survey of probability theory

CHAPTER

1 introduction

The science of statistics deals with the development of theories and techniques which are appropriate for making inferences under the conditions of uncertainty and partial ignorance that necessarily exist in a wide range of activities. Current statistical practice depends mainly on the formulation of probability models of various physical systems, on methods of collecting and analyzing numerical data, and on the design of efficient and informative experiments. The term *decision theory* refers to the class of statistical problems in which the statistician must gain information about certain critical parameters in order to be able to make effective decisions in situations where the consequences of his decisions will depend on the values of these parameters. The theory of optimal statistical decisions that will be developed in this book can be of value in many types of activities and has been successfully applied in a wide variety of problems. This theory is commonly called subjective statistical decision theory or Bayesian statistical decision theory.

In most of the book, the decision-making aspects of each problem can be formalized through an explicit specification covering the available decisions, the items of cost or gain involved in making these decisions, and the relevant probability distributions. The theory and techniques on which this specification is based will be discussed thoroughly. The dis-

cussions will include mathematical methods of characterizing the statistician's information and uncertainty relating to the values of the parameters and also methods of changing this characterization as additional information about the parameters is acquired.

Subjective, or Bayesian, statistical decision theory is applicable to those problems in which the information and uncertainty about the parameters can, at any time, be summarized by a probability distribution of their possible values. Therefore, this book will deal only with those statistical decision problems which meet the following two requirements: (1) The conditions can be formulated in terms of a manageable number of parameters. (2) Although the values of these parameters are not known exactly, any uncertainty about the values can be represented by a suitable probability distribution. In connection with these requirements, the following two points should be kept in mind. First, the number of parameters that would be manageable in a given situation is, to a large extent, a function of the current state of computer science and technology (obviously, this number is larger now than ever before). Second, there is considerable controversy among statisticians and other persons who study the foundations of probability as to whether or not uncertainty about the value of a particular parameter can be represented by a probability distribution. Some authorities believe that such a distribution is appropriate only when values of the parameter clearly have relative frequencies. Another group of authorities maintains that probability is a logical concept which can be applied to parameters in a much wider class of problems. Furthermore, these authorities believe that in each such problem there is a uniquely defined distribution which is appropriate for a particular parameter and must necessarily be assigned to that parameter. Authorities in a third group believe that essentially all probability distributions are subjective and that whenever anyone carries out a statistical investigation involving a parameter, he can represent his uncertainty pertaining to the values of that parameter by a suitable probability distribution. There are also other distinctive opinions and many modifications of the three here mentioned. Authors of some of the well-known books on these matters are, in alphabetical order, Carnap (1962), Fisher (1956), Good (1950), Jeffreys (1961), Keynes (1921), Nagel (1937), Reichenbach (1949), Savage (1954), and von Mises (1957).

It will be assumed in all problems in this book that each parameter can be assigned a particular probability distribution. Reasoning of the type given in Chap. 6 should be relevant in a broad class of decision problems, and this reasoning suggests how suitable distributions might be assigned in such problems. Even though there is a great deal of controversy in regard to certain aspects of the foundations of probability and statistics, the differences of opinion do not extend to the propriety of

assigning probability distributions to the parameters in many of the specific problems that will be studied here. Moreover, in problems where the parameters do have appropriate probability distributions, including many important problems of common occurrence, it is generally agreed that the theory and techniques of statistical decision making to be presented in this book are applicable, correct, and useful.

Since the modern theory of mathematical probability is the basic tool in the development of statistical methods, the next four chapters are devoted to a survey of the portions of that theory which will be used later in the book. In order that this survey may be somewhat condensed and still be essentially complete with respect to the results which will be needed, proofs and motivation will generally be omitted. Readers for whom this brief survey is insufficient should consult any of the standard texts on probability and statistics. Some recent texts at an introductory level are Brunk (1965), Feller (1957), Freeman (1963), Harris (1966), Hogg and Craig (1965), Lindgren (1962), McCord and Moroney (1964), Mood and Graybill (1963), Papoulis (1965), Parzen (1960), Pfeiffer (1965), and Tucker (1962). Some others at a more advanced level are Cramér (1946), Feller (1966), Fisz (1963), Gnedenko (1962), Krickeberg (1965), Loève (1963), Rao (1965), and Wilks (1962).