CHAPTER 1

INTRODUCTION

The finite element method (FEM) is a numerical technique for the approximate solution of boundary value problems arising from the differential equation-based mathematical modeling of physical phenomena. Early developments and applications of the method were prompted by problems in structural analysis in the 1960s and 1970s [1], [2]. Its introduction in the discipline of computational electromagnetics as a tool for the numerical solution of electromagnetic boundary value problems occurred in the 1970s. For a historical overview of the application of the method to the solution of electromagnetic boundary value problems the reader is referred to [3].

While the emphasis of early applications of finite elements to electromagnetics was on static, quasi-static, and guided-wave eigenvalue problems [4], the modeling versatility of the method led several researchers to pursue ways in which it could be used for the solution of electrodynamic boundary value problems, particularly those concerned with electromagnetic radiation and scattering phenomena in unbounded regions. However, it was not until the mid 1980s that FEM started becoming widely accepted by electromagnetic researchers and designers alike as an important computer-aided modeling tool for electromagnetic analysis.

Since then, advances in computing technology, combined with aggressive research into sparse matrix solution, mesh generation technology and the mathematical attributes of finite element approximations of electromagnetic boundary value problems, have helped establish FEM as one of the most versatile and effective numerical techniques for computer-aided electromagnetic analysis and design.

The maturity of the finite element method as a numerical tool for electromagnetic analysis becomes evident from a brief overview of its very rich literature that includes its application to quantitative electromagnetic waveguide analysis for the design and optimization of microwave, millimeter-wave and optical devices, components and systems, to electromagnetic scattering for target identification, to electromagnetic radiation for antenna design [3]-[6].

The purpose of this introductory chapter is three-fold. First, a brief overview is provided of the two most commonly used approaches in practice for the development of finite element approximations of a differential equation-based boundary value problem. Second, the electromagnetic boundary value problem of interest in this book is defined and discussed briefly. This is followed by the discussion of current and future challenges in finite element-based electromagnetic field modeling. This discussion helps motivate the methodologies and algorithms presented in the following chapters.

1.1 STATEMENT OF THE BOUNDARY VALUE PROBLEM

A typical boundary value problem (BVP) is stated in terms of its governing differential equation over a domain Ω,

$$L\phi = f \tag{1.1}$$

together with appropriate boundary conditions imposed on the boundary Γ that encloses the domain Ω. In (1.1), L is the governing differential operator, ϕ is the physical field quantity that is solved for, and f is the forcing term (or excitation).

The analytic solution of (1.1) is not possible in the general case where the domain Ω involves nonseparable geometries and materials exhibiting arbitrary position dependence and anisotropy. More specifically, in the case of electromagnetic BVPs, analytic solutions are available only for a few classes of BVPs involving separable geometries with mostly homogeneous materials [7]-[11].

For those cases for which an analytic solution of (1.1) is not possible, a numerical method must be used. Among the various techniques possible [12] [13], the method of finite elements is most attractive in the presence of substantial geometric and material complexity. The basic steps of two of the most commonly used methodologies for the development of a finite element approximation of (1.1), namely, the Ritz and the Petrov-Galerkin's method, are discussed next.

1.2 RITZ FINITE ELEMENT METHOD

The Ritz method is a variational method where the solution to (1.1) is formulated in terms of the minimization of a functional. More specifically, a functional is derived, the minimum of which corresponds to the solution of (1.1), subject to the given boundary conditions [13]. For the purpose of keeping the mathematical development simple, we refrain from using the Hilbert space and formal linear operator theory nomenclature in this brief overview of machinery of variational methods. The reader is referred to [13] and [14] for an exposure to the rigorous mathematical framework in which variational methods are founded and developed.

For the purposes of this introductory discussion we consider only the case where the operator and the unknown and forcing functions in (1.1) are scalar. Furthermore, the properties of L are further constrained in a manner best described with the aid of the *inner* *product* of two functions, ϕ and ψ, defined as follows:

$$\langle \phi, \psi \rangle = \iiint_{\Omega} \phi\, \psi dv. \tag{1.2}$$

The operator L is said to be *self-adjoint* if the following equality holds,

$$\langle L\phi, \psi \rangle = \langle \phi, L\psi \rangle. \tag{1.3}$$

The operator L is said to be *positive definite* if

$$\langle L\phi, \phi \rangle = \begin{cases} > 0, & \phi \neq 0, \\ = 0, & \phi = 0. \end{cases} \tag{1.4}$$

If L in (1.1) is self-adjoint and positive definite then the solution to (1.1) can be obtained through the minimization of the functional [3]

$$F(\phi) = \frac{1}{2}\langle L\phi, \phi \rangle - \frac{1}{2}\langle \phi, f \rangle - \frac{1}{2}\langle f, \phi \rangle. \tag{1.5}$$

To show that the function, ϕ, that minimizes (1.5) is also the solution to (1.1), let us consider the change, δF, in F, caused by an arbitrary variation, $\delta\phi$, in ϕ,

$$\begin{aligned} \delta F &= F(\phi + \delta\phi) - F(\phi) \\ &= \frac{1}{2}\langle L\delta\phi, \phi \rangle + \frac{1}{2}\langle L\phi, \delta\phi \rangle - \frac{1}{2}\langle \delta\phi, f \rangle - \frac{1}{2}\langle f, \delta\phi \rangle \\ &= \frac{1}{2}\langle \delta\phi, L\phi \rangle - \frac{1}{2}\langle \delta\phi, f \rangle + \frac{1}{2}\langle L\phi, \delta\phi \rangle - \frac{1}{2}\langle f, \delta\phi \rangle \\ &= \langle \delta\phi, (L\phi - f) \rangle. \end{aligned} \tag{1.6}$$

Since $\delta\phi$ is arbitrary, it is evident from the above result that the extreme point of the functional $F(\phi)$ corresponds to the solution of (1.1).

The FEM approximation and its solution are obtained from (1.5) in the following manner. The domain Ω is discretized into a number of smaller sub-domains, which are referred to as *elements*. For example, for a one-dimensional (1D) domain, short line segments, interconnected to cover the entire domain, may be used as elements. For a two-dimensional (2D) domain, the most commonly used elements are the triangle and the quadrilateral. Finally, for a three-dimensional (3D) domain, tetrahedra, triangular prisms, or hexahedron bricks are used as elements. In all cases, connectivity of the elements in a manner such that the entire domain is covered by the resulting *finite-element mesh* or *finite-element grid*, must be ensured. Inside each element the unknown physical field quantity ϕ is expanded in terms of known polynomial functions, w_j. Hence, an approximation is obtained for the unknown field over Ω,

$$\hat{\phi} = \sum_{j=1}^{N} w_j c_j = w^T c, \tag{1.7}$$

where the vector c contains the coefficients c_j, $j = 1, 2, \ldots, N$, which are the unknown degrees of freedom in the approximation, while the vector w contains the expansion functions w_j, $j = 1, 2, \ldots, N$. Their number, N, is dictated by various factors, among which geometric/material complexity and accuracy of the approximation are the most important. Substitution of (1.7) into (1.5) yields

$$F(c) = \frac{1}{2}c^T \left(\iiint_{\Omega} wLw^T dv \right) c - \left(\iiint_{\Omega} fw^T dv \right) c. \tag{1.8}$$

The minimization of $F(c)$ and, hence, the calculation of the expansion coefficients in (1.7), reduces to the solution of the linear system of equations that is obtained by setting all first derivatives of $F(c)$ with respect to c_j, $j = 1, 2, \ldots, N$, equal to zero,

$$\left(\iiint_{\Omega} wLw^T dv\right) c = \iiint_{\Omega} wf dv. \tag{1.9}$$

A more in-depth discussion of the mathematical attributes of the Ritz method can be found in [13]. Numerous examples from its utilization for the development of finite element approximations of both scalar and vector electromagnetic BVPs are given in [3].

1.3 PETROV-GALERKIN'S FINITE ELEMENT METHOD

The Petrov-Galerkin's method relies upon the idea that for $\hat{\phi}$ of (1.7) to be a good approximation to the solution of (1.1) the integration of the *residual* obtained by substituting $\hat{\phi}$ in (1.1) over the domain Ω, weighted by a set of appropriate *weighting* (or *testing*) functions, is zero. Because of this weighting process, the method is also called the *method of weighted residuals*, since it seeks an approximate solution to (1.1) by enforcing its weighted residual to be, on the average, zero over the domain of interest.

Substitution of (1.7) in (1.1) yields to the following residual error,

$$r = Lw^T c - f. \tag{1.10}$$

Let u_j, $j = 1, 2, \ldots, N$, be a set of testing functions. The linear system of equations for the calculation of the expansion coefficients in (1.7) is obtained by requiring that

$$\iiint_{\Omega} u_j r dv = 0, \quad j = 1, 2, \ldots, N. \tag{1.11}$$

The resulting system has the form

$$\left(\iiint_{\Omega} uLw^T dv\right) c = \iiint_{\Omega} uf dv, \tag{1.12}$$

where the vector u contains the N testing functions, u_j, $j = 1, 2, \ldots, N$.

The special case where the testing functions are taken to be the same with the expansion functions is most commonly used in practice and is referred to as *Galerkin's method.* Clearly, in this case (1.12) is identical to (1.9).

It is apparent that the development of the finite element approximation via the Petrov-Galerkin procedure is more straightforward than the Ritz process. The simplicity of the process is most attractive when the development of a functional for (1.1) is hindered by the complexity of the operator L. The Galerkin's method will be used throughout the book for the development of the finite element approximation to the electromagnetic BVPs of interest.

1.4 TIME-HARMONIC MAXWELL'S EQUATIONS AND BOUNDARY CONDITIONS

Of interest to this work is the solution of linear electromagnetic BVPs under the assumption of time-harmonic, sinusoidal excitation of angular frequency, $\omega = 2\pi f$. Thus, with the time convention $e^{j\omega t}$, $j = \sqrt{-1}$, assumed and suppressed for simplicity, the complex phasor form of the governing system of Maxwell's equations becomes

$$\begin{aligned}
&\nabla \times \vec{E} = -j\omega\vec{B} && \text{(Faraday's law)}\\
&\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}_c + \vec{J}_v && \text{(Ampere's law)}\\
&\nabla \cdot \vec{D} = \rho_v && \text{(Gauss' electric field law)}\\
&\nabla \cdot \vec{B} = 0 && \text{(Gauss' magnetic field law).}
\end{aligned} \tag{1.13}$$

In the above equation, the familiar notation $\vec{E}$, $\vec{H}$, $\vec{D}$, $\vec{B}$ has been used for the electric field intensity, magnetic field intensity, electric flux density, and magnetic flux density, respectively. The *conduction* electric current density, $\vec{J}_c$, accounts for induced current flow in the medium due to the presence of conductor and/or dielectric loss, while the electric current density $\vec{J}_v$ represents the impressed current sources in the domain of interest. Similarly, electric charge density ρ_v represents the impressed electric charge in the domain.

Maxwell's equations are consistent with the conservation of charge statement

$$\nabla \cdot \left(\vec{J}_c + \vec{J}_v\right) = -j\omega\rho_v. \tag{1.14}$$

This is easily seen by observing that this result is readily obtained from Ampere's law by taking the divergence of both sides and making use of Gauss' law for the electric field.

The properties of the media inside the domain in which a solution of (1.13) is sought enter the BVP through the *constitutive relations* between electromagnetic field quantities and their associated flux densities. The constitutive relations are dictated by the macroscopic electromagnetic properties of the media of interest. For the case of simple, linear media, these relations are simply given by

$$\vec{D} = \bar{\epsilon} \cdot \vec{E}, \quad \vec{B} = \bar{\mu} \cdot \vec{H}, \quad \vec{J}_c = \bar{\sigma} \cdot \vec{E}, \tag{1.15}$$

where the tensors $\bar{\epsilon}$, $\bar{\mu}$, and $\bar{\sigma}$ are, respectively, the electric permittivity, magnetic permeability, and electric conductivity tensors of the media. Their linearity is understood to mean that they are independent of the field intensities. However, they are allowed to exhibit frequency and position dependence.

1.4.1 Boundary conditions at material interfaces

As it will be discussed in detail in Chapter 3, all four equations in (1.13) are needed for the well-posed definition of a uniquely solvable electromagnetic BVP. In addition, for the electromagnetic BVP to be well-posed (1.13) must be complemented by appropriate boundary conditions at material interfaces, across which the constitutive parameters exhibit discontinuities, and at the enclosing boundary, Γ, of the domain Ω.

At material interfaces the following set of boundary conditions hold

$$\begin{aligned}
\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) &= 0\\
\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) &= \rho_s\\
\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) &= \vec{J}_s\\
\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) &= 0,
\end{aligned} \tag{1.16}$$

where $\hat{n}$ is the unit normal on the interface (taken to be in the direction from Medium 2 toward Medium 1), and $\vec{J}_s$ and ρ_s are, respectively, any surface electric current and charge densities present on the interface. These equations simplify for the case where one of the media is a *perfect* electric conductor. A perfect electric conductor is an idealization of a highly conducting medium, obtained in the limit where the electric conductivity is assumed to become infinite. In this limit, and under the assumption of zero-field initial conditions, both electric and magnetic fields are identically zero inside the conductor. Hence, with Medium 2 assumed to be a perfect electric conductor, (1.16) simplify as follows:

$$\begin{aligned} \hat{n} \times \vec{E}_1 &= 0 \\ \hat{n} \cdot \vec{D}_1 &= \rho_s \\ \hat{n} \times \vec{H}_1 &= \vec{J}_s \\ \hat{n} \cdot \vec{B}_1 &= 0. \end{aligned} \tag{1.17}$$

It is noted that in this case the electric current and charge densities on the perfectly conducting surface are induced by the presence of non-zero fields in the exterior of the conductor (i.e., in Medium 1).

1.4.2 Boundary conditions at the enclosing boundary

With regards to boundary conditions at the enclosing boundary Γ, the assignment of appropriate boundary conditions is dictated by the requirement that the solution of the resulting BVP is unique. The pertinent conditions are well-documented in the electromagnetic literature (e.g., [7], [8]), and require that, in the presence of some loss in Ω, the solution of (1.13) subject to (1.16) is uniquely defined provided that either the tangential components of the electric field are specified over Γ, or the tangential components of the magnetic field are specified over Γ, or the tangential components of the electric field are specified over part of Γ and the tangential components of the magnetic field are specified over the remaining part.

The final case of a boundary condition of relevance to the finite element-based solution of electromagnetic boundary value problems is the case of a *surface impedance condition*, where a relationship is imposed between the tangential components of the electric and the magnetic fields on the enclosing boundary, Γ, as follows:

$$\vec{E}_t = \hat{t}_1 E_{t_1} + \hat{t}_2 E_{t_2} = Z_s \vec{H} \times \hat{n}, \tag{1.18}$$

where, $\vec{E}_t$ denotes the tangential components of $\vec{E}$ on the conductor surface, $\hat{n}$ is the inward-pointing unit normal on the conductor surface, and the tangential unit vectors $\hat{t}_1$, $\hat{t}_2$, are such that $\hat{t}_1 \times \hat{t}_2 = \hat{n}$. The linear surface impedance Z_s is, in general, frequency and position dependent.

Surface impedance conditions are extremely useful in finite element applications since they serve as effective means for containing the complexity of the solution domain Ω. For example, consider the case where a portion of the domain of interest is occupied by a good conductor of conductivity high enough for magnitude of the displacement current density, $j\omega\epsilon\vec{E}$, to be negligible compared to that of the conduction current density, $\sigma\vec{E}$. This is recognized as the good conductor approximation [15], under which electromagnetic field penetration inside the good conductor decays exponentially from the surface to the interior with attenuation constant, α, given by

$$\alpha = \sqrt{\pi f \mu \sigma}. \tag{1.19}$$

For example, for the case of aluminum, with conductivity $\sigma = 4\times 10^7$ S/m, and permeability, $\mu = 4\pi \times 10^{-7}$ H/m, the attenuation constant is 40π Nepers/mm at 100 MHz. Clearly, under such conditions, field penetration inside the conductor is restricted to a thin layer below the conductor surface; hence, the term *skin effect* is used to refer to this situation. As elaborated in [15], the interior of the good conductor can be removed from the domain of solution, with its presence taken into account by imposing the surface impedance condition of 1.18 with

$$Z_s = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}. \tag{1.20}$$

A second example of a surface impedance boundary condition used frequently in practice is associated with the case where either a portion of or the entire enclosing boundary, Γ, is at infinity. Clearly, this is the case of an unbounded domain, characteristic of electromagnetic scattering and radiation problems. One approximate way for truncating the domain Ω such that a finite element solution becomes feasible is through the introduction of a mathematical (non-physical) boundary, placed at a finite distance from the electromagnetic device/structure under analysis, on which an *absorbing* or *radiation* surface boundary condition is imposed. The name reflects the fact that the boundary condition is constructed in a manner such that it supports one-way propagation of electromagnetic waves away from the domain Ω, with minimum spurious (non-physical) reflection.

From the numerous absorbing boundary conditions for time-harmonic electromagnetic BVPs [3], the simplest one is known as the first-order radiation boundary condition, which, under the assumption of lossless media of permittivity ϵ and permeability μ occupying the unbounded region, is given by

$$\vec{E}_t = \hat{t}_1 E_{t_1} + \hat{t}_2 E_{t_2} = \sqrt{\frac{\mu}{\epsilon}}\vec{H} \times \hat{n}. \tag{1.21}$$

In the above expression the unit normal, $\hat{n}$, on the truncation boundary is taken to be pointing in the outward direction, away from the domain Ω, and, as before, the tangential unit vectors $\hat{t}_1$, $\hat{t}_2$, are such that $\hat{t}_1 \times \hat{t}_2 = \hat{n}$.

1.4.3 Uniqueness in the presence of impedance boundaries

In the presence of surface impedance conditions on the boundary enclosing the domain of interest, uniqueness of the solution of the electromagnetic BVP is guaranteed only when the real part of the surface impedance is non-negative. Prior to giving a proof of this result, we note that this requirement is satisfied by both the good-conductor surface impedance given in (1.20) and the impedance coefficient in the first-order radiation boundary condition (1.21).

Without loss of generality, our proof is given for the case where an impedance boundary condition is imposed over the entire surface of the enclosing boundary Γ. Consider two possible solutions, $\left(\vec{E}_a, \vec{H}_a\right)$ and $\left(\vec{E}_b, \vec{H}_b\right)$. Both solutions satisfy Maxwell's equations for the given set of sources. Also, both satisfy the surface impedance boundary condition on Γ. Thus it follows that their difference, $\left(\delta\vec{E}, \delta\vec{H}\right)$, satisfies the source-free equations,

$$\begin{aligned} -\nabla \times \delta\vec{E} &= \hat{z}\delta\vec{H}, \\ \nabla \times \delta\vec{H} &= \hat{y}\delta\vec{E}, \end{aligned} \tag{1.22}$$

where the short-hand notation $\hat{z} = j\omega\mu$, $\hat{y} = \sigma + j\omega\epsilon$ has been adopted. Furthermore, it is on Γ,

$$\delta\vec{E}_t = Z_s \delta\vec{H} \times \hat{n}. \tag{1.23}$$

Taking the inner product of the first of (1.22) with the complex conjugate of $\delta\vec{H}$, and adding to it the inner product of the complex conjugate of the second with $\delta\vec{E}$ yields

$$\nabla \cdot \left(\delta\vec{E} \times \delta\vec{H}^*\right) + \hat{z}|\delta\vec{H}|^2 + \hat{y}^*|\delta\vec{E}|^2 = 0, \tag{1.24}$$

where the superscript $*$ denotes complex conjugation and use was made of the vector identity (A.6) from Appendix A. Integration of the above equation over Ω, followed by the application of the divergence theorem (A.14) yields

$$\oiint_{\Gamma} \left(\delta\vec{E} \times \delta\vec{H}^*\right) \cdot \hat{n} ds + \iiint_{\Omega} \left(\hat{z}|\delta\vec{H}|^2 + \hat{y}^*|\delta\vec{E}|^2\right) dv = 0. \tag{1.25}$$

Use of (1.23) in the integrand of the surface integral allows us to recast the above equation in the following form:

$$\oiint_{\Gamma} Z_s|\delta\vec{H} \times \hat{n}|^2 ds + \iiint_{\Omega} \left(\hat{z}|\delta\vec{H}|^2 + \hat{y}^*|\delta\vec{E}|^2\right) dv = 0. \tag{1.26}$$

Considering the real and imaginary parts of this equation separately, we have

$$\begin{aligned} &\oiint_{\Gamma} \mathrm{Re}(Z_s)|\delta\vec{H} \times \hat{n}|^2 ds + \iiint_{\Omega} \left(\mathrm{Re}(\hat{z})|\delta\vec{H}|^2 + \mathrm{Re}(\hat{y})|\delta\vec{E}|^2\right) dv = 0, \\ &\oiint_{\Gamma} \mathrm{Im}(Z_s)|\delta\vec{H} \times \hat{n}|^2 ds + \iiint_{\Omega} \left(\mathrm{Im}(\hat{z})|\delta\vec{H}|^2 - \mathrm{Im}(\hat{y})|\delta\vec{E}|^2\right) dv = 0. \end{aligned} \tag{1.27}$$

Since for lossy media $\mathrm{Re}(\hat{z})$ and $\mathrm{Re}(\hat{y})$ are non-negative, it is immediately evident that, with $\mathrm{Re}(Z_s) \geq 0$ everywhere on Γ, the equations above are satisfied only if $\delta\vec{E} = \delta\vec{H} = 0$ everywhere inside Ω. Hence, we conclude that uniqueness of the solution to Maxwell's equations in a domain involving surface impedance boundary conditions requires the real part of the surface impedance to be non-negative.

This concludes the discussion of the governing equations for the linear electromagnetic problem. A more in-depth discussion of the electromagnetic system, along with possible simplifications (or reductions) of the governing equations for the special cases of static and quasi-static conditions, can be found in Chapter 3.

1.5 PRESENT AND FUTURE CHALLENGES IN FINITE ELEMENT MODELING

To date, the application of FEM to electromagnetic field modeling has reached significant maturity. Evidence of this maturity is the availability of several commercially available finite element solvers, which are used extensively for the design of state-of-the-art electromagnetic devices of relevance to static, quasi-static and dynamic electromagnetic applications. However, as the community of finite element developers and users continues to grow and as the sophistication of the computational electromagnetics practitioner continues to advance, the expectations for the capabilities of electromagnetic computer-aided design (CAD) tools also continue to increase. In particular, irrespective of the engineering application of interest, ease of use, modeling versatility, robustness, solution accuracy, and computation expediency are all attributes that an increasing pool of users demands from state-of-the-art and future electromagnetic CAD tools.

Improvements in ease of use and modeling versatility are both driven by continuing advances in the sophistication of computer-aided design frameworks, geometric modeling and visualization software, and mesh generation algorithms. However, numerical robustness, solution accuracy and computation expediency, are all intimately related to the numerical method used for the solution of the BVP of interest. More specifically, in the context of finite element-based solution of electromagnetic BVPs, the primary issue that impacts numerical robustness, solution accuracy and computation expediency, is the large disparity in electrical size of geometric features in the computational domain. This is a pressing issue, already encountered in today's electronic/electromagnetic engineering applications, and stands out as one of the primary impeding factors in the routine application of state-of-the-art FEM technology to the solution of challenging electromagnetic BVPs at the system level. As such, it deserves some more discussion.

Finite element methods are best suited for the electromagnetic analysis of highly inhomogeneous structures that exhibit significant geometric complexity. Among the several application areas where such structures are encountered (e.g., scattering by large, multi-body structures, analysis of antenna arrays on platforms, analysis of integrated optical circuits, analysis of integrated electronic systems, system-level electromagnetic interference and electromagnetic compatibility analysis), the analysis of high-speed/high-frequency, mixed-signal integrated circuits (ICs) will be used to illustrate and discuss the impact of the large disparity in size in geometry features on the numerical robustness, accuracy and computation expediency of electrodynamic field solvers.

To begin with, *electrical length* is understood to mean length measured in terms of the wavelength, λ, at the operating frequency at which the electromagnetic analysis is carried out. For example, a thin wire, designed to operate as a half-wavelength dipole at 100 MHz, is (approximately) 1.5 m long. Its electrical length is $\lambda/2$ at 100 MHz, $\lambda/200$ at 1 MHz, 5λ at 1 GHz, and 50λ at 10 GHz.

A first indication of how electrical size impacts model complexity is provided by recalling that numerical solution accuracy requires the spatial sampling of the structure by the finite element mesh to be in the order of ten elements per wavelength [3]. Consequently, while a finite element mesh with ~ 5 elements along the wire length would suffice for an accurate finite element solution at 100 MHz, a mesh with ~ 500 elements along the wire length will be required for an accurate finite element solution at 10 GHz.

Furthermore, if the finite element analysis of the same wire at 1 MHz is required, it should be clear that, even though its electrical size at this frequency is $\lambda/200$, a finite element mesh of element size smaller than the wire length must be used for the accurate resolution of the electromagnetic field distribution in the immediate neighborhood of the wire, especially in view of the singular behavior of the electric field at the wire end points. Actually, one could argue that, instead of an electrodynamic model, a quasi-static model could be used in this case.

This observation suggests that, depending on the electrical size of the structure under analysis, a different finite element-based field solver may be used at different frequencies. Thus a static or quasi-static field solver may be used when the electrical size of the structure is small enough for electromagnetic retardation to have negligible effect on the field distribution and, thus the fields exhibit a predominantly static behavior. On the other hand, an electromagnetic field solver becomes necessary at higher frequencies at which geometric features are in the order of (or larger) than the wavelength and, thus electromagnetic retardation must be taken into account to ensure solution accuracy.

Next, we extend these preliminary observations to the case of electromagnetic field modeling in the context of high-speed/high-frequency IC analysis. We begin with the observation that the complexity of integrated electronics systems continues to escalate rapidly as the semiconductor industry aggressively moves toward reduced device feature size, in support of enhanced functionality integration and higher device speed and system operating bandwidths. The floor-planning of such systems and the design of the interconnection and power distribution networks for their functional blocks cannot be effected without the support of electromagnetic modeling for signal integrity assessment, interference prevention or mitigation, and electrical performance verification.

While the strongly heterogeneous environment of such a system calls for a finite element-based model, the large disparity in geometric feature size makes the use of such a numerical model computationally cumbersome if not prohibitive. More specifically, the several-orders-of magnitude variation in geometric feature size at all levels of circuit integration (e.g., at the chip level geometric features vary in size from sub-microns to centimeters; at the package level this variation is from several microns to centimeters; at the board and system level the variation in size is from the order of millimeters to tens of centimeters of even meters) would require an electromagnetic finite element model involving tens or even hundreds of millions of unknowns.

While the actual size of the model would depend on the portion of the IC under analysis and the frequency at which such analysis is required, in all cases of practical interest the dimension of the finite element matrix is large enough that the direct solution, despite the sparsity of the matrix, is, at least for today's computer technology, computationally overwhelming.

The alternative to a direct solution is an iterative solution. Over the past twenty years major advances have been made in the robustness and sophistication of iterative methods for the solution of sparse linear systems [16]. A prerequisite for a matrix to be suitable for solution through an iterative process is that the matrix is not ill-conditioned. The reason for this is that the convergence of the iterative solution process is critically dependent on the *condition number* of the matrix. We will discuss in detail the condition number of a matrix and its relevance to the convergence of the iterative solution in Chapter 4. For the purposes of this discussion it suffices to say that the condition number of a matrix is roughly given by the magnitude of the ratio of its largest to its smallest eigenvalue. The larger the condition number, the more ill-conditioned the matrix becomes. Severe ill-conditioning leads to numerical instability of the iterative solver, which manifests itself either in terms of stalled convergence or (if convergence is achieved) an inaccurate solution. Clearly, such lack of numerical robustness is unacceptable for a solver aimed for use as a computer-aided analysis and design tool.

The ill-conditioning of matrices resulting from the discretization of a structure exhibiting several-orders-of-magnitude variation in geometric feature size, is easily argued by considering the simple case of a one-dimensional, homogeneous domain of length d, discretized by means of a uniform grid of size h such that $h \ll d$; hence, the number of elements, $N = d/h$, in the domain, is very large. For simplicity, a finite-difference approximation to the one-dimensional Helmholtz operator, $d^2/dx^2 + k^2$, which governs electromagnetic wave propagation in the one-dimensional domain, will be considered. The wavenumber, k, depends on the angular frequency and the speed of light, $v_p = 1/\sqrt{\mu\epsilon}$, in the medium, and is given by

$$k = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}. \tag{1.28}$$

Assuming zero field values at the end points of the domain, the finite difference matrix of the discrete problem is of dimension $(N-1) \times (N-1)$, and is given by

$$\begin{pmatrix} 2-(kh)^2 & -1 & & & \\ -1 & 2-(kh)^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2-(kh)^2 & -1 \\ & & & -1 & 2-(kh)^2 \end{pmatrix}. \tag{1.29}$$

The availability of the elements of the matrix allows us to obtain rough estimates for the eigenvalues by making use of Gershgorin's theorem which states that any eigenvalue of a matrix A of dimension N is located inside one of the closed discs of the complex plane centered at A_{ii}, $i = 1, 2, \ldots, N$, and having radius

$$r_i = \sum_{j=1, j\neq i}^{j=N} |A_{ij}|. \tag{1.30}$$

Thus the following bounds can be obtained for the eigenvalues, s, of (1.29),

$$|s - (2 - (kh)^2)| \leq 2 \Rightarrow -(kh)^2 \leq s \leq 4 - (kh)^2 \tag{1.31}$$

or, written in terms of the number of elements, N, in the grid,

$$-\left(\frac{2\pi d_\lambda}{N}\right)^2 \leq s \leq 4 - \left(\frac{2\pi d_\lambda}{N}\right)^2, \tag{1.32}$$

where $d_\lambda = d/\lambda$ is the electrical length of the domain.

With $d_\lambda \sim \mathcal{O}(1)$ and $N \gg 1$, it is immediately evident from (1.32) that the smallest eigenvalue is of order N^{-2}, while the largest eigenvalue is about 4. Hence, the condition number grows, roughly, as $\mathcal{O}(N^2)$, making evident the convergence difficulties associated with the application of iterative solvers for the solution of finite element matrices of dimension in the order of tens of millions of unknowns.

One class of methods aimed at tackling this hurdle are the so-called multigrid and multi-level methods. It is the primary objective of this book to establish the necessary mathematical infrastructure and develop effective methodologies and computer algorithms for the utilization of multigrid and multi-level methods for robust iterative solution of finite element approximations of electromagnetic BVPs.

However, multigrid methods alone are not capable of tackling the escalating complexity of the electromagnetic BVP associated with the electrical analysis of state-of-the-art and future-generation ICs. To alleviate the complexity, *domain decomposition* strategies can be employed. As the name suggests, this class of methods calls for the decomposition of the structure into sub-domains. Each sub-domain is modeled independently of the others, possibly utilizing different models or even different numerical methods. The generated solutions for all sub-domains must be made consistent with each other through the enforcement of the appropriate field continuity conditions at the sub-domain interfaces.

The process used for stitching together the sub-domain models in a consistent manner can follow one of several alternatives within the framework of iterative methods (e.g., [16]-[19]). In the context of linear system analysis, one can view the sub-domain models as *multi-port networks*, the port variables of which are the shared degrees of freedom between sub-domain models. In this context, one can consider the modeling of each individual sub-domain as a process for generating the *transfer function matrix* for the sub-domain. Once the transfer function matrix has been constructed, general-purpose, network-analysis oriented simulation techniques can be utilized for the analysis of the entire structure.

Recognizing the importance of such capability, a systematic methodology is presented in this book for the expedient, broadband generation of matrix transfer function representations of portions of an electromagnetic structure. Combined with the multigrid/multi-level machinery, streamlined and optimized for electromagnetic BVPs, the proposed algorithms constitute a major step toward the establishment of a robust and computationally efficient finite element-based modeling framework for the analysis of any electromagnetic system that exhibits large disparity in geometry feature size over its spatial extent.

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