Chapter 1

Hippocrates' Quadrature of the Lune

(ca. 440 B.C.)

The Appearance of Demonstrative Mathematics

Our knowledge of the very early development of mathematics is largely speculative, pieced together from archaeological fragments, architectural remains, and educated guesses. Clearly, with the invention of agriculture in the years 15,000–10,000 B.C., humans had to address, in at least a rudimentary fashion, the two most fundamental concepts of mathematics: multiplicity and space. The notion of multiplicity, or "number," would arise when counting sheep or distributing crops; over the centuries, refined and extended by generations of scholars, these ideas evolved into arithmetic and later into algebra. The first farmers likewise would have needed insight into spatial relationships, primarily in regard to the areas of fields and pastures; such insights, carried down through history, became geometry. From the beginnings of civilization, these two great branches of mathematics—arithmetic and geometry—would have coexisted in primitive form.

This coexistence has not always been a harmonious one. A continuing feature of the history of mathematics has been the prevailing tension between the arithmetic and the geometric. There have been times when one branch has overshadowed the other and when one has been regarded as logically superior to its more suspect counterpart. Then a new discovery, a new point of view, would turn the tables. It may come as a surprise that mathematics, like art or music or literature, has been subject to such trends in the course of its long and illustrious history.

We find clear signs of mathematical development in the civilization of ancient Egypt. For the Egyptians, the emphasis was on the practical side of mathematics as a facilitator of trade, agriculture, and the other increasingly complex aspects of everyday life. Archaeological records indicate that by 2000 B.C. the Egyptians had a primitive numeral system as well as some geometric ideas about triangles, pyramids, and the like. There is a tradition, for instance, that Egyptian architects used a clever device for making right angles. They would tie 12 equally long segments of rope into a loop, as shown in Figure 1.1. Stretching five consecutive segments in a straight line from *B* to *C* and then pulling the rope taut at *A*, they thus formed a rigid triangle with a right angle *BAC*. This configuration, laid upon the ground, allowed the workers to construct a perfect right angle at the corner of a pyramid, temple, or other building.

Implicit in this construction is an understanding of the Pythagorean relationship of right triangles. That is, the Egyptians seemed to know that a triangle with sides of length 3, 4, and 5 must contain a right angle. Of course, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$, and so we catch an early glimpse of one of the most important relationships in all of mathematics (see Figure 1.2).

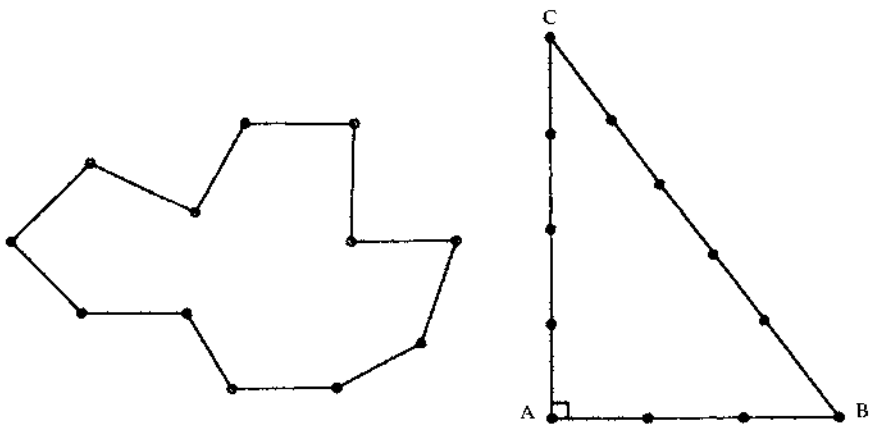

FIGURE 1.1

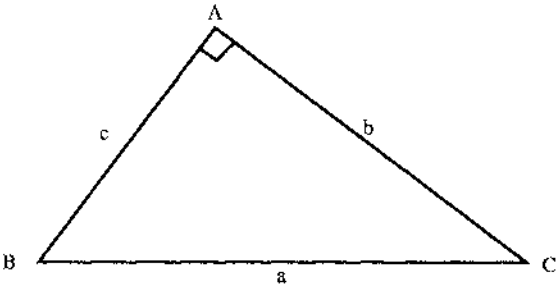

FIGURE 1.2

Technically, this Egyptian insight was not a case of the Pythagorean theorem itself, which states, "If $\triangle BAC$ is a right triangle, then $a^2 = b^2 + c^2$." Rather, it was an example of the *converse* of the Pythagorean theorem: "If $a^2 = b^2 + c^2$, then $\triangle BAC$ is a right triangle." That is, for a proposition of the form "If *P*, then *Q*," the related statement "If *Q*, then *P*" is called the proposition's "converse." As we shall see, a perfectly true statement may have a false converse, but in the case of the famous Pythagorean theorem, both the proposition and its converse are valid. In fact, these will be the "great theorems" in the next chapter.

Although the Egyptians seemed to have some insight into the geometry of 3-4-5 right triangles, it is doubtful they possessed the broader understanding that, for instance, a 5-12-13 triangle or a 65-72-97 triangle likewise contains a right angle (since in each case $a^2 = b^2 + c^2$). More critically, the Egyptians gave no indication of how they might *prove* this relationship. Perhaps they had some logical argument to support their observation about 3-4-5 triangles; perhaps they hit upon it purely by trial and error. In any case, the notion of proving a general mathematical result by a carefully crafted logical argument is nowhere to be found in Egyptian writings.

The following example of Egyptian mathematics may be illuminating: it is their approach to finding the volume of a truncated square pyramid—that is, a square pyramid with its top lopped off by a plane parallel to the base (see Figure 1.3). Such a solid is today called the frustum of a pyramid. The technique for finding its volume appears in the so-called "Moscow Papyrus" from 1850 B.C.:

> If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take a third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.

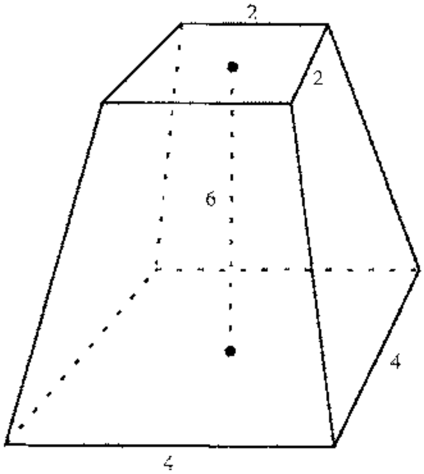

FIGURE 1.3

This is a most remarkable prescription, which indeed yields the correct answer for the frustum's volume. Notice, however, what it does not do. It does not give a general formula to cover frusta of other dimensions. Egyptians would have to generalize from this particular case in order to determine the volume of a different-sized frustum, a process that could be a bit confusing. Far simpler and more concise is our modern formula

$$V = \tfrac{1}{3}h(a^2 + ab + b^2)$$

where *a* is the side of the square on the bottom, *b* is the side of the square on the top, and *h* is the frustum's height. Worse, there was no indication of *why* this Egyptian recipe provided the correct answer. Instead, a simple "You will find it right" sufficed.

It is probably dangerous to draw sweeping conclusions from a particular example, yet historians have noted that a dogmatic approach to mathematics was certainly in keeping with the authoritarian society that was pharaonic Egypt. Inhabitants of that ancient land were conditioned to give unquestioned obedience to their rulers. By analogy, when presented with an authoritative mathematical technique that concluded "You will find it right," Egyptian subjects were hardly likely to demand a more thorough explanation of why it worked. In the land of the Pharaoh, you did what you were told, whether in erecting a colossal temple or in solving a math problem. Those adamantly questioning the system would end up as mummies before their time.

Another great ancient civilization—or, more precisely, civilizations flourished in Mesopotamia and produced mathematics significantly more advanced than that of Egypt. The Babylonians, for instance, solved fairly sophisticated problems with a definite algebraic character, and the existence of a clay tablet called Plimpton 322, dated roughly between 1900 and 1600 B.C., shows that they definitely understood the Pythagorean theorem in far more depth than their Egyptian counterparts; that is, the Babylonians recognized that a 5-12-13 triangle or a 65-72-97 triangle (and many more) was right. In addition, they developed a sophisticated place system for their numerals. We, of course, are accustomed to a base-10 numeral system, obviously derived from the 10 fingers of the human hand, so it may seem a bit odd that the Babylonians chose a base-60 system. While no one speculates that these ancient people had 60 fingers, their choice of base can still be seen in our measurement of time (60 seconds per minute) and angles ($6 \times 60° = 360°$ in a circle).

But for all of their achievements, the Mesopotamians likewise addressed only the question of "how" while avoiding the much more significant issue of "why." Those seeking the appearance of a demonstrative mathematics—a theoretical, deductive system in which emphasis was placed upon *proving* critical relationships—would have to look to a later time and a different place.

The time was the first millennium B.C., and the place was the Aegean coasts of Asia Minor and Greece. Here there arose one of the most significant civilizations of history, whose extraordinary achievements would forever influence the course of western culture. Engaged in a thriving commerce, both within their own lands and across the Mediterranean, the Greeks developed into a mobile, adventuresome people, relatively prosperous and sophisticated, and considerably more independent in thought and action than the western world had seen before. These curious, free-thinking merchants were much less likely to submit meekly to authority. Indeed, with the development of Greek democracy, the citizens *became* the authority (although it must be stressed that citizenship in the classical world was very narrowly defined). To such individuals, everything was open to debate and analysis, and ideas were not about to be accepted with a passive, unquestioning obedience.

By 400 B.C., this remarkable civilization could already boast a rich, some would say unsurpassed, intellectual heritage. The epic poet Homer, the historians Herodotus and Thucydides, the dramatists Aeschylus, Sophocles, and Euripides, the politician Pericles, and the philosopher Socrates—these individuals had all left their marks as the fourth century B.C. began. Inhabitants of the modern world, where fame can fade so quickly, may find it astonishing that these names have endured gloriously for over 2000 years. To this day, we admire their boldness in subjecting Nature and the human condition to the penetrating light of reason. Granted, it was reason still contaminated by large doses of superstition and ignorance, but the Greek thinkers were profoundly successful. If their conclusions were not always correct, the Greeks nonetheless sensed that theirs was the path that would lead from a barbarous past to an undreamed-of future. The term "awakening" is often used in describing this special moment in history, and it is apt. Humankind was indeed arising from the slumber of thousands of centuries to confront this strange, mysterious world with Nature's most potent weapon—the human mind.

Such was certainly the case with mathematics. Around 600 B.C. in the town of Miletus on the western coast of Asia Minor, there lived the great Thales (ca. 640–ca. 546 B.C.), one of the so-called "Seven Wise Men" of antiquity. Thales of Miletus is generally credited with being the father of demonstrative mathematics, the first scholar who supplied the "why" along with the "how." As such, he is the earliest known mathematician.

We have very little hard evidence about his life. Indeed, he emerges from the mists of the past as a pseudo-mythical figure, and it is anybody's guess as to the truth of the exploits and discoveries attributed to him. Looking back seven centuries, the biographer Plutarch (A.D. 46–120) wrote that ". . . at that time Thales alone had raised philosophy above mere practice into speculation." A noted mathematician and astronomer who somehow predicted the solar eclipse in 585 B.C., Thales, like the stereotypical scientist, was chronically absent-minded and incessantly preoccupied—according to legend, he once was strolling along, gazing upward at his beloved stars, when he tumbled into an open well.

His "fatherhood" of demonstrative mathematics notwithstanding, Thales never married. When Solon, a contemporary, asked why, Thales arranged a cruel ruse whereby a messenger brought Solon news of his son's death. According to Plutarch, Solon then

> . . . began to beat his head and to do and say all that is usual with men in transports of grief. But Thales took his hand, and, with a smile, said, "These things, Solon, keep me from marriage and rearing children, which are too great for even your constancy to support; however, be not concerned at the report, for it is a fiction."

Clearly, Thales was not the kindest of people. A similar impression emerges from the story of a farmer who routinely tied heavy bags of salt on the back of his donkey when driving the beast to market. The clever animal quickly learned to roll over while fording a particular stream, thereby dissolving much of the salt and making his burden far lighter. Exasperated, the farmer went to Thales for advice, and Thales recommended that on the next trip to market the farmer load the donkey with sponges.

It was certainly not kindness to man or beast that earned Thales his high reputation in mathematics. Rather, it was his insistence that geometric statements not be accepted simply because of their intuitive plausibility; instead they had to be subjected to rigorous, logical proof. This is no small legacy to leave the discipline of mathematics.

What, precisely, are some of his theorems? Tradition holds that it was Thales who first *proved* the following geometric results:

- Vertical angles are equal.
- The angle sum of a triangle equals two right angles.
- The base angles of an isosceles triangle are equal.
- An angle inscribed in a semicircle is a right angle.

In none of these cases do we have any record of his proofs, but we can speculate on their nature. For instance, consider the last proposition above. The proof given below is taken from Euclid's *Elements*, Book III, Proposition 31, but it is simple and direct enough to be a prime candidate for Thales' own.

THEOREM An angle inscribed in a semicircle is a right angle.

PROOF Let a semicircle be drawn with center O and diameter BC, and choose any point A on the semicircle (Figure 1.4). We must prove that $\angle BAC$ is right. Draw line OA and consider $\triangle AOB$. Since OB and OA are radii of the semicircle, they have the same length, and so $\triangle AOB$ is isosceles. Hence, as Thales had previously proved, $\angle ABO$ and $\angle BAO$ are equal (or, in modern terminology, congruent); call them both α. Like-

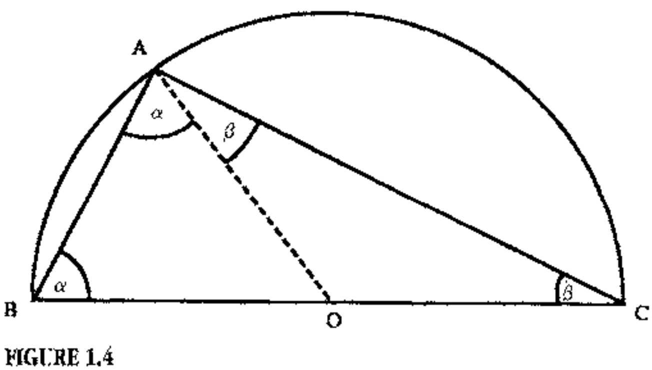

FIGURE 1.4

wise, in $\triangle AOC$, OA and OC have the same length, and so $\angle OAC = \angle OCA$; call them both β. But, from the large triangle BAC, we see that

$$\begin{aligned}\text{2 right angles} &= \angle ABC + \angle ACB + \angle BAC\\ &= \alpha + \beta + (\alpha + \beta)\\ &= 2\alpha + 2\beta = 2(\alpha + \beta)\end{aligned}$$

Hence, one right angle $= \frac{1}{2}$[2 right angles] $= \frac{1}{2}[2(\alpha + \beta)] = \alpha + \beta = \angle BAC$. This is exactly what we were to prove.

Q.E.D.

(*Note:* It has become customary, upon the completion of a proof, to insert the letters "Q.E.D.," which abbreviate the Latin *Quod erat demonstrandum* [Which was to be proved]. This alerts the reader to the fact that the argument is over and we are about to set off in new directions.)

After Thales, the next major figure in Greek mathematics was Pythagoras. Born in Samos around 572 B.C., Pythagoras lived and worked in the eastern Aegean, even, according to some legends, studying with the great Thales himself. But when the tyrant Polycrates assumed power in this region, Pythagoras fled to the Greek town of Crotona in southern Italy, where he founded a scholarly society now known as the Pythagorean brotherhood. In their contemplation of the world about them, the Pythagoreans recognized the special role of "whole number" as the critical foundation of all natural phenomena. Whether in music, or astronomy, or philosophy, the central position of "number" was everywhere evident. The modern notion that the physical world can be understood by "mathematization" owes more than a little to this Pythagorean viewpoint.

In the world of mathematics proper, the Pythagoreans gave us two great discoveries. One, of course, was the incomparable Pythagorean theorem. As with all other results from this distant time period, we have no record of the original proof, although the ancients were unanimous in attributing it to Pythagoras. In fact, legend says that a grateful Pythagoras sacrificed an ox to the gods to celebrate the joy his proof brought to all concerned (except, presumably, the ox).

The other significant contribution of the Pythagoreans was received with considerably less enthusiasm, for not only did it defy intuition, but it also struck a blow against the pervasive supremacy of the whole number. In modern parlance, they discovered irrational quantities, although their approach had the following geometric flavor:

Two line segments, AB and CD, are said to be *commensurable* if there exists a smaller segment EF that goes evenly into both AB and CD.

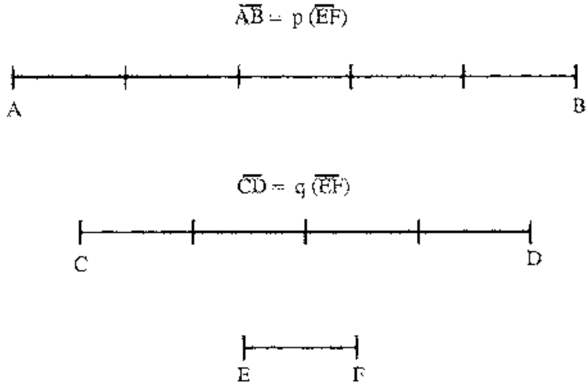

FIGURE 1.5

That is, for some whole numbers *p* and *q*, *AB* is composed of *p* segments congruent to *EF* while *CD* is composed of *q* such segments (Figure 1.5). Consequently, $\overline{AB}/\overline{CD} = p(\overline{EF})/q(\overline{EF}) = p/q$. (Here we are using the notation $\overline{AB}$ to stand for the length of segment *AB*). Since p/q is the *ratio* of two positive integers, we say that the ratio of the lengths of commensurable segments is a "rational" number.

Intuitively, the Pythagoreans felt that *any* two magnitudes are commensurable. Given two line segments, it seemed preposterous to doubt the existence of another segment *EF* dividing evenly into both, even if it took an extremely tiny *EF* to do the job. The presumed commensurability of segments was critical to the Pythagoreans, not only because they used this idea in their proofs about similar triangles but also because it seemed to support their philosophical stance on the central role of whole numbers.

However, tradition credits the Pythagorean Hippasus with discovering that the side of a square and its diagonal (*GH* and *GI* in Figure 1.6) are not commensurable. That is, no matter how small one goes, there is no magnitude *EF* dividing *evenly* into both the square's side and its diagonal.

This discovery had a number of profound consequences. Obviously, it shattered those Pythagorean proofs that rested upon the supposed commensurability of all segments. It would be almost two centuries before the mathematician Eudoxus found a way to patch up the theory of similar triangles by devising alternative proofs that did not rely upon the concept of commensurability. Secondly, it had an unsettling impact upon the supremacy of whole numbers, for if not all quantities were commensurable, then whole numbers were somehow inadequate to represent the ratios of all geometric lengths. Consequently, the discovery

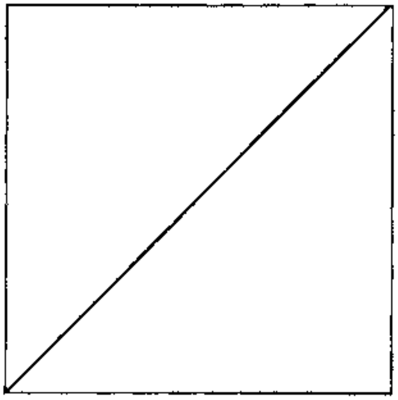

FIGURE 1.6

firmly established the superiority of geometry over arithmetic in all subsequent Greek mathematics. In the Figure 1.6, for instance, the side and diagonal of the square are beyond suspicion as *geometric* objects. But, as *numbers,* they presented a major problem. For, if we imagine that the side of the square above has length 1, then the Pythagorean theorem tells us that the length of the diagonal is $\sqrt{2}$; and, since side and diagonal are not commensurable, we see that $\sqrt{2}$ cannot be written as a rational number of the form p/q. Numerically, then, $\sqrt{2}$ is an "irrational," whose arithmetic character is quite mysterious. Far better, thought the Greeks, to avoid the numerical approach altogether and concentrate on magnitudes simply as geometric entities. This preference for geometry over arithmetic would dominate a thousand years of Greek mathematics.

A final result of the discovery of irrationals was that the Pythagoreans, incensed at all the trouble Hippasus had caused, supposedly took him far out upon the Mediterranean and tossed him overboard to his death. If true, the story indicates the dangers inherent in free thinking, even in the relatively austere discipline of mathematics.

Thales and Pythagoras, while prominent in legend and tradition, are obscure, shadowy figures from the distant past. Our next individual, Hippocrates of Chios (ca. 440 B.C.) is a little more solid. In fact, it is to him that we attribute the earliest mathematical proof that has survived in reasonably authentic form. This will be the subject of our first great theorem.

Hippocrates was born on the island of Chios sometime in the fifth century B.C. This was, of course, the same region that produced his illustrious predecessors mentioned earlier. (Note in passing that Chios is not far from the island of Cos, where another "Hippocrates" was born about this time; it was Hippocrates of Cos—not our Hippocrates—who became the father of Greek medicine and originator of the physicians' Hippocratic oath.)

Of the mathematical Hippocrates, we have scant biographical information. Aristotle wrote that, while a talented geometer, he ". . . seems in other respects to have been stupid and lacking in sense." This is an early example of the stereotype of the mathematician as being somewhat overwhelmed by the demands of everyday life. Legend has it that Hippocrates earned this reputation after being defrauded of his fortune by pirates, who apparently took him for an easy mark. Needing to make a financial recovery, he traveled to Athens and began teaching, thus becoming him one of the few individuals ever to enter the teaching profession for its financial rewards.

In any case, Hippocrates is remembered for two signal contributions to geometry. One was his composition of the first *Elements,* that is, the first exposition developing the theorems of geometry precisely and logically from a few given axioms or postulates. At least, he is credited with such a work, for nothing remains of it today. Whatever merits his book had were to be eclipsed, over a century later, by the brilliant *Elements* of Euclid, which essentially rendered Hippocrates' writings obsolete. Still, there is reason to believe that Euclid borrowed from his predecessor, and thus we owe much to Hippocrates for his great, if lost, treatise.

The other significant Hippocratean contribution—his quadrature of the lune—fortunately has survived, although admittedly its survival is tenuous and indirect. We do not have Hippocrates' own work, but Eudemus' account of it from around 335 B.C., and even here the situation is murky, because we do not really have Eudemus' account either. Rather, we have a summary by Simplicius from A.D. 530 that discussed the writings of Eudemus, who, in turn, had summarized the work of Hippocrates. The fact that the span between Simplicius and Hippocrates is almost a thousand years—roughly the time between us and Leif Erikson—indicates the immense difficulty historians face when considering the mathematics of the ancients. Nonetheless, there is no reason to doubt the general authenticity of the work in question.

Some Remarks on Quadrature

Before examining Hippocrates' lunes, we need to address the notion of "quadrature." It is obvious that the ancient Greeks were enthralled by the symmetries, the visual beauty, and the subtle logical structure of geometry. Particularly intriguing was the manner in which the simple and elementary could serve as foundation for the complex and intricate. This will become quite apparent in the next chapter as we follow Euclid through the development of some very sophisticated geometric propositions beginning with just a few basic axioms and postulates.

This enchantment with building the complex from the simple was also evident in the Greeks' geometric constructions. For them, the rules of the game required that all constructions be done only with compass and (unmarked) straightedge. These two fairly unsophisticated tools—allowing the geometer to produce the most perfect, uniform one-dimensional figure (the straight line) and the most perfect, uniform two-dimensional figure (the circle)—must have appealed to the Greek sensibilities for order, simplicity, and beauty. Moreover, these constructions were within reach of the technology of the day in a way that, for instance, constructing a parabola was not. Perhaps it is accurate to suggest that the aesthetic appeal of the straight line and circle reinforced the central position of straightedge and compass as geometric tools while, conversely and simultaneously, the physical availability of these tools enhanced the role to be played by straight lines and circles in the geometry of the Greeks.

The ancient mathematicians were consequently committed to, and limited by, the output of these tools. As we shall see, even the seemingly unsophisticated compass and straightedge can produce, in the hands of ingenious geometers, a rich and varied set of constructions, from the bisection of lines and angles, to the drawing of parallels and perpendiculars, to the creation of regular polygons of great beauty. But a considerably more challenging problem in the fifth century B.C. was that of the quadrature or squaring of a plane figure. To be precise:

□ The *quadrature* (or squaring) of a plane figure is the construction—using only compass and straightedge—of a square having area equal to that of the original plane figure. If the quadrature of a plane figure can be accomplished, we say that the figure is *quadrable* (or squarable).

That the quadrature problem appealed to the Greeks should come as no surprise. From a purely practical viewpoint, the determination of the area of an irregularly shaped figure is, of course, no easy matter. If such a figure could be replaced by an equivalent square, then determining the original area would have been reduced to the trivial matter of finding the area of that square.

Undoubtedly the Greeks' fascination with quadrature went far beyond the practical. For, if successfully accomplished, quadrature would impose the symmetric regularity of the square onto the asymmetric irregularity of an arbitrary plane figure. To those who sought a natural world governed by reason and order, there was much appeal in the process of replacing the asymmetric by the symmetric, the imperfect by the perfect, the irrational by the rational. In this sense, quadrature represented not only the triumph of human reason, but also the inherent simplicity and beauty of the universe itself.

Devising quadratures was thus a particularly fascinating problem for Greek mathematicians, and they produced clever geometric constructions to that end. As is often the case in mathematics, solutions can be approached in stages, by first squaring a reasonably "tame" figure and moving from there to the quadrature of more irregular, bizarre ones. The key initial step in this process is the quadrature of the rectangle, the procedure for which appears as Proposition 14 of Book II of Euclid's *Elements*, although it was surely known well before Euclid. We begin with this.

STEP 1 Quadrature of the rectangle (Figure 1.7)

Let *BCDE* be an arbitrary rectangle. We must construct, with compass and straightedge only, a square having area equal to that of *BCDE*. With the straightedge, extend line *BE* to the right, and use the compass to mark off segment *EF* with length equal to that of *ED*—that is, $\overline{EF} = \overline{ED}$. Next, bisect *BF* at *G* (an easy compass and straightedge construction), and with center *G* and radius $\overline{BG} = \overline{FG}$, describe a semicircle as shown. Finally, at *E*, construct line *EH* perpendicular to *BF*, where *H* is the point of intersection of the perpendicular and the semicircle, and from there construct square *EKLH*.

We now claim that the shaded square having side of length $\overline{EH}$—a figure we have just *constructed*—has area equal to that of the original rectangle *BCDE*.

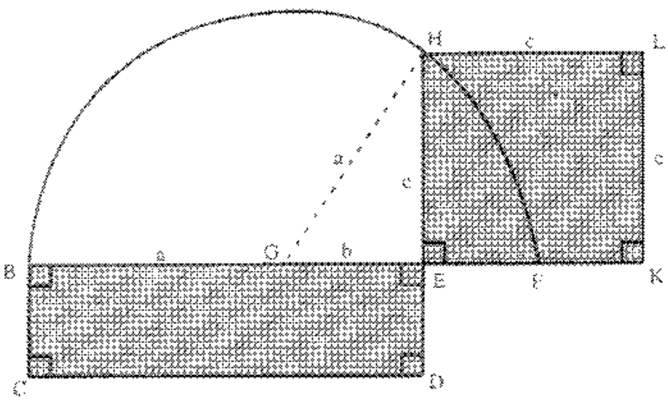

FIGURE 1.7

To verify this claim requires a bit of effort. For notational convenience, let a, b, and c be the lengths of segments HG, EG, and EH, respectively. Since $\triangle GEH$ is a right triangle by construction, the Pythagorean theorem gives us $a^2 = b^2 + c^2$, or equivalently $a^2 - b^2 = c^2$. Now clearly $\overline{FG} = \overline{BG} = \overline{HG} = a$, since all are radii of the semicircle. Thus, $\overline{EF} = \overline{FG} - \overline{EG} = a - b$ and $\overline{BE} = \overline{BG} + \overline{GE} = a + b$. It follows that

$$\begin{aligned} \text{Area (rectangle } BCDE) &= (\text{base}) \times (\text{height}) \\ &= (\overline{BE}) \times (\overline{ED}) \\ &= (\overline{BE}) \times (\overline{EF}), \text{ since we constructed } \overline{EF} = \overline{ED} \\ &= (a + b)(a - b) \text{ by the observations above} \\ &= a^2 - b^2 \\ &= c^2 = \text{Area (square } EKLH) \end{aligned}$$

Consequently, we have proved that the original rectangular area equals that of the shaded square which we *constructed* with compass and straightedge, and this completes the rectangle's quadrature.

With this done, the steps toward squaring more irregular regions come quickly.

STEP 2 Quadrature of the triangle (Figure 1.8)

Given $\triangle BCD$, construct a perpendicular from D meeting BC at point E. Of course, we call $\overline{DE}$ the triangle's "altitude" or "height" and know that the area of the triangle is ½(base) × (height) = ½$(\overline{BC}) \times (\overline{DE})$. If we bisect DE at F and construct a rectangle with $\overline{GH} = \overline{BC}$ and $\overline{HJ} = \overline{EF}$, we know that the rectangle's area is $(\overline{HJ}) \times (\overline{GH}) = (\overline{EF}) \times (\overline{BC}) = \frac{1}{2}(\overline{DE}) \times (\overline{BC}) = \text{area } (\triangle BCD)$. But we then apply Step 1 to construct a square equal in area to this rectangle, and so the square's area is also that of $\triangle BCD$. This completes the quadrature of the triangle.

We next move to the following very general situation.

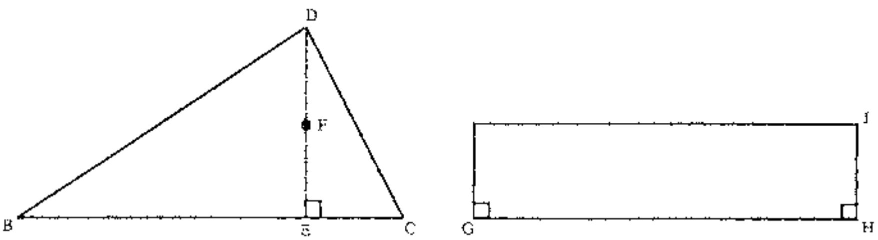

FIGURE 1.8

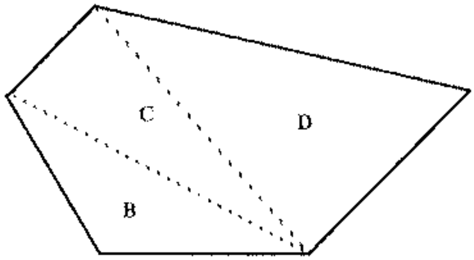

FIGURE 1.9

STEP 3 Quadrature of the polygon (Figure 1.9)

This time we begin with a general polygon, such as the one shown. By drawing diagonals, we subdivide it into a collection of triangles with areas **B**, **C**, and **D**, so that the total polygonal area is **B** + **C** + **D**.

Now triangles are known to be quadrable by Step 2, so we can construct squares with sides b, c, and d and areas **B**, **C**, and **D** (Figure 1.10). We then construct a right triangle with legs of length b and c, whose hypotenuse is of length x, where $x^2 = b^2 + c^2$. Next, we construct a right triangle with legs of length x and d and hypotenuse y, where we have $y^2 = x^2 + d^2$, and finally, the shaded square of side y (Figure 1.11).

Combining our facts, we see that

$$y^2 = x^2 + d^2 = (b^2 + c^2) + d^2 = \mathbf{B} + \mathbf{C} + \mathbf{D}$$

so that the area of the original polygon equals the area of the square having side y.

This procedure clearly could be adapted to the situation in which the polygon was divided by its diagonals into four, five, or any number of triangles. No matter what polygon we are given (see Figure 1.12), we can subdivide it into a set of triangles, square each one by Step 2, and use these individual squares and the Pythagorean theorem to build a

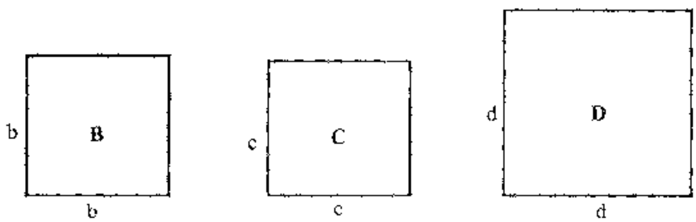

FIGURE 1.10

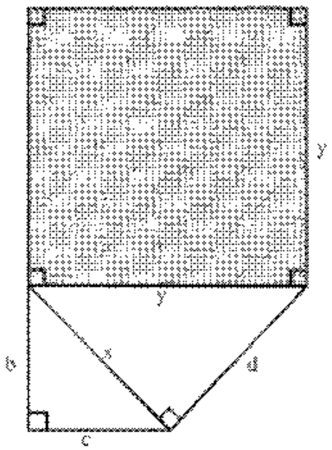

FIGURE 1.11

large square with area equal to that of the polygon. In short, polygons are quadrable.

By an analogous technique we could likewise square a figure whose area was the *difference* between—and not the sum of—two quadrable areas. That is, suppose we knew that area **E** was the difference between areas **F** and **G**, and we had already constructed squares of sides f and g with areas as shown in Figure 1.13. Then we would construct a right triangle with hypotenuse f and leg g. We let e be the length of the other leg and construct a square with side e. We then have

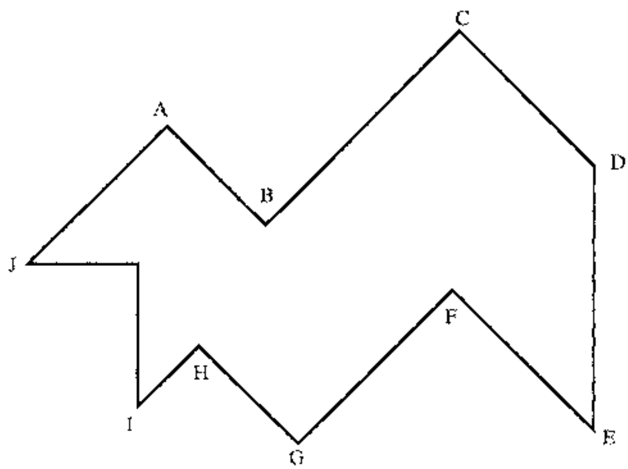

FIGURE 1.12

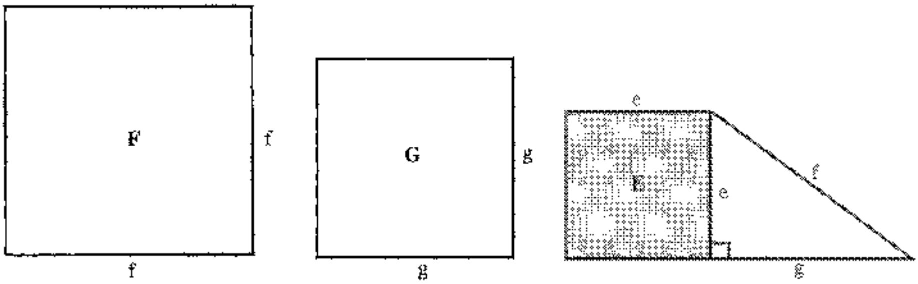

FIGURE 1.13

$$\text{Area (square)} = e^2 = f^2 - g^2 = \mathbf{F} - \mathbf{G} = \mathbf{E}$$

so that area **E** is likewise quadrable.

With the foregoing techniques, the Greeks of Hippocrates' day could square wildly irregular polygons. But this triumph was tempered by the fact that such figures are *rectilinear*—that is, their sides, although numerous and meeting at all sorts of strange angles, are merely straight lines. Far more challenging was the issue of whether figures with curved boundaries—the so-called *curvilinear* figures—were likewise quadrable. Initially, this must have seemed unlikely, for there is no obvious means to straighten out curved lines with compass and straightedge. It must therefore have been quite unexpected when Hippocrates of Chios succeeded in squaring a curvilinear figure known as a "lune" in the fifth century B.C.

Great Theorem: The Quadrature of the Lune

A lune is a plane figure bounded by two circular arcs—that is, a crescent. Hippocrates did not square all such figures but rather a particular lune he had carefully constructed. (As will be shown in the Epilogue, this distinction seemed to be the source of some misunderstanding in later Greek geometry.) His argument rested upon three preliminary results:

- The Pythagorean theorem
- An angle inscribed in a semicircle is right.
- The areas of two circles or semicircles are to each other as the squares on their diameters.

$$\frac{\text{Area (semicircle 1)}}{\text{Area (semicircle 2)}} = \frac{d^2}{D^2}$$

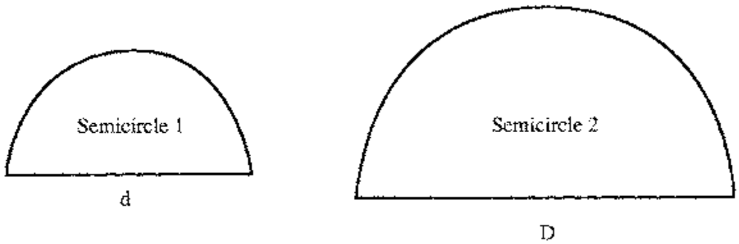

FIGURE 1.14

The first two of these results were well known long before Hippocrates came upon the scene. The last proposition, on the other hand, is considerably more sophisticated. It gives a comparison of the areas of two circles or semicircles based on the relative areas of the squares constructed on their diameters (see Figure 1.14). For instance, if one semicircle has five times the diameter of another, the former has 25 times the area of the latter. This proposition presents math historians with a problem, for there is widespread doubt that Hippocrates actually had a valid proof. He may well have *thought* he could prove it, but modern scholars generally feel that this theorem—which later appeared as the second proposition in Book XII of Euclid's *Elements*—presented logical difficulties far beyond what Hippocrates would have been able to handle. (A derivation of this result is presented in Chapter 4.)

That aside, we now consider Hippocrates' proof. Begin with a semicircle having center O and radius $\overline{AO} = \overline{OB}$, as shown in Figure 1.15. Construct OC perpendicular to AB, with point C on the semicircle, and draw lines AC and BC. Bisect AC at D, and using $\overline{AD}$ as a radius and D as center, draw semicircle AEC, thus creating lune $AECF$, which is shaded in the diagram.

Hippocrates' plan of attack was simple yet brilliant. He first had to establish that the lune in question had *precisely* the same area as the shaded $\triangle AOC$. With this behind him, he could then apply the known fact that triangles can be squared to conclude that the lune can be squared as well. The details of the classic argument follow:

THEOREM Lune $AECF$ is quadrable.

PROOF Note that $\angle ACB$ is right since it is inscribed in a semicircle. Triangles AOC and BOC are congruent by the "side-angle-side" congruence scheme, and consequently $\overline{AC} = \overline{BC}$. We thus apply the Pythagorean theorem to get

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 = 2(\overline{AC})^2$$

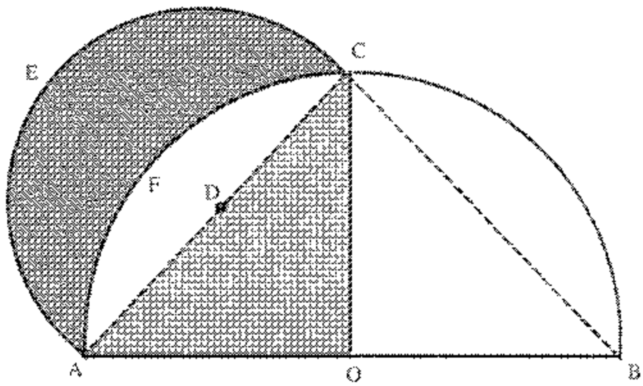

FIGURE 1.15

Because AB is the diameter of semicircle ACB, and AC is the diameter of semicircle AEC, we can apply the third principle above to get

$$\frac{\text{Area (semicircle } AEC)}{\text{Area (semicircle } ACB)} = \frac{(\overline{AC})^2}{(\overline{AB})^2} = \frac{(\overline{AC})^2}{2(\overline{AC})^2} = \frac{1}{2}$$

In other words, semicircle AEC has half the area of semicircle ACB.

But we now look at quadrant $AFCO$ (a "quadrant" is a quarter of a circle). Clearly this quadrant also has half the area of semicircle ACB, and we immediately conclude that

$$\text{Area (semicircle } AEC) = \text{Area (quadrant } AFCO)$$

Finally, we need only subtract from each of these figures their shared region $AFCD$, as in Figure 1.16. This leaves

$$\text{Area (semicircle } AEC) - \text{Area (region } AFCD) = \text{Area (quadrant } AFCO) - \text{Area (region } AFCD)$$

and a quick look at the diagram verifies that this amounts to

$$\text{Area (lune } AECF) = \text{Area } (\triangle ACO)$$

But, as we have seen, we can construct a square whose area equals that of the triangle, and thus equals that of the lune as well. This is the quadrature we sought.

Q.E.D.

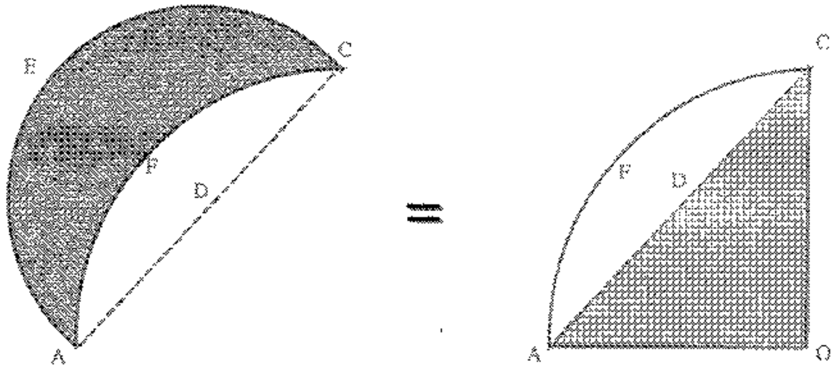

FIGURE 1.16

Here indeed was a mathematical triumph. Looking back from his fifth century vantage point, the commentator Proclus (A.D. 410–485) would write that Hippocrates of Chios ". . . squared the lune and made many other discoveries in geometry, being a man of genius when it came to constructions, if ever there was one."

Epilogue

With Hippocrates' success at squaring the lune, Greek mathematicians must have been optimistic about squaring that most perfect curvilinear figure, the circle. The ancients devoted much time to this problem, and some later writers attributed an attempt to Hippocrates himself, although the matter is again clouded by the difficulties of assessing commentaries upon commentaries. Nonetheless, Simplicius, writing in the fifth century, quoted his predecessor Alexander Aphrodisiensis (ca. A.D. 210) as saying that Hippocrates had claimed that he could square the circle. Piecing together the evidence, we gather that this is the sort of argument Alexander had in mind:

Begin with an arbitrary circle with diameter *AB*. Construct a large circle with center *O* and a diameter *CD* that is *twice AB*. Within the larger circle, inscribe a regular hexagon by the known technique of letting each side be the circle's radius. That is,

$$\overline{CE} = \overline{EF} = \overline{FD} = \overline{DG} = \overline{GH} = \overline{HC} = \overline{OC}$$

It is important to note that each of these segments, being the radius of the larger circle, also has length $\overline{AB}$. Then, using the six segments as diameters, construct the six semicircles shown in Figure 1.17. This gen-

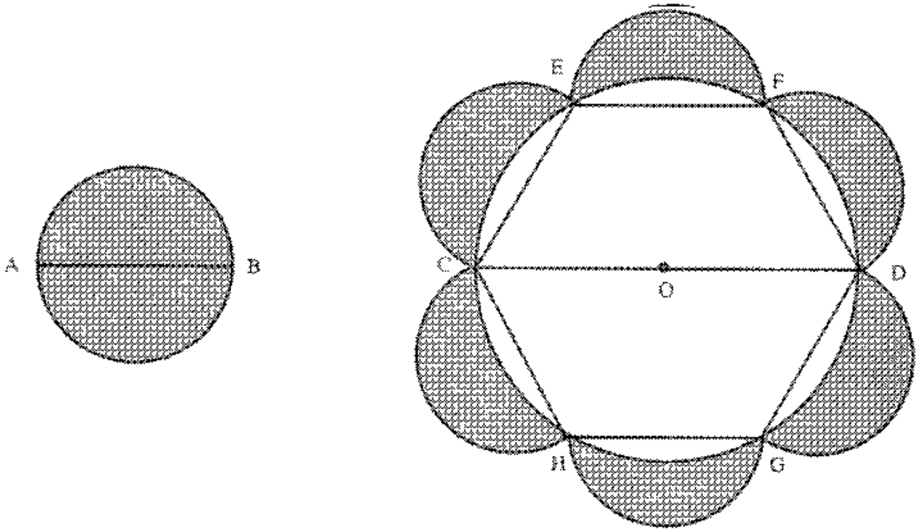

FIGURE 1.17

erates the shaded region composed of the six lunes and the circle upon *AB*.

Next imagine decomposing the figure on the right in two different ways: first, as the regular hexagon *CEFDGH* plus the six semicircles; second, as the large circle plus the six lunes. Obviously these yield the same overall area since they arise from the decomposition of the same figure. But the six semicircles amount to three full circles, each with diameter equal to $\overline{AB}$. Thus,

$$\text{Area (hexagon)} + 3 \text{ Area (circle on } AB) = \text{Area (large circle)} + \text{Area (six lunes)}$$

Now the large circle, having twice the diameter, must have $2^2 = 4$ times the area of its smaller counterpart. Hence,

$$\text{Area (hexagon)} + 3 \text{ Area (circle on } AB) = 4 \text{ Area (circle on } AB) + \text{Area (six lunes)}$$

and, subtracting "3 Area (circle on *AB*)" from both sides of this equation, we get

$$\text{Area (hexagon)} = \text{Area (circle on } AB) + \text{Area (six lunes)} \qquad \text{or}$$

$$\text{Area (circle on } AB) = \text{Area (hexagon)} - \text{Area (six lunes)}$$

According to Alexander, Hippocrates then reasoned as follows: The hexagon, being a polygon, can be squared; each lune, from the preceding argument, can likewise be squared, and so, by the additive process, a square whose area is the sum of the half-dozen lunar areas can be constructed. Thus, the circle on *AB* can be squared by the simple process of subtracting areas that we noted earlier.

Unfortunately, there is a glaring flaw in this argument, as Alexander was quick to point out: the lune that Hippocrates squared in our great theorem was not constructed along the side of a regular inscribed *hexagon* but rather along the side of an inscribed *square*. In other words, Hippocrates never provided a process for squaring the kind of lune that arose here.

Most modern scholars doubt that a mathematician of Hippocrates' stature could have bumbled into such an error. It is more likely that Alexander or Simplicius or any of the other intermediaries who passed along Hippocrates' original argument garbled it in some manner. We will probably never know the whole story. Nonetheless, it is likely that this kind of reasoning supported the idea that the quadrature of the circle should somehow be possible. If the preceding argument did not quite do the job, then maybe just a little more effort and a little more insight might have brought success.

But it was not to be. For generations, for centuries, the challenge to square the circle went unmet, although not for any lack of trying. Countless solutions were proposed involving a multitude of ingenious twists and turns. Yet in the end, each was found to contain an error. Gradually, mathematicians began to suspect that there was an intrinsic impossibility in the circle's quadrature with compass and straightedge. Of course, the mere lack of a correct argument, even after 2000 years of trying, did not establish its impossibility; perhaps mathematicians had just not been clever enough to find their way through the geometric thickets. Further, if the quadrature of the circle was impossible, this fact would have to be *proved* with all the logical rigor of any other theorem, and it was by no means clear how to go about such a proof.

One point should be stressed. No one doubted that, given a circle, there *exists* a square of equal area. For instance, consider a given, fixed circle and a small square spot of light projecting on the page beside it, the square's area being substantially less than that of the circle. If we continuously move the projector away from the page, thereby gradually increasing the area of the square image, we eventually arrive at a square whose area exceeds that of the circle. Appealing to the intuitive notion of "continuous growth," we can correctly conclude that at some intermediate instant, the area of the square exactly equaled the area of the circle.

But this is all beside the point. Remember that the crucial issue is not whether such a square *exists*, but whether it can be *constructed* with compass and straightedge. It is here that the difficulties appeared, for the geometer was limited to these two particular tools; moving spotlights around was simply against the rules.

The problem of squaring the circle remained unresolved from the time of Hippocrates until just over a century ago. At last, in 1882, the German mathematician Ferdinand Lindemann (1852–1939) succeeded in proving unequivocally that the quadrature of the circle was an impossibility. The technical details of his proof are quite advanced and go well beyond the scope of this book. However, the following is a brief synopsis of how it was that Lindemann answered this age-old question.

He did it by translating the issue from the realm of geometry to the realm of number. If we imagine the collection of all real numbers, depicted in the schematic diagram in Figure 1.18 as being contained within the large rectangle, we can subdivide them into two exhaustive and mutually exclusive categories—the algebraic numbers and the transcendental numbers.

By definition, a real number is ***algebraic*** if it is the solution to some polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

where all the coefficients a_n, a_{n-1}, . . . , a_2, a_1, and a_0 are integers. Thus, the rational number $\frac{2}{3}$ is algebraic since it is the solution of the polynomial equation $3x - 2 = 0$; the irrational $\sqrt{2}$ is likewise algebraic since it satisfies $x^2 - 2 = 0$; and even $\sqrt[3]{1 + \sqrt{5}}$ is algebraic since it satisfies $x^6 - 2x^3 - 4 = 0$. Note that, in each case, these polynomials have integer coefficients.

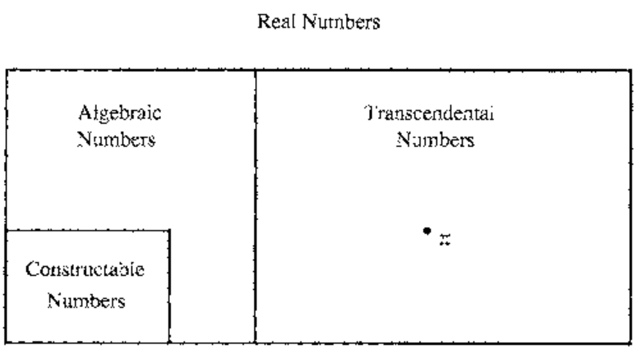

FIGURE 1.18

Less formally, we can think of the algebraic numbers as the "easy" or "familiar" quantities encountered in arithmetic and elementary algebra. For instance, all whole numbers are algebraic, as are all fractions and their square roots, cube roots, and so on.

By contrast, a number is ***transcendental*** if it is not algebraic—that is, if it is not the solution of *any* polynomial equation with integer coefficients. Such numbers are much more complicated than their relatively simple algebraic cousins. By the very definition, it is clear that any real number is either algebraic or transcendental but not both. This is a stark dichotomy, rather like any person's being either a man or a woman, with no middle ground.

Now begin with a unit length (that is, a length to represent the number "1") and keep track of what other lengths we can produce by straightedge and compass construction. It turns out that the totality of all possible constructible lengths, while vast, does not include every real number. For instance, starting from a length of 1, we can construct lengths of 2, 3, 4, and so on, as well as rational lengths like $\frac{1}{2}$, $\frac{2}{5}$, $\frac{13}{10}$ and even irrational lengths involving only square roots, like $\sqrt{2}$ or $\sqrt{5}$. Further, if we can construct two magnitudes, we can construct their sum, difference, product, or quotient. Putting all of these operations together, we see that more complex expressions such as

$$\sqrt{\frac{6-2\sqrt{2}}{1+\sqrt{4+\sqrt{23-\sqrt{7}}}}}$$

are actually constructible lengths.

This vast array of constructible numbers forms a subset of the algebraic numbers, even as the collection of all bald men forms a subset of all men. As Figure 1.18 suggests, these constructible quantities are strictly embedded within the algebraic numbers. The crucial point is that no member of the transcendental numbers can be constructed with compass and straightedge. (If we stretch our analogy one step further, this corresponds to the statement that no woman will be found among the bald men.)

All of this was known at the time when Lindemann took up the problem. Building on the efforts of his predecessors, particularly the brilliant French mathematician Charles Hermite (1822–1901), Lindemann attacked the famous number π. (In elementary geometry we encounter π as the ratio of a circle's circumference to its diameter; we shall have much more to say about this critical constant in Chapter 4.) Lindemann's triumph was to prove that π is transcendental. In other words, π is *not* algebraic and thus is not constructible. This, in turn, tells us that $\sqrt{\pi}$ is not constructible either, since if we could construct $\sqrt{\pi}$, we could, with a few more swipes of the compass and straightedge, construct π as well.

At first, this numerical discovery may seem to have little bearing on the geometry of circle-squaring, but we shall see that it provided the missing piece of the puzzle.

THEOREM The quadrature of the circle is impossible

PROOF Let us assume, for the sake of eventual contradiction, that circles *can* be squared. We get out our compass and easily construct a circle having radius $r = 1$. Its area is thus $\pi r^2 = \pi$. If circles are quadrable, as we have temporarily assumed, then we employ our compass and straightedge, work feverishly slashing arcs and drawing lines, and eventually, after only a finite number of such steps, end up with a square that also has area π, as indicated in Figure 1.19. In this process, we would have had to *construct* the square, which of course would require us to have constructed each of its four sides. Call the length of the square's side x. Then we see that

$$\pi = \text{Area of circle} = \text{Area of square} = x^2$$

and so the length $x = \sqrt{\pi}$ would be constructible with compass and straightedge. But, as we have noted, no such construction for $\sqrt{\pi}$ is possible.

What went wrong? Tracing back through the argument and looking for the source of our contradiction, we find it can only be the initial assumption, namely, that circles can be squared. As a consequence, we must reject this and conclude, once and for all, that the quadrature of the circle is a logical impossibility!

Q.E.D.

Lindemann's discovery, then, showed that squaring the circle—a quest that occupied mathematicians from Hippocrates' day until modern

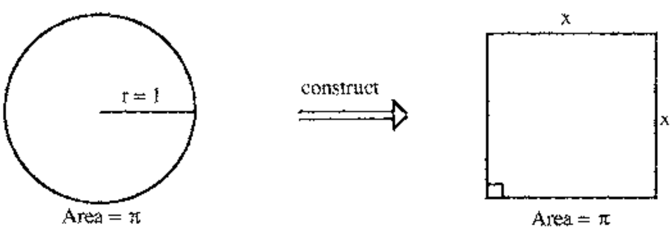

FIGURE 1.19

times—was a lost cause. All of the suggestive proofs, all of the promising clues starting with the quadrature of the lune, turned out to be illusory. Compass and straightedge alone are inadequate for turning circles into squares.

And what did history have to say about lunes? Our great theorem above showed Hippocrates squaring a particular lune, and he managed to do two other kinds as well. Thus, as of 440 B.C., three types of lunes were known to be quadrable. At this point, progress stopped for over two millennia until, in 1771, the great Leonhard Euler (1707–1783)—who will be the object of our attention in Chapters 9 and 10—found two more kinds of lunes that were squarable. There the matter rested until the twentieth century when N. G. Tschebatorew and A. W. Dorodnow proved that these five are the *only* squarable lunes! All other lunes, such as the one that generated Alexander's harsh criticism cited earlier, share with the circle the impossibility of being squared.

So the final chapter in the story of Hippocrates and his lunes has been written, and it has been a rather perverse story at that. At first, intuition suggested that curved figures could not be squared with compass and straightedge. Hippocrates' lunes turned intuition upside down, and the search was on for quadratures galore. But, in the end, the negative results of Lindemann, Tschebatorew, and Dorodnow showed that intuition had not been so flawed after all. The quadrature of curvilinear figures, far from being the norm, must forever remain the exception.