + 1$
(b) $a_n = 5n - 2$
(c) $a_n = 2n + 7$
(d) $a_n = 3^n + 1$
(e) $a_n = 7n - 6$
(f) $a_n = 3n^2 + n$
(g) $a_n = 2^{2n-1} - 1$

**8.** For each of these sequences given recursively, write out the first five (or more if necessary) terms of the sequence. Use your answer to discover a closed formula for the sequence.
(a) $a_1 = 5; a_n = 5 \cdot a_{n-1}$
(b) $a_1 = 2; a_n = 5 \cdot a_{n-1}$
(c) $a_1 = 3; a_n = 2 \cdot a_{n-1}$
(d) $a_1 = 4; a_n = 4 + a_{n-1}$
(e) $a_1 = 1; a_n = 4 + a_{n-1}$
(f) $a_1 = 6; a_n = 3 + a_{n-1}$

**9.** For each of these sequences given recursively, write out the first five (or more if necessary) terms of the sequence. Use the idea developed in Example 8 to discover a closed formula for each sequence.
(a) $a_1 = 5; a_n = a_{n-1} + (n+4)$
(b) $a_1 = 1; a_n = a_{n-1} + (2n-1)$
(c) $a_1 = -3; a_n = a_{n-1} + 4n$

**10.** Give an algebraic expression that describes the $n^{\text{th}}$ whole number larger than 1,964.

**11.** Give an algebraic expression that describes the $n^{\text{th}}$ odd three-digit number.

**12.** Give an algebraic expression that describes the $n^{\text{th}}$ even two-digit number.

**13.** Give an algebraic expression that describes the $n^{\text{th}}$ odd perfect square.

**14.** Form a sequence using the relationship, "$a_n$ is the ones' digit of $(a_{n-1} + a_{n-2})$." Thus, all your terms will be from the set $\{0, 1, 2, \ldots, 9\}$, which means that this sequence will have to repeat itself at some point. Try different pairs of starting numbers from $\{0, 1, 2, \ldots, 9\}$ and see how long it takes for the sequence to repeat itself. (For example, the sequence 6, 8, 4, 2 follows this rule and would repeat itself if continued.) What is the longest possible such cycle?

**15.** What does the equation

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

say when $a = 2$ and $b = -1$?

**16.** What does the equation

$$C(n,k) = C(n-1,k) + C(n-1,k-1)$$

say when $n = 2k$?

**17.** Give an algebraic expression for the $n^{\text{th}}$ term in the arithmetic progression

$$a, a+3, a+6, a+9, \ldots$$

**18.** Give an algebraic expression for the $n^{\text{th}}$ term in the arithmetic progression

$$a, a+d, a+2\cdot d, a+3\cdot d, \ldots$$

**19.** For the sequence whose closed formula is $a_n = 2^n$, form the new sequence $s_n$ by the definition

$$s_n = a_1 + a_2 + \cdots + a_n$$

Fill in the blanks in Table 1-5, and make a conjecture about the relationship between the bottom two rows.

**20.** For the sequence whose closed formula is $a_n = 2^n - 1$, form the new sequence $s_n$ by the definition

$$s_n = a_1 + a_2 + \cdots + a_n$$

Fill in the blanks in Table 1-6, and make a conjecture about the relationship between the bottom two rows.

**21.** Evaluate each of the following sums:

**(a)** $\sum_{k=1}^{7} 3k$

**(b)** $\sum_{k=1}^{3} 7k$

**(c)** $\sum_{k=1}^{9} 4$

**(d)** $\sum_{k=0}^{4} \frac{1}{2^k}$

**(e)** $\sum_{k=-1}^{3} 3-2k$

| **n** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| $a_n$ | 2 | 4 | 8 | | | | | |
| $s_n$ | 2 | 6 | 14 | | | | | |

**Table 1-5** Table for Exercise 19

| **n** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| $a_n$ | 1 | 3 | | | | |
| $s_n$ | 1 | 4 | | | | |

**Table 1-6** Table for Exercise 20

**22.** Consider the sequence in Practice Problem 3 with recursive description $a_n = a_{n-1} + 2n$ with $a_1 = 2$.

**(a)** Show that $a_4 = \sum_{k=1}^{4} 2k$

**(b)** Show that $a_7 = \sum_{k=1}^{7} 2k$

**(c)** Show that $a_1 = \sum_{k=1}^{1} 2k$

**23.** Consider the sequence $s_n = \sum_{k=1}^{n} 3$.

**(a)** Evaluate $s_5$ and $s_{10}$.

**(b)** Explain in words why $s_n = s_{n-1} + 3$.

**(c)** Give a closed formula for $s_n$ as an algebraic expression using the variable $n$.

**24.** Express each of the following sums using sigma notation:

**(a)** $4+8+12+16+20+24+28+32+36$

**(b)** $2+3+4+\cdots+16+17+18$

**(c)** $5+5+5+5+5+5$

**(d)** $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$

**(e)** $1+9+17+25+33+41+49$

**25.** Consider the Josephus game from the previous section.

**(a)** Suppose we use $J(p,s)$ to mean the Josephus game with $p$ persons and a "skip" amount of $s$. For example, $J(10,2)$ means the game with 10 people and with the elimination starting out 2, 4, 6. For each value of $p$ from 2 to 12, determine the winner for the $J(p,2)$ game, and fill in Table 1-7.

**(b)** If you did this correctly, there should be a discernible pattern in the answers. Describe that

| **p** | **Winner of Game J (p, 2)** |
|---|---|
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| 12 | |

**Table 1-7** Table for Exercise 25

pattern in words. Then use the pattern to predict the answer for 13, 14, 15, and 16 people.

**(c)** Predict the answer for 32 people; for 31 people; for 30 people.

**26.** These are somewhat more challenging number sequences than those given in Exercise 1. For each of these sequences, give a characterization of the sequence. For each, we suggest the type that may be easiest to use, but you may give characterizations of other types.

**(a)** 1, 9, 25, 49, 81, __________ (Formula for $n^{\text{th}}$ term)

**(b)** 1, 2, 2, 4, 8, 32, 256, __________ (Relate term to previous terms)

**(c)** 1, 1, 4, 36, 1764, __________ (Relate term to previous terms)

**(d)** 1, 4, 27, 256, 3125, __________ (Formula for $n^{\text{th}}$ term)

**(e)** 1, 5, 7, 17, 31, 65, __________ (Formula for $n^{\text{th}}$ term)

**(f)** 1, 3, 4, 7, 11, 18, __________ (Relate term to previous terms)

**(g)** 2, 3, 5, 7, 11, 101, 131, 151, __________ (Recognizable set of integers)

**(h)** 6, 8, 10, 14, 15, 21, 22, __________ (Recognizable set of integers)

**27.** Refer to Exercise 25, where we introduced the notation $J(p, s)$ for the Josephus game with $p$ persons and a skip amount of $s$.

**(a)** Suppose five people wearing name tags labeled 1, 2, 3, 4, and 5 line up in that order, and play the game with a skip amount of 4. It is easy to verify that Player 1 wins the game. (Try it!) Then answer these questions without actually playing the game.

**i.** If five people wearing name tags Anne, Sue, Matt, Tom, and Linda line up in that order and play the game with a skip amount of 4, who will win?

**ii.** If five people wearing name tags labeled 5, 6, 1, 2, 3 line up in that order and play the game with a skip amount of 4, who will win?

**(b)** Suppose six people wearing name tags labeled 1, 2, 3, 4, 5, 6 line up in that order and play the game with a skip amount of 4.

**i.** Who is the first person eliminated?

**ii.** After the first person is eliminated, who is left (and in what order)?

**iii.** Without finishing the game, use the answer to a previous part of this question to tell who will win.

**(c)** The preceding exercise establishes that, knowing that $J(5, 4)$ is 1, we can easily determine that $J(6, 4)$ is 5. Use this fact, and similar reasoning, to determine $J(7, 4)$.

**(d)** Suppose you know that $J(15, 4)$ is 13. Use this to calculate $J(16, 4)$.

**28.** In this exercise, the value of each term depends on a term that appears further back in the sequence, and the rule for $a_n$ depends on whether $n$ is even or odd. Give the first seven terms of each sequence.

**(a)** $a_1 = 1$ and $a_n = \begin{cases} a_{n/2} + n, & \text{if } n \text{ is even} \\ a_{(n-1)/2} + n, & \text{if } n \text{ is odd} \end{cases}$

**(b)** $a_1 = 1$ and $a_n = \begin{cases} 3a_{n/2} + 2, & \text{if } n \text{ is even} \\ 3a_{(n-1)/2} + 2, & \text{if } n \text{ is odd} \end{cases}$

**(c)** $a_1 = 1$ and $a_n = \begin{cases} 3a_{n/2} + n, & \text{if } n \text{ is even} \\ 3a_{(n-1)/2} + n, & \text{if } n \text{ is odd} \end{cases}$

**29.** In this exercise, we build a sequence whose values are *strings* of A's and B's, similar to those used in Section 1.1 to represent tennis matches. We define the sequence recursively by the following rule:

$$a_1 = B$$
$$a_n = A\, a_{n-1}\, BA$$

That is, $a_n$ consists of the letter $A$, followed by the string for $a_{n-1}$, followed by the string $BA$. For example, $a_2$ is the string consisting of $A$ followed by $a_1$ (which is $B$) followed by $BA$, making $a_2 = ABBA$.

**(a)** Calculate terms $a_3$ through $a_6$ of this sequence.

**(b)** Modify the recursive definition to yield the sequence $AB$, $AABB$, $AAABBB$, $AAAABBBB$, and so on. That is, $a_n$ should consist of $n$ occurrences of $A$ followed by $n$ occurrences of $B$.

**(c)** Modify the recursive definition to yield the sequence $ABB$, $AABB$, $AAABB$, $AAAABB$, $AAAAABB$, and so on. That is, $a_n$ should consist of $n$ occurrences of $A$ followed by $BB$.

**30.** This exercise combines the ideas in Exercises 28 and 29. Define the sequence as $a_1 = A$ and

$$a_n = \begin{cases} a_{n/2}\, B, & \text{if } n \text{ is even} \\ a_{(n-1)/2}\, A, & \text{if } n \text{ is odd} \end{cases}$$

For example, $a_2 = a_{2/2}\, B = a_1\, B = AB$ and $a_5 = a_{(5-1)/2}\, A = a_2\, A = ABA$.

**(a)** Write down the first ten strings in this sequence.

**(b)** Use the results from part (a) to determine $a_{17}$ and $a_{21}$.

**(c)** Suppose you already know that $a_{315} = ABBAAABAA$. Give the values of $a_{630}$ and $a_{631}$.

## 1.3 Truth-tellers, Liars, and Propositional Logic

Raymond Smullyan (1919– ) has written several books of puzzles about conversations held on a fictional island where inhabitants are either truth-tellers or liars. The point of the puzzles is to discern which inhabitant is of which type, using only the statements you hear them make, if possible.

**Example 1** *You meet two inhabitants of Smullyan's Island. A says, "Exactly one of us is lying," and B says, "At least one of us is telling the truth." Who (if anyone) is telling the truth?*

To solve this puzzle, we will consider all possibilities for each person's status as a liar or truth-teller. In the table below, we will use $p$ to stand for the phrase, "$A$ is truthful," and $q$ to stand for the phrase, "$B$ is truthful." With this convention, we can organize our consideration of all the possibilities as follows: Either $p$ is true or $p$ is false, and either $q$ is true or $q$ is false. In each case, we can determine the truth or fallacy of what $A$ and $B$ said.

The secret to avoiding dizziness with these puzzles is to **first consider each statement on its own instead of worrying about who said it.** For example, we can tell that the phrase, "At least one of us is telling the truth" is true if either of $p$ or $q$ is true. We can make this conclusion without knowing who made the statement.

In this same way, we can use a table to analyze the truth value of each spoken statement for every possible combination of truth values of the simpler statements $p$ and $q$.

| | | **Statement 1** | **Statement 2** |
|---|---|---|---|
| $p$ | $q$ | **Exactly One is Lying** | **At Least One is Truthful** |
| T | T | F | T |
| T | F | T | T |
| F | T | T | T |
| F | F | F | F |

We call this analytic device a **truth table**. It is a valuable tool for analyzing the truth value of any compound statement based on the truth values of its simpler parts, and we will see later in this section that truth tables are useful for far more than solving logic puzzles. Now that we have this representation of the problem, we just need to think about what a solution should look like.

SOLUTION We can think about the meaning of each row in the truth table given above.

- The first row of the table represents the hypothetical situation where both $A$ and $B$ are truth-tellers (because both $p$ and $q$ are true). This cannot be the real situation, however, because Statement 1 is false in this row, so a truth-teller like $A$ could not have said it.

- The second row of the table cannot represent the real situation, since it cannot be that the liar $B$ would have made a true statement.
- The third row is ruled out, because in this row the liar $A$ has made a true statement.
- In the final row, both $A$ and $B$ are liars and both statements are false.

We conclude that the fourth row of the truth table describes the only possible real situation, and so the solution to the puzzle is, *Both inhabitants are lying*. □

In this example, we found rows where the truth value describing each inhabitant's type matches the truthfulness of his or her statement. Another way to describe this is to say that the part of the row to the left of the double vertical lines should match the part to the right of those lines. In this way, the use of a truth table helps organize our thinking about the problem, making the search for a solution more routine.

Before we proceed to our next example, let us say a few words about building truth tables. For the two variables $p$ and $q$ there are a total of four possible combinations. We could list them in some random order, but if we proceed systematically, we can be sure we don't leave any out. Most people do this by thinking analogously to our counting system. When we count from 20 to 39, for example, we write

$$20, 21, 22, 23, \ldots, 29, 30, 31, 32, \ldots, 39$$

The tens' digit stays fixed while we cycle through all the possible ones' digits. Then we go to the next tens digit and repeat the cycle of the ones' digit. Likewise, we wrote the truth table in the order TT, TF, FT, FF—with the right-hand value cycling though its values for T, then again for F.

Those who built "game trees" for exercises in Section 1.1 will be interested to know that this method of listing the rows is actually the same thing, as we see in Figure 1-9. For just two variables this is admittedly not that important, but as we add more variables, it becomes more important to be consistent. For example, for four variables this order would be TTTT, TTTF, TTFT, TTFF, TFTT, TFTF, TFFT, TFFF, FTTT, FTTF, FTFT, FTFF, FFTT, FFTF, FFFT, FFFF.

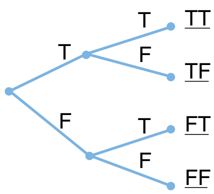


**Figure 1-9** The order of rows in a truth table

**Example 2** *Suppose you meet three inhabitants of Smullyan's Island and have the following conversation. Can you tell which inhabitants (if any) are lying?*

- A *says, "Exactly one of us is telling the truth."*
- B *says, "We are all lying."*
- C *says, "The other two are lying."*

SOLUTION The analysis of this puzzle is the same once we add a symbol $r$ to represent the statement "$C$ is truthful." The truth table below begins with TTT and follows the order we discussed earlier. In the table, we mark the row where the truth values of $p$, $q$, and $r$ match the truth values of what $A$, $B$, and $C$ say, respectively.

| | $p$ | $q$ | $r$ | Statement 1<br>Exactly One is Truthful | Statement 2<br>All are Lying | Statement 3<br>$A$ and $B$ are Lying |
|---|---|---|---|---|---|---|
| | T | T | T | F | F | F |
| | T | T | F | F | F | F |
| | T | F | T | F | F | F |
| ⋆ | T | F | F | T | F | F |
| | F | T | T | F | F | F |
| | F | T | F | T | F | F |
| | F | F | T | T | F | T |
| | F | F | F | F | T | T |

We conclude that the only possible situation is where $p$ is true while $q$ and $r$ are false. Therefore, $A$ is the only inhabitant telling the truth. □

**Practice Problem 1** *You meet two inhabitants of Smullyan's Island. A says, "We are both telling the truth." B says, "A is lying." Who (if anyone) is telling the truth?*

In each of the previous examples, exactly one row of the truth table has been consistent, and we can conclude that this one row gives *the solution*. However, it is possible to have situations where more than one row of the table is consistent, in which case we cannot determine the status of all the speakers. On the other hand, if no row is consistent, then we have arrived at a paradox of sorts. Here are the simplest examples of each of these situations.

## Example 3

1. *You meet an inhabitant A, who says, "I am telling the truth." Is she?*
2. *You meet an inhabitant A, who says, "I am lying." Is he?*

SOLUTION In each part we need only one variable $p$ to represent the statement "$A$ is a truth-teller."

1. The table for this statement looks like this:

| $p$ | Statement 1<br>I am Telling the Truth |
|---|---|
| T | T |
| F | F |

We cannot conclude anything, since both rows are consistent.

2. The table for this statement looks like this:

| $p$ | Statement 1<br>I am Lying |
|---|---|
| T | F |
| F | T |

Since neither row is consistent, we have a paradox.

□

## The Logic of Propositions

Before we look at more examples, let's agree on some notation to make writing these things a little easier. We will refer to this notation as *propositional logic notation*.

> ***Definition*** We call a sentence a **proposition** if it is unambiguously true or false. A *propositional variable* is simply a variable name that stands for a proposition.
>
> A *formal proposition* will mean a proposition written using propositional logic notation according to the following rules:
>
> 1. Any propositional variable alone is a formal proposition.
> 2. Given formal propositions $p$ and $q$, the compound statement $p \wedge q$ is a formal proposition. The proposition $p \wedge q$ stands for, "Both $p$ and $q$ are true," and we read this as "$p$ **and** $q$."
> 3. Given formal propositions $p$ and $q$, the compound statement $p \vee q$ is a formal proposition. The proposition* $p \vee q$ stands for, "Either $p$ or $q$ is true," and we read this as "$p$ **or** $q$." In mathematics, when we say that either $p$ or $q$ is true, we mean that at least one is true. Saying "or" always allows for the possibility that both are true.
> 4. Given a formal proposition $p$, the compound statement $\neg p$ is a formal proposition. The proposition $\neg p$ stands for, "It is not the case that $p$ is true," and we read this as "**not** $p$." We refer to $\neg p$ as the **negation** of $p$.

This system allows for the creation of complicated expressions by repeatedly using these basic rules.

**Example 4** *Show that the expression $(p \vee q) \wedge \neg(p \wedge q)$ is a formal proposition using the definition above.*

SOLUTION As usual in mathematics, we perform our analysis from inside the parentheses out.

- Since $p$ and $q$ are propositional variables, then they are each formal propositions by (1) in the definition.
- Since $p$ and $q$ are formal propositions, then $p \vee q$ and $p \wedge q$ are formal propositions by (2) and (3) in the definition, respectively.
- Since $p \wedge q$ is a formal proposition, then $\neg(p \wedge q)$ is a formal proposition by (4) in the definition.
- Since $p \vee q$ and $\neg(p \wedge q)$ are formal propositions, then so is $(p \vee q) \wedge \neg(p \wedge q)$ by (2) in the definition.

□

* One way to keep $\wedge$ and $\vee$ straight in your head is that the word "AND" begins with the letter A, and the $\wedge$ symbol looks sort of like capital A.

It is customary to consider these operations as having precedence rules (much as $+$, $-$, $\cdot$, and/in arithmetic). The "not" operation ($\neg$) has highest precedence, followed by "and" ($\wedge$) and then "or" ($\vee$). For example, the expression $\neg p \wedge \neg q \vee p$ is the same as $((\neg p) \wedge (\neg q)) \vee p$. As with arithmetic operations, use parentheses if you are worried about ambiguity in your expressions.

**Practice Problem 2** *You meet two inhabitants of Smullyan's Island, A and B. Using p to represent the proposition "A is truthful" and q to represent the proposition "B is truthful," how would you write each of these statements as formal propositions?*

*(a)* *A is lying.*
*(b)* *At least one of us is truthful.*
*(c)* *Either B is lying or A is.*
*(d)* *Exactly one of us is lying.*

We should make a comment here about the meaning of the word "or" in mathematics. This can be a slightly confusing issue since many times in English we use the word "or" to imply a choice must be made. In conversational English, we must rely on context to resolve the issue, and we often have to ask the speaker to clarify the meaning. You may have had conversations like this:

**Waiter:** Do you want soup or salad?
**You:** Yes, I'll have both.
**Waiter:** No, which one do you want?

In this situation, the waiter meant for the word "or" to convey that a choice was to be made, while perhaps you thought that neither soup nor salad was included in the price of the meal and the waiter is simply asking if you would like an appetizer, in which case it is reasonable to get both things. To avoid this ambiguity of the English language, in mathematics when we say, "$p$ *OR* $q$ is true," we will always mean that either one or both statements are true.

**Example 5** *You meet two inhabitants of Smullyan's Island. A says, "Either B is lying or I am," and B says, "A is lying." Who (if anyone) is telling the truth?*

SOLUTION As before we build a truth table to consider all possiblities for the status of the two speakers. Notice that the statement "Either $B$ is lying or $A$ is" is the same as saying, "At least one is lying."

| | | | Statement 1 | Statement 2 |
|---|---|---|---|---|
| | $p$ | $q$ | Either $B$ is Lying or $A$ is | $A$ is Lying |
| | T | T | F | F |
| ⋆ | T | F | T | F |
| | F | T | T | T |
| | F | F | T | T |

Only row 2 of the table is consistent, so we can conclude that $A$ is truthful and $B$ is lying. □

Using the operations $\wedge$, $\vee$, and $\neg$, we can write various combinations of propositions in a concise manner. For example, suppose we use $e$ to represent the statement "Sue is an English major" and $j$ to represent the statement "Sue is a junior." Here are several possible combinations we could write.

**Example 6** *How would you write each of these propositions using combinations of e (meaning "Sue is an English major") and j (meaning "Sue is a junior") with the operations $\wedge$, $\vee$, and $\neg$?*

1. *Sue is a junior English major.*
2. *Sue is either an English major or she is a junior.*
3. *Sue is a junior, but she is not an English major.*
4. *Sue is neither an English major nor a junior.*
5. *Sue is exactly one of the following: an English major or a junior.*

SOLUTION

1. The sentence implies that *both* conditions are true. That is, she is a junior *and* she is an English major. So we write $j \wedge e$.
2. Although in English there might be some ambiguity about this, in mathematics the word *or* always means either one or both. So we write $j \vee e$.
3. In English when we say "but," it means the same as saying, "and," so the statement is $j \wedge (\neg e)$. Because of the precedence conventions, we do not need the parentheses, and we could simply write $j \wedge \neg e$.
4. There are two ways to think of "neither-nor" in English. The first is that it means the "opposite" (or negation) of either-or. This leads to the statement $\neg(j \vee e)$. Another choice is to reason that "neither this nor that" means "not this and also not that." This leads to the statement $\neg j \wedge \neg e$. These two statements are equivalent.
5. This is sometimes called the **exclusive or** of the two conditions—one or the other is true, but not both. Again, there are two equivalent solutions to the problem. One solution comes from writing the "one or the other is true, but not both" in symbols, as $(j \vee e) \wedge \neg(j \wedge e)$. The other solution comes from reasoning, "She could either be a junior but not an English major, or she could be an English major but not a junior." This gives the symbolic statement $(j \wedge \neg e) \vee (e \wedge \neg j)$.

□

**Practice Problem 3** *You meet three inhabitants, A, B, and C, of Smullyan's Island. Using p to represent the statement "A is truthful," q to represent the statement "B is truthful," and r to represent the statement "C is truthful," how would you write each of these phrases?*

(a) *A and B are lying.*

(b) *All three are lying.*

*(c)* *One of us is lying.*

*(d)* *Exactly one of the three is truthful.*

## Truth Tables for Formal Propositions

In addition to saving time and space, another reason for creating propositional logic notation is so we can understand the relationship between the truth value of a complex statement and the truth values of its simpler component propositions. We can illustrate this relationship for each of the basic propositional connectives **and**, **or**, and **not** using truth tables.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

As in the solutions to the logic puzzles, a truth table shows us the truth value of a compound statement for every possible combination of truth values of its simple components.

**Example 7** *Give the truth table for the formal proposition* $p \wedge \neg q$.

SOLUTION

| $p$ | $q$ | $\neg q$ | $p \wedge \neg q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | F | F |
| F | F | T | F |

□

In the solution above, we created a column in the table for $\neg q$ in order to get the column for $p \wedge \neg q$ by simply applying the truth table rule for $\wedge$ to the $p$ column and $\neg q$ column. Specifically, since $A \wedge B$ is true only when both propositions $A$ and $B$ are true, then $p \wedge \neg q$ is true only when both propositions $p$ and $\neg q$ are true, so the $p$ and $\neg q$ columns in the truth table are all we need in order to form the column for $p \wedge \neg q$ .

By making the construction of a truth table a formal process, we can easily build tables for complicated expressions without becoming mired in their grammatical or logical muck.

*Explore more on the Web.*

**Practice Problem 4** *Complete the following truth table*:

| $p$ | $q$ | $p \vee q$ | $\neg (p \vee q)$ |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

As the statements get complicated, it is necessary to have a step-by-step procedure for building a truth table. Once again, we will work from inside the parentheses out.

**Example 8** *Find the truth table for the formal proposition* $(p \vee q) \wedge \neg(p \wedge q)$

SOLUTION This is the same proposition we looked closely at in Example 4. We can use the order in which the formation rules were used to get the order in which we should build the truth table. The final column is formed from the two columns that precede it using the truth table rule for $\wedge$.

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $p \vee q$ | $(p \vee q) \wedge \neg(p \wedge q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | F |
| T | F | F | T | T | T |
| F | T | F | T | T | T |
| F | F | F | T | F | F |

□

## Negation and Logical Equivalence

As we indicated earlier, we call $\neg p$ the *negation* of $p$. We now examine this concept in a little more detail for two situations: (1) propositions involving comparisons, and (2) propositions involving the $\wedge$ and $\vee$ connectives.

**Example 9** *Let p stand for the proposition "Tammy has more than two children." Express the negated proposition as an English-language sentence.*

SOLUTION A common mistake is to use "less than" as the negation of "more than." The correct negation is "less than or equal to." We write, "Tammy has less than or equal to 2 children," or "The number of children Tammy has is less than or equal to 2."

If we use the symbol $c$ to indicate how many children Tammy has, then we may write the original proposition $p$ mathematically as "$c > 2$." The negation $\neg p$ can be written as "$c \leq 2$." □

**Example 10** *In the previous example, if someone tells us that the value of c is actually 3, what does this mean? Which is true, the original proposition p or its negation* $\neg p$*?*

SOLUTION This means that Tammy has three children, and we conclude that $p$ is a true statement. □

**Practice Problem 5** *Let g stand for John's current grade point average. To be admitted to graduate school, a person needs at least a 3.0 grade point average. Use g to express the proposition p, "John's grade point average is high enough to be admitted to graduate school." Then use g to express the negated proposition* $\neg p$.

We now address the issue of forming the negation of propositions involving the $\wedge$ and $\vee$ connectives. In Example 6 we found two different ways to express the proposition "Sue is neither an English major nor a junior," using the symbols $e$ for "Sue is an English major" and $j$ for "Sue is a junior." We stated that the two solutions were "equivalent." To explore exactly what we mean by this, we build a truth table for our first statement "$\neg(j \vee e)$." In the truth table that follows, we calculate the final result, $\neg(j \vee e)$, by first calculating $j \vee e$ and then negating that column.

| $j$ | $e$ | $j \vee e$ | $\neg(j \vee e)$ |
|---|---|---|---|
| T | T | T | F |
| T | F | T | F |
| F | T | T | F |
| F | F | F | T |

We see that the statement "$\neg(j \vee e)$" is true precisely when both $j$ and $e$ are false. Next we build a truth table for the second statement "$\neg j \wedge \neg e$." We calculate the final result by first calculating the intermediate values $\neg j$ and $\neg e$.

| $j$ | $e$ | $\neg j$ | $\neg e$ | $\neg j \wedge \neg e$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

We see that the statement "$\neg j \wedge \neg e$" is true precisely when both $j$ and $e$ are false. The two statements ("$\neg(j \vee e)$" and "$\neg j \wedge \neg e$") are true under exactly the same circumstances—they have the same truth table. This is what we mean when we say the statements are equivalent.

**Definition** Two statements are said to be **logically equivalent** if they have the same truth value for every row of the truth table.

There is another important point to be made—the logical equivalence does not depend on the particular meaning of the symbols $j$ and $e$ in this example. The fact that these two statements are equivalent is just a special instance of a general property. For any statements $p$ and $q$, it is always true that $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$. This is one of the two facts that have come to be known as DeMorgan's laws.

**Proposition 1** ***(DeMorgan's laws)*** *Let $p$ and $q$ be any propositions. Then*

1. *$\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$.*
2. *$\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$.*

*In words, to negate a condition containing* **and** *($\wedge$) or* **or** *($\vee$), negate each part and change the* **and** *to* **or** *or vice versa.*

PROOF The first of these is established in the previous discussion (for propositional variables $j$ and $e$), and the second is left for Exercise 16. ■

**Practice Problem 6** *One way to write, "Sue is not both a junior and an English major" is to write the negation of "Sue* ***is*** *both a junior and an English major." This gives the statement "$\neg(j \wedge e)$." Use DeMorgan's laws to write an equivalent statement, and explain what this statement says in words.*

There are several other situations where we can easily see that two statements are logically equivalent. As a simple example, for any statements $p$ and $q$, it is easy to see that $p \wedge q$ is equivalent to $q \wedge p$. Similarly, we know from our use of ordinary English that any statement $p$ is logically equivalent to its "double negative" $\neg(\neg p)$. To see this formally, we just calculate the truth table:

| $p$ | $\neg p$ | $\neg(\neg p)$ |
|---|---|---|
| T | F | T |
| F | T | F |

The fact that the first and third columns are identical establishes the equivalence.

**Example 11** *Write the negation of the phrase "Sue is a junior, but she is not an English major" two different ways. What does it mean for this phrase to be false?*

SOLUTION In an earlier example, we wrote, "Sue is a junior, but she is not an English major," as $j \wedge \neg e$. The simplest way to write the negation is to enclose this in parentheses, preceded by the **not** operation: "$\neg(j \wedge \neg e)$." By DeMorgan's laws and the double negative property, this is equivalent to "$\neg j \vee e$" (change **and** to **or** and negate each part). In words, "Either Sue is *not* a junior, or she *is* an English major." □

**Example 12** *Using $g$ to indicate John's score on the recent math test, we can express the statement "John got a B on the test" as "$(g \geq 80) \wedge (g < 90)$." Write the negation as an expression in symbolic logic notation.*

SOLUTION This example uses both ideas we have considered in the section. To negate the statement, we must change the $\wedge$ to $\vee$ and negate each part. To negate each part, we must take care to properly negate the comparison operations. The solution is "$(g < 80) \vee (g \geq 90)$." In words, "John either got below a B or above a B." □

**Practice Problem 7**

(a) *Using $y$ to indicate yesterday's high temperature and $t$ for today's high temperature, negate the proposition $p$ written as "$(y < 0) \vee (t < 0)$." Express both $p$ and $\neg p$ as English-language sentences.*

*(b)* *You meet three inhabitants of Smullyan's Island, $A$, $B$ and $C$. Using $p$ to represent the statement "A is truthful," $q$ to represent the statement "B is truthful," and $r$ to represent the statement "C is truthful," how would you write each of these phrases in symbolic logic notation? Use the double negative property and DeMorgan's laws to make your answer as simple as possible.*

*i.* *Not all of us are lying. This is the negation of "All of us are lying."*

*ii.* *Not one of us is lying. This is the negation of "At least one of us is lying."*

There are many other examples of the logical equivalence of statements that can be established using truth tables. To make these easy to state, we will use the notation $p \equiv q$ to represent the statement "Propositions $p$ and $q$ are logically equivalent."

**Example 13** *Use a truth table to establish the following logical equivalences:*

*1.* $\neg(\neg p \vee q) \equiv p \wedge \neg q$

*2.* $p \wedge (p \vee q) \equiv p$

SOLUTION

1. In the table below, because the fifth and eighth columns are identical, we see that $\neg(\neg p \vee q)$ is logically equivalent to $p \wedge \neg q$.

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $\neg(\neg p \vee q)$ | $p$ | $\neg q$ | $p \wedge \neg q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | T | F | T | F | F |
| T | F | F | F | T | T | T | T |
| F | T | T | T | F | F | F | F |
| F | F | T | T | F | F | T | F |

2. The equivalence follows from the fact that, in the table below, the fourth column is the same as the first.

| $p$ | $q$ | $p \vee q$ | $p \wedge (p \vee q)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | T | F |
| F | F | F | F |

□

Explore more on the Web.

**Practice Problem 8** *Use a truth table to show that $p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$.*

**Example 14** *Examine the statement $(p \vee \neg q) \vee (\neg p \vee q)$ with a truth table.*

SOLUTION Here is the truth table:

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \vee \neg q$ | $\neg p \vee q$ | $(p \vee \neg q) \vee (\neg p \vee q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | T | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | T | T | T |

□

In this example, the column containing the final result (for the complete statement) consists of all T's. This means that no matter what truth values we assign to the variables $p$ and $q$, the overall statement is true. It is also possible to have just the opposite situation, where all combinations of truth values yield a false result. This is significant enough to warrant special names for these kinds of propositions.

### Definition

1. A **tautology** is a proposition whose value is True for all possible combinations of the truth values of the propositional variables.
2. A **contradiction** is a proposition whose value is False for all possible combinations of the truth values of the propositional variables.

DeMorgan's laws and the double negation property were our first examples of logical equivalence, but there are many more. The following theorem summarizes several that will be important later in this course. The proof of each of these statements consists of simply creating a truth table for the two given propositions and verifying they are the same.

**Theorem 2** *Let p, q, and r stand for any propositions. Let t indicate a tautology, and c indicate a contradiction. Then all the logical equivalences shown in Table 1–8 hold.*

PROOF DeMorgan's laws, the double negative property, one of the absorption properties, and one of the distributive properties are established throughout this section. See the exercises at the end of this section for the rest. ■

| | | | |
|---|---|---|---|
| (a) | Commutative | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| (b) | Associative | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| (c) | Distributive | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| (d) | Identity | $p \wedge t \equiv p$ | $p \vee c \equiv p$ |
| (e) | Negation | $p \vee \neg p \equiv t$ | $p \wedge \neg p \equiv c$ |
| (f) | Double negative | $\neg(\neg p) \equiv p$ | |
| (g) | Idempotent | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| (h) | DeMorgan's laws | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ |
| (i) | Universal bound | $p \vee t \equiv t$ | $p \wedge c \equiv c$ |
| (j) | Absorption | $p \wedge (p \vee q) \equiv p$ | $p \vee (p \wedge q) \equiv p$ |
| (k) | Negations of $t$ and $c$ | $\neg t \equiv c$ | $\neg c \equiv t$ |

**Table 1-8** Logical Equivalences

Observe that each property (except (f)) actually consists of two properties. For example, the "commutative" property has two versions, one for the $\wedge$ operation and one for the $\vee$ operation.

Since each of these statements is true under all truth assignments, we can think of these logical equivalences as providing us with a substitution rule of sorts. For example, since $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$, we can replace any expression of the form $(p \wedge q) \wedge r$ with the expression $p \wedge (q \wedge r)$ without changing the truth value. This is an instance of the **substitution rule** for propositional logic. We can use this rule along with the facts in Theorem 2 to demonstrate the logical equivalence of two propositions through what appears to be sheer algebraic manipulation.

**Example 15** *Verify the logical equivalence $p \vee (\neg p \wedge q) \equiv p \vee q$ using the substitution rule and quoting the appropriate parts of Theorem 2.*

SOLUTION Each proposition below is equivalent to the previous proposition by the cited part of Theorem 2 and the substitution rule when necessary:

$$\begin{aligned}
p \vee (\neg p \wedge q) &\equiv (p \vee \neg p) \wedge (p \vee q) && \text{part (c), distributive}\\
&\equiv t \wedge (p \vee q) && \text{part (e), negation}\\
&\equiv (p \vee q) \wedge t && \text{part (a), commutative}\\
&\equiv p \vee q && \text{part (d), identity}
\end{aligned}$$

In each step, we have replaced part of the expression with an equivalent expression. For example, in the first step we use the second of the two *distributive* properties to replace the entire expression by the equivalent expression. In the second step, we replace $(p \vee \neg p)$ with $t$ by quoting the first of the two *negation* properties. □

**Practice Problem 9** *Verify $p \wedge (\neg p \vee q) \equiv p \wedge q$ using the substitution rule and quoting the appropriate parts of Theorem 2.*

## Solutions to Practice Problems

1 Here is the truth table:

| $p$ | $q$ | Statement 1: We are Both Telling the Truth | Statement 2: A is Lying |
|---|---|---|---|
| T | T | T | F |
| T | F | F | F |
| F | T | F | T |
| F | F | F | T |

The third row is the only one that is consistent, so we conclude that $A$ is lying and $B$ is telling the truth.

2 (a) "$A$ is lying" is represented by $\neg p$.

(b) "At least one of us is truthful" is represented by $p \vee q$.

(c) "Either $B$ is lying or $A$ is" is represented by $\neg q \vee \neg p$.

(d) "Exactly one of $A$ and $B$ is lying" is represented by $(p \wedge \neg q) \vee (q \wedge \neg p)$.

3 (a) "$A$ and $B$ are lying" is represented by $\neg p \wedge \neg q$.

(b) "All are lying" is represented by $\neg p \wedge \neg q \wedge \neg r$.

(c) "One of us is lying" (take this to mean "at least one of us is lying") is represented by $\neg p \vee \neg q \vee \neg r$.

(d) "Exactly one is truthful" is represented by $(p \wedge \neg q \wedge \neg r) \vee (q \wedge \neg p \wedge \neg r) \vee (r \wedge \neg p \wedge \neg q)$.

4 The complete truth table is shown below.

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ |
|---|---|---|---|
| T | T | T | F |
| T | F | T | F |
| F | T | T | F |
| F | F | F | T |

5 We write $p$ as "$g \geq 3.0$" and $\neg p$ as "$g < 3.0$."

6 By the second of DeMorgan's laws, "$\neg(j \wedge e)$" is equivalent to "$\neg j \vee \neg e$." This says that (since she isn't both) it must be true either that Sue is not a junior or that she is not an English major.

7 (a) We write $\neg p$ as "$(y \geq 0) \wedge (t \geq 0)$." In English, $p$ says that the temperature was below 0 at least one of the two days, and $\neg p$ says it was at or above 0 both days.

(i) "All are lying" is represented by $\neg p \wedge \neg q \wedge \neg r$. The negation is $\neg(\neg p \wedge \neg q \wedge \neg r)$. By DeMorgan's laws, to negate we change **and** to **or** and negate each part, giving $p \vee q \vee r$. Observe that the negation of "All are lying" is "At least one is telling the truth."

(ii) "At least one of us is lying" is represented by $\neg p \vee \neg q \vee \neg r$. The negation is $\neg(\neg p \vee \neg q \vee \neg r)$, which simplifies to $p \wedge q \wedge r$. The negation of "At least one of us is lying" is "All are telling the truth."

8 One way to lay out the truth table is given below. Because the fifth and eighth columns are identical, the two expressions at the top of those columns are logically equivalent.

9 This problem is the "dual" of the previous example—each $\wedge$ has been changed to $\vee$ and vice versa. It is not surprising that the solution mimics the solution to the example. The duality principle will be addressed further in Section 3.4.

$$\begin{aligned} p \wedge (\neg p \vee q) &\equiv (p \wedge \neg p) \vee (p \wedge q) && \text{part (c), distributive} \\ &\equiv c \vee (p \wedge q) && \text{part (e), negation} \\ &\equiv (p \wedge q) \vee c && \text{part (a), commutative} \\ &\equiv p \wedge q && \text{part (d), identity} \end{aligned}$$

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

Truth Table for Practice Problem 8.

## Exercises for Section 1.3

**1.** Solve each of these logic puzzles by using truth tables.

**(a)** You come across two inhabitants of Smullyan's Island. $A$ says, "We are both telling the truth," and $B$ says, "$A$ is lying." Who if anyone is telling the truth?

**(b)** You come across three inhabitants of Smullyan's Island. $A$ says, "$B$ or $C$ is lying," $B$ says, "$C$ is lying," and $C$ says, "$A$ and I are both telling the truth." Who if anyone is telling the truth?

**(c)** You come across three inhabitants of Smullyan's Island. $A$ says, "$B$ and $C$ are both lying," $B$ says, "Only one of the other two is lying," and $C$ says, "At least one of us is lying." Who if anyone is telling the truth?

**2.** Give an example of what two people might say to create a paradox, where each person's individual statement does not on its own create a paradox.

**3.** Suppose you meet two inhabitants $A$ and $B$ of Smullyan's Island. Let $p$ represent the statement "$A$ is truthful" and $q$ represent the statement "$B$ is truthful." Write each of the following in symbolic logic notation:

**(a)** $A$ is lying or $B$ is telling the truth.

**(b)** Neither $A$ nor $B$ is lying.

**(c)** $A$ is truthful but $B$ is not.

**4.** Upon meeting a third island inhabitant $C$, you can continue the previous exercise, adding $r$ to represent the statement "$C$ is truthful." Use these names along with the basic logic operations to write each of the following English sentences in symbolic logic notation:

**(a)** $A$ is lying and $B$ or $C$ is truthful.

**(b)** $A$ and $B$ are lying, or $A$ and $C$ are truthful.

**(c)** At least two people are telling the truth.

**(d)** Exactly two people are telling the truth.

**5.** Use the letter $f$ to represent the statement "this person is female," $a$ for "this person is over age 30," and $m$ for "this person is a math major." Use these names along with the basic operations **and** ($\wedge$), **or** ($\vee$), and **not** ($\neg$) to write each of the following English sentences with symbolic logic:

(a) This person is female but not a math major.

**(b)** This person is a female math major over age 30.

(c) This person is neither female nor a math major.

**(d)** Either this person is not female or this person is over age 30.

**6.** Use the phrases in the previous exercise to write an English statement equivalent to each of the following propositions:

(a) $\neg f \wedge m$

**(b)** $\neg f \wedge \neg a$

(c) $f \wedge (a \vee m)$

**(d)** $(f \wedge a) \vee (\neg f \wedge m)$

**7.** Use the letter $t$ to represent the statement "Bill is tall," $d$ for "Bill is dark," and $h$ for "Bill is handsome." As in Exercise 5, use these names along with the basic logic operations to write each of the following English sentences in symbolic logic notation:

(a) Bill is tall, dark, and handsome

**(b)** Bill is tall and dark, but not handsome.

(c) Either Bill is tall or he is handsome, but not both.

**(d)** Bill is neither tall nor handsome.

**8.** Use the phrases in the previous exercise to write an English statement equivalent to each of the following propositions:

(a) $(t \vee d \vee h) \wedge \neg(t \wedge d \wedge h)$

**(b)** $\neg t \wedge \neg d$

(c) $d \wedge \neg(t \wedge \neg d)$

**(d)** $(t \wedge d) \vee (\neg t \wedge d)$

**9.** Use the letter $s$ to represent the statement "Chris likes to play soccer," $r$ for "Chris likes to read," and $p$ for "Chris likes to eat pizza." As before, use these names along with the basic logic operations to write each of the following English sentences in symbolic logic notation:

**(a)** Chris likes pizza but he does not like soccer.

**(b)** Chris likes to read and eat pizza, or he likes to play soccer.

**(c)** Chris does not like to eat pizza, but he likes to play soccer or read.

**(d)** Chris likes to do two of these things but not all three.

**10.** Suppose $x$ and $y$ indicate particular real numbers. Write conditions that express the following, by using comparisons (such as $x > 0$) and the basic operations of logic.

(a) Both $x$ and $y$ are positive.

**(b)** At least one of $x$ and $y$ is positive.

(c) Exactly one of $x$ and $y$ is positive.

**(d)** Neither $x$ nor $y$ is positive.

**11.** Complete the following truth tables for the given compound expressions.

(a)

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \wedge (\neg p \vee q)$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

**(b)**

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

**(c)**

| $p$ | $q$ | $p \vee q$ | $\neg p \vee q$ | $(p \vee q) \wedge (\neg p \vee q)$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

(d)

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ |
|---|---|---|---|---|
| T | T | T | | |
| T | T | F | | |
| T | F | T | | |
| T | F | F | | |
| F | T | T | | |
| F | T | F | | |
| F | F | T | | |
| F | F | F | | |

**(e)**

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \vee r$ |
|---|---|---|---|---|
| T | T | T | | |
| T | T | F | | |
| T | F | T | | |
| T | F | F | | |
| F | T | T | | |
| F | T | F | | |
| F | F | T | | |
| F | F | F | | |

**12.** Rewrite each of the following statements in propositional logic notation, making the meaning of your propositional variables clear, and then create a truth table for the sentence. The first one is done for you as an example.

**(ex)** *Either the food is terrific, or everyone is tired and hungry at mealtimes.* Let $f$ represent "the food is terrific," $t$ represent "everyone is tired," and $h$ represent "everyone is hungry." Then the given statement can be written as $f \vee (t \wedge h)$, and it has the following truth table:

| $f$ | $t$ | $h$ | $t \wedge h$ | $f \vee (t \wedge h)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | F | T |
| T | F | T | F | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | T | F | F | F |
| F | F | T | F | F |
| F | F | F | F | F |

**(a)** Either everyone is not hungry at mealtime, or everyone is tired and the snackbar makes a profit.

**(b)** The staff is friendly, or else they are not friendly but very well paid.

**(c)** The staff is not very well paid, and they are friendly.

**(d)** You play sports or you play mahjong or nobody knows your name.

**(e)** You do not play mahjong or you do not play sports, and nobody knows your name.

**13.** For each of the statements in Exercise 12, obtain a simple symbolic logic expression for the **negation** of that statement, and then rewrite this as a proper English sentence.

**14.** Use the number variable $b$ to represent Trina's board exam math score and $m$ to represent Trina's math placement test score. To take calculus the first semester, a student must have a board exam math score of at least 600 or a math placement test score of at least 25. Use the variables $b$ and $m$ to express the statement "Trina may take calculus in her first semester." Use the variables $b$ and $m$ to express the negation of this statement as well.

**15.** Use the number variable $a$ to represent Fred's age and $v$ to represent the number of years Fred has worked at his job. To be eligible for the company's retirement plan, an employee must be at least 62 years of age and have worked for at least 15 years. Use the variables $a$ and $v$ to express the statement "Fred is eligible for the company's retirement plan." Use the variables $a$ and $v$ to express the negation of this statement as well.

**16.** Use truth tables to verify that $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$, establishing the second of DeMorgan's laws in Theorem 2.

**17.** Use truth tables to verify that $p \vee (p \wedge q)$ is logically equivalent to $p$, establishing the second absorption property in Theorem 2.

**18.** Use truth tables to verify that $p \wedge (q \vee r)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$, establishing the first distributive property in Theorem 2.

**19.** Use truth tables to verify the first versions of the commutative and associative properties from Theorem 2.

**20.** Use truth tables to verify the first versions of the identity, idempotent, and universal bound properties from Theorem 2.

**21.** Use truth tables to check if each of the given pairs of symbolic logic statements are equivalent.

**(a)** $p \wedge (\neg p \vee q)$ and $\neg p \wedge (p \vee q)$

**(b)** $(p \vee q) \wedge (\neg p \vee q)$ and $q$

**(c)** $(\neg q \wedge p) \vee (\neg p \wedge q)$ and $\neg p \vee \neg q$

**(d)** $p \wedge (q \vee r)$ and $(p \wedge q) \vee r$

**(e)** $(p \vee q) \wedge (q \vee r)$ and $(p \wedge r) \vee q$

**22.** For each of the following statements, rewrite them in propositional logic notation, making the meaning of your propositional variables clear. Use truth tables to find any pairs of logically equivalent statements.

**(a)** Jillian likes playing in the sand or volleyball, but she does not like sailing.

**(b)** Jillian likes playing in the sand, and she likes sailing or volleyball.

**(c)** Jillian likes playing in the sand and volleyball, or she likes sailing.

**(d)** Jillian likes playing in the sand and sailing, or she likes volleyball and sailing.

**23.** Use the double negative property and DeMorgan's laws to rewrite each of the following as an equivalent statement that never has the **not** symbol ($\neg$) outside of a parenthesized expression. The first one is done for you as an example.

**(ex)** $\neg(p \wedge \neg q)$ is equivalent to $\neg p \vee \neg(\neg q)$ by DeMorgan's law, and this is equivalent to $\underline{\neg p \vee q}$ by the double negative property.

**(a)** $\neg(\neg p \vee \neg q)$

**(b)** $\neg(\neg(\neg p))$

**(c)** $\neg(\neg(\neg p \wedge q))$

**(d)** $\neg(p \wedge (q \vee \neg p))$

**24.** By quoting the parts of Theorem 2, verify the following logical equivalences. In each case, start from the left side and use parts of the theorem to change the problem, ending up with the right side. (See Example 15 and the solution to Practice Problem 9.)

(a) $(p \wedge \neg q) \vee p \equiv p$
(b) $(p \vee t) \wedge (p \vee c) \equiv p$
(c) $q \wedge (p \vee r) \equiv (p \wedge q) \vee (q \wedge r)$

**25.** By using parts of Theorem 2 as algebraic rules, simplify the expression $(p \vee q) \wedge \neg(\neg p \wedge r)$ as much as you can.

## 1.4 Predicates

In Section 1.3, we worked with propositions such as “Sue is an English major” and “Person $A$ is lying.” In each case, the subject is a particular person or other entity. However, in many circumstances the subject is not a particular fixed entity. For example, consider the question “How many students in my discrete math class are English majors?” To answer the question, you would check the truth value of the sentence “__________ is an English major,” filling in the blank with the name of each person in the class in turn.

In mathematics, we call a statement of this form a *predicate*. One way to relate to the concept is to think of the meaning of the word *predicate* in an English class. In the English sentence “Mr. Morton wrote Pearl a poem,” the predicate is “wrote Pearl a poem,” and it tells what Mr. Morton did.* Since predicates do not have subjects, they are not complete sentences. However, with the addition of any subject, you will get some sentence. For example, the predicate “__________ is an English major” can become the propositions “Tom Myers is an English major” or “Mary Johnson is an English major” by adding the appropriate subjects.

In mathematics, it is customary to use a variable rather than an empty blank to represent the missing subject. If we use the variable $s$ (short for *student*), the sentence above would become “$s$ is an English major.” Each time we replace $s$ by the name of a student in the class, we get a statement that is either true or false. Sentences of this form, containing one or more variables, are discussed in this section.

### Simple Predicates and Their Negations

The preceding discussion leads us to our formal definition.

> ***Definition*** A *predicate* $P(x)$ is a statement that incorporates a variable $x$, such that whenever $x$ is replaced by a value, the resulting proposition is unambiguously true or false. If the predicate uses several variables $x_1, x_2, \ldots$, then we will extend the notation to $P(x_1, x_2, \ldots)$ accordingly.

**Example 1** *If $P(n)$ is the predicate “$n$ is even,” write the proposition that results when the variable $n$ is replaced by each of the values* 2, 17, *and* 240. *Which of the resulting statements are true?*

* Those of you who are of a certain age will recognize this character from Saturday morning’s *Schoolhouse Rock*.

SOLUTION We will write $P(2)$ to denote the proposition resulting from replacing $n$ with 2 in the predicate $P(n)$. In this case, $P(2)$ is the true proposition "2 is even." Similarly, $P(17)$ denotes replacing $n$ by 17, giving us the false statement "17 is even." Finally, $P(240)$ is the true statement "240 is even." □

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**Practice Problem 1** *In each of the predicates below, replace x with each of the values 2, 23, –5, and 15. Decide if each resulting proposition is true or false.*

*(a)* $P(x)$ *is the predicate* "$x > 15$."

*(b)* $Q(x)$ *is the predicate* "$x \leq 15$."

*(c)* $R(x)$ *is the predicate* "$(x > 5) \wedge (x < 20)$."

Most of what you have learned about propositions has an obvious analogue for predicates. For example, we can combine predicates using the operation symbols $\wedge$ (**and**), $\vee$ (**or**), $\neg$ (**not**) to create new predicates. We can talk about the **negation** of a predicate, and we can make sense of the notion of **equivalent** predicates. For example, here are two equivalent ways to express the negation of the predicate "$x < 0$":

- $\neg(x < 0)$: in English, "$x$ is not less than 0."
- $x \geq 0$: in English, "$x$ is greater than or equal to 0."

These statements are equivalent because, no matter what number is substituted for the variable $x$, the resulting propositions have the same truth value. In addition, the "double negative" property and DeMorgan's laws apply to predicates just as they do to propositions.

**Example 2** *Table 1-9 shows some predicates and their negations. The final example shows a predicate with more than one variable.*

In our next example, we use a simple notation for describing a collection of numbers, written as what we call a *set*. We will study sets in detail in Chapter 3. For now, all you need to know is that our sets are collections of objects, and that the objects in that collection are called *elements* of the set or *members* of the set.

**Example 3** *Consider the set* $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. *For each predicate below, make a list of the members of the set D that make the statement true. Also,*

| Predicate | Negation of Predicate |
|---|---|
| $x > 5$ | $x \leq 5$ |
| $(x > 0) \wedge (x < 10)$ | $(x \leq 0) \vee (x \geq 10)$ |
| $\neg(x = 8)$ | $x = 8$ |
| $(x \geq 0) \vee (y \geq 0)$ | $(x < 0) \wedge (y < 0)$ |

**Table 1-9** Table for Example 2

*form the negation of the predicate and list the members of D that make the negation true.*

*1.* $x \geq 8$

*2.* $(x > 5) \wedge (x \text{ is even})$

*3.* $x^2 = x$

*4.* $x > 0$

*5.* $x > x^2$

SOLUTION We present the solution in Table 1-10. The negation is true for precisely those numbers in the set $D$ for which the original statement is not true. □

## Truth and Quantifiers

Unlike propositions, we almost never talk about predicates being simply true or false. What is more important is the truth or fallacy of the proposition we get *after* substituting a value for the variable in a predicate. Hence, our discussion of the "truth" of predicates must include the set that these values come from. We will call this set the *domain* of the predicate, and we will never discuss the truth of predicates without making the domain clear.

Given a predicate and domain, the natural question to ask is, "Does the predicate become a true proposition when its variable is replaced by members of the domain?" The three most common answers to this question are **always, sometimes**, or **never**.

1. In Example 3, the answer is "always" for the fourth predicate but not for the others. When the answer is "always," we might use phrases such as "for all members of the set, the statement is true," or "for every element of the set, the statement is true" in describing the situation.
2. In Example 3, the answer is "sometimes" for all but the fifth predicate. When we answer "sometimes" to this question, we are likely to use phrases similar to "there is an element of the set that makes the statement true" or "there exists a member of the set such that the predicate is satisfied."
3. In Example 3, the answer is "never" for the fifth predicate. This is not really necessary to treat separately since "never" is simply the negation of "sometimes." We will address this case in the exercises at the end of this section.

### Example 4 *Here are two statements that use this terminology:*

*1.* *"For every k that is a member of the set* $\{1, 2, 3, 4, 5\}$*, it is true that* $k < 20$.*"*

| | True for ... | Negation | True for ... |
|---|---|---|---|
| 1. | 8, 9, 10 | $x < 8$ | 1, 2, 3, 4, 5, 6, 7 |
| 2. | 6, 8, 10 | $(x \leq 5) \vee (x \text{ is odd})$ | 1, 2, 3, 4, 5, 7, 9 |
| 3. | 1 | $x^2 \neq x$ | 2, 3, 4, 5, 6, 7, 8, 9, 10 |
| 4. | All the members | $x \leq 0$ | None of the members |
| 5. | None of the members | $x \leq x^2$ | All the members |

**Table 1-10** Table for Example 3

| Predicate | True for These Members of $D\ldots$ | True for At Least One? | True for All? |
|---|---|---|---|
| $x < 0$ | | | |
| $x > -3$ | | | |
| $x^2 < x$ | | | |
| $x^2 \geq x$ | | | |

**Table 1-11** Table for Practice Problem 2

2. *"There exists a member $m$ of the set $\{-1, 0, 1\}$ with the property that $m^2 = m$." Notice that "there exists" always means "there exists (at least one)."*

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**Practice Problem 2** *In Table 1-11, list the members of the set $D = \{-1, 0, 1, 2\}$ that make each predicate true. In addition, answer the two questions: (1) Is the predicate true for at least one member of the set? and (2) Is the predicate true for all the members of the set? When the answer to either question is "yes," write the answer as a complete sentence using phrases similar to "there exists" or "for all."*

The phrases "for all," "there exists," "in the set," and so on are used so often that special notation has been developed to make them shorter to write.

### *Definition*

1. The symbol $\in$ indicates membership in a set. For example, "$k \in D$" means that $k$ is a member of the set $D$.
2. The symbol $\forall$ means "for all" or "for every." Notice that this is an upside-down letter A, and that the letter A is the first letter in the word "all."
3. The symbol $\exists$ means "there is (at least one)" or "there exists (at least one)." Notice that this is a backwards letter E, and that the letter E is the first letter in the word "exists."
4. The symbols $\forall$ and $\exists$ (and the corresponding phrases such as "for all" and "there exists") are called **quantifiers**. When we use quantifiers with a predicate, we refer to the resulting statement as a **quantified predicate**.
5. A **counterexample** is an example illustrating that a "for all" statement is false.

A quantified predicate is unambiguously either true or false. If the domain is small, we can usually determine whether the sentence is true or false by simply substituting each domain element for the variable(s) in the predicate. For larger domains, including infinite sets, it can be more difficult to determine.

**Example 5** *Let $D = \{3, 4, 5, 10, 20, 25\}$. Express each sentence using the special symbols $\in$, $\forall$, and $\exists$, and decide whether or not it is true.*

1. *For every n that is a member of D, $n < 20$.*
2. *For all n in the set D, $n < 5$ or n is a multiple of 5.*
3. *There is (at least one) k in the set D with the property that $k^2$ is also in the set D.*
4. *There exists m a member of the set D such that $m \geq 3$.*

SOLUTION

1. $\forall n \in D, n < 20$. This statement is false, since 20 and 25 are counterexamples.
2. $\forall n \in D, n < 5$ or $n$ is a multiple of 5. This statement is true.
3. $\exists k \in D$ such that $k^2 \in D$. This statement is true, since $5^2 \in D$. Sometimes we replace the words "such that" with a comma, as in $\exists k \in D,\ k^2 \in D$. When this is read aloud, words similar to "such that" or "with the property that" are usually added to make it read more like ordinary English.
4. $\exists m \in D$ such that $m \geq 3$, or more simply, just $\exists m \in D, m \geq 3$. This statement is true.

□

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**Practice Problem 3** *Translate each symbolic quantified predicate into an English-language sentence about the set $D = \{-2, -1, 0, 1, 2\}$, and then decide whether or not it is true.*

(a) $\forall d \in D, d > -2$
(b) $\exists d \in D, d > -2$
(c) $\forall n \in D, (n > -3) \wedge (n < 3)$
(d) $\forall x \in D, x^2 \leq 4$
(e) $\exists m \in D, m > 10$

When we write a quantified predicate, there is always a set $D$, called the domain, that we have in mind. If you are being informal, you might not specify the set, or you might specify the set without using the set notation. A useful practice, when faced with a quantified statement written in this less formal style, is to identify the set and rewrite the statement in terms of that set.

The clear specification of the domain is very important in determining whether a quanitified statement is true. The next example illustrates this issue as well as the practical skill of making informal quantified statements more formal.

**Example 6** *For each quantified statement, determine the domain D and rewrite the predicate formally in terms of that set D. If the domain is ambiguous, give examples of how different domains can change the truth value of the statement.*

1. *For all x, $x^2 \geq x$.*
2. *$\forall$ even integer m, m ends in the digit 0, 2, 4, 6, or 8.*
3. *There is an integer n whose square root is also an integer.*
4. *Every integer larger than 0 has a square that is larger than 0.*

SOLUTION

1. There are several possibilities. One possibility is that the person writing the predicate was intending to describe a property of the set of all real numbers,

frequently written as $\mathbb{R}$. If we use this notation for the domain, we can write, "$\forall x \in \mathbb{R}, x^2 \geq x$." This statement is false since $x = 0.5$ is a counterexample.

If instead we take the domain to be the set of integers (commonly written $\mathbb{Z}$ after the German word for numbers, *Zahlen*), then we have the *true* statement, "$\forall x \in \mathbb{Z}, x^2 \geq x$."

2. Here $D$ is the set of even integers, and we write

$$\forall m \in D, m \text{ ends in the digit } 0, 2, 4, 6, \text{ or } 8$$

3. Using $\mathbb{Z}$ once again to indicate the set of all integers, we can write, "$\exists k \in \mathbb{Z}, \sqrt{k} \in \mathbb{Z}$."
4. Here the domain $D$ is the set of positive integers, and we write, "$\forall n \in D, n^2 > 0$."

## Negating Quantified Statements

In Practice Problem 3, two of the statements are false. As before, when a statement is false, the *negation* of that statement is true. Looking more closely at these two statements in this Practice Problem will help us determine the formal meaning of "negation" for a quantified statement.

**Example 7** *For the set $D = \{-2, -1, 0, 1, 2\}$, explain why each of these quantified predicates is false. Use your explanation to write the negation of the statement, first in English and then with more formal notation.*

*1.* $\forall d \in D, d > -2$

*2.* $\exists m \in D, m > 10$

SOLUTION

1. If we write $P(d)$ for the predicate "$d > -2$," then $P(-1), P(0), P(1)$, and $P(2)$ are all true. However, $P(-2)$ is false, so we cannot say that *all* the elements of $D$ make $P(d)$ true. That is, *there exists (at least one) element $d$ of $D$ for which $P(d)$ is false*. This negation can be formally written as "$\exists d \in D, \neg P(d)$," or "$\exists d \in D, d \leq -2$."
2. If we write $Q(m)$ for the predicate "$m > 10$," then $Q(-2)$, $Q(-1)$, $Q(0)$, $Q(1)$, and $Q(2)$ are all false, so we cannot find an element of $D$ that makes $Q(m)$ true. That is, *for all elements $m$ of $D$, $Q(m)$ is false*. This negation can be formally written as "$\forall m \in D, \neg Q(m)$," or "$\forall m \in D, m \leq 10$."

□

In words, we can say that the opposite of "true for all" is "false for at least one." Likewise, the opposite of "true for at least one" is "false for all." Using the notation of this section, we can summarize the relationship between a quantified statement and its negation in the following proposition.

***Proposition 1*** *For any predicates $P$ and $Q$ over a domain $D$,*

- *The negation of* $\underline{\forall x \in D, P(x)}$ *is* $\underline{\exists x \in D, \neg P(x)}$.
- *The negation of* $\underline{\exists x \in D, Q(x)}$ *is* $\underline{\forall x \in D, \neg Q(x)}$.

Of course, when we negate the predicates themselves, we use the same ideas we learned about earlier. In particular, there are two important points to remember:

1. Techniques such as the double negative property and DeMorgan's laws apply to predicates.
2. Particular care must be taken when negating comparisons. For example, the negation of a comparison involving $\leq$ will involve $>$.

*Explore more on the Web.*

**Practice Problem 4** *Let $D = \{-1, 0, 1, 2\}$. For each statement, write the negation, and then decide which is true, the original statement or the negation.*

(a) $\forall x \in D, (x \leq 0) \vee (x \geq 2)$

(b) $\exists x \in D, (x < 0) \wedge (x^2 > 0)$

(c) *For all $x \in D$, $x^2 < x$.*

(d) *There exists $x \in D$ such that $x^2 < x$.*

## Multiple Quantifiers and Their Negations

We have said that a predicate can have more than one variable. For example, consider the predicate $P(x, y)$ given as $x + 2y = 3$, where $x$ and $y$ stand for integers. That is, $x$ and $y$ are to be taken from the set $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. If we wish to substitute particular values (say, 10 and 4) for the variables, we use the notation $P(10, 4)$. This notation indicates that we replace $x$ by 10 and $y$ by 4 in the predicate, obtaining the (false) proposition $10 + 2 \cdot 4 = 3$. As usual, when we use two variables, we do not imply that the variables *must* indicate different quantities, only that they *can* indicate different quantities. For example, it is perfectly correct to write $P(-5, -5)$ to indicate the statement $(-5) + 2 \cdot (-5) = 3$.

**Example 8** *Let $P(x, y)$ be the predicate $x \cdot y = 36$.*

1. *Identify which of the following statements are true: $P(3, 4)$, $P(-9, -4)$, $P(12, -1)$.*
2. *Assuming the domain is the set of integers, find other values for $x$ and $y$ that make the statement $P(x, y)$ true.*

SOLUTION

1. The statements are $3 \cdot 4 = 36$, $(-9) \cdot (-4) = 36$, and $12 \cdot (-1) = 36$. Only the second statement is true, the first and third are false.
2. Here are some additional true statements: $P(36, 1)$, $P(6, 6)$, and $P(-2, -18)$. There are several more that are true. Notice that if the variables $x$ and $y$ are taken from the domain of all real numbers rather than all integers, then there are an infinite number of choices for $x$ and $y$.

□

When a predicate contains more than one variable, the question of quantifiers becomes more complicated since each variable can be quantified separately. Fortunately, some situations are relatively simple. If all the quantifiers are the same, there is no difficulty in making the proper interpretation.

## Example 9

1. *Consider the quantified statement "There exist integers x and y such that* $x \cdot y = 36$*." Write this using symbolic logic notation, and decide if it is true or not.*
2. *Do the same for the quantified statement "For all integers x and y, it is true that* $x \cdot y = 36$*."*

### SOLUTION

1. We can write this in two equivalent ways:
   - $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 36$
   - $\exists x, y \in \mathbb{Z}, x \cdot y = 36$

   The latter is a shortcut that means the same as the former. In either case, we can use $P(x, y)$ to indicate the predicate $x \cdot y = 36$, and write $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x, y)$, or $\exists x, y \in \mathbb{Z}, P(x, y)$. The quantified predicate is true since, for example, $P(-2, -18)$ is true.
2. Similarly, we can write the "for all" quantifiers in two equivalent ways:
   - $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y = 36$
   - $\forall x, y \in \mathbb{Z}, x \cdot y = 36$

   Since $P(10, 2)$ is the false statement "$10 \cdot 2 = 36$," we see that the quantified predicate is false, and we say that $x = 10$, $y = 2$ is a counterexample to the statement $\forall x, y \in \mathbb{Z}, x \cdot y = 36$.

□

**Practice Problem 5** *Write the following as quantified statements using the symbols* $\exists$ *and* $\forall$*, and decide whether the statement is true:*

(a) *There are odd integers m and n whose product is* 35.

(b) *There are even integers m and n whose product is* 35.

(c) *For every choice of integers s and t, it is true that* $s^2 + t^2 \geq 0$.

(d) *For every choice of real numbers x, y, and z,* $x + y + z > 1$.

The example and practice problem illustrate two important points. First, the *domain* of the variables is very important. If the domain $D$ is the set of odd integers, the quantified statement $\exists m, n \in D, m \cdot n = 35$ is true. However, the same statement is false if the domain $D$ is the set of even integers. Second, there are some situations where establishing the truth of a quantified predicate is easy, and others where it takes more work.

A more difficult situation occurs when a quantified predicate contains *both* $\exists$ and $\forall$ quantifiers in the same statement. Even with only two variables, there are several different possibilities for the arrangement of the quantifiers and variables. The key issues can be understood by thinking carefully about the difference between these two quantified statements:

- $\forall x, \exists y, P(x, y)$
- $\exists y, \forall x, P(x, y)$

The first thing to note is that, in mathematics, we always interpret these from left to right, while ordinary English is often ambiguous. For example, "For every problem there is a solution," and "There is a solution for every problem," are generally taken to mean the same thing. The listener forms the meaning from the context and, to a certain extent, from the speaker's intonations. In a mathematical setting, we are always more precise in our speaking, writing, and interpretation.

**Example 10** *Let $\mathbb{Z}$ indicate the set of all integers. Which of the following quantified predicates statements are true? Explain.*

1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3$
2. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 15$
3. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 15$

SOLUTION

1. For every possible value of $x$, you must be able to find a corresponding value for $y$ that makes the predicate "$x + 2y = 3$" true. To see if this is can be done, think of it as a game: Your opponent chooses $x$, then you try to find a $y$ that makes $P(x, y)$ true.
   (a) The opponent chooses $x = 1$, and you choose $y = 1$, making $P(x, y)$ the true statement $1 + 2 \cdot 1 = 3$.
   (b) The opponent chooses $x = -3$, and you choose $y = 3$, making $P(x, y)$ the true statement $-3 + 2 \cdot 3 = 3$.
   (c) The opponent chooses $x =1{,}337$, and you choose $y = -667$, making $P(x, y)$ the true statement $1{,}337 + 2 \cdot (-667) = 3$.
   (d) The opponent chooses $x = 0$, and you give up. By algebra you can see that you would have to choose $y = \frac{3}{2}$, but this is not a member of the set $\mathbb{Z}$.

   The quantified statement is false, and $x = 0$ is a counterexample. If $x$ has the value 0, no possible choice for $y$ can make "$x + 2y = 3$" true.
2. Again, think of this as a game. Your opponent chooses a value for $x$, and you must find a corresponding value for $y$. If you play the game with an opponent, you may discover a pattern for how you can choose your value for $y$, once your opponent has chosen his or her value for $x$. The strategy is "Always choose $y = 15 - x$." This quantified predicate is a true statement.
3. To the casual reader, this may appear the same as the previous example. However, because we read left to right, the rules of the game have changed in a subtle but crucial way. This time, you must go first, so you must try to find a $y$ value that will work no matter how your opponent chooses his or her value for $x$. Clearly, this is impossible to do—no matter what $y$ you choose, your opponent will have an infinite number of $x$ values to choose from that make the predicate $x + y = 15$ false. For example, if you choose $y = 73$, your opponent only has to make the predicate $x + 73 = 15$ false, and that is easy to do. The quantified predicate is false.

□

**Practice Problem 6** *Let $\mathbb{Z}$ indicate the set of all integers and $\mathbb{R}$ the set of real numbers. Which of the following quantified predicates are true? Explain.*

*(a)* $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, x + 2y = 3$

*(b)* $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x \cdot y = 1$

*(c)* $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y = x$

*(d)* $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y = y$

We now turn to the question of negating predicates that have multiple quantifiers. Actually, we already know how to do this, if we just keep in mind that we read the statements left to right. Recall from Proposition 1 that

$$\begin{array}{llll} \text{The negation of} & \forall x \in D, P(x) & \text{is} & \exists x \in D, \neg P(x). \\ \text{The negation of} & \exists x \in D, Q(x) & \text{is} & \forall x \in D, \neg Q(x). \end{array}$$

When there is more than one quantifier, we apply this same process to each quantifier in turn, proceeding from left to right. The following example and practice problem illustrate the process.

**Example 11** *Write the negation of each of these statements, simplified so as not to require the $\neg$ symbol to the left of any quantifier.*

*1.* $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3$

*2.* $\exists x > 0, \forall y > 0, x \cdot y < x$

*3.* $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 13,$ *and* $x \cdot y = 36$

SOLUTION In each part, we proceed in several steps. First we simply put a negation symbol ($\neg$) in front of the entire statement. Then we apply the negation process to each quantifier in turn and finally to the predicate itself.

**1.**

| | |
|---|---|
| $\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3)$ | negation of original statement |
| $\exists x \in \mathbb{Z}, \neg(\exists y \in \mathbb{Z}, x + 2y = 3)$ | by Proposition 1 |
| $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x + 2y = 3)$ | by Proposition 1 |
| $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + 2y \neq 3$ | equivalent form of "not equal" |

**2.**

| | |
|---|---|
| $\neg(\exists x > 0, \forall y > 0, x \cdot y < x)$ | negation of original statement |
| $\forall x > 0, \neg(\forall y > 0, x \cdot y < x)$ | by Proposition 1 |
| $\forall x > 0, \exists y > 0, \neg(x \cdot y < x)$ | by Proposition 1 |
| $\forall x > 0, \exists y > 0, x \cdot y \geq x$ | equivalent form of "not less than" |

**3.**

| | |
|---|---|
| $\neg(\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x + y = 13) \wedge (x \cdot y = 36))$ | negation of original statement |
| $\forall x \in \mathbb{Z}, \neg(\exists y \in \mathbb{Z}, (x + y = 13) \wedge (x \cdot y = 36))$ | by Proposition 1 |
| $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg((x + y = 13) \wedge (x \cdot y = 36))$ | by Proposition 1 |
| $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (\neg(x + y = 13) \vee \neg(x \cdot y = 36))$ | by DeMorgan's laws |
| $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + y \neq 13) \vee (x \cdot y \neq 36)$ | equivalent form of "not equal" |

□

**Practice Problem 7** *Negate each quantified predicate. Which is true, the original statement or its negation? The domain for the first statement is* $\mathbb{R}$, *the set of real numbers.*

*(a)* $\forall x > 0, \exists y \in \mathbb{R}, (y > x) \wedge (x + y = 2x)$

*(b)* $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y \leq 0$

*(c)* $\forall x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 \geq 0$

## Solutions to Practice Problems

1 **(a)** When $x$ is replaced by 2, the statement becomes "$2 > 15$," a false statement. When $x$ is replaced by 23, it becomes "$23 > 15$," which is true. Similarly, "$-5 > 15$" and "$15 > 15$" are both false. Thus, only $P(23)$ is true.

**(b)** We see that $Q(2)$, $Q(-5)$, and $Q(15)$ are true. Observe that "$x \leq 15$" is true for precisely the values of $x$ for which the predicate "$x > 15$" is false.

**(c)** For the listed numbers, only $x = 15$ makes this predicate true. (Only $R(15)$ is true.)

2 The completed table is given as Table 1-12. Here are some sentences that express the true cases shown in this table.

- There exists an element $x$ of $D$ such that $x < 0$.
- There is an element $x$ in $D$ with the property that $x > -3$.
- There is a number in the set $D$ whose square is bigger than or equal to itself.
- For every element $x$ of the set $D$, it is true that $x > -3$.
- For all $x$ in $D$, $x^2 \geq x$.

3 **(a)** For every $d$ in the set $D$, $d > -2$. This is false since $-2$ is a counterexample.

**(b)** There is an element $d$ in $D$ such that $d > -2$. This is true.

**(c)** For every element $n$ of the set $D$, $n > -3$ and $n < 3$. This is true.

**(d)** For all $x$ in $D$, $x^2 \leq 4$. This is true.

**(e)** There exists an $m$ in $D$ with the property that $m > 10$. This is false.

4 Table 1-13 shows each original statement along with its negation.

5 **(a)** $\exists m, n$ odd integers, such that $m \cdot n = 35$. This is true, since $5 \cdot 7 = 35$.

**(b)** $\exists m, n$ even integers, such that $m \cdot n = 35$. This is false. It is possible to prove (we will learn how in Chapter 2) that whenever two even integers are multiplied, the result is even.

**(c)** $\forall s, t \in \mathbb{Z}, s^2 + t^2 \geq 0$. This is true. One can prove this by showing that when an integer is squared, the result cannot be negative, and by showing that the sum of two nonnegative numbers is never negative.

**(d)** $\forall x, y, z \in \mathbb{R}, x + y + z > 1$. This is false. The choice $x = -2, y = 0, z = 1$ is a counterexample.

6 **(a)** This is true. If your opponent chooses a value for $y$, the pattern $x = 3 - 2y$ can be used to find a corresponding value for $x$.

**(b)** This is false, since $x = 0$ is a counterexample—that is, there is no choice of $y$ that makes $0 \cdot y = 1$ true. However, if we change the domain for $x$ to the set of nonzero real numbers, the statement would be true. If it is clear we are talking about real numbers, this might be written as $\forall x \neq 0, \exists y \in \mathbb{R}, x \cdot y = 1$.

**(c)** This is true, since we can choose $x = 0$.

**(d)** This is also true, since we can choose $x = 1$.

7 **(a)** The negation is the true statement $\exists x > 0, \forall y \in \mathbb{R}, (y \leq x) \vee (x + y \neq 2x)$.

**(b)** The negation is the true statement $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y > 0$.

**(c)** The negation is the false statement $\exists x, y, z \in Z, x^2 + y^2 + z^2 < 0$. The original statement is true.

| Predicate | True for These Members of $D$... | True for at Least One? | True for All? |
|---|---|---|---|
| $x < 0$ | $-1$ | T | F |
| $x > -3$ | $-1, 0, 1, 2$ | T | T |
| $x^2 < x$ | None | F | F |
| $x^2 \geq x$ | $-1, 0, 1, 2$ | T | T |

**Table 1-12** Solution for Practice Problem 2

| | Statement | Negation | Which is True? |
|---|---|---|---|
| 1. | $\forall x \in D, (x \leq 0) \vee (x \geq 2)$ | $\exists x \in D, (x > 0) \wedge (x < 2)$ | Negation |
| 2. | $\exists x \in D, (x < 0) \wedge (x^2 > 0)$ | $\forall x \in D, (x \geq 0) \vee (x^2 \leq 0)$ | Statement |
| 3. | For all $x \in D$, $x^2 < x$. | $\exists x \in D$ such that $x^2 \geq x$ | Negation |
| 4. | There exists $x \in D$ such that $x^2 < x$. | $\forall x \in D, x^2 \geq x$ | Negation |

**Table 1-13** Table for Practice Problem 4

## Exercises for Section 1.4

**1.** Write each of the following predicates using the simple predicates $x > 0$ and $y > 0$ along with the propositional connectives $\wedge$, $\vee$, and $\neg$:

(a) Both $x$ and $y$ are positive.

**(b)** At least one of $x$ and $y$ is positive.

(c) Exactly one of $x$ and $y$ is positive.

**(d)** Neither $x$ nor $y$ is positive.

**2.** For each of the given values of $x$ and $y$, determine which predicates from Exercise 1 become true statements.

**(a)** If $x = 8$ and $y = 3$, which of (a–d) are true?

**(b)** If $x = -5$ and $y = 0$, which of (a–d) are true?

**(c)** If $x = 7$ and $y = -7$, which of (a–d) are true?

**(d)** If $x = 0$ and $y = 0$, which of (a–d) are true?

**(e)** If $x = -3$ and $y = -10$, which of (a–d) are true?

**(f)** If $x = -8$ and $y = 2$, which of (a–d) are true?

3. For each predicate given in the first column of Table 1-14, list the members of the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ that make the statement true. (Recall that $\mathbb{Z}$ is the set of all integers.)

**4.** Which elements of the set $D = \{2, 4, 6, 8, 10, 12\}$ make each of the predicates from Exercise 3 true?

5. Which elements of the set $D = \{2, 4, 6, 8, 10, 12\}$ make the **negation** of each of these predicates true?

**(a)** $Q(n)$ is the predicate, "$n > 10$."

**(b)** $R(n)$ is the predicate, "$n$ is even."

**(c)** $S(k)$ is the predicate, "$k^2 < 1$."

**(d)** $T(m)$ is the predicate, "$m - 2$ is an element of $D$."

| Predicate | True for These Members of *S* . . . |
|---|---|
| $x$ is even | |
| $n > 5$</p> | |
| $m$ is even and $m > 5$ | |
| $10 < x^2 + 1 < 25$ | |
| $x/3 \in \mathbb{Z}$ | |
| $10/x \in \mathbb{Z}$ | |

**Table 1-14** Table for Exercise 3

**6.** Based on your answers to Exercise 5, circle the true statements in the list given below.

7. Let $D = \{1, 3, 5, 7, 8, 9\}$. Decide whether each of the following statements is true for all the elements of $D$. For each that is not, give a counterexample. That is, provide a number in $D$ for which the statement is not true.

**(a)** $x$ is even and $x > 7$.

**(b)** $x$ is odd or $x > 7$.

**(c)** $x$ is not odd and $x \leq 7$.

**(d)** $x$ is odd.

| | | | |
|---|---|---|---|
| $\forall n \in D, Q(n)$ | $\exists n \in D, Q(n)$ | $\forall n \in D, \neg Q(n)$ | $\exists n \in D, \neg Q(n)$ |
| $\forall n \in D, R(n)$ | $\exists n \in D, R(n)$ | $\forall n \in D, \neg R(n)$ | $\exists n \in D, \neg R(n)$ |
| $\forall k \in D, S(k)$ | $\exists k \in D, S(k)$ | $\forall k \in D, \neg S(k)$ | $\exists k \in D, \neg S(k)$ |
| $\forall m \in D, T(m)$ | $\exists m \in D, T(m)$ | $\forall m \in D, \neg T(m)$ | $\exists m \in D, \neg T(m)$ |

List for Exercise 6

8. Write each of the following statements using quantifiers and predicates. In each case, you must specify the domain and define the predicates you use.
   **(a)** Every biology major is required to take geometry.
   **(b)** There are computer science majors who do not minor in mathematics.
   **(c)** There is no math major who is required to take a business course.
   **(d)** There are puzzles that have no solution.
9. Write each of the following statements using quantifiers and predicates. In each case, you must specify the domain and define the predicates you use.
   **(a)** For every integer $n$, $2n \neq 9$.
   **(b)** There exists a triangle $T$ that is equilateral and has perimeter 10.
   **(c)** Every circle has an integer diameter or an integer area.
   **(d)** Every two real numbers has an integer in between.
10. Let $B$ be the set of all biology majors and let $G(x)$ be the predicate "$x$ is required to take geometry." Write each statement below using quantifiers over the domain $B$ and the predicate $G(x)$, and then match any statements that are equivalent in meaning.
   **(a)** There is no biology major who is required to take geometry.
   **(b)** There is a biology major who is not required to take geometry.
   **(c)** There is no biology major who is not required to take geometry.
   **(d)** Every biology major is not required to take geometry.
11. The lesson of Exercise 10 is that a statement of the form "No member of $D$ makes $P(x)$ true" can be formally written as either "$\neg\exists x \in D, P(x)$" or "$\forall x \in D, \neg P(x)$." For each of the following English sentences, specify a domain and a predicate, and write the statement symbolically using both of these forms.
   **(a)** Friends of Alaina never get tired of playing at the beach.
   **(b)** No friend of Alaina dislikes doing cartwheels.
   **(c)** No math course is too hard for Jennica.
   **(d)** The meals at the camp are never too bad.
12. For each of the following English sentences, specify a domain and a predicate, and write the quantified statement symbolically using either form discussed in Exercise 11.
   **(a)** Even numbers are never prime.
   **(b)** Triangles never have four sides.
   **(c)** There are no integers $a$ and $b$ for which $a^2/b^2 = 2$.
   **(d)** No square number immediately follows a prime number.
13. Recall that $\mathbb{Z}$ denotes the set of all integers.
   **(a)** Consider the statement "$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3$."
      **i.** If $x$ is 27, what value can be chosen for $y$ to make the equation true?
      **ii.** Find two values of $x$ for which it is impossible to find a corresponding $y$ value that makes the equation true.
      **iii.** Is there a pattern for the $x$ values that serve as counterexamples to the given statement?
   **(b)** Consider the statement "$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, x + 2y = 3$."
      **i.** Identify the $x$ value that makes the equation true for each of the following values of $y$: 3, $-10$, 0, 17.
      **ii.** Describe a general strategy for choosing $x$ once your opponent has chosen the value for $y$.
14. Write the negation of each of the following statements, using Proposition 1 and the rules for negating propositions to simplify each to the point that no $\neg$ symbol occurs to the left of a quantifier. (Recall that $\mathbb{Z}$ denotes the set of all integers, and $\mathbb{R}$ denotes the set of all real numbers.)
   **(a)** $\forall a \in \mathbb{R}, \forall b \in \mathbb{Z}, a^2 + b \in \mathbb{Z}$
   **(b)** $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = x$
   **(c)** $\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, x = 2y$
   **(d)** $\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, \frac{x}{y} = 2$
15. For each statement in Exercise 14, is the original statement or its negation true?
16. Write the negation of each of the following statements as an English sentence. You might find it helpful to write a symbolic expression as an intermediate step.
   **(a)** Every time you roll a "6," you have to take a card.
   **(b)** There is a day in your life that is better than every other day.
   **(c)** In every good book, there is a plot twist or a surprise ending.
   **(d)** Every math course has a topic that everyone finds easy to do.
17. Write the negation of each of the following statements as an English sentence:
   **(a)** For every integer $x$, there is an integer $y$ that is bigger than $x$.
   **(b)** In every set of integers, there is a smallest number.
   **(c)** For every positive integer $x$, there is a positive integer $y$ such that $y$ is smaller than $x$ and $y$ is a factor of $x$.

**18.** For each statement in Exercise 17, is the original statement or its negation true?

**19.** Write the negation of each of the following statements as an English sentence:

**(a)** For all odd integers $b$, there is no real number $x$ such that $x^2 + bx + 15 = 0$.

**(b)** For every two real numbers $x$ and $y$, there is an integer $n$ such that $x < n < y$.

**(c)** For every pair of integers that sum to 5, at least one of the numbers must be bigger than 2.

**20.** For each statement in Exercise 19, is the original statement or its negation true?

## 1.5 Implications

Many statements in mathematics, as well as in ordinary conversation, refer to the logical connection between two simpler statements rather than to the actual truth of either one. For example,

**(i)** If Bob has an 8:00 class today, then it is a Tuesday.

**(ii)** If it is raining, then the street is wet.

**(iii)** If you are a computer science major, then you must take discrete mathematics.

**(iv)** If a real number $x$ satisfies $x^2 > 4$, then $x > 2$.

### *Definition*

1. A statement of the form "if $p$ is true, then $q$ is true" is called an **implication**.
2. We write an implication as $p \to q$, which is read, "$p$ **implies** $q$." The $\to$ operator is taken as having lower precedence than $\wedge$, $\vee$, and $\neg$. For example, the proposition $\neg p \wedge q \to r$ means $((\neg p) \wedge q) \to r$.
3. In the statement "if $p$, then $q$," we call $p$ the **hypothesis** and $q$ the **conclusion**.

In this definition, $p$ and $q$ can indicate either propositions or predicates. Of our four examples, the first two involve propositions and the last two involve predicates.

When an implication involves predicates, there might be some parts of the sentence that are left unsaid. For example, when the speaker says, "If you are a computer science major, then you must take discrete mathematics," she no doubt has in mind all the students at her particular school. So even though she didn't say so, there is an unspoken domain the speaker has in mind, and there is an unspoken quantifier. Here are more precise versions of examples (iii) and (iv) that make the domain and the quantifier more explicit:

**(iii′)** For all students $s$ at this school, if $s$ is a computer science major, then $s$ must take discrete mathematics.

**(iv′)** For all real numbers $x$, if $x^2 > 4$, then $x > 2$.

**Example 1** *In each of the following implicational statements, identify the domain $D$ and the predicates $P(x)$ and $Q(x)$ so that the implication is of the form "For all $x \in D$, if $P(x)$, then $Q(x)$." (You may use variables other than $x$ if you wish.)*

1. *If a triangle has three equal sides, then it has three equal angles.*
2. *If an integer ends with a "2," then it is a multiple* of 2.*
3. *If a quadrilateral can be inscribed in a circle, then the opposite angles in the quadrilateral sum to* 180°.

SOLUTION

1. $D$ is the set of all *triangles*, $P(t)$ is "$t$ has three equal sides," and $Q(t)$ is "$t$ has three equal angles."
2. $D$ is the set of *integers*, $P(n)$ is "$n$ ends with 2," and $Q(n)$ is "$n$ is a multiple of 2."
3. $D$ is the set of *quadrilaterals*, $P(d)$ is "$d$ can be inscribed in a circle," and $Q(d)$ is "$d$ has opposite angles summing to 180°."

□

**Practice Problem 1** *As in the preceding example, identify the domain D and the predicates* $P(x)$ *and* $Q(x)$ *so that the implication is of the form "For all* $x \in D$*, if* $P(x)$*, then* $Q(x)$*."*

(a) *If a real number x has a real square root, then x is not negative.*
(b) *If a real number x satisfies* $x^2 - x = 6$*, then* $x = 3$.
(c) *If an integer n is even, then* $2^n - 1$ *is a multiple of 3.*
(d) *If an integer n ends with a 3, then n is a multiple of 3.*

## The Logic of Implications

To help us understand the logic of implicational statements, we begin with a concrete example.

**Example 2** †*Trooper Jones walks into the Goldilocks Pub and sees four Boatsville College students (Al, Betty, Cindy, and Dan) enjoying various beverages. She asks the bartender, "Is anyone breaking the drinking law?" The bartender replies, "Everyone in here is obeying the law."*

*In front of each person, there is a card which has the person's age on one side and what he or she is drinking on the other side. Trooper Jones sees that the face-up sides of the cards look like Figure 1-10.*

*The drinking age law states in effect*

$$\textit{\textbf{If you are drinking beer, then you are at least 21 years of age.}} \tag{1.1}$$

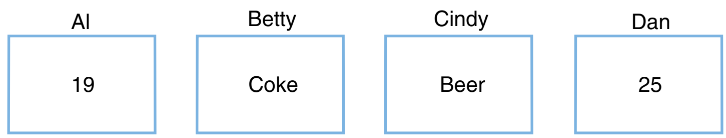


**Figure 1-10** Age-drink cards.

* The integers resulting from the product of 2 and an integer are the *multiples* of 2.

† This example is based on a psychology experiment carried out by Griggs and Cox [32] in 1982.

1. *Identify a set $D$ and predicates $P(x)$ and $Q(x)$ so that the bartender's statement is of the form "For all $x \in D$, if $P(x)$, then $Q(x)$."*
2. *Whose cards does Trooper Jones need to turn over to check that everyone is obeying the law?*

SOLUTION

1. $D$ is the set of four Boatsville College students currently in the bar, $P(x)$ is the predicate "$x$ is drinking beer," and $Q(x)$ is the predicate "$x$ is at least 21 years of age."
2. She should turn over the cards for Al and Cindy. She is looking for a counterexample to the bartender's claim that everyone is obeying the law, and these are the only two students who could possibly be counterexamples.

□

This example illustrates the basic logic behind an implication or "if. . . , then. . . " statement. It will help to analyze why Trooper Jones does not turn over Betty's card and Dan's.

- *P(Betty)* is the statement "Betty is drinking beer" and *Q(Betty)* is "Betty is at least 21 years of age." Because *P(Betty)* is false, Trooper Jones doesn't care whether *Q(Betty)* is true or not. In ordinary English, Trooper Jones knows that Betty is not drinking beer, so Betty is obeying the law no matter what her age is.
- *P(Dan)* is the statement "Dan is drinking beer" and *Q(Dan)* is "Dan is at least 21 years of age." Because *Q(Dan)* is true, Trooper Jones doesn't care whether *P(Dan)* is true or not. In ordinary English, Trooper Jones knows that Dan is of legal age, so Dan is obeying the law no matter what he is drinking.

If the law is being broken, it must be because someone *is drinking beer* and *is not at least 21 years old*. That is, **the only time that a statement of the form "If $p$, then $q$" is false is when the hypothesis (statement $p$) is true, but the conclusion (statement $q$) is false.** At all other times, we would have to say that the whole statement is true since the law is not being broken. The truth table shown in Table 1-15 sums up our analysis of an implicational statement.

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Table 1-15** Truth Table for Implication

The last two rows of this table are the hardest for most people to swallow. How can a statement of the form "If $p$, then $q$" be true when the statement $p$ is false, and *especially* when both statements $p$ and $q$ are false? For our concrete example, we can see that the last two rows correspond to the situation for Betty, for whom the hypothesis "$x$ is drinking beer" is false. Betty would match the third row if she is 23, and the fourth row if she is 18. In either case, though, she is obeying the law, so the implication $p \to q$ is true.

As we will see, this same logic applies to the more abstract setting of mathematical implications.

**Example 3** *Identify the hypothesis and conclusion for the following statement:*

*For every positive integer $n$, if $n$ is odd, then $n^3 - n$ is divisible by 4.*

*Do you think the statement is true or false?*

| $n$ | $n^3 - n$ | Divisible by 4? |
|---|---|---|
| 1 | 0 | Yes, since $0 = (4)(0)$. |
| 3 | 24 | Yes, since $24 = (4)(6)$. |
| 5 | 120 | Yes, since $120 = (4)(30)$. |
| 7 | 336 | Yes, since $336 = (4)(84)$. |

**Table 1-16** Analysis of the Statement in Example 3

SOLUTION The domain is the set of positive integers. The hypothesis is the predicate "$n$ is odd," and the conclusion is the predicate "$n^3 - n$ is divisible by 4." To explore the truth of the implication, it is natural to think about examples first, so we do this in Table 1-16. When considering examples, we only listed values for $n$ for which the hypothesis is true. We never considered the values of 2, 4, 6, or 8 for $n$. This omission is analogous to the Trooper's behavior toward the people whose cards indicated that they were not drinking beer. Whether $n^3 - n$ is divisible by 4 for even values of $n$ has no relevance on the truth of the implication, just as people who are not drinking beer are not breaking the law regardless of their age.

If we *do* look at these other (even) values of $n$, we see that sometimes they make the conclusion true and sometimes they make the conclusion false, just as some people who are not drinking beer might be under 21 and some might be 21 or older.

Table 1-17 shows some examples of this. Since the values of $n$ in Table 1-17 never make the hypothesis true, they cannot be counterexamples to the implication. It turns out that there *are* no counterexamples, and that the statement is therefore true.* □

**Example 4** *Why do we consider the following statement to be true?*

$$\textit{For all integers } n, \textit{ if } 3n = 9, \textit{ then } n^2 = 9.$$

SOLUTION As in the previous example, each of the component predicates "$3n = 9$" and "$n^2 = 9$" can be true or false depending on the value of $n$ as Table 1-18 illustrates. Once again, when the hypothesis is false, we really don't care about the conclusion. To establish whether the implicational statement is true, we must ask, "Could there possibly be a counterexample—that is, is there a value of $n$ for which the hypothesis is true but the conclusion is false?" Since the only value of $n$ that makes the hypothesis true in this example is $n = 3$, and since this value of $n$ also makes the conclusion true, we conclude that the implicational statement is true. □

* We will learn how to prove this type of statement in the next chapter.

| $n$ | $n^3 - n$ | Divisible by 4? |
|---|---|---|
| 2 | 6 | Not divisible by 4. |
| 4 | 60 | Yes, since $60 = (4)(15)$ |
| 6 | 210 | Not divisible by 4. |
| 8 | 504 | Yes, since $504 = (4)(126)$. |

**Table 1-17** More Analysis of the Statement in Example 3

| $n$ | Hypothesis ($3n = 9$) | Conclusion ($n^2 = 9$) |
|---|---|---|
| $-3$ | False | True |
| 0 | False | False |
| 3 | True | True |
| 10 | False | False |

**Table 1-18** Analysis of the Statement in Example 4

### Example 5 *Why do we consider the following statement to be false?*

*For all integers n, if* $n^2 > 9$, *then* $n > 3$.

SOLUTION It is possible to find a counterexample, if we remember that "integers" includes both positive and negative numbers. For example, the value $n = -4$ makes the hypothesis ($n^2 > 9$) true while making the conclusion ($n > 3$) false. □

It is only the presence of values of $n$ that make the hypothesis true and the conclusion false, which causes the implicational statement to be false.

> ***Summary*** For a statement of the form "if **hypothesis**, then **conclusion**" to be *FALSE*, it must be the case that the **hypothesis** is true while the **conclusion** is false. Otherwise, the statement is *TRUE*.
>
> For a quantified statement "$\forall x$, if $P(x)$, then $Q(x)$" to be *FALSE*, it must be the case that at least one value of $x$ is a counterexample—that is, there is at least one value of $x$ that makes the **hypothesis** $P(x)$ true but makes the **conclusion** $Q(x)$ false. Otherwise, the quantified statement is *TRUE*.

**Practice Problem 2** *Decide whether each of the following quantified statements is true or false. For each that is false, give a counterexample. Remember that* $\mathbb{R}$ *is the set of real numbers, and* $\mathbb{Z}$ *is the set of integers.*

*(a)* $\forall x \in \mathbb{R}$, *if* $x^2 - 5x + 4 = 0$, *then* $x > 0$.

*(b)* $\forall n \in \mathbb{Z}$, *if* $n^2 = 1$, *then* $n^3 = 1$.

*(c)* $\forall$ *positive integers a and b, if a and b are both odd, then* $a + b$ *is also odd.*

*(d)* $\forall$ *positive integers a and b, if a and b are both odd, then ab is also odd.*

## Negating Implications

Since the negation of a statement captures what it means for the statement to be false, we can take our understanding of when an implication is false and turn it into a formal rule for forming the negation of an implicational statement.

***Proposition 1*** *The negation of the implication* $p \to q$ *is the statement* $p \wedge (\neg q)$.

PROOF The proof consists of showing that the propositions $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent using truth tables.

| $p$ | $q$ | $p \to q$ | $\neg(p \to q)$ | $\neg q$ | $p \wedge \neg q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | F | F |
| F | F | T | F | T | F |

Since the columns for $\neg(p \to q)$ and $p \wedge \neg q$ are identical, we conclude that these statements are logically equivalent. **Notice that the negation of an implication is not an implication!** ■

If a quantified implicational statement has the form $\forall x \in D, P(x) \to Q(x)$ for some domain $D$, then the statement is false if there is a value for $x$ in $D$ that makes the hypothesis $P(x)$ true but the conclusion $Q(x)$ false. Hence, the negation of a quantified implicational statement can itself be expressed as a quantified statement.

***Proposition 2*** *The negation of the implication* $\forall x \in D, P(x) \to Q(x)$ *is the statement* $\exists x \in D, P(x) \wedge (\neg Q(x))$.

PROOF This follows from Proposition 1 of this section and Proposition 1 of Section 1.4. ■

**Example 6** *Write the negation of each of the following statements:*

1. *If Bob has an 8:00 class today, then it is a Tuesday.*
2. *If Jessica gets chocolate, then she has a happy birthday.*
3. *For all real numbers x, if* $x > 2$, *then* $x^2 > 4$.
4. $\forall$ *real numbers* $x > 0$, *if* $x^2 = 1$, *then* $x^3 = 1$.

SOLUTION Each negation is given.

1. Bob has an 8:00 class today, and it is not Tuesday.
2. Jessica gets chocolate, but she doesn't have a happy birthday.
3. There exists a real number $x$ such that $x > 2$ but $x^2 \leq 4$.
4. $\exists$ real number $x > 0$, $(x^2 = 1) \wedge (x^3 \neq 1)$. Observe that the domain is still the set of real numbers that are greater than 0.

□

**Practice Problem 3** *Write the negation of each of the following statements*:

1. *If you buy the extended warranty, then nothing will go wrong with your television.*
2. *If Christopher gets a flu shot, then he will not get the flu.*
3. *For all triangles t, if t has three equal sides, then t has three equal angles.*
4. $\forall x \in \{1, 2, 3, 4, 5\}$, $x^2$ *is positive.*

## Contrapositives, Converses, and Inverses

We have determined how the truth value of an implicational statement depends on the truth value of its component parts. However, there are many ways to form an implication with a given pair of component statements. Trying different combinations for hypothesis and conclusion is a fairly natural thing to do when considering a new mathematics problem. It is often the case that there are important properties to be studied, and you are first trying to figure out how the properties are related to each other.

**Example 7** *Let $P(n)$ stand for the predicate "n ends in a digit 2," and $Q(n)$ for the predicate "n is divisible by 2." We can use these predicates to form many different implications.*

1. $P(n) \rightarrow Q(n)$, *that is, "If n ends in a digit 2, then n is divisible by 2."*
2. $Q(n) \rightarrow P(n)$, *that is, "If n is divisible by 2, then n ends in a digit 2."*
3. $\neg P(n) \rightarrow \neg Q(n)$, *that is, "If n does not end in a digit 2, then n is not divisible by 2."*
4. $\neg Q(n) \rightarrow \neg P(n)$, *that is, "If n is not divisible by 2, then n does not end in a digit 2."*

*As is often the case, each statement has an implied domain and quantifier—in this case, "For all integers n, . . . . " Decide if each of the quantified statements is true or false. For each that is false, give a counterexample.*

SOLUTION

1. This statement is true.
2. The value $n = 14$ is a counterexample since it makes the hypothesis true (14 is divisible by 2) and the conclusion false (14 does not end in 2).
3. The value $n = 14$ is a counterexample since it makes the hypothesis true (14 does not end in 2) and the conclusion false (14 is divisible by 2).
4. This statement is true.

□

You might have noticed that we used the same counterexample for the second and third statements above. If you think about it, any counterexample to the second statement must be a counterexample to the third statement and vice versa. The same relationship holds between the first and fourth statements as well, so since the first statement has no counterexamples, then neither does the fourth statement. Because of these connections, the relationships illustrated by these four statements have special names in the mathematics literature.

**Definition** Consider the implication $\forall x \in D, P(x) \rightarrow Q(x)$.

1. The **converse** of the implication is $\forall x \in D, Q(x) \rightarrow P(x)$.
2. The **inverse** of the implication is $\forall x \in D, \neg P(x) \rightarrow \neg Q(x)$.
3. The **contrapositive** of the implication is $\forall x \in D, \neg Q(x) \rightarrow \neg P(x)$.

We have defined the terms in the context of quantified statements, but the same terms can be applied in an obvious way to propositions:

- The converse of $p \rightarrow q$ is $q \rightarrow p$.
- The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

Since we have a mechanism for proving propositions to be logically equivalent, we can formally establish the relationships mentioned in the discussion following Example 7.

## Proposition 3

1. *An implication and its contrapositive are logically equivalent.*
2. *The converse and inverse of an implication are logically equivalent.*
3. *An implication is not logically equivalent to its converse (and thus not to its inverse).*

PROOF We will prove the first statement by comparing the truth tables of $p \rightarrow q$ and $\neg q \rightarrow \neg p$.

| $p$ | $q$ | $p \rightarrow q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | T | F | F |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

Since the columns for $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are identical, we conclude that these statements are logically equivalent. The other two parts of this proposition are addressed in Exercise 5 at the end of this section. ■

### Example 8

1. *Give an example of a true implication whose converse is false.*
2. *Give an example of a true implication whose converse is also true.*

SOLUTION

1. The implication "For all integers $a$ and $b$, if $a$ and $b$ are odd, then $a + b$ is even" is true.* Its converse "For all integers $a$ and $b$, if $a + b$ is even, then $a$ and $b$ are odd" is false. One counterexample is $a = 2$, $b = 14$.
2. "If $n$ is even, then $n^2$ is even" is true and has a true converse.

□

Since it is possible for a true implicational statement to have a false converse, but it is also possible for a true statement to have a true converse, we must always treat implications and their converses as two entirely different statements that must each be analyzed on its own merits.

## The Language of Implication

One of the frustrations of the English language is that there are many ways to state the same thought. This phenomenon occurs not only in ordinary conversation, but also in mathematical expression. This is one reason mathematicians are so fond of symbolism—a completely symbolic expression of a theorem is less likely to contain the ambiguity inherent in spoken and written language. In this short section, we will give you a few examples of some common sources of confusion in the English language.

One possible source of confusion is our tendency in English to rearrange the parts of an implication without changing its meaning. For example, these sentences convey exactly the same idea:

- I will pass the course if I ace the final.
- If I ace the final, then I will pass the course.

Switching the location of the hypothesis and conclusion within the sentence does not change the meaning. The hypothesis is the part that goes with the word *if*, and in each sentence that is the phrase "I ace the final." Do not confuse this with forming the converse, which involves interchanging the hypothesis and conclusion. Here is a mathematical example:

| | |
|---|---|
| Statement: | If an integer $m$ ends in the digit 0, then $m$ is a multiple of 5. |
| Same statement: | An integer $m$ is a multiple of 5 if it ends in the digit 0. |
| Converse: | If an integer $m$ is a multiple of 5, then $m$ ends in the digit 0. |

Another possible source of confusion is the variety of ways in which we can express quantification and implication. We have discussed earlier our tendency to

* In the next chapter, we will learn how to prove statements of the type in this example.

leave out domains and quantifiers when we speak. Another issue when formalizing statements is the choice of writing a "for all" statement as an implication or not. Given a statement of the form "$\forall x \in D, Q(x)$" where $D$ is a subset of a natural larger set $U$, we can equivalently write the statement as "$\forall x \in U, (x \in D) \rightarrow Q(x)$." For now, the choice is a matter of taste, but we will learn to prefer the latter in the next chapter when we discuss formal proofs of this type of statement.

**Example 9** *Here is one way to express the idea that every computer science major at this university must take discrete mathematics: "Let D be the set of computer science majors at this university. For all $s \in D$, s must take discrete mathematics."*

1. *Rewrite this as an implication.*
2. *Form the negation of the original statement, and the statement written as an implication.*

SOLUTION

1. Here is one possible solution: The set $U$ of all students at this university is a natural set that includes all of $D$. So an equivalent implicational statement is "For all $s \in U$, if $s$ is a computer science major, then $s$ must take discrete mathematics."
2. The negation of the original statement is "There exists a computer science major at this university who does not have to take discrete mathematics." For the implicational form, the negation is "There exists a student at this university who is a computer science major but does not have to take discrete mathematics." Both say the same thing, but in a slightly different way.

□

**Practice Problem 4** *Rewrite each quantified predicate as an implication. Unless otherwise indicated, assume that variables stand for real numbers. Use $\mathbb{R}$ and $\mathbb{Z}$ to indicate the sets of reals and integers, if you wish.*

*(a)* $\forall$ *even integer m, m ends in the digit 0, 2, 4, 6, or 8.*

*(b)* $\forall x > 0, x^2 > x$

*(c)* *For every positive odd integer n, $n^3 - n$ is divisible by 4.*

Finally, it is sometimes the case that both an implication and its converse are true. This is the strongest possible relationship between the properties that make up the hypothesis and conclusion of the implication. Because of the importance of this situation, mathematicians have some common ways to convey the idea that an implication and its converse are both true. For example, the statement "If $n$ is even, then $n^2$ is even" is true, and it has a true converse. A mathematician might say, "If $n$ is even, then $n^2$ is even, and conversely," or "$n$ is even if and only if $n^2$ is even."

In general, for any statements $p$ and $q$ (with or without variables), the phrase "$p$ if and only if $q$" means that both $p \rightarrow q$ and $q \rightarrow p$ are true. For this reason, we use the notation $p \leftrightarrow q$ to express this *biconditional statement*. When considering a statement of this form, it is often easiest to consider each of these two statements separately.

**Practice Problem 5** *What are the two implicational statements expressed by the statement "The integer n is a multiple of* 10 *if and only if n is even"? Is each statement true? If not, give a counterexample.*

## Logic Puzzles Revisited

Since now we know how to make truth tables for implicational statements, we end this section with some more logic puzzles, involving the use of implication. Remember to make your truth tables by first considering the statements themselves without worrying about who said them.

**Example 10** *You meet two inhabitants. A says, "If B is truthful, then so am I," and B says, "At least one of us is lying." Who (if anyone) is telling the truth?*

SOLUTION Here is the truth table. Notice that $A$'s statement is $q \to p$, which is false only when $q$ is true but $p$ is false.

| | $p$ | $q$ | Statement 1<br>If B is truthful, then so is A | Statement 2<br>At least one of us is lying |
|---|---|---|---|---|
| | T | T | T | F |
| | T | F | T | T |
| ⋆ | F | T | F | T |
| | F | F | T | T |

We conclude that only $B$ is a truth-teller. □

**Example 11** *You meet two inhabitants. A says, "If B is truthful, then so am I," and B says, "A is lying." Who (if anyone) is telling the truth?*

SOLUTION This is similar to the previous example. $A$'s statement (if $q$, then $p$) is false only in the situation where $q$ is true but $p$ is false.

| | $p$ | $q$ | Statement 1<br>If $B$ is truthful, then so is $A$ | Statement 2<br>$A$ is lying |
|---|---|---|---|---|
| | T | T | T | F |
| ? | T | F | T | F |
| ? | F | T | F | T |
| | F | F | T | T |

In this case there is an ambiguous answer since either of the middle two rows could be solutions. We can conclude that exactly one of $A$ and $B$ is a truth-teller, but we cannot determine which one from what they said. □

**Example 12** *You meet three inhabitants. A says, "If B is lying, then so is C," B says, "C is truthful," and C says, "At least one of us is lying." Who (if anyone) is telling the truth?*

SOLUTION To build the table for $A$'s statement, we analyze it as being $\neg q \to \neg r$. The only time it is false is when $\neg q$ is true but $\neg r$ is false. Put another way, the only time Statement 1 is false is when $q$ is false but $r$ is true.

| | | | Statement 1 | Statement 2 | Statement 3 |
|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | If $B$ is lying, then so is $C$ | $C$ is truthful | One of us is lying |
| T | T | T | T | T | F |
| T | T | F | T | F | T |
| T | F | T | F | T | T |
| T | F | F | T | F | T |
| F | T | T | T | T | T |
| F | T | F | T | F | T |
| F | F | T | F | T | T |
| F | F | F | T | F | T |

In this example there are no solutions, so this is a paradox. In other words, given the rules of the problem, no three people could have uttered those particular phrases. □

## Solutions to Practice Problems

1 (a) $D$ is the set of real numbers, $P(x)$ is "$x$ has a real square root," and $Q(x)$ is "$x$ is not negative."

(b) $D$ is the set of real numbers, $P(x)$ is "$x$ satisfies the equation $x^2 - x = 6$," and $Q(x)$ is "$x = 3$."

(c) $D$ is the set of integers, $P(n)$ is "$n$ is even," and $Q(n)$ is "$2^n - 1$ is divisible by 3."

(d) $D$ is the set of integers, $P(n)$ is "$n$ ends with a 3," and $Q(n)$ is "$n$ is divisible by 3."

2 (a) There are only two values for $x$ that make the hypothesis true ($x = 1$ and $x = 4$), and for both these values the conclusion is also true. Therefore, the "if, then" statement is true.

(b) Although there is a value of $n$ (namely $n = 1$) that makes both the hypothesis and conclusion true, there is also a value ($n = -1$) where the hypothesis is true and the conclusion is false. So the "if, then" statement is false.

(c) This statement is false, and it is easy to find a counterexample—just pick any odd integers for $a$ and $b$. For example, with $a = 1$ and $b = 7$, we have $a + b = 8$. Since for this choice of $a$ and $b$ the hypothesis is true but the conclusion is false, the original "if, then" statement is false.

(d) Table 1-19 shows some possible values of $a$ and $b$. The fact that some entries make the conclusion false is irrelevant, since these same values make the hypothesis false as well. It turns out that any time we make the hypothesis true, the conclusion is also true, and so the "if, then" statement is true. In the next chapter we will see how to prove this fact.

3 (a) You buy the extended warranty, and something does go wrong with your television.

(b) Christopher gets the flu shot, but he does get the flu.

(c) There is a triangle $t$ where $t$ has three equal sides but $t$ does not have three equal angles.

(d) $\exists x \in \{1, 2, 3, 4, 5\}$ such that $x^2 \le 0$.

4 (a) $\forall m \in Z$, if $m$ is even, then $m$ ends in the digit 0, 2, 4, 6, or 8.

(b) $\forall x \in R$, if $x > 0$, then $x^2 > x$.

(c) $\forall n \in Z$, $(n > 0) \wedge (n \text{ odd}) \to n^3 - n$ is divisible by 4.

5 The statement "If $n$ is a multiple of 10, then $n$ is even" is a true statement, but the converse "If $n$ is even, then $n$ is a multiple of 10" is false, having $n = 2$ as a counterexample. Hence, the given "if and only if" statement is false.

| $a$ | $b$ | Hypothesis $a$ and $b$ are Both Odd | Conclusion $ab$ is Odd |
|---|---|---|---|
| 3 | 5 | True | True |
| 3 | 2 | False | False |
| 8 | 4 | False | False |
| 31 | 11 | True | True |

**Table 1-19** Table for Practice Problem 2(d)

## Exercises for Section 1.5

1. Write each statement as a statement of formal propositional logic. That is, assign variable names to the simple phrases, and write the statement using those variables along with the logical connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$.
   (a) If you don't attend the concert, you will get an F for the course.
   **(b)** We will go if you go.
   (c) I ate my lunch but I did not eat breakfast.
   **(d)** If you don't eat your breakfast, you will be hungry.
   (e) It is false that this triangle has both a 30° angle and a 60° angle.
   **(f)** If a quadrilateral is a square, it has four equal sides and four equal angles.
   **(g)** If a triangle has either two equal sides or two equal angles, then it is an isosceles triangle.
2. For each statement in Exercise 1, give a truth table for the statement, and use it to explain what conditions would make the statement true and what would make it false.
3. In a certain board game played with a pair of dice, if you roll "doubles" three times in a row, you must place your piece on the board square marked "Jail." After playing the game for two grueling hours, no one ever rolled "doubles" three times in a row. Can we conclude that no piece was ever placed on the "Jail" square during the game? Refer specifically to the hypothesis and conclusion of the rule above in your explanation.
4. Complete the following truth tables for the given compound expressions:

(a)

| $p$ | $q$ | $p \wedge q$ | $(p \wedge q) \rightarrow q$ |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

**(b)**

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \rightarrow (p \vee q)$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

(c)

| $p$ | $q$ | $p \vee q$ | $(p \vee q) \rightarrow q$ |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

**(d)**

| $p$ | $q$ | $\neg p$ | $\neg q$ | $\neg q \rightarrow p$ | $\neg p \wedge (\neg q \rightarrow p)$ |
|---|---|---|---|---|---|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

(e)

| $p$ | $q$ | $r$ | $q \rightarrow r$ | $p \wedge (q \rightarrow r)$ |
|---|---|---|---|---|
| T | T | T | | |
| T | T | F | | |
| T | F | T | | |
| T | F | F | | |
| F | T | T | | |
| F | T | F | | |
| F | F | T | | |
| F | F | F | | |

**(f)**

| $p$ | $q$ | $r$ | $p \wedge q$ | $p \vee r$ | $(p \wedge q) \rightarrow (p \vee r)$ |
|---|---|---|---|---|---|
| T | T | T | | | |
| T | T | F | | | |
| T | F | T | | | |
| T | F | F | | | |
| F | T | T | | | |
| F | T | F | | | |
| F | F | T | | | |
| F | F | F | | | |

**5.** Use truth tables to check if each of the given pairs of symbolic logic statements are equivalent.

(a) $p \rightarrow q$ and $q \rightarrow p$ (NOTE: This is a generic implication and its converse.)

**(b)** $\neg p \rightarrow \neg q$ and $q \rightarrow p$ (NOTE: This is the inverse and converse of a generic implication $p \rightarrow q$.)

(c) $p \wedge (p \rightarrow q)$ and $p \wedge q$

**(d)** $p \rightarrow (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$

**(e)** $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \vee (q \rightarrow r)$

**6.** It is sometimes useful to realize that the implication $p \rightarrow q$ is logically equivalent to $\neg p \vee q$. This exercise explores this equivalence.

**(a)** What combination of truth values for $p$ and $q$ makes the implication $p \rightarrow q$ false?

**(b)** Explain how you know that the negation of $\neg p \vee q$ is $p \wedge \neg q$. (HINT: Refer to Theorem 2 of Section 1.3.)

**(c)** What combination of truth values makes $p \wedge \neg q$ true? (This same combination would make $\neg p \vee q$ false.)

**(d)** Use a truth table to formally demonstrate that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$.

7. For each of the following statements, rewrite them in propositional logic notation, making the meaning of your propositional variables clear. Use truth tables to find any pairs of logically equivalent statements.

**(a)** If Alaina likes basketball, then she likes swimming and gymnastics.

**(b)** If Alaina likes gymnastics, then she likes swimming and basketball.

**(c)** If Alaina dislikes gymnastics or dislikes swimming, then she dislikes basketball.

**(d)** Alaina dislikes basketball or she likes both swimming and gymnastics.

8. Write each of the following predicates using the simple predicates $x > 0$ and $y > 0$ along with the propositional connectives $\wedge$, $\vee$, $\neg$, and $\rightarrow$:

**(a)** If $x$ is positive, then $y$ is positive.

**(b)** If $x$ is positive, then $y$ is not positive.

**(c)** If $x$ is not positive, then $y$ is positive.

**(d)** If $x$ is not positive, then $y$ is not positive.

**9.** For each of the given values of $x$ and $y$, determine which predicates from Exercise 8 become true statements.

**(a)** If $x = 8$ and $y = 3$, which of (a–d) are true?

**(b)** If $x = -5$ and $y = 0$, which of (a–d) are true?

**(c)** If $x = 7$ and $y = -7$, which of (a–d) are true?

**(d)** If $x = 0$ and $y = 0$, which of (a–d) are true?

**(e)** If $x = -3$ and $y = -10$, which of (a–d) are true?

**(f)** If $x = -8$ and $y = 2$, which of (a–d) are true?

10. Let $D$ be the set $\{1, 3, 5, 7, 8, 10, 11, 12\}$. For each of the following, decide whether it is true for all the elements of $D$. If it is not, give a counterexample.

**(a)** If $x$ is even, then $x > 7$.

**(b)** If $x$ is odd, then $x > 7$.

**(c)** If $x$ is even, then $x \leq 12$.

**(d)** If $x$ is odd, then $x \leq 12$.

**(e)** If $x > 12$, then $x$ is even.

**11.** Let $D$ be the set $\{1, 3, 5, 7, 8, 10, 11, 12\}$. For each of the following, decide whether it is true for all the elements of $D$. If it is not, give a counterexample.

**(a)** If $x$ is even and $x > 7$, then $x < 20$.

**(b)** If $x$ is odd, then $x < 10$.

**(c)** If $x < 10$ and $x \neq 8$, then $x$ is odd.

**(d)** If $x$ is odd or $x < 5$, then $x - 1$ is even.

**(e)** If $x > 5$ and $x < 7$, then $x$ is negative.

12. For each of the following statements, describe the predicates $P$ and $Q$ that make the formal statement

For every positive integer $n$, $P(n) \rightarrow Q(n)$

correctly represent the statement.

**(a)** If $n$ is even, then $n^2 + n$ is even.

**(b)** If $n$ is a multiple of 5, then $n$ has ones' digit of 5.

**(c)** If $n$ is prime, then $2^n - 1$ is prime.

**13.** For each of the following statements, describe the predicates $P$ and $Q$ that make the formal statement

There exists an integer $n$, $P(n) \wedge Q(n)$

correctly represent the statement.

**(a)** Some odd numbers $n$ make $2^n - 1$ a multiple of 7.

**(b)** At least one multiple of 5 does not have a ones' digit of 0.

**(c)** It is possible for a multiple of 3 to have a ones' digit of 7.

14. Express each of the following statements using predicates and the quantifier $\forall$:

**(a)** If $n$ ends in 5, then $n$ is a multiple of 5.

**(b)** If $m$ ends in 3, then $m$ is a multiple of 3.

**(c)** If $n$ is a multiple of 5, then $n^2 - 1$ is a multiple of 3.

**(d)** For every positive real number $x$, if $x < \sqrt{2}$, then $2/x > \sqrt{2}$.

**15.** Which of the statements in Exercise 14 are true?

**16.** For each of the statements in Exercise 14, express the negation using predicates and the quantifier $\exists$.

**17.** Express each of the following statements using predicates and the quantifiers $\forall$ and $\exists$:

**(a)** If $n$ is a multiple of 5, then $n$ ends in 5 or $n$ ends in 0.

**(b)** If $n$ is not a multiple of 3, then $n^2 - 1$ is a multiple of 3.

**(c)** For all odd integers $a$ and $b$, there is no real number $x$ such that $x^2 + ax + b = 0$.

**(d)** For every real number $y$, if $y \geq 0$, then there exists $x \in \mathbb{R}$ such that $x^2 = y$.

**18.** Which of the statements in Exercise 17 are true?

**19.** For each of the statements in Exercise 17, express the negation using predicates and the quantifiers $\exists$ and $\forall$.

**20.** Give a counterexample to each of the following to show that it is a false implicational statement. If you think the statement is actually true, write one sentence explaining why you think no counterexample exists.

**(a)** If $n^2 = 4$, then $n^3 = 8$.

**(b)** If $\sin(x) = 0$, then $\cos(x) = 1$.

**(c)** If $\cos(x) = 0$, then $\sin(x) = 1$.

**(d)** If $x^3 = x$, then $x^2 = 1$.

**21.** Consider the statement "If $n$ is even, then $2^n - 1$ is a multiple of 3." If we let $P(x)$ be the predicate "$x$ is even," and we let $Q(x)$ be the predicate "$2^x - 1$ is a multiple of 3," then we can represent the original statement as

For every natural number $n$, $P(n) \to Q(n)$.

Create an organized table of your attempts to find a counterexample to this statement. Why do you think that no counterexample exists?

**22.** A consequence of Proposition 2 is that the negation of the statement $\exists x \in D, P(x) \wedge Q(x)$ is the statement $\forall x \in D, P(x) \to \neg Q(x)$. Use this fact to write the negation of each of the following statements, using careful wording and an "if, then" structure when possible:

**(a)** There exists a positive integer $n$ such that $n$ is even and $\frac{1}{n} > 1$.

**(b)** There exist positive integers $a$ and $b$ such that $a - b$ is odd and $a^2 = 2b^2$.

**(c)** There exist integers $a$ and $b$ such that $a$ and $b$ are positive and $a/b = 1 + b/a$.

**(d)** There exists a right triangle with perimeter equal to three times the length of one leg.

**23.** Use the idea in Exercise 22 to write the negation of each of the following statements, using careful wording and an "if, then" structure when possible:

**(a)** There exist integers $m$ and $n$ such that $m \geq n$ and $m^2 + n^2 = 11$.

**(b)** There exists a real number $z$ such that $z > 0$ and $z + \frac{1}{z} = 1$.

**(c)** If $x$ is a positive real number, then there is a positive integer $n$ with $nx > 1$.

**(d)** If $S$ is a set of three people, then there exist two people in $S$ with the same sex.

**24.** Which of the original statements in the previous exercise are true?

**25.** Form the contrapositive of each of these statements:

**(a)** If you don't attend the concert, you will get an F for the course.

**(b)** We will go if you go.

**(c)** If you don't eat your breakfast, you will be hungry.

**(d)** If a quadrilateral is a square, it has four equal sides and four equal angles.

**(e)** If a triangle has either two equal sides or two equal angles, then it is an isosceles triangle.

**26.** Form the converse of each of the statements in Exercise 25.

**27.** Form the inverse of each of the statements in Exercise 25.

**28.** Solve each of these logic puzzles by using truth tables.

**(a)** You come across two inhabitants. $A$ says, "I am lying if $B$ is," and $B$ says, "$A$ is lying if I am." Can you tell who if anyone is telling the truth?

**(b)** You come across three inhabitants. $A$ says, "If $B$ is lying, then so is $C$," $B$ says, "If $C$ is lying, then so is $A$," and $C$ says, "If $A$ is lying, then so is $B$." Who if anyone is telling the truth?

**29.** If there was a third person in Example 11, what could she have said that would have determined everyone's truthfulness?

**30.** Express each of the following using quantified statements over the domain $S$ of all college students and the predicates $C(x)$ meaning "$x$ is a computer science major," and $D(x)$ meaning "$x$ takes discrete mathematics." Which statements are equivalent to one another? Which statements are negations of one another?

**(a)** Every computer science major takes discrete mathematics.

**(b)** Some computer science majors take discrete mathematics.

**(c)** No computer science major takes discrete mathematics.

**(d)** Not every computer science major takes discrete mathematics.

**(e)** All computer science majors do not take discrete mathematics.

**(f)** You must take discrete mathematics if you are a computer science major.

**(g)** Some computer science majors do not take discrete mathematics.

## 1.6 Excursion: Validity of Arguments

Charles Sanders Peirce wrote [41]

> *. . . Bad reasoning as well as good reasoning is possible; and this fact is the foundation of the practical side of logic.*

Logic has been an important part of intellectual endeavors since the time of the ancient Greeks. Aristotle (384–322 B.C.E.) wrote about the importance of precise reasoning in discourse, and Euclid's famous book *The Elements* has been used in teaching the art of rigorous thought and the skill of persuasive argument for nearly 2,000 years. Since so many decisions in our lives are based on the persuasive use of language and logic, the field of formal logic is an active, important discipline of philosophy. In the next chapter, we will consider the question of mathematical theorems and their proofs. These mathematical proofs are one example of a general concept—using valid arguments to persuade others that something is true. In the current section, we will examine the general notion of arguments in everyday life, and how to detect some common misuses of logic. We begin with an example of a phenomenon that we sometimes see in advertising: *What you think you hear may not be what the advertisement really said.*

**Example 1** *A car company advertises, "If you didn't buy from us, you paid too much." Write this as a statement of formal propositional logic, and examine it with a truth table.*

SOLUTION Let $p$ stand for "You bought a car from us," and $q$ stand for "You paid too much." The statement in the ad is represented by $\neg p \to q$, and its truth table looks like this:

| $p$ | $q$ | $\neg p$ | $\neg p \to q$ |
|---|---|---|---|
| T | T | F | T |
| T | F | F | T |
| F | T | T | T |
| F | F | T | F |

Hence, the only circumstances under which the statement is false are if you do not buy your car from this company and you do not pay too much for it. Notice that if you do buy your car from this company, the advertised statement is true regardless of the truth value of statement $q$. □

The car company that sponsored this ad is probably hoping you hear something different from what they actually say. Two possible misinterpretations are:

- If you paid too much for your car, you didn't buy from us. This is the *converse* of the advertiser's claim.
- If you bought a car from us, you did not pay too much. This is the *inverse* of the advertiser's claim.

As we indicated in the previous section, a statement and its converse (or inverse) do not necessarily have the same truth value. Truth in advertising may force the car company to make true statements in its advertisements—but it is up to the consumer to resist inferring that the inverse or converse is also true.

Why are we so likely to misinterpret the advertiser's claim? One reason is that, in ordinary conversation, we frequently make statements of the form "if $p$, then $q$" when our true meaning *does* include both the statement and its converse (or inverse).

**Example 2** *When you were a child, your parents said, "If you don't eat your peas, you can't have dessert." You promptly ate your peas and asked for dessert. What statement did you hear? Is it the same statement your parents made? Do you think you misinterpreted their statement?*

SOLUTION You are acting under the assumption that their statement was "If you *do* eat your peas, you *can* have dessert." This is NOT the same statement. It is the inverse. However, most parents who make a statement such as this really mean both what they actually say *and* its inverse. □

When we are dealing with our parents or our friends, there is perhaps no harm in reading into their statements more than was actually said. However, in a more formal setting such as a debate, a mathematical proof, or a false advertising suit, it is important to establish the exact meaning of each statement, and to analyze whether each successive statement follows logically from the previous statements.

**Example 3** *Using $p$ to stand for "You bought a car from us" and $q$ to stand for "You paid too much," analyze the* **inverse** *of the advertiser's claim: "If you bought a car from us, you did not pay too much." Under what circumstances is the inverse of this statement false?*

SOLUTION The statement is written formally as $p \to \neg q$, and it has the following truth table:

| $p$ | $q$ | $\neg q$ | $p \to \neg q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | F | T |
| F | F | T | T |

The first row is the only row where the result is false. Hence, if there is a customer who buys from the dealer and pays too much, then this implicational statement is false. In every other case, the implicational statement is true. □

If the car dealer's ad contained this inverse statement, there would be a good chance that it would be false, and they could be sued for false advertising. The original ad is carefully phrased to avoid false advertising, counting on you the consumer to misinterpret what it says.

**Practice Problem 1** *Create the truth tables for each of the following, and relate them to the truth tables for the original claim and the inverse of the claim:*

*(a)* *The converse of the advertiser's claim: "If you did pay too much, you didn't buy from us."*

*(b)* *The contrapositive of the advertiser's claim: "If you didn't pay too much, you bought from us."*

## More on the Language of Implication

One obstacle to analyzing the truth of English statements is that English as a language is somewhat complicated and often confusing. One could focus a large part of an entire course on the pitfalls of ambiguity in spoken language. We will not pursue that route since mathematics rarely suffers from this problem. Precision in language is one of the hallmarks of mathematics. Even in mathematics, though, there are a variety of ways to state the basic logical relationships between properties, so it is worth spending some time on the most common ways to express the basic implication relationship.

**Proposition 1** *Let $p$ and $q$ stand for statements. The implication "if $p$ is true, then $q$ is true" can be expressed in each of the following ways:*

1. ***If** $p$ is true, then $q$ is true.*
2. *$p$ is true **only if** $q$ is true.*
3. *For $q$ to be true, it is **sufficient** that $p$ is true.*
4. *For $p$ to be true, it is **necessary** that $q$ is true.*

**Example 4** *Each of the following is an equivalent way to express the basic implication "If you live in Pittsburgh, then you live in Pennsylvania:"*

- *You live in Pittsburgh only if you live in Pennsylvania.*
- *It is necessary to live in Pennsylvania in order to live in Pittsburgh.*
- *To live in Pennsylvania, it is sufficient to live in Pittsburgh.*

At least partly because of the possible rearrangements of the clauses within the sentences, implications that are phrased using *only if*, *sufficient*, or *necessary* can be more difficult to understand than those that use the word *if*. Here are some hints that might prove helpful:

- If the order of the clauses is confusing to you, then before doing anything else, rewrite the statement ordered the way with which you are more comfortable.
- A statement that uses "only if" is the converse of the same statement with "if" in its place.

- If a statement uses "sufficient," the condition that is sufficient is the *hypothesis* of the corresponding "if, then" statement.
- If a statement uses "necessary," the condition that is necessary is the *conclusion* of the corresponding "if, then" statement. Hence, a statement that uses "necessary" is the converse of the same statement with "sufficient" in its place.

**Practice Problem 2** *Rewrite each of the following sentences in "if, then" form:*

*(a) You will pass the test only if you study for at least four hours.*
*(b) Attending class regularly is a necessary condition for passing the course.*
*(c) In order to be a square, it is sufficient that the quadrilateral have four equal angles.*
*(d) In order to be a square, it is necessary that the quadrilateral have four equal angles.*
*(e) An integer is an odd prime only if it is greater than 2.*

## Valid and Invalid Forms of Reasoning

Early in life, children develop reasoning skills. Some of these skills are so fundamental that they seem almost trivial. For example, suppose you tell a child, "Your mother has gone to work, and so has your father." If you later ask, "Where is your mother?" the child will likely state that she is at work. The child has used a form of reasoning that we might write as

Statements $p$ and $q$ are both true.
Therefore, I can conclude that $p$ is true,

where $p$ is the proposition "Mother is at work," and $q$ is the proposition "Father is at work." We can write forms of reasoning like this using two different brief representations:

$$\frac{p \wedge q}{\therefore\ p} \qquad \text{or} \qquad p \wedge q, \therefore\ p$$

Similarly, if you tell the child, "Either you must eat your peas or you must eat your carrots," the child will instinctively eat the carrots and announce, "I'm all done." The child has used the form of reasoning represented as follows:

$$\frac{q}{\therefore\ p \vee q} \qquad \text{or} \qquad q, \therefore\ p \vee q$$

where $p$ indicates "Peas have been eaten," and $q$ indicates "Carrots have been eaten." In both vertical and horizontal representation formats, we list one or more statements that are presumed to be true, and then we list a conclusion that can be inferred from these premises. Here is another simple example of a valid form of reasoning consisting of two premises followed by the conclusion:

$$\begin{array}{c} p \\ q \\ \hline \therefore\ p \wedge q \end{array} \qquad \text{or} \qquad p, q, \therefore\ p \wedge q$$

This example points out the rather obvious fact that if you know $p$ is true, and you also know $q$ is true, it is valid to conclude that the statement "$p \wedge q$" is also true.

In future examples and exercises, we will explore a variety of other valid forms of reasoning, some trivial and some not so trivial. For now, we will concentrate our attention on some important forms of reasoning related to *implicational statements*.

**Example 5** *A parent says, "If you clean your room, we will go play miniature golf." The child cleans her room, and announces, "Let's go golfing." Give a formal description of the reasoning process used by the child.*

SOLUTION We use $p$ for "You clean your room," and $q$ for "We play miniature golf." We can write the given argument structure in vertical or horizontal format as follows:

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore q \end{array} \qquad \text{or} \qquad p \to q,\ p,\ \therefore q$$

□

In this example, the child has applied the form of reasoning known by the Latin name **modus ponens**, which means "in a manner that asserts." In words, it simply states that if an implication is true, and its hypothesis is also true, then its conclusion must be true.

This direct method of reasoning is related to another valid form of reasoning known as **modus tollens**, which means "in a manner that negates." Simply stated, if an implication statement is true, and its conclusion is false, then its hypothesis must be false. This is represented symbolically as follows:

$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad\quad} \\ \therefore \neg p \end{array} \qquad \text{or} \qquad p \to q, \neg q, \therefore \neg p$$

**Example 6** *Here is an example of a valid argument using modus tollens:*

*If this quadrilateral is a square, then it has four equal sides.*
*This quadrilateral does not have four equal sides.*
*Therefore, this quadrilateral is not a square.*

Notice that with *modus tollens*, one is simply applying the reasoning process of *modus ponens* to the contrapositive statement $\neg q \to \neg p$. That is, the following methods of reasoning are the same because they only differ in the first line, and those lines represent equivalent implicational statements:

$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad\qquad\qquad\qquad\quad} \\ \therefore \neg p \quad \text{by } \textit{modus tollens} \end{array}, \qquad \begin{array}{l} \neg q \to \neg p \\ \underline{\neg q \qquad\qquad\qquad\qquad\quad} \\ \therefore \neg p \quad \text{by } \textit{modus ponens} \end{array}$$

Of course, not every form of reasoning applied by a person in an argument is a valid form of reasoning. Before we discuss how we tell a valid form from an invalid form of reasoning, we will look at two common erroneous methods of argument. These and other invalid reasoning forms are referred to as **fallacies**. The particular names of the two fallacies below stem from the tendency to confuse an implication with its converse or inverse, as we discussed earlier in this section.

### Definition

1. The **converse fallacy** is the invalid reasoning form given by $p \to q, q, \therefore p$.
2. The **inverse fallacy** is the invalid reasoning form given by $p \to q, \neg p, \therefore \neg q$.

**Example 7** *For each of the following, identify the reasoning as either valid (modus ponens), valid (modus tollens), invalid (converse fallacy), or invalid (inverse fallacy):*

1. *If $n$ is even, then $n^2$ is even. I know $n^2$ is even. Therefore, $n$ is even.*
2. *If this triangle has three equal sides, then it has three equal angles. This triangle has three equal sides. Therefore, it has three equal angles.*
3. *If you eat your supper, we will play miniature golf. You didn't eat your supper. Therefore, we will not play miniature golf.*
4. *If I get an A on the final exam, I get an A for the course. I didn't get an A for the course. Therefore, I didn't get an A on the final.*
5. *If I don't get an A on the final exam, I won't get an A for the course. I got an A for the course. Therefore, I got an A on the final.*

SOLUTION

1. Let $p$ stand for "$n$ is even" and $q$ for "$n^2$ is even." The reasoning follows the form $p \to q, q, \therefore p$. This is the converse fallacy.
2. Let $p$ be "this triangle has three equal sides" and $q$ "this triangle has three equal angles." We write the reasoning as $p \to q$, $p, \therefore q$, which is valid by *modus ponens*.
3. With $p$ as "eat supper" and $q$ as "miniature golf," this is $p \to q, \neg p, \therefore \neg q$, the inverse fallacy.
4. If $p$ indicates "I get an A on the final exam" and $q$ "I get an A for the course," we have $p \to q, \neg q, \therefore \neg p$, which is valid by *modus tollens*.
5. We can choose our symbols to make the implication read as $p \to q$, so $p$ would be "I don't get an A on the final exam" and $q$ would be "I don't get an A for the course." The second sentence, "I got an A for the course," is the negation of $q$, that is, $\neg q$. Likewise, the third sentence, "I got an A on the final," is $\neg p$. The reasoning is $p \to q, \neg q, \therefore \neg p$, valid by *modus tollens*.

□

**Practice Problem 3** *For each of the following, identify the reasoning as either valid (modus ponens), valid (modus tollens), invalid (converse fallacy), or invalid (inverse fallacy):*

(a) *If you don't eat your supper, we won't play miniature golf. You ate your supper. Therefore, we will play miniature golf.*

(b) *If you fail the final, you will fail the course. You passed the final. Therefore, you will pass the course.*

(c) *If you fail the final, you will fail the course. You failed the final. Therefore, you will fail the course.*

(d) *If the sky is red at morning, it will not rain. It rained. Therefore, the sky was not red at morning.*

## Analysis of Arguments

In order to use logic effectively in discourse, we need to see how to analyze the validity of persuasive arguments. We have already looked at some specific types of arguments that use *modus ponens* and *modus tollens*, and we have also learned to detect invalid arguments caused by the converse and inverse fallacies. At this point we have all the tools to analyze more complex arguments.

Recall from Section 1.3 that an propositional logic expression is a **tautology** if its truth value is true for all possible combinations of the truth values of the individual components. With this piece of terminology, we can give a formal definition to a **valid argument structure**.

**Definition** An *argument structure* is a list of propositions

$$p_1, p_2, \ldots, p_n, \therefore q$$

where statements $p_1, p_2, \ldots, p_n$ are called the *premises* of the argument and the statement $q$ is called the *conclusion* of the argument. Such an argument structure is said to be *valid* if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

is a tautology, and it is said to be *invalid* otherwise.

To be picky, we should always use the phrase "valid argument structure" rather than "valid argument." Technically, a valid argument is an argument (with actual content) that follows a valid argument structure. For example, $p \rightarrow q, p, \therefore q$ is an argument structure. An argument that follows this structure is "If it is raining, the streets are wet. It is raining. Therefore, the streets are wet." However, many people use the two terms interchangeably.

### Example 8

1. *Show that the modus ponens argument structure is valid.*
2. *Decide whether the following argument is valid. If not, give an assignment of truth values that makes all premises true and the conclusion false.*

   *You must pass the final exam or pass all the tests if you pass the course. You failed the final exam and the course. Therefore, you must not have passed all the tests.*

SOLUTION

1. Recall that *modus ponens* is formalized as $p \rightarrow q, p, \therefore q$. We build a truth table for the compound expression $((p \rightarrow q) \wedge p) \rightarrow q$:

| $p$ | $q$ | $p \to q$ | $(p \to q) \wedge p$ | $q$ | $((p \to q) \wedge p) \to q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | F | T |
| F | T | T | F | T | T |
| F | F | T | F | F | T |

Since the final result has all $T$ truth values, the expression $((p \to q) \wedge p) \to q$ is a tautology, so the argument structure is valid.

2. Let $e$ stand for passing the final exam, $a$ for passing all the tests, and $p$ for passing the course. Do not be fooled by the placement of the word "if" in the middle of the statement. This means that $p$ is the hypothesis of this statement, so we can write this statement as $p \to (e \vee a)$.

So we can write the argument as $p \to (e \vee a), \neg e \wedge \neg p, \therefore \neg a$. The truth table for this statement is shown below. The "final result" column is for the expression $(p \to (e \vee a)) \wedge (\neg e \wedge \neg p)) \to \neg a$.

| $e$ | $a$ | $p$ | $e \vee a$ | $p \to (e \vee a)$ | $\neg e$ | $\neg p$ | $\neg e \wedge \neg p$ | $(p \to (e \vee a)) \wedge (\neg e \wedge \neg p)$ | $\neg a$ | Final Result |
|---|---|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | F | F | F | T |
| T | T | F | T | T | F | T | F | F | F | T |
| T | F | T | T | T | F | F | F | F | T | T |
| T | F | F | T | T | F | T | F | F | T | T |
| F | T | T | T | T | T | F | F | F | F | T |
| F | T | F | T | T | T | T | T | T | F | F |
| F | F | T | F | F | T | F | F | F | T | T |
| F | F | F | F | T | T | T | T | T | T | T |

Since there is an F in the final column, the reasoning is faulty. A person who fails the final, passes all the tests, but fails the course demonstrates the fallacy. □

## Practice Problem 4

*(a)* *Show that modus tollens is a valid argument structure.*

*(b)* *The argument structure $p \vee q, p \to r, q \to r, \therefore r$ is called "division into cases." We know one of two things is true, and we prove that no matter which is true, the conclusion $r$ follows. Show that this argument structure is valid.*

## Solutions to Practice Problems

1 We use $p$ to stand for "You bought a car from us" and $q$ to stand for "You paid too much."

(a) The converse statement is written formally as $q \to \neg p$. Here is the truth table. The final result is the same as for the inverse.

| $p$ | $q$ | $\neg p$ | $q \to \neg p$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | T |
| F | T | T | T |
| F | F | T | T |

(b) The contrapositive statement is written formally as $\neg q \to p$. Here is the truth table. The final result is the same as for the original statement.

| $p$ | $q$ | $\neg q$ | $\neg q \to p$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | T |
| F | F | T | F |

2 (a) If you pass the test, then you studied for at least four hours.

(b) If you pass the course, then you attended class regularly.

(c) If a quadrilateral has four equal angles, then it is a square.

(d) If a quadrilateral is a square, then it has four equal angles.

(e) If an integer is an odd prime, then it is greater than 2.

3 (a) Inverse fallacy

(b) Inverse fallacy

(c) Valid by *modus ponens*

(d) Valid by *modus tollens*

4 (a) *Modus tollens* is formalized as $p \to q, \neg q, \therefore \neg p$. See the first truth table below, for the compound expression $((p \to q) \wedge \neg q) \to \neg p$. The final column entries are all true, so the argument structure is valid.

(b) The second truth table shows $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \to r$):

| $p$ | $q$ | $p \to q$ | $\neg q$ | $(p \to q) \wedge \neg q$ | $\neg p$ | $((p \to q) \wedge \neg q) \to \neg p$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | T | T | T |

| $p$ | $q$ | $r$ | $p \vee q$ | $p \to r$ | $q \to r$ | $(p \vee q) \wedge (p \to r)$ $\wedge (q \to r)$ | $r$ | $((p \vee q) \wedge (p \to r)$ $\wedge (q \to r)) \to r)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | F | T |
| T | F | T | T | T | T | T | T | T |
| T | F | F | T | F | T | F | F | T |
| F | T | T | T | T | T | T | T | T |
| F | T | F | T | T | F | F | F | T |
| F | F | T | F | T | T | F | T | T |
| F | F | F | F | T | T | F | F | T |

## Exercises for Section 1.6

1. A parent says, "If you don't eat your supper, you can't have dessert." The child eats his supper and the parent says, "No dessert for you." If the child takes his parent to court, does he have a valid case?

2. Each argument is either correct or it has a fallacy. Write the argument in symbols, then determine whether the argument is valid. If it is valid, determine whether it uses *modus ponens* or *modus tollens*. If it isn't valid, identify the fallacy as either the converse or the inverse fallacy.

**(a)** If both numbers are even, then the sum is even. They are not both even. Therefore, the sum is not even.

**(b)** If this number is a perfect square, then the equation has a rational solution. The equation has a rational solution. Therefore, this number is a perfect square.

**(c)** If you didn't buy from us, you paid too much. You did buy from us. Therefore, you didn't pay too much.

**(d)** If this prime number is even, then it is less than 5. This prime number is even. Therefore, it is less than 5.

**(e)** If this city is large, then it has large buildings. This city has large buildings. Therefore, it is large.

**3.** For each of the following, identify the reasoning as either valid (*modus ponens*), valid (*modus tollens*), invalid (converse fallacy), or invalid (inverse fallacy).

**(a)** If you pass the test, you studied for at least four hours. You studied for at least four hours. Therefore, you will pass the test.

**(b)** If you pass the course, you attend class regularly. You did not attend class regularly. Therefore, you will not pass the course.

**(c)** The quadrilateral is a square if it has four equal angles. The quadrilateral has four equal angles. Therefore, it is a square.

**4.** Recall that you can phrase "if, then" statements with words like "necessary," "sufficient" and "only if." Following the model from the text, convert the following statements to "if, then " statements.

**(a)** In order to pass this course, it is sufficient to read the book.

**(b)** In order to pass this course, it is necessary that you read the book.

**(c)** You will pass this course only if you read the book.

**5.** Write each of the following as an implication in "if, then" form:

**(a)** In order for you to get a refund from the store, it is necessary that you have your sales receipt.

**(b)** To grasp someone's mathematics background, it is sufficient to have them pronounce the name "Euler."

**(c)** You are legally driving in Pennsylvania only if you are 16 years old.

**(d)** To understand the history of calculus, it is necessary to study Descartes.

**(e)** It is sufficient to carry an umbrella to stay dry outside.

**(f)** Stephen King is fun to read if you like horror stories.

**6.** Use truth tables to characterize each of the following propositions as a tautology, a contradiction, or neither:

**(a)** $p \to \neg p$

**(b)** $(p \to q) \vee (q \to p)$

**(c)** $(p \wedge q) \vee (q \to \neg p)$

**(d)** $(p \vee \neg q) \to (q \wedge \neg p)$

**7.** Use truth tables to characterize each of the following propositions as a tautology, a contradiction, or neither:

**(a)** $(p \to (q \wedge r)) \vee ((p \wedge q) \to r)$

**(b)** $((p \to q) \wedge (q \to \neg p)) \to \neg p$

**(c)** $((p \to q) \wedge (\neg p \to r)) \to (q \vee r)$

**8.** Some of the valid reasoning forms are so obvious that they come automatically to us. For example, if we know two things are true, then we certainly know that the first is true. We can formalize this as $p \wedge q, \therefore\ p$. Use a truth table to formally demonstrate the validity of this and the other "obvious" forms of reasoning. For each, give a concrete example of this form of reasoning in daily life.

**(a)** $p \wedge q, \therefore\ p$

**(b)** $p, \therefore\ p \vee q$

**(c)** $p, q, \therefore\ p \wedge q$

**(d)** $p \vee q, \neg p, \therefore\ q$

**(e)** $p \to q, q \to r, \therefore\ p \to r$

**9.** Use a truth table to decide whether each of the following argument structures is valid. If it is not, give an assignment of truth values to the propositional variables that makes each premise true and the conclusion false.

**(a)** $p \to q, q \to p, \therefore\ p \wedge q$

**(b)** $p \to (\neg q \wedge r), q, \therefore\ \neg p$

**(c)** $p \to q, r \to p, \therefore\ r \to q$

**(d)** $p \vee (q \wedge r), p \to r, \therefore\ \neg q \to r$

**(e)** $p \wedge (q \vee r), r \to \neg p, \therefore\ r \to q$

**10.** Use truth tables to show that each of the following arguments is invalid. Write one sentence explaining a situation (corresponding to an assignment of truth values) that illustrates this.

**(a)** If Newton is not considered a great mathematician and Leibniz's work is not ignored, then calculus would not be the centerpiece of the modern math curriculum. Newton is considered the greatest mathematician only if Leibniz's work is ignored. Therefore, calculus is the centerpiece of the modern math curriculum and Leibniz's work is not ignored.

**(b)** If the weather forecast is good, then we have a picnic if and only if we have bread for sandwiches. If we have a picnic, then the weather forecast is good. Therefore, we have bread for sandwiches. (NOTE: The statement "$p$ if and only if $q$" can be expressed symbolically as $(p \to q) \wedge (q \to p)$.)

**(c)** If I have a good round of golf, then the wind is calm or the weather is dry. The wind is calm and the weather is dry. Therefore, I have a good round of golf.

**(d)** Either I read a novel or I both lie on the couch and watch a baseball game on TV. I read a novel only if I lie on the couch. Therefore, if I do not watch a baseball game, then I do not lie on the couch.

**11.** Use a truth table to show that each of the following arguments is valid:

(a) For our camping trip, we take extra blankets if we take our gas heater, and if we do not take extra blankets, we do not take our air mattress. Therefore, we take our air mattress or gas heater only if we take extra blankets.

**(b)** Steve votes for a Libertarian candidate if and only if both his wife (Stella) votes for a Democrat and his father (Stan) votes for a Republican. For Stella to vote for a Democrat, it is necessary that Stan not vote for a Republican. Therefore, Steve does not vote Libertarian.

**(c)** If the prime interest rate goes up, then it is sufficient that unemployment goes down for prices to rise. However, unemployment goes down only if the prime interest rate goes up. Therefore, if prices do not go up, then unemployment does not go down.

**(d)** Each summer either I visit my family or I take a car trip and take off some time from work. I visit my family only if I do not take off time from work. Therefore, if I take off time from work, then I take a car trip.

# Chapter 1 Summary

## 1.1 First Examples

You should have tried one or more of the games and puzzles in this section:

- Perhaps you have tried the card trick, using different sets of actions by the person following the instructions, to see if the club is always the card facing the opposite way from the others.
- Perhaps you have played the Josephus game with different numbers of people, and different rules for how many people are "skipped" at each step, looking for patterns in the results.
- Perhaps you have looked for organized ways to represent the outcomes of tennis matches, individual tennis games, and other similar situations.
- Perhaps you have tried to find a pattern for what kind of figures can be traced without lifting the pencil from the paper and without retracing any lines.
- Perhaps you have looked for winning strategies in the game played on the $4 \times 4$ grid.

## 1.2 Number Puzzles and Sequences

### Terms and concepts

- You should recognize the use of the term *sequence* to describe a list of numbers, and feel comfortable with the notation $a_n$ to indicate the $n^{\text{th}}$ term of the sequence.
- You should be able to distinguish between a *recursive formula* and a *closed formula* for a sequence.
- You should recognize the *Fibonacci numbers* and their recursive formula.
- You should feel comfortable with *sigma notation*. In particular, you should be able to interpret $\sum_{k=1}^{n} a_k$ as the sum of the first $n$ terms of a sequence.

### Working with sequences

- Given a recursive formula for a sequence, you should be able to:
  - Calculate the first several terms of the sequence.
  - Give recursive formulas for specific terms in the sequence (for example, the $20^{\text{th}}$ term, or the $(3k-1)^{\text{st}}$ term).
  - Calculate particular terms of the sequence, given information about earlier terms in the sequence.
- Given a closed formula for a sequence, you should be able to:
  - Calculate the first several terms of the sequence.
  - Give values or formulas for specific terms in the sequence (for example, the $20^{\text{th}}$ term, or the $(3k-1)^{\text{st}}$ term).
- Given the first few terms of a sequence, you should be able to:
  - Discover a pattern in those terms.
  - Apply that pattern to calculate additional terms.
  - Apply that pattern to discover recursive and/or closed formulas for the sequence.
- You should know how to verify that a sequence satisfies a given recursive formula.
- You should be able to convert from sigma notation to the usual summation notation, and back.

## 1.3 Truth-Tellers, Liars, and Propositional Logic

### Terms and concepts

- You should be familiar with *propositions* and the related terms *propositional variable* and *formal proposition*.
- You should understand the concept of the *negation* of a proposition.
- You should understand the meaning of the operations $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*not*).
- You should know the terminology "*exclusive or*" and how it differs from the $\vee$ (*or*) operation.
- You should know what it means to say that two statements are *logically equivalent*.
- You should understand the terms *tautology* and *contradiction*.

### Operations with propositions

- You should be able to create new propositions from existing ones using the operations $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*not*).
- You should be able to interpret and formalize English-language constructions such as "neither-nor" and "exactly one."
- You should be able to form the negation of a proposition using the double negative property and DeMorgan's laws.
- You should be able to apply truth tables to:
  - Analyze the statements made by Smullyan's *Truth-tellers* and *Liars*.
  - Analyze the truth value of compound statements based on the truth value of their components.
  - Determine whether two statements are logically equivalent.
- You should be able to use Theorem 2 to:
  - Simplify compound statements.
  - Verify that two statements are equivalent.

## 1.4 Predicates

### Terms and concepts

- You should be familiar with *predicates* and the notation $P(x)$.
- You should understand the concept of *domain* of a predicate, and the related terms *set*, *element* and *member*, as well as the notation $\in$.
- You should know the purpose of *quantifers* and understand the notation $\forall$ (*for all*) and $\exists$ (*there exists*).
- You should know the circumstances in which a quantified statement is true, and in particular the meaning of the word *counterexample*.

### Operations with predicates

- You should be able to create new predicates from existing ones using the operations $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*not*).
- You should be able to form the negation of a predicate using the double negative property and DeMorgan's laws.
- Given a predicate over a specific domain, you should be able to determine the truth value of $P(x)$ for particular elements of the domain.
- Given a quantified statement over a specific domain, you should be able to determine the truth value of the statement.
- You should be able to *negate a quantified statement* both symbolically and in natural language:
  - The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$.
  - The negation of $\exists x \in D, Q(x)$ is $\forall x \in D, \neg Q(x)$.
- With *multiple quantifiers*, you should know the difference between $\forall x, \exists y,\ P(x, y)$ and $\exists y, \forall x,\ P(x, y)$.

## 1.5 Implications

### Terms and concepts

- You should recognize *implications*, whether written as "if $p$ is true, then $q$ is true," as "$p$ *implies* $q$," or using the notation $p \rightarrow q$.
- You should be able to identify the *hypothesis* and the *conclusion* of an implication.
- You should understand the meaning of the word *counterexample* as applied to implications.
- You should know the meaning of the terms *converse*, *inverse*, and *contrapositive*.
- You should know the term *biconditional* and its notation $p \leftrightarrow q$.

### Interpreting implications

- Given an implication involving predicates, you should be able to identify the domain, the hypothesis, and the conclusion, writing the implication in the form "For all $x \in D$, if $P(x)$, then $Q(x)$."
- You should understand the logic of the implication $p \rightarrow q$.

- You should be able to give a truth table for $p \to q$.
- You should know that the only time $p \to q$ is false is when the hypothesis (statement $p$) is true but the conclusion (statement $q$) is false.

- For an implication in the form, "For all $x \in D$, if $P(x)$, then $Q(x)$," you should be able to determine if you think the implication is true, and be able to give a counterexample if it is not true.
- You should be able to interpret a variety of English-language constructions that explicitly or implicitly involve quantified implications:
  - Convert such sentences to formal notation.
  - Recognize which sentences have the same meaning as each other, and which do not.

### Operations with implications

- You should be able to negate implications, including quantified implications.
- You should be able to form the converse, inverse, and contrapositive of an implication.
- You should know that the original implication and its contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.
- You should be able to give examples illustrating that a statement and its converse may not have the same truth value.
- You should be able to apply the logic of implications to logic puzzles (Smullyan's truth-tellers and liars).

## 1.6 Excursion: Validity of Arguments

### Terms and concepts

- You should understand the use of the phrases "*if*," "*only if*," "*necessary*," and "*sufficient*," in English-language implications.
- You should know that an *argument structure* has the general form:

$$p_1, p_2, \ldots, p_n, \therefore q$$

  and recognize the *premises* of the argument $(p_1, p_2, \ldots, p_n)$ and the *conclusion* of the argument $(q)$.
- You should recognize *modus ponens* and *modus tollens* as two valid argument structures.
- You should recognize the *converse fallacy* and the *inverse fallacy* as two invalid argument structures.

### Interpreting arguments

- You should be able to distinguish among a statement, its converse, and its inverse, and apply this to discourse of various types.
- You should know how to analyze what a person has actually said, avoiding the inappropriate substitution of a converse or inverse for the person's statement.
- You should be able to interpret English-language implications stated in a variety of ways (if, only if, necessary, sufficient), converting each to standard "if $p$, then $q$" form.
- You should be able to use a truth table to determine whether or not a given argument structure is valid.
- You should be able to analyze English-language arguments using a two-step process:
  - Convert the argument to a formal argument structure.
  - Analyze the validity of that structure. In many situations, you may recognize the structure as:
    * One of the valid argument forms *modus ponens*, *modus tollens*; or
    * One of the invalid argument forms *converse fallacy*, *inverse fallacy*.