PREFACE

The central theme of this book is the development of robust preconditioners for the iterative solution of electromagnetic field boundary value problems (BVPs) discretized by means of finite methods. More precisely, the book provides a detailed presentation of our successful attempts to utilize concepts from multigrid and multilevel methods for the preconditioning of matrices resulting from the approximation of electromagnetic BVPs using finite elements.

Solution by iteration is the reluctant compromise of modern scientific computing. A brief review of the young history of numerical computation offers unequivocal support for Edward Teller's bleak conjecture that "*A state-of-the-art calculation requires 100 hours of CPU time on the state-of-the-art computer, independent of the decade.*" Hidden in this statement is recognition and praise of our innate drive to advance our understanding of the physical world and, eventually, through its predictive modeling, establish our dominance over it. Yet, the deeper our understanding becomes the higher the complexity of the mysteries we need to unravel. And this complexity pushes our computing technology to its current limits.

In our desire to stay one step ahead of this seemingly perpetual complexity barrier, iteration seems to be an invaluable ally. Loosely speaking an iterative process should be interpreted as the attempt to solve a given problem by trial and error. In the context of the theme of this book, the pertinent problem is the solution of the system of linear equations resulting from the discretization of the electromagnetic BVP using finite elements. A guess for the unknown vector is made and the error (residual) is calculated between the forcing vector and the vector resulting from the multiplication of the trial vector by the matrix. A convergent iterative process is one through which the residual is iteratively reduced below some a-priori defined threshold.

Thanks to continuing advances in iterative matrix-solving methods, we are now equipped with solid knowledge about the right ingredients for the construction of an effective iterative solver. Among them, the availability of a good preconditioner stands out. Effective preconditioning of the iteration matrix expedites convergence and can enhance solution accuracy. Multigrid and multilevel methods have been shown to be most effective as preconditioners. Their success stems from the fact that the convergence of standard relaxation techniques can be accelerated by utilizing different spatial samplings for the damping of different components in the decomposition of the error in terms of the eigenvectors of the iteration matrix. In the context of finite methods spatial sampling is most typically effected through the size of the element in the numerical grid or the order of the interpolating polynomial.

The construction of an effective multigrid/multilevel preconditioner requires the understanding of the properties of the eigenvectors and eigenvalues of the iteration matrix. This, in turn, calls for a thorough insight in the way the development of the finite element approximation of the boundary value problem maps the properties of the eigenpairs of the governing mathematical operator onto those of its finite element matrix approximation.

It was primarily due to lack of such insight and understanding that the application of multigrid/multilevel ideas to the development of effective preconditioners for finite element matrix approximations of vector electromagnetic BVPs was slow in coming. However, over the past ten years significant progress has been made toward filling this void. This book attempts to provide the reader with an elucidating description of the path that led us to today's state-of- the-art in multigrid/multilevel preconditioners for finite element-based iterative electromagnetic field solvers.

Chapter 1 begins with a brief overview of the finite element method and its application to the numerical solution of the electromagnetic (BVP). This overview is followed by a discussion of current and future challenges in finite element-based electromagnetic field modeling, which helps motivate the methodologies and algorithms presented in the following chapters.

Chapter 2 presents a complete and systematic development of spaces of functions suitable for the expansion of scalar potential fields, tangentially continuous vector fields, normally continuous vector fields, and scalar charge densities on the triangles and tetrahedra used for the discretization of the computational domain. The corresponding spaces for these four classes of quantities are often referred to as Whitney-0, Whitney-1, Whitney-2, and Whitney-3 forms. They are the natural choices for the representation of the scalar potential, the field intensities ($\vec{E}$ and $\vec{H}$), the flux densities ($\vec{D}$ and $\vec{B}$), and the scalar charge density, respectively. The reason for this is that their development is guided by their physical attributes, as described mathematically by the governing equations and associated boundary conditions. It is shown in Chapter 2 that explicit, hierarchical basis functions of arbitrary order can be developed for all four spaces. In addition, the mathematical relations of the four spaces are presented and discussed. The proposed basis functions can be used, either directly or after a further partial orthogonalization post-processing, to improve the condition number of the finite element matrix.

Chapter 3 is devoted to the development of the finite element formulations of various classes of electromagnetic BVPs. In particular, formulations pertinent to static, quasi-static, and dynamic problems are presented, along with their subsequent reduction into matrix equation statements of their discrete approximation using the Galerkin process. The emphasis of the chapter is on the appropriate choice of basis functions for the expansion of pertinent unknown field quantities and their sources, in a manner such that the requirements for solvability and uniqueness of the solution of the continuous problem are mapped correctly onto the finite element matrix approximation of the BVP. The chapter concludes with an investigation of the source of the so-called low-frequency numerical instability of integral equation-based approximations of electrodynamic problems, in the context of the properties of the basis functions used for the approximation of the unknown electric current and charge densities.

In Chapter 4 an overview of iterative methods and preconditioning techniques for sparse linear systems of equations is provided. The discussion leads to the motivation for multigrid methods and a brief overview of the fundamental steps in the development of a multigrid algorithm. This is followed, in Chapter 5, by the demonstration of the application of the nested multigrid process as an effective preconditioner for enhancing the convergence of the iterative solution of finite element approximations of the scalar wave equation.

Chapter 6 extends the application of multigrid processes as preconditioners for enhancing the convergence of the iterative solution of finite element approximations of vector electromagnetic BVPs. Applications of the proposed nested multigrid preconditioners to the solution of numerous electromagnetic BVPs are used to demonstrate the superior numerical convergence and efficient memory use of these algorithms.

The nested multigrid algorithms discussed in Chapters 5 and 6 can be viewed as an h-adaptive finite element method, since use is made of the lowest order basis functions, while the finite element mesh is refined in a nested fashion. However, p-adaptive techniques, where the order of the basis functions is increased while the finite element mesh is kept unchanged, are oftentimes more effective in reducing the discretization error, especially when used in regions in the computational domain where the field quantities exhibit smooth spatial variation. Thus, in Chapter 7 we present a robust, hierarchical, multilevel preconditioning technique for the fast finite element analysis of electromagnetic BVPs.

As a precursor to the discussion of electromagnetic eigenvalue problems, Chapter 8 briefly introduces various Krylov-based techniques for the solution of large matrix eigenvalue problems. In Chapter 9 the Krylov-based matrix eigenvalue solution machinery is streamlined for application to the solution of matrix eigenvalue problems resulting from the finite element approximation of two-dimensional electromagnetic eigenvalue problems. In particular, the emphasis is on the finite element-based eigenvalue analysis of uniform, inhomogeneous, anisotropic electromagnetic waveguides.

In Chapter 10 an efficient algorithm is developed for the robust solution of sparse matrix eigenvalue problems resulting from the finite element approximation of three-dimensional electromagnetic cavities, as well as unbounded and lossy electromagnetic resonators. The proposed algorithm is based on a field-flux formulation of the finite element approximation of Maxwell's equations. The special relationship between the vector bases used for the expansion of the electric field vector $\vec{E}$ and the magnetic flux density vector $\vec{B}$ is used to reduce the computational complexity of the numerical solution of the finite element matrices resulting from the proposed formulation.

The formulation introduced in Chapter 10 is further exploited in Chapter 11 for the establishment of a systematic methodology and associated computer algorithms for the generation of reduced-order matrix transfer function representations of multi-port electromagnetic systems. The emphasis of the presentation in Chapter 10 is on two topics. First, it is shown that the field-flux formulation of the electromagnetic system leads to a passive discrete, state-space model, which is compatible with the well-established Krylov-subspace model order reduction techniques for reduced-order macromodeling of linear systems. Second, it is shown that through the proper selection of the expansion functions for the finite element approximation of the electric field and the magnetic flux density, a computationally efficient algorithm can be devised for the calculation of the broadband reduced-order matrix transfer function of the electromagnetic system.

Finally, in Chapter 12, we consider the application of finite elements to the modeling of periodic structures. In addition to the discussion of how the finite element formulation of the electromagnetic BVP must be modified to incorporate the results of Floquet's theory, the way the convergence of the iterative solution of the resulting finite element matrix can be enhanced via a multigrid/multilevel preconditioner is presented also.

To facilitate the reader with the computer implementation of the various methodologies presented in this book we have provided their algorithmic description in pseudo-code form. It is hoped that this feature will help expedite the incorporation of these algorithms in existing finite element-based solvers for wave problems.

The numerical examples presented in the applications section of each chapter are used to validate the presented methodologies and demonstrate their computational efficiency and robustness. As such, they tend to involve rather simple electromagnetic problems for which the answer is already known or can be obtained by other means. We hope the reader will appreciate the important role that such simple exercises play in building confidence in the tools we develop to tackle harder problems with yet unknown answers.

In all, this is a book about making iterative methods work well when applied to solving wave problems using finite methods. Our hope is that its readers will benefit from using some of the presented ideas to enrich their bag of tricks with new ones, and advance their wave field solvers to a new plateau of modeling versatility, computational efficiency and numerical robustness.

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