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Photons and the Electromagnetic Field

1.1 Particles and Fields

The concept of photons as the quanta of the electromagnetic field dates back to the beginning of the twentieth century. In order to explain the spectrum of black-body radiation, Planck, in 1900, postulated that the process of emission and absorption of radiation by atoms occurs discontinuously in quanta. Einstein, by 1905, had arrived at a more drastic interpretation. From a statistical analysis of the Planck radiation law and from the energetics of the photoelectric effect, he concluded that it was not merely the atomic mechanism of emission and absorption of radiation which is quantized, but that electromagnetic radiation itself consists of photons. The Compton effect confirmed this interpretation.

The foundations of a systematic quantum theory of fields were laid by Dirac in 1927 in his famous paper on 'The Quantum Theory of the Emission and Absorption of Radiation'. From the quantization of the electromagnetic field one is naturally led to the quantization of any classical field, the quanta of the field being particles with well-defined properties. The interactions between these particles are brought about by other fields whose quanta are other particles. For example, we can think of the interaction between electrically charged particles, such as electrons and positrons, as being brought about by the electromagnetic field or as due to an exchange of photons. The electrons and positrons themselves can be thought of as the quanta of an electron–positron field. An important reason for quantizing such particle fields is to allow for the possibility that the number of particles changes as, for example, in the creation or annihilation of electron–positron pairs.

These and other processes of course only occur through the interactions of fields. The solution of the equations of the quantized interacting fields is extremely difficult. If the interaction is sufficiently weak, one can employ perturbation theory. This has been outstandingly successful in quantum electrodynamics, where complete agreement exists between theory and experiment to an incredibly high degree of accuracy. Perturbation

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theory has also very successfully been applied to weak interactions, and to strong interactions at short distances, where they become relatively weak.

The most important modern perturbation-theoretic technique employs Feynman diagrams, which are also extremely useful in many areas other than relativistic quantum field theory. We shall later develop the Feynman diagram technique and apply it to electromagnetic, weak and strong interactions. For this a Lorentz-covariant formulation will be essential.

In this introductory chapter we employ a simpler non-covariant approach, which suffices for many applications and brings out many of the ideas of field quantization. We shall consider the important case of electrodynamics for which a complete classical theory – Maxwell's – exists. As quantum electrodynamics will be re-derived later, we shall in this chapter, at times, rely on plausibility arguments rather than fully justify all steps.

1.2 The Electromagnetic Field in the Absence of Charges

1.2.1 The classical field

Classical electromagnetic theory is summed up in Maxwell's equations. In the presence of a charge density $\rho(\mathbf{x}, t)$ and a current density $\mathbf{j}(\mathbf{x}, \mathrm{t})$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satisfy the equations

$$\boldsymbol{\nabla} \bullet \mathbf{E} = \rho \tag{1.1a}$$

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \frac{1}{c}\,\mathbf{j} + \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t} \tag{1.1b}$$

$$\boldsymbol{\nabla} \bullet \mathbf{B} = 0 \tag{1.1c}$$

$$\boldsymbol{\nabla} \wedge \mathbf{E} = -\frac{1}{c}\,\frac{\partial \mathbf{B}}{\partial t} \tag{1.1d}$$

where, as throughout this book, rationalized Gaussian (c.g.s.) units are being used.[1]

From the second pair of Maxwell's equations [Eqs. (1.1c) and (1.1d)] follows the existence of scalar and vector potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, defined by

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}, \quad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t}. \tag{1.2}$$

Eqs. (1.2) do not determine the potentials uniquely, since for an arbitrary function $f(\mathbf{x}, t)$ the transformation

$$\phi \rightarrow \phi' = \phi + \frac{1}{c}\,\frac{\partial f}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla} f \tag{1.3}$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unaltered. The transformation (1.3) is known as a gauge transformation of the second kind. Since all observable quantities can be expressed in

[1] They are also called rationalized Lorentz–Heaviside units. In these units, the fine structure constant is given by $\alpha = e^2/(4\pi\hbar c) \approx 1/137$, whereas in unrationalized gaussian units $\alpha = e^2_{\mathrm{unrat}}/\hbar c$, i.e. $e = e_{\mathrm{unrat}}\sqrt{(4\pi)}$. Correspondingly for the fields $\mathbf{E} = \mathbf{E}_{\mathrm{unrat}}/\sqrt{(4\pi)}$, etc.

terms of **E** and **B**, it is a fundamental requirement of any theory formulated in terms of potentials that it is gauge-invariant, i.e. that the predictions for observable quantities are invariant under such gauge transformations.

Expressed in terms of the potentials, the second pair of Maxwell's equations [Eqs. (1.1c) and (1.1d)] are satisfied automatically, while the first pair [Eqs. (1.1a) and (1.1b)] become

$$-\boldsymbol{\nabla}^2\,\phi - \frac{1}{c}\,\frac{\partial}{\partial t}\,(\boldsymbol{\nabla}\bullet\,\mathbf{A}) = \Box\phi\; - \frac{1}{c}\,\frac{\partial}{\partial t}\left(\frac{1}{c}\,\frac{\partial\phi}{\partial t}\; + \boldsymbol{\nabla}\bullet\,\mathbf{A}\right) = \rho \tag{1.4a}$$

$$\Box\mathbf{A}\; + \boldsymbol{\nabla}\left(\frac{1}{c}\,\frac{\partial\phi}{\partial t}\; + \boldsymbol{\nabla}\bullet\,\mathbf{A}\right) = \frac{1}{c}\,\mathbf{j} \tag{1.4b}$$

where

$$\Box\;\equiv\;\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\;-\;\boldsymbol{\nabla}^2. \tag{1.5}$$

We now go on to consider the case of the free field, i.e. the absence of charges and currents: $\rho=0, \mathbf{j}=0$. We can then choose a gauge for the potentials such that

$$\boldsymbol{\nabla}\bullet\,\mathbf{A} = 0. \tag{1.6}$$

The condition (1.6) defines the Coulomb or radiation gauge. A vector field with vanishing divergence, i.e. satisfying Eq. (1.6), is called a transverse field, since for a wave

$$\mathbf{A}(\mathbf{x},\,t) = \mathbf{A}_0\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}\,-\,\omega t)}$$

Eq. (1.6) gives

$$\mathbf{k}\bullet\,\mathbf{A} = 0, \tag{1.7}$$

i.e. **A** is perpendicular to the direction of propagation **k** of the wave. In the Coulomb gauge, the vector potential is a transverse vector. In this chapter we shall be employing the Coulomb gauge.

In the absence of charges, Eq. (1.4a) now becomes $\boldsymbol{\nabla}^2\phi = 0$ with the solution, which vanishes at infinity, $\phi \equiv 0$. Hence Eq. (1.4b) reduces to the wave equation

$$\Box\mathbf{A} = 0. \tag{1.8}$$

The corresponding electric and magnetic fields are, from Eqs. (1.2), given by

$$\mathbf{B} = \boldsymbol{\nabla}\wedge\,\mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\,\frac{\partial\mathbf{A}}{\partial t}, \tag{1.9}$$

and, like **A**, are transverse fields. The solutions of Eq. (1.8) are the transverse electromagnetic waves in free space. These waves are often called the radiation field. Its energy is given by

$$H_{\mathrm{rad}} = \frac{1}{2}\,\int\!\left(\mathbf{E}^2\;+\;\mathbf{B}^2\right)\mathrm{d}^3\mathbf{x}. \tag{1.10}$$

In order to quantize the theory, we shall want to introduce canonically conjugate coordinates (like x and p_x in non-relativistic quantum mechanics) for each degree of

freedom and subject these to commutation relations. At a given instant of time t, the vector potential $\mathbf{A}$ must be specified at every point $\mathbf{x}$ in space. Looked at from this viewpoint, the electromagnetic field possesses a continuous infinity of degrees of freedom. The problem can be simplified by considering the radiation inside a large cubic enclosure, of side L and volume $V = L^3$, and imposing periodic boundary conditions on the vector potential $\mathbf{A}$ at the surfaces of the cube. The vector potential can then be represented as a Fourier series, i.e. it is specified by the denumerable set of Fourier expansion coefficients, and we have obtained a description of the field in terms of an infinite, but denumerable, number of degrees of freedom. The Fourier analysis corresponds to finding the normal modes of the radiation field, each mode being described independently of the others by a harmonic oscillator equation. (All this is analogous to the Fourier analysis of a vibrating string.) This will enable us to quantize the radiation field by taking over the quantization of the harmonic oscillator from non-relativistic quantum mechanics.

With the periodic boundary conditions

$$\mathbf{A}(0,\ y,\ z,\ t) = \mathbf{A}(L,\ y,\ z,\ t), \text{etc.}, \tag{1.11}$$

the functions

$$\frac{1}{\sqrt{V}}\ \boldsymbol{\varepsilon}_r(\mathbf{k})\ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \quad r = 1,\ 2, \tag{1.12}$$

form a complete set of transverse orthonormal vector fields. Here the wave vectors $\mathbf{k}$ must be of the form

$$\mathbf{k} = \frac{2\pi}{L}\ (n_1,\ n_2,\ n_3), \quad n_1,\ n_2,\ n_3 = 0,\ \pm 1,\ \ldots, \tag{1.13}$$

so that the fields (1.12) satisfy the periodicity conditions (1.11). $\boldsymbol{\varepsilon}_1(\mathbf{k})$ and $\boldsymbol{\varepsilon}_2(\mathbf{k})$ are two mutually perpendicular real unit vectors which are also orthogonal to $\mathbf{k}$:

$$\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_s\ (\mathbf{k}) = \delta_{rs}, \quad \boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r,\ s = 1,\ 2. \tag{1.14}$$

The last of these conditions ensures that the fields (1.12) are transverse, satisfying the Coulomb gauge condition (1.6) and (1.7).[2]

We can now expand the vector potential $\mathbf{A}(\mathbf{x}, t)$ as a Fourier series

$$\mathbf{A}(\mathbf{x},\ t) = \sum_{\mathbf{k}} \sum_{r} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) \left[a_r(\mathbf{k},\ t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + a_r^*\ (\mathbf{k},\ t)\mathrm{e}^{-\ \mathrm{i}\mathbf{k}\cdot\mathbf{x}} \right], \tag{1.15}$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}|$. The summations with respect to r and $\mathbf{k}$ are over both polarization states $r = 1, 2$ (for each $\mathbf{k}$) and over all allowed momenta $\mathbf{k}$. The factor to the left of $\boldsymbol{\varepsilon}_r(\mathbf{k})$ has been introduced for later convenience only. The form of the series (1.15) ensures that the vector potential is real: $\mathbf{A} = \mathbf{A}^*$. Eq. (1.15) is an expansion of $\mathbf{A}(\mathbf{x}, t)$ at each instant of time t. The time-dependence of the Fourier expansion coefficients follows, since $\mathbf{A}$ must satisfy the

[2] With this choice of $\boldsymbol{\varepsilon}_r(\mathbf{k})$, Eqs. (1.12) represent linearly polarized fields. By taking appropriate complex linear combinations of $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ one obtains circular or, in general, elliptic polarization.

wave equation (1.8). Substituting Eq. (1.15) in (1.8) and projecting out individual amplitudes, one obtains

$$\frac{\partial^2}{\partial t^2}\, a_r(\mathbf{k},\ t) = -\ \omega_{\mathbf{k}}^2 a_r(\mathbf{k},\ t). \tag{1.16}$$

These are the harmonic oscillator equations of the normal modes of the radiation field. It will prove convenient to take their solutions in the form

$$a_r(\mathbf{k},\ t) = a_r(\mathbf{k})\ \exp\left(-\mathrm{i}\omega_{\mathbf{k}} t\right), \tag{1.17}$$

where the $a_r(\mathbf{k})$ are initial amplitudes at time $t = 0$.

Eq. (1.15) for the vector potential, with Eq. (1.17) and its complex conjugate substituted for the amplitudes a_r and a_r^*, represents our final result for the classical theory. We can express the energy of the radiation field, Eq. (1.10), in terms of the amplitudes by substituting Eqs. (1.9) and (1.15) in (1.10) and carrying out the integration over the volume V of the enclosure. In this way one obtains

$$H_{\mathrm{rad}} = \sum_{\mathbf{k}} \sum_{r} \hbar\omega_{\mathbf{k}} a_r^*(\mathbf{k}) a_r(\mathbf{k}). \tag{1.18}$$

Note that this is independent of time, as expected in the absence of charges and currents; we could equally have written the time-dependent amplitudes (1.17) instead, since the time dependence of a_r and of a_r^* cancels.

As already stated, we shall quantize the radiation field by quantizing the individual harmonic oscillator modes. As the interpretation of the quantized field theory in terms of photons is intimately connected with the quantum treatment of the harmonic oscillator, we shall summarize the latter.

1.2.2 Harmonic oscillator

The harmonic oscillator Hamiltonian is, in an obvious notation,

$$H_{\mathrm{osc}} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2,$$

with q and p satisfying the commutation relation $[q, p] = \mathrm{i}\hbar$. We introduce the operators

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \frac{1}{(2\hbar m\omega)^{1/2}}\ (m\omega q \pm \mathrm{i}p).$$

These satisfy the commutation relation

$$\left[a,\ a^\dagger\right] = 1, \tag{1.19}$$

and the Hamiltonian expressed in terms of a and $a^\dagger$ becomes:

$$H_{\mathrm{osc}} = \frac{1}{2}\hbar\omega\left(a^\dagger a\ +\ a a^\dagger\right) = \hbar\omega\left(a^\dagger a\ +\ \frac{1}{2}\right). \tag{1.20}$$

This is essentially the operator

$$N\ \equiv\ a^\dagger a, \tag{1.21}$$

which is positive definite, i.e. for any state $|\Psi\rangle$

$$\langle\Psi|N|\Psi\rangle = \langle\Psi|a^\dagger a|\Psi\rangle = \langle a\Psi\ |\ a\Psi\rangle\ \geq 0.$$

Hence, N possesses a lowest non-negative eigenvalue

$$\alpha_0 \geq 0.$$

It follows from the eigenvalue equation

$$N|\alpha\rangle = \alpha|\alpha\rangle$$

and Eq. (1.19) that

$$Na|\alpha\rangle = (\alpha\ -\ 1)a|\alpha\rangle, \quad Na^\dagger|\alpha\rangle = (\alpha\ +\ 1)a^\dagger|\alpha\rangle, \tag{1.22}$$

i.e. $a|\alpha\rangle$ and $a^\dagger|\alpha\rangle$ are eigenfunctions of N belonging to the eigenvalues $(\alpha - 1)$ and $(\alpha + 1)$, respectively. Since α_0 is the lowest eigenvalue we must have

$$a|\alpha_0\rangle = 0, \tag{1.23}$$

and since

$$a^\dagger a|\alpha_0\rangle = \alpha_0|\alpha_0\rangle,$$

Eq. (1.23) implies $\alpha_0 = 0$. It follows from Eqs. (1.19) and (1.22) that the eigenvalues of N are the integers $n = 0, 1, 2, \ldots$, and that if $\langle n|n\rangle = 1$, then the states $|n \pm 1\rangle$, defined by

$$a|n\rangle = n^{1/2}|n\ -\ 1\rangle, \quad a^\dagger|n\rangle = (n\ +\ 1)^{1/2}|n\ +\ 1\rangle, \tag{1.24}$$

are also normed to unity. If $\langle 0|0\rangle = 1$, the normed eigenfunctions of N are

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle, \quad n = 0,\ 1,\ 2, \ldots. \tag{1.25}$$

These are also the eigenfunctions of the harmonic oscillator Hamiltonian (1.20) with the energy eigenvalues

$$E_n = \hbar\omega\left(n\ +\ \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots. \tag{1.26}$$

The operators a and $a^\dagger$ are called lowering and raising operators because of the properties (1.24). We shall see that in the quantized field theory $|n\rangle$ represents a state with n quanta. The operator a(changing $|n\rangle$ into $|n\ -\ 1\rangle$) will annihilate a quantum; similarly, $a^\dagger$, will create a quantum.

So far we have considered one instant of time, say $t = 0$. We now discuss the equations of motion in the Heisenberg picture.[3] In this picture, the operators are functions of time. In particular

$$\mathrm{i}\hbar\ \frac{\mathrm{d}a(t)}{\mathrm{d}t} = [a(t),\ H_{\mathrm{osc}}] \tag{1.27}$$

[3] See the appendix to this chapter (Section 1.5) for a concise development of the Schrödinger, Heisenberg and interaction pictures.

with the initial condition $a(0) = a$, the lowering operator considered so far. Since H_{osc} is time-independent, and $a(t)$ and $a^\dagger(t)$ satisfy the same commutation relation (1.19) as a and $a^\dagger$, the Heisenberg equation of motion (1.27) reduces to

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -\mathrm{i}\omega a(t)$$

with the solution

$$a(t) = a\,\mathrm{e}^{-\,\mathrm{i}\omega t}. \tag{1.28}$$

1.2.3 The quantized radiation field

The harmonic oscillator results we have derived can at once be applied to the radiation field. Its Hamiltonian, Eq. (1.18), is a superposition of independent harmonic oscillator Hamiltonians (1.20), one for each mode of the radiation field. [The order of the factors in (1.18) is not significant and can be changed, since the a_{r} and a_r^* are classical amplitudes.] We therefore introduce commutation relations analogous to Eq. (1.19)

$$\left.\begin{aligned} \left[a_r(\mathbf{k}),\, a_s^\dagger\left(\mathbf{k}'\right)\right] &= \delta_{rs}\,\delta_{\mathbf{k}\mathbf{k}'} \\ \left[a_r(\mathbf{k}),\, a_s\left(\mathbf{k}'\right)\right] &= \left[a_r^\dagger(\mathbf{k}),\, a_s^\dagger\left(\mathbf{k}'\right)\right] = 0 \end{aligned}\right\} \tag{1.29}$$

and write the Hamiltonian (1.18) as

$$H_{\text{rad}} = \sum_{\mathbf{k}} \sum_{r} \hbar\omega_{\mathbf{k}} \left(a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) + \frac{1}{2} \right). \tag{1.30}$$

The operators

$$N_r(\mathbf{k}) = a_r^\dagger(\mathbf{k}) a_r(\mathbf{k})$$

then have eigenvalues $n_r(\mathbf{k}) = 0, 1, 2, \ldots$, and eigenfunctions of the form (1.25)

$$|n_r(\mathbf{k})\rangle = \frac{\left[a_r^\dagger(\mathbf{k})\right]^{n_r(\mathbf{k})}}{\sqrt{n_r(\mathbf{k})!}}\,|0\rangle. \tag{1.31}$$

The eigenfunctions of the radiation Hamiltonian (1.30) are products of such states, i.e.

$$|\ldots\, n_r(\mathbf{k})\, \ldots\rangle = \prod_{\mathbf{k}_i} \prod_{\boldsymbol{r}_i} |n_{\boldsymbol{r}_i}(\mathbf{k}_i)\rangle, \tag{1.32}$$

with energy

$$\sum_{\mathbf{k}} \sum_{r} \hbar\omega_{\mathbf{k}} \left(n_r(\mathbf{k}) + \frac{1}{2} \right). \tag{1.33}$$

The interpretation of these equations is a straightforward generalization from one harmonic oscillator to a superposition of independent oscillators, one for each radiation mode ($\mathbf{k}$, r). $a_r(\mathbf{k})$ operating on the state (1.32) will reduce the *occupation number* $n_r(\mathbf{k})$

of the mode ($\mathbf{k}$, r) by unity, leaving all other occupation numbers unaltered, i.e. from Eq. (1.24):

$$a_r(\mathbf{k})|\ldots n_r(\mathbf{k})\ldots\rangle = [n_r(\mathbf{k})]^{1/2}\,|\ldots,\ n_r(\mathbf{k})-1,\ \ldots\rangle. \tag{1.34}$$

Correspondingly the energy (1.33) is reduced by $\hbar\omega_{\mathbf{k}} = \hbar c|\mathbf{k}|$. We interpret $a_r(\mathbf{k})$ as an annihilation (or destruction or absorption) operator, which annihilates one photon in the mode ($\mathbf{k}$, r), i.e. with momentum $\hbar\mathbf{k}$, energy $\hbar\omega_{\mathbf{k}}$ and linear polarization vector $\boldsymbol{\varepsilon}_r(\mathbf{k})$. Similarly, $a_r^\dagger(\mathbf{k})$ is interpreted as a creation operator of such a photon. The assertion that $a_r(\mathbf{k})$ and $a_r^\dagger(\mathbf{k})$ are absorption and creation operators of photons with momentum $\hbar\mathbf{k}$ can be justified by calculating the momentum of the radiation field. We shall see later that the momentum operator of the field is given by

$$\mathbf{P} = \sum_{\mathbf{k}}\sum_{r}\hbar\mathbf{k}\left(N_r(\mathbf{k}) + \frac{1}{2}\right), \tag{1.35}$$

which leads to the above interpretation. We shall not consider the more intricate problem of the angular momentum of the photons, but only mention that circular polarization states obtained by forming linear combinations

$$-\frac{1}{\sqrt{2}}\,[\boldsymbol{\varepsilon}_1(\mathbf{k}) + \mathrm{i}\boldsymbol{\varepsilon}_2(\mathbf{k})], \qquad \frac{1}{\sqrt{2}}\,[\boldsymbol{\varepsilon}_1(\mathbf{k}) - \mathrm{i}\boldsymbol{\varepsilon}_2(\mathbf{k})], \tag{1.36}$$

are more appropriate for this. Remembering that ($\boldsymbol{\varepsilon}_1(\mathbf{k})$, $\boldsymbol{\varepsilon}_2(\mathbf{k})$, $\mathbf{k}$) form a right-handed Cartesian coordinate system, we see that these two combinations correspond to angular momentum $\pm\hbar$ in the direction $\mathbf{k}$ (analogous to the properties of the spherical harmonics $Y_1^{\pm1}$), i.e. they represent right- and left-circular polarization: the photon behaves like a particle of spin 1. The third spin component is, of course, missing because of the transverse nature of the photon field.

The state of lowest energy of the radiation field is the vacuum state $|0\rangle$, in which all occupation numbers $n_r(\mathbf{k})$ are zero. According to Eqs. (1.30) or (1.33), this state has the energy $\frac{1}{2}\,\Sigma_{\mathbf{k}}\Sigma_r\hbar\omega_{\mathbf{k}}$. This is an infinite *constant*, which is of no physical significance: we can eliminate it altogether by shifting the zero of the energy scale to coincide with the vacuum state $|0\rangle$. This corresponds to replacing Eq. (1.30) by

$$H_{\mathrm{rad}} = \sum_{\mathbf{k}}\sum_{r}\hbar\omega_{\mathbf{k}}a_r^\dagger(\mathbf{k})\,a_r(\mathbf{k}). \tag{1.37}$$

[The 'extra' term in Eq. (1.35) for the momentum will similarly be dropped. It actually vanishes in any case due to symmetry in the $\mathbf{k}$ summation.]

The representation (1.32) in which states are specified by the occupation numbers $n_r(\mathbf{k})$ is called the *number representation.* It is of great practical importance in calculating transitions (possibly via intermediate states) between initial and final states containing definite numbers of photons with well-defined properties. These ideas are, of course, not restricted to photons, but apply generally to the particles of quantized fields. We shall have to modify the formalism in one respect. We have seen that the photon occupation numbers $n_r(\mathbf{k})$ can assume all values 0,1,2, ... Thus, photons satisfy Bose–Einstein statistics. They are *bosons.* So a modification will be required to describe particles obeying Fermi–Dirac

statistics (*fermions*), such as electrons or muons, for which the occupation numbers are restricted to the values 0 and 1.

We have quantized the electromagnetic field by replacing the classical amplitudes a_r and a_r^* in the vector potential (1.15) by operators, so that the vector potential and the electric and magnetic fields become operators. In particular, the vector potential (1.15) becomes, in the Heisenberg picture [cf. Eqs. (1.28) and (1.17)], the time-dependent operator

$$\mathbf{A}(\mathbf{x},\ t) = \mathbf{A}^{+}(\mathbf{x},\ t)\ +\ \mathbf{A}^{-}(\mathbf{x},\ t), \tag{1.38a}$$

with

$$\mathbf{A}^{+}(\mathbf{x},\ t) = \sum_{\mathbf{k}} \sum_{r} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) a_r(\mathbf{k})\ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}\ -\ \omega_{\mathbf{k}} t)}, \tag{1.38b}$$

$$\mathbf{A}^{-}(\mathbf{x},\ t) = \sum_{\mathbf{k}} \sum_{r} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) a_r^{\dagger}(\mathbf{k})\ \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}\ -\ \omega_{\mathbf{k}} t)}. \tag{1.38c}$$

The operator $\mathbf{A}^{+}$ contains only absorption operators, $\mathbf{A}^{-}$ only creation operators. $\mathbf{A}^{+}$ and $\mathbf{A}^{-}$ are called the positive and negative frequency parts of $\mathbf{A}$.[4] The operators for $\mathbf{E}(\mathbf{x},\ t)$ and $\mathbf{B}(\mathbf{x},\ t)$ follow from Eqs. (1.9). There is an important difference between a quantized field theory and non-relativistic quantum mechanics. In the former it is the amplitudes (and hence the fields) which are operators, and the position and time coordinates $(\mathbf{x},\ t)$ are ordinary numbers, whereas in the latter the position coordinates (but not the time) are operators.

Finally, we note that a state with a definite number v of photons (i.e. an eigenstate of the total photon number operator $N = \Sigma_{\mathbf{k}} \Sigma_r N_r(\mathbf{k})$) cannot be a classical field, not even for $v \to \infty$. This is a consequence of the fact that $\mathbf{E}$, like $\mathbf{A}$, is linear in the creation and absorption operators. Hence the expectation value of $\mathbf{E}$ in such a state vanishes. It is possible to form so-called coherent states $|c\rangle$ for which $\langle c|\mathbf{E}|c\rangle$ represents a transverse wave and for which the relative fluctuation $\Delta\mathbf{E}/\langle c|\mathbf{E}|c\rangle$ tends to zero as the number of photons in the state, $\langle c|N|c\rangle$, tends to infinity, i.e. in this limit the state $|c\rangle$ goes over into a classical state of a well-defined field.[5]

1.3 The Electric Dipole Interaction

In the last section we quantized the radiation field. Since the occupation number operators $a_r^{\dagger}(\mathbf{k}) a_r(\mathbf{k})$ commute with the radiation Hamiltonian (1.37), the occupation numbers $n_r(\mathbf{k})$ are constants of the motion for the free field. For anything 'to happen' requires interactions with charges and currents so that photons can be absorbed, emitted or scattered.

The complete description of the interaction of a system of charges (for example, an atom or a nucleus) with an electromagnetic field is very complicated. In this section we shall consider the simpler and, in practice, important special case of the interaction occurring via

[4] This is like in non-relativistic quantum mechanics where a time-dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ with $\omega = E/\hbar > 0$ corresponds to a positive energy, i.e. a positive frequency.

[5] For a discussion of coherent states see R. Loudon, *The Quantum Theory of Light*, Clarendon Press, Oxford, 1973, pp. 148–153. See also Problem 1.1.

the electric dipole moment of the system of charges. The more complete (but still non-covariant) treatment of Section 1.4 will justify some of the points asserted in this section.

We shall consider a system of N charges $e_1, e_2, \ldots, e_N$ which can be described non-relativistically, i.e. the position of e_i, $i = 1, \ldots, N$, at time t is classically given by $\mathbf{r}_i = \mathbf{r}_i(t)$. We consider transitions between definite initial and final states of the system (e.g. between two states of an atom). The transitions are brought about by the electric dipole interaction if two approximations are valid.

Firstly it is permissible to neglect the interactions with the magnetic field.

Secondly, one may neglect the spatial variation of the electric radiation field, causing the transitions, across the system of charges (e.g. across the atom). Under these conditions the electric field

$$\mathbf{E}_{\mathrm{T}}(\mathbf{r},\, t) = -\,\frac{1}{c}\,\frac{\partial \mathbf{A}(\mathbf{r},\, t)}{\partial t}\,, \tag{1.39}$$

resulting from the transverse vector potential (1.38) of the radiation field (we are again using the Coulomb gauge $\boldsymbol{\nabla}\bullet\mathbf{A} = 0$), can be calculated at *one* point somewhere inside the system of charges, instead of at the position of each charge.[6] Taking this point as the origin of coordinates $\mathbf{r} = 0$, we obtain for the interaction causing transitions, the electric dipole interaction H_{I} given by

$$H_{\mathrm{I}} = -\mathbf{D} \bullet \mathbf{E}_{\mathrm{T}}(0,\, t) \tag{1.40}$$

where the electric dipole moment is defined by

$$\mathbf{D} = \sum_i e_i \mathbf{r}_i. \tag{1.41}$$

Transitions brought about by the interaction (1.40) in first-order perturbation theory are called *electric dipole transitions.* Since $\mathbf{E}_{\mathrm{T}}$, like $\mathbf{A}$ [Eq. (1.38)], is linear in the photon absorption and creation operators, so is H_{I}. It follows that, in electric dipole transitions, one photon is emitted or absorbed. In the next section it will be shown that the electric dipole approximation is valid, provided the wavelength $\lambda = 2\pi/k$ of the radiation emitted or absorbed in the transition is very large compared to the linear dimensions R of the system of charges: $\lambda >> R$. For example, for optical transitions in atoms, R is of the order of 1 Å and λ lies in the range 4000–7500 Å. Similarly, for gamma rays emitted by nuclei, R is of the order of a few fermis (1 f $= 10^{-15}$ m) and since $\lambda / 2\,\pi = [197/(E \text{ in MeV})]$ f for a gamma ray of E MeV, the electric dipole approximation is valid up to quite high gamma-ray energies.

If there are selection rules forbidding a transition in the electric dipole approximation, it might still occur via the magnetic interactions or via parts of the electric interactions which are neglected in the dipole approximation. It may happen that a transition is strictly forbidden, i.e. cannot occur in first-order perturbation theory, even when the exact interaction is used as perturbation instead of H_{I} [Eq. (1.40)]. In such cases, the transition can still occur in higher orders of perturbation theory or, possibly, by some quite different mechanism.[7]

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[6] In Eq. (1.39) we have written $\mathrm{E_T}$, since we now also have the Coulomb interaction between the charges, which makes a contribution $-\boldsymbol{\nabla}\phi$ to the electric field. [See Eqs. (1.2) and (1.4a) and Section 1.4.]

[7] For selection rules for radiative transitions in atoms, see H. A. Bethe and R. W. Jackiw, *Intermediate Quantum Mechanics*, 2nd edn, Benjamin, New York, 1968, Chapter 11.

Let us now consider in some detail the emission and absorption of radiation in electric dipole transitions in atoms. The atom will make a transition from an initial state $|A\rangle$ to a final state $|B\rangle$ and the occupation number of one photon state will change from $n_r(\mathbf{k})$ to $n_r(\mathbf{k}) \pm 1$. The initial and final states of the system will be

$$\left.\begin{aligned} |A, n_r(\mathbf{k})\rangle &= |A\rangle|n_r(\mathbf{k})\rangle \\ |B, n_r(\mathbf{k})\pm 1\rangle &= |\mathrm{B}\rangle|n_r(\mathbf{k})\pm 1\rangle \end{aligned}\right\}, \tag{1.42}$$

where the occupation numbers of the photon states which are not changed in the transition are not shown. The dipole operator (1.41) now becomes:

$$\mathbf{D} = -e \sum_i \mathbf{r}_i \equiv -e\mathbf{x}, \tag{1.43}$$

where the summation is over the atomic electrons and we have introduced the abbreviation $\mathbf{x}$. The transverse electric field $\mathbf{E}_\mathrm{T}(0, t)$ which occurs in the interaction (1.40) is from Eqs. (1.38)

$$\begin{aligned} \mathrm{E}_\mathrm{T}(0, t) &= -\frac{1}{c}\frac{\partial \mathbf{A}(0, t)}{\partial t} \\ &= \mathrm{i} \sum_{\mathbf{k}} \sum_r \left(\frac{\hbar\omega_\mathbf{k}}{2V}\right)^{1/2} \boldsymbol{\varepsilon}_r(\mathbf{k}) \left[a_r(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}\omega_\mathbf{k} t} - a_r^\dagger(\mathbf{k}) \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t}\right]. \end{aligned}$$

Let us consider radiative emission. The transition matrix element of the interaction (1.40) between the states (1.42) then is given by

$$\begin{aligned} &\langle B,\ n_r(\mathbf{k}) + 1|H_\mathrm{I}|A,\ n_r(\mathbf{k})\rangle \\ &\quad = \mathrm{i}\left(\frac{\hbar\omega_\mathbf{k}}{2V}\right)^{1/2} \langle n_r(\mathbf{k}) + 1\big|a_r^\dagger(\mathbf{k})\big|n_r(\mathbf{k})\rangle \langle B|\boldsymbol{\varepsilon}_r(\mathbf{k}) \bullet \mathbf{D}|A\rangle\ \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t} \\ &\quad = \mathrm{i}\left(\frac{\hbar\omega_\mathbf{k}}{2V}\right)^{1/2} [n_r(\mathbf{k}) + 1]^{1/2}\ \langle B|\boldsymbol{\varepsilon}_r(\mathbf{k}) \bullet \mathbf{D}|A\rangle\ \mathrm{e}^{\mathrm{i}\omega_\mathbf{k} t}, \end{aligned} \tag{1.44}$$

where the last line follows from Eq. (1.24).

The transition probability per unit time between initial and final states (1.42) is given by time-dependent perturbation theory as

$$w = \frac{2\pi}{\hbar}\ |\langle B,\ n_r(\mathbf{k}) + 1|H_\mathrm{I}|A,\ n_r(\mathbf{k})\rangle|^2\ \delta(E_A - E_B - \hbar\omega_\mathbf{k}) \tag{1.45}$$

where E_A and E_B are the energies of the initial and final atomic states $|A\rangle$ and $|B\rangle$.[8] The delta function ensures conservation of energy in the transition, i.e. the emitted photon's energy $\hbar\omega_\mathbf{k}$ must satisfy the Bohr frequency condition

$$\omega_\mathbf{k} = \omega \equiv (E_A - E_B)/\hbar. \tag{1.46}$$

[8] Time-dependent perturbation theory is, for example, developed in A. S. Davydov, *Quantum Mechanics*, 2nd edn, Pergamon, Oxford, 1976, see Section 93 [Eq. (93.7)]; E. Merzbacher, *Quantum Mechanics*, 2nd edn, John Wiley & Sons, Inc., New York, 1970, see Section 18.8; L. I. Schiff, Quantum Mechanics, 3rd edn, McGraw-Hill, New York, 1968, see Section 35.

The delta function is eliminated in the usual way from Eq. (1.45) by integrating over a narrow group of final photon states. The number of photon states in the interval $(\mathbf{k}, \mathbf{k} + \mathrm{d}\mathbf{k})$, all in the same polarization state ($\boldsymbol{\varepsilon}_1(\mathbf{k})$ or $\boldsymbol{\varepsilon}_2(\mathbf{k})$), is[9]

$$\frac{V\,\mathrm{d}^3\,\mathbf{k}}{(2\pi)^3} = \frac{Vk^2\,\mathrm{d}k\,\mathrm{d}\Omega}{(2\pi)^3}. \tag{1.47}$$

From Eqs. (1.44)–(1.47) we obtain the probability per unit time for an atomic transition $|A\rangle \rightarrow |B\rangle$ with emission of a photon of wave vector in the range $(\mathbf{k}, \mathbf{k} + \mathrm{d}\mathbf{k})$ and with polarization vector $\boldsymbol{\varepsilon}_r(\mathbf{k})$:

$$\begin{aligned} w_r\,\mathrm{d}\Omega = \int \frac{Vk^2\,\mathrm{d}k\,\mathrm{d}\Omega}{(2\pi)^3}\;\frac{2\pi}{\hbar}\;\delta(E_A - E_B - \hbar\omega_{\mathbf{k}}) \\ \times\left(\frac{\hbar\omega_{\mathbf{k}}}{2\,V}\right)[n_r(\mathbf{k})+1]\,|\langle B|\boldsymbol{\varepsilon}_r(\mathbf{k})\bullet\mathbf{D}|A\rangle\,|^2. \end{aligned} \tag{1.49}$$

If we perform the integration with respect to $k\,(=\omega_{\mathbf{k}}/c)$ and substitute (1.43) for $\mathbf{D}$, the last expression reduces to

$$w_r\,\mathrm{d}\Omega = \frac{e^2\omega^3\,\mathrm{d}\Omega}{8\pi^2\hbar c^3}\;[n_r(\mathbf{k})+1]\,|\boldsymbol{\varepsilon}_r(\mathbf{k})\bullet\mathbf{x}_{BA}|^2 \tag{1.50}$$

where $\mathbf{x}_{BA}$ stands for the matrix element

$$\mathbf{x}_{BA} \equiv \langle B|\mathbf{x}|A\rangle = \langle B|\sum_i \mathbf{r}_i|A\rangle. \tag{1.51}$$

The most interesting feature of Eq. (1.50) is the occurrence of the factor $[n_r(\mathbf{k}) + 1]$. $n_r(\mathbf{k})$ is the occupation number of photons in the $(\mathbf{k}, r)$ mode present initially, and thus the part of (1.50) proportional to $n_r(\mathbf{k})$ represents induced (or stimulated) emission, i.e. radiation which results from the radiation incident on the atom; classically, we can think of it as resulting from the forced oscillations of the electrons, and this term can be produced from a semiclassical theory of radiation.[10] However, even with no radiation present initially ($n_r(\mathbf{k}) = 0$), the transition probability (1.50) is different from zero. This corresponds to the spontaneous emission of radiation from an atom, and this cannot be derived from a semiclassical theory of radiation.

Eqs. (1.50) and (1.51) represent the basic result about emission of radiation in electric dipole transitions, and we only briefly indicate some consequences.

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[9] Since we are using a finite normalization volume V, we should be summing over a group of allowed wave vectors $\mathbf{k}$ [see Eq. (1.13)]. For large V (strictly $V \rightarrow \infty$)

$$\frac{1}{V}\sum_{\mathbf{k}} \rightarrow \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}. \tag{1.48}$$

The normalization volume V must of course drop out of all physically significant quantities such as transition rates etc.

[10] See, for example, L. I. Schiff, *Quantum Mechanics*, 3rd edn, McGraw-Hill, New York, 1968, Chapter 11, or Bethe and Jackiw, referred to earlier in this section, Chapter 10.

To sum over the two polarization states for a given $\mathbf{k}$, we note that $\boldsymbol{\varepsilon}_1(\mathbf{k})$, $\boldsymbol{\varepsilon}_2(\mathbf{k})$ and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ form an orthonormal coordinate system. Hence,

$$\begin{aligned}\sum_{r=1}^{2} |\boldsymbol{\varepsilon}_r(\mathbf{k}) \bullet \mathbf{x}_{BA}|^2 &= \mathbf{x}_{BA} \bullet \mathbf{x}_{BA}^* - (\hat{\mathbf{k}} \bullet \mathbf{x}_{BA})(\hat{\mathbf{k}} \bullet \mathbf{x}_{BA}^*) \\ &= (\mathbf{x}_{BA} \bullet \mathbf{x}_{BA}^*)(1 - \cos^2\theta) \\ &= |\mathbf{x}_{BA}|^2 \sin^2\theta,\end{aligned}$$

where the last line but one defines the angle θ which the complex vector $\mathbf{x}_{BA}$ makes with $\hat{\mathbf{k}}$. Hence, from Eq. (1.50)

$$\sum_{r=1}^{2} w_r \, \mathrm{d}\Omega = \frac{e^2 \omega^3}{8\pi^2 \hbar c^3} \, \mathrm{d}\Omega [n_r(\mathbf{k}) + 1] |\mathbf{x}_{BA}|^2 \sin^2\theta. \tag{1.52}$$

For spontaneous emission, the total transition probability per unit time is obtained from the last equation, with $n_r(\mathbf{k}) = 0$, by integrating over all directions. Since

$$\int \sin^2\theta \, \mathrm{d}\Omega = \frac{8\pi}{3},$$

we obtain

$$w_{\text{total}}(A \rightarrow B) = \frac{e^2 \omega^3}{3\pi\hbar c^3} |\mathbf{x}_{BA}|^2. \tag{1.53}$$

The lifetime τ of an excited atomic state $|A\rangle$ is defined as the reciprocal of the total transition probability per unit time to *all* possible final states $|B_1\rangle$, $|B_2\rangle$, ..., i.e.

$$\frac{1}{\tau} = \sum_n w_{\text{total}}(A \rightarrow B_n). \tag{1.54}$$

In particular, if the state $|A\rangle$ can decay to states with non-zero total angular momentum, Eq. (1.54) must contain a summation over the corresponding magnetic quantum numbers.

The selection rules for electric dipole transitions follow from the matrix element (1.51). For example, since $\mathbf{x}$ is a vector, the states $|A\rangle$ and $|B\rangle$ must have opposite parity, and the total angular momentum quantum number J of the atom and its z-component M must satisfy the selection rules

$$\Delta J = 0, \pm 1, \quad \text{not } J = 0 \rightarrow J = 0, \quad \Delta M = 0, \pm 1.$$

The second selection rule (not $J = 0 \rightarrow J = 0$) applies strictly to one-photon processes, not only in the electric dipole approximation. It is a consequence of the fact that there are no one-photon states with zero angular momentum. To form such a state from the spin 1 of the photon and a unit of orbital angular momentum requires all three components of the spin angular momentum, but because of the transversality of the radiation field, only two of the spin components are available [compare Eq. (1.36)].

Finally, we note that very similar results hold for the absorption of radiation in electric dipole transitions. The matrix element

$$\langle B, n_r(\mathbf{k}) - 1 | H_{\text{I}} | A, n_r(\mathbf{k}) \rangle$$

corresponding to Eq. (1.44) now involves the factor $[n_r(\mathbf{k})]^{1/2}$ instead of $[n_r(\mathbf{k})+1]^{1/2}$. Our final result for emission, Eq. (1.50), also holds for absorption, with $[n_r(\mathbf{k})+1]$ replaced by $[n_r(\mathbf{k})]$, $d\Omega$ being the solid angle defining the incident radiation, and the matrix element $\mathbf{x}_{BA}$, Eq. (1.51), representing a transition from an atomic state $|A\rangle$ with energy E_A to a state $|B\rangle$ with energy $E_B > E_A$. Correspondingly, the frequency ω is defined by $\hbar\omega = E_B - E_A$ instead of Eq. (1.46).

1.4 The Electromagnetic Field in the Presence of Charges

After the special case of the electric dipole interaction, we now want to consider the general interaction of moving charges and an electromagnetic field. As this problem will later be treated in a relativistically covariant way, we shall not give a rigorous complete derivation, but rather stress the physical interpretation. As in the last section, the motion of the charges will again be described non-relativistically. In Section 1.4.1 we shall deal with the Hamiltonian formulation of the classical theory. This will enable us very easily to go over to the quantized theory in Section 1.4.2. In Sections 1.4.3 and 1.4.4 we shall illustrate the application of the theory for radiative transitions and Thomson scattering.

1.4.1 Classical electrodynamics

We would expect the Hamiltonian of a system of moving charges, such as an atom, in an electromagnetic field to consist of three parts: a part referring to matter (i.e. the charges), a part referring to the electromagnetic field and a part describing the interaction between matter and field.

For a system of point masses m_i, $i = 1, \ldots, N$, with charges e_i and position coordinates $\mathbf{r}_i$, the Hamiltonian is

$$H_{\mathrm{m}} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + H_{\mathrm{C}} \tag{1.55a}$$

where H_{C} is the Coulomb interaction

$$H_{\mathrm{C}} \equiv \frac{1}{2} \sum_{\substack{i,j \\ (i \neq j)}} \frac{e_i e_j}{4\pi \left| \mathbf{r}_i - \mathbf{r}_j \right|} \tag{1.55b}$$

and $\mathbf{p}_i = m_i \, d\mathbf{r}_i/dt$ is the kinetic momentum of the ith particle. This is the usual Hamiltonian of atomic physics, for example.

The electromagnetic field in interaction with charges is described by Maxwell's equations [Eqs. (1.1)]. We continue to use the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, so that the electric field (1.2) decomposes into transverse and longitudinal fields

$$\mathbf{E} = \mathbf{E}_{\mathbf{T}} + \mathbf{E}_{\mathbf{L}},$$

where

$$\mathbf{E}_{\mathbf{T}} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{E}_{\mathbf{L}} = -\nabla\phi.$$

(A longitudinal field is defined by the condition $\boldsymbol{\nabla} \wedge \mathbf{E}_{\mathbf{L}} = 0$.) The magnetic field is given by $\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}$.

The total energy of the electromagnetic field

$$\frac{1}{2} \int \left(\mathbf{E}^2 + \mathbf{B}^2\right) \mathrm{d}^3\mathbf{x}$$

can be written

$$\frac{1}{2} \int \left(\mathbf{E}_{\mathbf{T}}^2 + \mathbf{B}^2\right) \mathrm{d}^3\mathbf{x} + \frac{1}{2} \int \mathbf{E}_{\mathbf{L}}^2 \, \mathrm{d}^3\mathbf{x}.$$

The last integral can be transformed, using Poisson's equation $\boldsymbol{\nabla}^2 \phi = -\rho$, into

$$\frac{1}{2} \int \mathbf{E}_{\mathbf{L}}^2 \, \mathrm{d}^3\mathbf{x} = \frac{1}{2} \int \frac{\rho(\mathbf{x}, t)\rho\left(\mathbf{x}', t\right)}{4\pi|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}^3\mathbf{x} \, \mathrm{d}^3\mathbf{x}'. \tag{1.56}$$

Thus the energy associated with the longitudinal field is the energy of the *instantaneous* electrostatic interaction between the charges. With

$$\rho(\mathbf{x}, t) = \sum_i e_i \, \delta(\mathbf{x} - \mathbf{r}_i(t))$$

Eq. (1.56) reduces to

$$\begin{aligned} \frac{1}{2} \int \mathbf{E}_{\mathbf{L}}^2 \, \mathrm{d}^3\mathbf{x} &= \frac{1}{2} \sum_{i,j} \frac{e_i e_j}{4\pi \left| \mathbf{r}_i - \mathbf{r}_{\mathrm{j}} \right|} \\ &= \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{e_i e_j}{4\pi \left| \mathbf{r}_i - \mathbf{r}_j \right|} \equiv H_{\mathrm{C}}, \end{aligned} \tag{1.57}$$

where, in the last line, we have dropped the infinite self-energy which occurs for point charges. The term H_{C} has already been included in the Hamiltonian H_{m}, Eqs. (1.55), so we must take as additional energy of the electromagnetic field that of the transverse radiation field

$$H_{\mathrm{rad}} = \frac{1}{2} \int \left(\mathbf{E}_{\mathbf{T}}^2 + \mathbf{B}^2\right) \mathrm{d}^3\mathbf{x}. \tag{1.58}$$

Eqs. (1.55) allow for the instantaneous Coulomb interaction of charges. To allow for the interaction of moving charges with an electromagnetic field, one must replace the matter-Hamiltonian (1.55a) by

$$H'_{\mathrm{m}} = \sum_i \frac{1}{2m_i} \left(\mathbf{p}_i - \frac{e_i}{c} \mathbf{A}_i\right)^2 + H_{\mathrm{C}} \tag{1.59}$$

where $\mathbf{A}_i, = \mathbf{A}(\mathbf{r}_i, t)$ denotes the vector potential at the position $\mathbf{r}_i$ of the charge e_i at time t. In Eq. (1.59) $\mathbf{p}_i$ is the momentum coordinate canonically conjugate to the position coordinate $\mathbf{r}_i$, in the sense of Lagrangian mechanics, and it is related to the velocity $\mathbf{v}_i = \mathrm{d}\mathbf{r}_i/\mathrm{d}t$ of the ith particle by

$$\mathbf{p}_i = m_i \mathbf{v}_i + \frac{e_i}{c} \mathbf{A}_i.$$

It is only for $\mathbf{A} = 0$ that this conjugate momentum reduces to the kinetic momentum $m_i\mathbf{v}_i$. The justification for the form (1.59) for H'_{m} is that it gives the correct equations of motion for the charges (see Problem 1.2):

$$m_i \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = e_i \left[\mathbf{E}_i + \frac{\mathbf{v}_i}{c} \wedge \mathbf{B}_i\right], \tag{1.60}$$

where $\mathbf{E}_i$ and $\mathbf{B}_i$ are the electric and magnetic fields at the instantaneous position of the ith charge.[11]

We can regroup the terms in Eq. (1.59) as

$$H'_{\mathrm{m}} = H_{\mathrm{m}} + H_{\mathrm{I}} \tag{1.61}$$

where H_{I}, the interaction Hamiltonian of matter and field, is given by

$$\begin{aligned} H_{\mathrm{I}} &= \sum_i \left\{ -\frac{e_i}{2m_i c} \left(\mathbf{p}_i \cdot \mathbf{A}_i + \mathbf{A}_i \cdot \mathbf{p}_i\right) + \frac{e_i^2}{2m_i c^2} \mathbf{A}_i^2 \right\} \\ &= \sum_i \left\{ -\frac{e_i}{m_i c} \mathbf{A}_i \cdot \mathbf{p}_i + \frac{e_i^2}{2m_i c^2} \mathbf{A}_i^2 \right\} . \end{aligned} \tag{1.62}$$

In the quantum theory $\mathbf{p}_i$, the momentum canonically conjugate to $\mathbf{r}_i$, will become the operator $-\mathrm{i}\hbar\boldsymbol{\nabla}_i$. Nevertheless, the replacement of $\mathbf{p}_i \cdot \mathbf{A}_i$ by $\mathbf{A}_i \cdot \mathbf{p}_i$ in the second line of Eq. (1.62) is justified by our gauge condition $\boldsymbol{\nabla}_i \cdot \mathbf{A}_i = 0$. Eq. (1.62) represents the general interaction of moving charges in an electromagnetic field (apart from $\mathrm{H_C}$). It does not include the interaction of the magnetic moments, such as that due to the spin of the electron, with magnetic fields.

Combining the above results (1.55), (1.58), (1.59) and (1.62), we obtain for the complete Hamiltonian

$$H = H'_{\mathrm{m}} + H_{\mathrm{rad}} = H_{\mathrm{m}} + H_{\mathrm{rad}} + H_{\mathrm{I}}. \tag{1.63}$$

Just as this Hamiltonian leads to the correct equations of motion (1.60) for charges, so it also leads to the correct field equations (1.4), with $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, for the potentials.[12]

1.4.2 Quantum electrodynamics

The quantization of the system described by the Hamiltonian (1.63) is carried out by subjecting the particles' coordinates $\mathbf{r}_i$ and canonically conjugate momenta $\mathbf{p}_i$ to the usual commutation relations (e.g. in the coordinate representation $\mathbf{p}_i \rightarrow -i\hbar\boldsymbol{\nabla}_i$), and quantizing the radiation field, as in Section 1.2.3. The longitudinal electric field $\mathbf{E_L}$ does not provide any additional degrees of freedom, being completely determined via the first Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E_L} = \rho$ by the charges.

The interaction H_{I} in Eq, (1.63) is usually treated as a perturbation which causes transitions between the states of the non-interacting Hamiltonian

$$H_0 = H_{\mathrm{m}} + H_{\mathrm{rad}}. \tag{1.64}$$

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[11] For the Lagrangian and Hamiltonian formulations of mechanics which are here used see, for example, H. Goldstein, *Classical Mechanics*, 2nd edn, Addison-Wesley, Reading, Mass., 1980, in particular pp. 21–23 and 346.

[12] See W. Heitler, *The Quantum Theory of Radiation*, 3rd edn, Clarendon Press, Oxford, 1954, pp. 48–50.

The eigenstates of H_0 are again of the form

$$|A,\ \ldots n_r(\mathbf{k})\ldots\rangle = |A\rangle\,|\ldots n_r(\mathbf{k})\ldots\rangle,$$

with $|A\rangle$ and $|\ldots n_r(\mathbf{k})\ldots\rangle$ eigenstates of $H_{\rm m}$ and $H_{\rm rad}$.

Compared with the electric dipole interaction (1.40), the interaction (1.62) differs in that it contains a term quadratic in the vector potential. This results in two-photon processes in first-order perturbation theory (i.e. emission or absorption of two photons or scattering). In addition, the first term in (1.62) contains magnetic interactions and higher-order effects due to the spatial variation of $\mathbf{A}(\mathbf{x}, t)$, which are absent from the electric dipole interaction (1.40). These aspects are illustrated in the applications to radiative transitions and Thomson scattering which follow.

1.4.3 Radiative transitions in atoms

We consider transitions between two states of an atom with emission or absorption of one photon. This problem was treated in Section 1.3 in the electric dipole approximation, but now we shall use the interaction (1.62).

We shall consider the emission process between the initial and final states (1.42). Using the expansion (1.38) of the vector potential, we obtain the matrix element for this transition [which results from the term linear in $\mathbf{A}$ in Eq. (1.62)]

$$\begin{aligned}&\langle B,\ n_r(\mathbf{k})\ +\ 1|H_{\rm I}|A,\ n_r(\mathbf{k})\rangle\\ &\quad=-\frac{e}{m}\left(\frac{\hbar}{2V\omega_{\mathbf{k}}}\right)^{1/2}[n_r(\mathbf{k})+1]^{1/2}\ \langle B|\boldsymbol{\varepsilon}_r(\mathbf{k})\bullet\sum_i \mathrm{e}^{-i\mathbf{k}\bullet\mathbf{r}_i}\ \mathbf{p}_i|A\rangle\ \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}t}.\end{aligned}\tag{1.65}$$

Using this matrix element, one calculates the transition probability per unit time as in Section 1.3. Instead of Eqs. (1.50) and (1.51), one obtains:

$$w_r\ \mathrm{d}\Omega = \frac{e^2\omega\ \mathrm{d}\Omega}{8\pi^2m^2\hbar c^3}\ [n_r(\mathbf{k})+1]\left|\boldsymbol{\varepsilon}_r(\mathbf{k})\bullet\langle B|\sum_i \mathrm{e}^{-\ \mathrm{i}\mathbf{k}\bullet\mathbf{r}_i}\mathbf{p}_i\ |A\rangle\right|^2.\tag{1.66}$$

These results go over into the electric dipole approximation if in the matrix elements in Eqs. (1.65) and (1.66) we can approximate the exponential functions by unity:

$$\mathrm{e}^{-\ \mathrm{i}\mathbf{k}\ \bullet\ \mathbf{r}_i}\approx 1.\tag{1.67}$$

This is justified provided the wavelength $\lambda = 2\pi/k$ of the radiation emitted in the transition is very large compared to the linear dimensions R of the system of charges (in our case, of the atom): $\lambda >> R$. The atomic wavefunctions $|A\rangle$ and $|B\rangle$ restrict the effective values of $\mathbf{r}_i$ to $r_i \lesssim R$, so that $\mathbf{k}\bullet\mathbf{r}_i \lesssim kR << 1$. We saw in Section 1.3 that this inequality is generously satisfied for optical atomic transitions. From the equation of motion $\mathrm{i}\hbar\dot{\mathbf{r}}_i = [\mathbf{r}_i,\ H]$ and Eq. (1.46)

$$\langle B|\ \mathbf{p}_i|A\rangle = m\langle B|\ \dot{\mathbf{r}}_i|A\rangle = -\ \mathrm{i}m\omega\langle B|\ \mathbf{r}_i|A\rangle.$$

Hence, in the approximation (1.67), Eqs. (1.65) and (1.66) reduce to the electric dipole form, Eqs. (1.44) and (1.50).

If selection rules forbid the transition $|A\rangle$ to $|B\rangle$ via the electric dipole interaction, it may in general still occur via higher terms in the expansion of the exponentials

$$e^{-i\mathbf{k}\cdot\mathbf{r}_i} = 1 - i\mathbf{k}\bullet\mathbf{r}_i + \cdots.$$

With the second term, the expression within the modulus sign in Eq. (1.66) becomes

$$\boldsymbol{\varepsilon}_r(\mathbf{k})\bullet\langle B|\sum_i(-i\mathbf{k}\bullet\mathbf{r}_i)\mathbf{p}_i|A\rangle = -i\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{\varepsilon}_{r\alpha}(\mathbf{k})k_\beta\langle B|\sum_i r_{i\beta}\,p_{i\alpha}|A\rangle,$$

where α, β (= 1, 2, 3) label the Cartesian components of the vectors $\boldsymbol{\varepsilon}_r$, $\mathbf{k}$, $\mathbf{r}_i$ and $\mathbf{p}_i$. The matrix element can be written as the sum of an antisymmetric and a symmetric second-rank tensor

$$\langle B|\sum_i r_{i\beta}p_{i\alpha}\,|A\rangle = \frac{1}{2}\left\{\langle B|\sum_i\left(r_{i\beta}p_{i\alpha} - r_{i\alpha}p_{i\beta}\right)|A\rangle + \langle B|\sum_i\left(r_{i\beta}p_{i\alpha} + r_{i\alpha}p_{i\beta}\right)|A\rangle\right\}.$$

The first term contains the antisymmetric angular momentum operator and corresponds to the magnetic dipole interaction. (In practice this must be augmented by the spin part.) The symmetric term corresponds to the electric quadrupole interaction. The parity and angular momentum selection rules for the transitions brought about by these matrix elements are easily determined from their forms. We obtain in this way an expansion into electric and magnetic multipoles, i.e. photons of definite parity and angular momentum. As usual, a better procedure for such an expansion, except in the simplest cases, is to use spherical rather than Cartesian coordinates.[13]

The result (1.66) can again be adapted to the case of absorption of radiation by replacing the factor $[n_r(\mathbf{k}) + 1]$ by $n_r(\mathbf{k})$ and the appropriate re-interpretation of the matrix element, etc.

1.4.4 Thomson scattering

As a second illustration, we consider Thomson scattering, i.e. the scattering of photons of energy $\hbar\omega$ by atomic electrons, with $\hbar\omega$ large compared to the binding energies of the electrons, so that they can be considered as free electrons, but $\hbar\omega$ very small compared to the electron rest energy mc^2. In this case the energy $\hbar\omega'$ of the scattered photon is not changed: $\hbar\omega' = \hbar\omega$, since for small recoil momenta the recoil energy may be neglected.

The scattering from an initial state with one photon of momentum $\hbar\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}_\alpha(\mathbf{k})$ (with $\alpha = 1$ or 2) to a final state with one photon of momentum $\hbar\mathbf{k}'$ and polarization $\boldsymbol{\varepsilon}_\beta(\mathbf{k}')$ (with $\beta = 1$ or 2) can occur in first-order perturbation theory via the term in $\mathbf{A}^2$ in the interaction (1.62). It can also occur in second-order perturbation theory via the term linear in $\mathbf{A}$ in Eq. (1.62), but one can show that

[13] See A. S. Davydov, *Quantum Mechanics*, 2nd edn, Pergamon, Oxford, 1976, Sections 81 and 95.

under our conditions the contribution of the second-order process is negligible.[14] The operator $\mathbf{A}^2(0,\ t)$ can, from Eq. (1.38), be written

$$\mathbf{A}^2(0,\ t) = \sum_{\mathbf{k}_1\mathbf{k}_2}\sum_{r,\ s}\frac{\hbar c^2}{2V(\omega_1\ \omega_2)^{1/2}}\ (\boldsymbol{\varepsilon}_r(\mathbf{k}_1)\bullet\boldsymbol{\varepsilon}_s(\mathbf{k}_2)) \times\ \left[a_r(\mathbf{k}_1)\mathrm{e}^{-\mathrm{i}\omega_1 t} + a_r^{\dagger}(\mathbf{k}_1)\ \mathrm{e}^{+i\omega_1 t}\right]\ \left[a_s(\mathbf{k}_2)\mathrm{e}^{-\mathrm{i}\omega_2 t} + a_s^{\dagger}(\mathbf{k}_2)\ \mathrm{e}^{+i\omega_2 t}\right], \tag{1.68}$$

where $\omega_r \equiv c|\mathbf{k}_r|$, $r = 1, 2$. This operator can bring about the transition from the initial state $|\mathbf{k},\ \alpha\rangle$ to the final state $|\mathbf{k}',\ \beta\rangle$ (we use a somewhat simplified, but unambiguous, notation) in two ways: either of the factors in square parentheses can act to absorb the initial photon, and the other factor then creates the final photon. One then obtains the matrix element for this transition from Eq. (1.62)

$$\langle\mathbf{k}',\ \beta|\ \frac{e^2}{2mc^2}\ \mathbf{A}^2(0,\ t)\ |\mathbf{k},\ \alpha\rangle = \frac{e^2\hbar}{2mV(\omega\omega')^{1/2}}\ \boldsymbol{\varepsilon}_\alpha(\mathbf{k})\bullet\boldsymbol{\varepsilon}_\beta(\mathbf{k}')\ \mathrm{e}^{\mathrm{i}(\omega'\ -\ \omega)\mathrm{t}}$$

where $\omega = c|\mathbf{k}|$ and. $\omega' = c|\mathbf{k}'|$. The transition probability per unit time for a photon, initially in the state $|\mathbf{k},\ \alpha\rangle$, to be scattered into an element of solid angle $\mathrm{d}\boldsymbol{\Omega}$ in the direction $\mathbf{k}'$, and with polarization $\boldsymbol{\varepsilon}_\beta(\mathbf{k}')$, is given by

$$\begin{aligned} w_{\alpha\ \rightarrow\ \beta}(\mathbf{k}')\ \mathrm{d}\Omega &= \frac{2\pi}{\hbar}\ \int\frac{Vk'^2\,\mathrm{d}k'\,\mathrm{d}\Omega}{(2\pi)^3}\ \delta(\hbar\omega'\ -\ \hbar\omega) \\ &\quad\times\ \left(\frac{e^2\hbar}{2mV}\right)^2\left(\frac{1}{\omega\omega'}\right)\ \left[\boldsymbol{\varepsilon}_\alpha(\mathbf{k})\bullet\boldsymbol{\varepsilon}_\beta(\mathbf{k}')\right]^2 \\ &= \frac{c}{V}\left(\frac{e^2}{4\pi mc^2}\right)^2\left[\boldsymbol{\varepsilon}_\alpha(\mathbf{k})\bullet\boldsymbol{\varepsilon}_\beta(\mathbf{k}')\right]^2\ \mathrm{d}\Omega \end{aligned}$$

where $|\mathbf{k}'|=|\mathbf{k}|$. Dividing this transition probability per unit time by the incident photon flux *(c/V)*, one obtains the corresponding differential cross-section

$$\sigma_{\alpha\ \rightarrow\ \beta}\ (\mathbf{k}')\ \mathrm{d}\Omega = r_0^2\left[\boldsymbol{\varepsilon}_\alpha(\mathbf{k})\bullet\boldsymbol{\varepsilon}_\beta(\mathbf{k}')\right]^2\ \mathrm{d}\Omega, \tag{1.69}$$

where the classical electron radius has been introduced by

$$r_0 = \frac{e^2}{4\pi mc^2}\ = 2.818\quad \mathrm{fm}. \tag{1.70}$$

For an unpolarized incident photon beam, the unpolarized differential cross-section (i.e. the final polarization state is not observed) is obtained from Eq. (1.69) by summing over final and averaging over initial polarization states. We introduce the abbreviations $\boldsymbol{\varepsilon}_\alpha\ \equiv \boldsymbol{\varepsilon}_\alpha\ (\mathbf{k})$ and $\boldsymbol{\varepsilon}'_\beta\ \equiv \boldsymbol{\varepsilon}_\beta(\mathbf{k}')$. Since $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ form an orthonormal coordinate system,

$$\sum_{\alpha\ =\ 1}^{2}\left(\boldsymbol{\varepsilon}_\alpha\bullet\boldsymbol{\varepsilon}'_\beta\right)^2 = 1\ -\ \left(\hat{\mathrm{k}}\bullet\boldsymbol{\varepsilon}'_\beta\right)^2.$$

[14] See J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, Reading, Mass., 1967, p. 51.

Similarly

$$\sum_{\beta=1}^{2}\left(\hat{\mathbf{k}}\bullet\boldsymbol{\varepsilon}'_{\beta}\right)^2 = 1 - \left(\hat{\mathbf{k}}\bullet\hat{\mathbf{k}}'\right)^2 = \sin^2\theta$$

where θ is the angle between the directions $\mathbf{k}$ and $\mathbf{k}'$ of the incident and scattered photons, i.e. the angle of scattering. From the last two equations

$$\frac{1}{2}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\left(\boldsymbol{\varepsilon}_{\alpha}\bullet\boldsymbol{\varepsilon}'_{\beta}\right)^2 = \frac{1}{2}\left(2 - \sin^2\theta\right) = \frac{1}{2}\left(1 + \cos^2\theta\right) \tag{1.71}$$

and hence the unpolarized differential cross-section for scattering through an angle θ is from Eq. (1.69) given as

$$\sigma(\theta)\,\mathrm{d}\Omega = \frac{1}{2}r_0^2\left(1 + \cos^2\theta\right)\mathrm{d}\Omega. \tag{1.69a}$$

Integrating over angles, we obtain the total cross-section for Thomson scattering

$$\sigma_{\text{total}} = \frac{8\pi}{3}\,r_0^2 = 6.65\times10^{-25}\ \text{cm}^2. \tag{1.72}$$

1.5 Appendix: The Schrödinger, Heisenberg and Interaction Pictures

These three pictures (abbreviated S.P., H.P. and I.P.) are three different ways of describing the time development of a system. In this Appendix, we shall derive the relationships between the three pictures. Quantities in these three pictures will be distinguished by the labels S, H and I.

In the S.P., the time-dependence is carried by the states according to the Schrödinger equation

$$\mathrm{i}\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,|A,\ t\rangle_{\mathrm{S}} = H\,|A,\ t\rangle_{\mathrm{S}}, \tag{1.73}$$

where H is the Hamiltonian of the system in the S.P. This can formally be solved in terms of the state of the system at an arbitrary reference time t_0

$$|A,\ t\rangle_{\mathrm{S}} = U_{\mathrm{S}}(t)\,|A,\ t_0\rangle_{\mathrm{S}} \tag{1.74}$$

where $U_{\mathrm{S}}(t)$ is the unitary operator:

$$U_{\mathrm{S}}(t) = \mathrm{e}^{-\mathrm{i}H(t\,-\,t_0)/\hbar}. \tag{1.75}$$

By means of $U_{\mathrm{S}}(t)$ we can carry out a unitary transformation of states and operators (O) from the S.P. to the H.P., in which we define

$$|A\rangle_{\mathrm{H}} = U_{\mathrm{S}}^{\dagger}(t)\,|A,\ t\rangle_{\mathrm{S}} = |A,\ t_0\rangle_{\mathrm{S}} \tag{1.76}$$

and

$$O^{\mathrm{H}}(t) = U_{\mathrm{S}}^{\dagger}(t)\, O^{\mathrm{S}}\, U_{\mathrm{S}}(t). \tag{1.77}$$

At $t = t_0$, states and operators in the two pictures are the same. We see from Eq. (1.76) that in the H.P. state, vectors are constant in time; the time-dependence is carried by the Heisenberg operators. From Eq. (1.77)

$$H^{\mathrm{H}} = H^{\mathrm{S}} \equiv H. \tag{1.78}$$

Since the transformation from the S.P. to the H.P. is unitary, it ensures the invariance of matrix elements and commutation relations:

$${}_{\mathrm{S}}\langle B,\ t|O^{\mathrm{S}}\big|A,\ t\rangle_{\mathrm{S}} = {}_{\mathrm{H}}\langle B, t\big|O^{\mathrm{H}}(t)\big|A,\ t\rangle_{\mathrm{H}}, \tag{1.79}$$

and if O and P are two operators for which $[O^{\mathrm{S}}, P^{\mathrm{S}}] = \text{const.}$, then $[O^{\mathrm{H}}(t), P^{\mathrm{H}}(t)]$ equals the same constant.

Differentation of Eq. (1.77) gives the Heisenberg equation of motion

$$\mathrm{i}\hbar\, \frac{\mathrm{d}}{\mathrm{d}t}\, O^{\mathrm{H}}(t) = \big[O^{\mathrm{H}}(t),\ H\big]. \tag{1.80}$$

For an operator which is time dependent in the S.P. (corresponding to a quantity which classically has an explicit time dependence), Eq. (1.80) is augmented to

$$\mathrm{i}\hbar\, \frac{\mathrm{d}}{\mathrm{d}t}\, O^{\mathrm{H}}(t) = \mathrm{i}\hbar\, \frac{\partial}{\partial t}\, O^{\mathrm{H}}(t) \ + \ \big[O^{\mathrm{H}}(t),\ H\big]. \tag{1.81}$$

We shall not be considering such operators.

The I.P. arises if the Hamiltonian is split into two parts

$$H = H_0 \ + \ H_{\mathrm{I}}. \tag{1.82}$$

In quantum field theory, H_{I} will describe the interaction between two fields, themselves described by H_0. [Note that the suffix I on H_{I} stands for 'interaction'. It does not label a picture. Eq. (1.82) holds in any picture.] The I.P. is related to the S.P. by the unitary transformation

$$U_0(t) = \mathrm{e}^{-\mathrm{i}H_0(t\ -\ t_0)\,/\,\hbar} \tag{1.83}$$

i.e.

$$|A,\ t\rangle_{\mathrm{I}} = U_0^{\dagger}(t)\, |A,\ t\rangle_{\mathrm{S}} \tag{1.84}$$

and

$$O^{\mathrm{I}}(t) = U_0^{\dagger}(t)\, O^{\mathrm{S}}\, U_0(t). \tag{1.85}$$

Thus the relation between I.P. and S.P. is similar to that between H.P. and S.P., but with the unitary transformation U_0 involving the non-interacting Hamiltonian H_0, instead of U involving the total Hamiltonian H. From Eq. (1.85):

$$H_0^{\mathrm{I}} = H_0^{\mathrm{S}} \ \equiv H_0. \tag{1.86}$$

Differentiating Eq. (1.85) gives the differential equation of motion of operators in the I.P.:

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} O^{\mathrm{I}}(t) = \left[O^{\mathrm{I}}(t),\ H_0\right]. \tag{1.87}$$

Substituting Eq. (1.84) into the Schrödinger equation (1.73), one obtains the equation of motion of state vectors in the I.P.

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \left|A,\ t\right\rangle_{\mathrm{I}} = H_{\mathrm{I}}^{\mathrm{I}}(t) \left|A,\ t\right\rangle_{\mathrm{I}} \tag{1.88}$$

where

$$H_{\mathrm{I}}^{\mathrm{I}}(t) = \mathrm{e}^{\mathrm{i}H_0(t\ -\ t_0)/\hbar}\ H_{\mathrm{I}}^{\mathrm{S}}\ \mathrm{e}^{-\mathrm{i}H_0(t\ -\ t_0)/\hbar}. \tag{1.89}$$

Finally, from the above relations, one easily shows that the I.P. and H.P. are related by

$$O^{\mathrm{I}}(t) = U(t) O^{\mathrm{H}}(t) U^{\dagger}(t) \tag{1.90}$$

$$|A,\ t\rangle_{\mathrm{I}} = U(t)|A\rangle_{\mathrm{H}} \tag{1.91}$$

where the unitary operator $U(t)$ is defined by

$$U(t) = \mathrm{e}^{\mathrm{i}H_0(t\ -\ t_0)/\hbar} \quad \mathrm{e}^{-\mathrm{i}H(t\ -\ t_0)/\hbar}. \tag{1.92}$$

The time development of the I.P. states follows from Eq. (1.91). From this equation

$$|A,\ t_1\rangle_{\mathrm{I}} = U(t_1)|A\rangle_{\mathrm{H}} = U(t_1)U^{\dagger}(t_2)|A,\ t_2\rangle_{\mathrm{I}}.$$

Hence

$$|A,\ t_1\rangle_{\mathrm{I}} = U(t_1,\ t_2)|A,\ t_2\rangle_{\mathrm{I}} \tag{1.93}$$

where the unitary operator $U(t_1, t_2)$, defined by

$$U(t_1,\ t_2) = U(t_1)U^{\dagger}(t_2), \tag{1.94}$$

satisfies the relations

$$U^{\dagger}(t_1,\ t_2) = U(t_2,\ t_1) \tag{1.95a}$$

$$U(t_1,\ t_2)U(t_2,\ t_3) = U(t_1,\ t_3). \tag{1.95b}$$

Problems

1.1. The radiation field inside a cubic enclosure, which contains no charges, is specified by the state

$$|c\rangle = \exp\left(-\frac{1}{2}|c|^2\right) \sum_{n\ =\ 0}^{\infty} \frac{c^n}{\sqrt{n!}}\ |n\rangle$$

where $c = |c|\ \mathrm{e}^{\mathrm{i}\delta}$ is any complex number and $|n\rangle$ is the state (1.31) in which there are n photons with wave vector $\mathbf{k}$ and polarization vector $\boldsymbol{\varepsilon}_r(\mathbf{k})$ present, and no others. Derive the following properties of the state $|c\rangle$.

(i) $|c\rangle$ is normalized: $\langle c|c\rangle = 1$.

(ii) $|c\rangle$ is an eigenstate of the destruction operator $a_r(\mathbf{k})$ with the complex eigenvalue c:

$$a_r(\mathbf{k})|c\rangle = c|c\rangle.$$

(iii) The mean number $\overline{N}$ of photons in the enclosure in the state $|c\rangle$ is given by

$$\overline{N} = \langle c|N|c\rangle = |c|^2 \tag{A}$$

where N is the total photon number operator.

(iv) The root-mean-square fluctuation ΔN in the number of photons in the enclosure in the state $|c\rangle$ is given by

$$(\Delta N)^2 = \langle c|N^2|c\rangle - \overline{N}^2 = |c|^2. \tag{B}$$

(v) The expectation value of the electric field $\mathbf{E}$ in the state $|c\rangle$ is given by

$$\langle c|\mathrm{E}|c\rangle = -\boldsymbol{\varepsilon}_r(\mathbf{k})2\left(\frac{\hbar\omega_{\mathbf{k}}}{2V}\right)^{1/2}|c|\sin(\mathbf{k}\bullet\mathbf{x} - \omega_{\mathbf{k}}t + \delta) \tag{C}$$

where V is the volume of the enclosure.

(vi) The root-mean-square fluctuation ΔE of the electric field in the state $|c\rangle$ is given by

$$(\Delta E)^2 = \langle c|\mathbf{E}^2|c\rangle - \langle c|\mathbf{E}|c\rangle^2 = \frac{\hbar\omega_{\mathbf{k}}}{2V}. \tag{D}$$

We noted in Section 1.2.3 that the expectation value of $\mathbf{E}$ in a state with a definite number of photons is zero, so that such a state cannot represent a classical field, even for very large photon numbers. In contrast, it follows from Eqs. (A)–(D) that the relative fluctuation in photon numbers

$$\frac{\Delta N}{\overline{N}} = \overline{N}^{-1/2}$$

tends to zero as $\overline{N} \to \infty$, and that the fluctuation ΔE becomes negligible for large field strengths, i.e. $|c\rangle$ goes over into a classical state in which the field is well defined as $\overline{N} \to \infty$. The state $|c\rangle$ is called a coherent state and represents the closest quantum-mechanical approach to a classical electromagnetic field. (For a full discussion, see the book by Loudon, quoted at the end of Section 1.2.)

1.2. The Lagrangian of a particle of mass m and charge q, moving in an electromagnetic field, is given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{q}{c}\mathbf{A}\cdot\dot{\mathbf{x}} - q\phi$$

where $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$ and $\phi = \phi(\mathbf{x}, t)$ are the vector and scalar potentials of the electromagnetic field at the position $\mathbf{x}$ of the particle at time t.

(i) Show that the momentum conjugate to $\mathbf{x}$ is given by

$$\mathbf{p} = m\dot{\mathbf{x}} + \frac{q}{c}\mathbf{A} \tag{A}$$

(i.e. the conjugate momentum $\mathbf{p}$ is not the kinetic momentum $m\dot{\mathbf{x}}$, in general) and that Lagrange's equations reduce to the equations of motion of the particle [compare Eq. (1.60)]

$$m\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{x}} = q\left[\mathbf{E} + \frac{1}{c}\dot{\mathbf{x}} \wedge \mathbf{B}\right], \tag{B}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields at the instantaneous position of the charge.

(ii) Derive the corresponding Hamiltonian [compare Eq. (1.59)]

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi,$$

and show that the resulting Hamilton equations again lead to Eqs. (A) and (B).

1.3. For Thomson scattering of an unpolarized beam of photons, obtain the differential cross-section for scattering through an angle θ, with the scattered radiation being linearly polarized in a given direction. By considering two mutually perpendicular such directions, use your result to re-derive Eq. (1.69a) for the unpolarized differential cross-section.

Show that for $\theta = 90°$, the scattered beam is 100% linearly polarized in the direction of the normal to the plane of scattering.